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# Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations

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### ABSTRACT

The paper aims to present the inverse scattering transforms and soliton solutions for nonlocal reverse-time nonlinear Schrödinger equations. The inverse scattering problems are formulated via Riemann–Hilbert problems, and their solutions are determined by the Sokhotski–Plemelj formula, which close the systems for the Jost solutions. Soliton solutions, corresponding to the reflectionless transforms, are generated from zeros and kernel vectors of the Riemann–Hilbert problems with the identity jump matrix.

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## 1. Introduction

Nonlocal integrable nonlinear Schrödinger (NLS) equations were presented and analyzed by symmetry reductions [1], and their inverse scattering transforms were established under zero or nonzero boundary conditions [2,3]. The N-soliton solutions were generated from the Riemann–Hilbert problems associated with the reflectionless transforms [4] and through the Hirota bilinear method [5]. A few multicomponent generalizations [6] and other nonlocal integrable equations [7] were also proposed.

The Riemann–Hilbert technique is among the most powerful methods to explore integrable equations and particularly construct soliton solutions [8]. Various integrable equations have been investigated through analyzing the associated Riemann–Hilbert problems. In this letter, we would like to construct a general class

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of multicomponent nonlocal reverse-time NLS equations and present their inverse scattering transforms and soliton solutions within the formulation of Riemann–Hilbert problems.

The rest of the letter is organized as follows. In Section 2, we make a kind of nonlocal group reductions to generate nonlocal reverse-time NLS equations. In Section 3, we formulate the inverse scattering transforms via Riemann–Hilbert problems. In Section 4, we construct soliton solutions from the reflectionless transforms. In the final section, we give a few concluding remarks.

#### 2. Nonlocal reverse-time NLS equations

Let  $n \in \mathbb{N}$  be arbitrary;  $\lambda$ , a spectral parameter; and u, a 2*n*-dimensional potential

$$u = u(x,t) = (p,q^T)^T, \ p = (p_1, p_2, \dots, p_n), \ q = (q_1, q_2, \dots, q_n)^T.$$
 (2.1)

The multicomponent AKNS matrix spectral problems read (see, e.g., [9]):

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \ -i\phi_t = V\phi = V(u,\lambda)\phi, \ U = \lambda\Lambda + P, \ V = \lambda^2\Omega + Q.$$
(2.2)

The involved four matrices are defined by  $\Lambda = \text{diag}(\alpha_1, \alpha_2 I_n), \ \Omega = \text{diag}(\beta_1, \beta_2 I_n)$ , and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \ Q = \frac{\beta}{\alpha} \lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix},$$
(2.3)

where  $\alpha = \alpha_1 - \alpha_2$  and  $\beta = \beta_1 - \beta_2$ . Obviously, if  $p_j = q_j = 0$ ,  $2 \le j \le n$ , the spectral problems in (2.2) reduce to the original AKNS ones [10]. The compatibility condition of (2.2),

$$U_t - V_x + i[U, V] = 0, (2.4)$$

presents the multicomponent standard NLS equations

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \ q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq).$$

$$(2.5)$$

Let us take a specific kind of nonlocal group reductions for the spectral matrix:

$$U^{T}(x, -t, -\lambda) = -CU(x, t, \lambda)C^{-1}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \ \Sigma^{T} = \Sigma,$$
(2.6)

which equivalently leads to that

$$P^{T}(x, -t) = -CP(x, t)C^{-1}, (2.7)$$

where  $\Sigma$  is a constant invertible symmetric matrix. It follows from this constraint (2.7) that

$$q(x,t) = -\Sigma^{-1} p^T(x,-t),$$
(2.8)

$$V^{T}(x, -t, -\lambda) = CV(x, t, \lambda)C^{-1}, \ Q^{T}(x, -t, -\lambda) = CQ(x, t, \lambda)C^{-1}.$$
(2.9)

These also imply that the nonlocal reductions in (2.7) agree with (2.4). Thus, the standard NLS equations (2.5) reduce to the following multicomponent nonlocal reverse-time NSL equations

$$ip_t(x,t) = \frac{\beta}{\alpha^2} [p_{xx}(x,t) - 2p(x,t)\Sigma^{-1}p^T(x,-t)p(x,t)].$$
(2.10)

If p(x,t) is a solution to (2.10), so are  $p^*(x,-t)$  and p(-x,t). Hence, (2.10) is PT-symmetric. When n = 1, we can obtain two scalar examples:  $ip_t(x,t) = p_{xx}(x,t) + 2\sigma p^2(x,t)p(x,-t)$ ,  $\sigma = \pm 1$ .

#### 3. Inverse scattering transforms

**Distribution of eigenvalues:** Assume that all the potentials rapidly vanish when  $x \to \pm \infty$  or  $t \to \pm \infty$ . Upon setting  $\check{P} = iP$  and  $\check{Q} = iQ$ , the equivalent pair of matrix spectral problems to (2.2) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \dot{P}\psi, \tag{3.1}$$

$$\psi_t = i\lambda^2[\Omega, \psi] + \check{Q}\psi. \tag{3.2}$$

Applying a generalized Liouville's formula, we can have det  $\psi = 1$ , since  $\operatorname{tr}(\check{P}) = \operatorname{tr}(\check{Q}) = 0$ . The adjoint equation of the *x*-part of (2.2) and the adjoint equation of (3.1) are given by

$$i\tilde{\phi}_x = \tilde{\phi}U,\tag{3.3}$$

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P. \tag{3.4}$$

Let  $\psi(\lambda)$  be a matrix eigenfunction of the spatial spectral problem (3.1) associated with an eigenvalue  $\lambda$ . Then,  $C\psi^{-1}(x,t,\lambda)$  is a matrix adjoint eigenfunction associated with the same eigenvalue  $\lambda$ . Moreover, under the nonlocal group reductions in (2.7), we see that

$$\tilde{\psi}(x,t,\lambda) = \psi^T(x,-t,-\lambda)C \tag{3.5}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue  $\lambda$ , i.e.,  $\psi^T(x, -t, -\lambda)C$  solves the adjoint spectral problem (3.4).

Therefore, upon observing the asymptotic properties for  $\psi$  at infinity of  $\lambda$ , the uniqueness of solutions tells that

$$\psi^{T}(x, -t, -\lambda) = C\psi^{-1}(x, t, \lambda)C^{-1},$$
(3.6)

if  $\psi \to I_{n+1}$ ,  $x, t \to \pm \infty$ . It follows that if  $\lambda$  is an eigenvalue of (3.1) (or (3.4)), then  $-\lambda$  will be another eigenvalue of (3.1) (or (3.4)), and the property (3.6) holds.

**Riemann–Hilbert problems:** Let us now formulate a class of associated Riemann–Hilbert problems with the variable x. In order to facilitate the expression, we also assume that

$$\alpha = \alpha_1 - \alpha_2 < 0, \ \beta = \beta_1 - \beta_2 < 0. \tag{3.7}$$

In the scattering problem, we first introduce the two matrix eigenfunctions  $\psi^{\pm}(x,\lambda)$  of (3.1) with the asymptotic conditions

$$\psi^{\pm} \to I_{n+1}, \text{ when } x \to \pm \infty,$$
(3.8)

respectively. It follows from det  $\psi = 1$  that det  $\psi^{\pm} = 1$  for all  $x \in \mathbb{R}$ . Since both

$$\phi^{\pm} = \psi^{\pm} E, \ E = \mathrm{e}^{i\lambda\Lambda x},\tag{3.9}$$

solve (2.2), they must be linearly dependent, and consequently, one has

$$\psi^{-}E = \psi^{+}ES(\lambda), \ S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}, \ \lambda \in \mathbb{R},$$
(3.10)

where  $S(\lambda)$  is called the scattering matrix. Note that det  $S(\lambda) = 1$  due to det  $\psi^{\pm} = 1$ .

First, to determine the two Jost solutions,  $P^+$  and  $P^-$ , which are analytic in  $\mathbb{C}^+$  and  $\mathbb{C}^-$  (the upper and lower half-planes) and continuous in  $\overline{\mathbb{C}}^+$  and  $\overline{\mathbb{C}}^-$  (the closed upper and lower half-planes), respectively, we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \dots, \psi_{n+1}^{\pm}), \tag{3.11}$$

where  $\psi_i^{\pm}$  denotes the *j*th column of  $\phi^{\pm}$ . Then we can take the matrix Jost solution  $P^+$  as

$$P^{+} = P^{+}(x,\lambda) = (\psi_{1}^{-},\psi_{2}^{+},\dots,\psi_{n+1}^{+}) = \psi^{-}H_{1} + \psi^{+}H_{2}, \qquad (3.12)$$

which is analytic in  $\lambda \in \mathbb{C}^+$  and continuous in  $\lambda \in \overline{\mathbb{C}}^+$ , and

$$P^+(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \overline{\mathbb{C}}^+ \to \infty.$$
 (3.13)

Here we denote  $H_1 = \text{diag}(1, \underbrace{0, \dots, 0}_{n})$  and  $H_2 = \text{diag}(0, \underbrace{1, \dots, 1}_{n})$ . Second, to determine the other Jost solution  $P^-$ , we construct the analytic counterpart of  $P^+$  in the lower half-plane  $\mathbb{C}^-$  from the adjoint matrix spectral problems. Note that the inverse matrices  $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$  and  $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$  solve those two adjoint equations, respectively. Similarly, upon expressing  $\tilde{\psi}^{\pm}$  by

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \cdots, \tilde{\psi}^{\pm,n+1})^T,$$
(3.14)

where  $\tilde{\psi}^{\pm,j}$  denotes the *j*th row of  $\tilde{\psi}^{\pm}$ , we can take the Jost solution  $P^-$  as the adjoint matrix solution of (3.4), i.e.,

$$P^{-} = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \cdots, \tilde{\psi}^{+,n+1})^{T} = H_{1}\tilde{\psi}^{-} + H_{2}\tilde{\psi}^{+} = H_{1}(\psi^{-})^{-1} + H_{2}(\psi^{+})^{-1},$$
(3.15)

which is analytic for  $\lambda \in \mathbb{C}^-$  and continuous for  $\lambda \in \overline{\mathbb{C}}^-$ , and that

$$P^{-}(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \overline{\mathbb{C}}^{-} \to \infty.$$
 (3.16)

Based on the two matrix Jost solutions, we introduce

$$G^{+}(x,\lambda) = P^{+}(x,\lambda), \ \lambda \in \overline{\mathbb{C}}^{+}, \ (G^{-})^{-1}(x,\lambda) = P^{-}(x,\lambda), \ \lambda \in \overline{\mathbb{C}}^{-},$$
(3.17)

and formulate the required matrix Riemann-Hilbert problems on the real line for the nonlocal reverse-time NLS equations (2.10):

$$G^{+}(x,\lambda) = G^{-}(x,\lambda)G_{0}(x,\lambda), \ \lambda \in \mathbb{R},$$
(3.18)

where by (3.10), the jump matrix  $G_0$  is given by

$$G_{0}(x,\lambda) = E(H_{1} + H_{2}S(\lambda))(H_{1} + S^{-1}(\lambda)H_{2})E^{-1}$$

$$= E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ s_{n+1,1} & 0 & \cdots & 0 & 1 \end{bmatrix} E^{-1}, \ S^{-1}(\lambda) = (\hat{s}_{jl})_{(n+1)\times(n+1)}.$$
(3.19)

The asymptotic properties of  $P^{\pm}$  provide the canonical normalization conditions

$$G^{\pm}(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \mathbb{C}_0^{\pm} \to \infty,$$
(3.20)

for the presented Riemann-Hilbert problems. The jump matrix  $G_0$  carries basic scattering data from the scattering matrix  $S(\lambda)$  and has the property  $G_0^T(x, -t, -\lambda) = CG_0(x, t, \lambda)C^{-1}$ .

**Evolution of the scattering data:** To complete the direct scattering transforms, we compute the derivative of (3.10) with time t and use the vanishing conditions of the potentials at infinity of t. It follows that the scattering matrix S needs to satisfy an evolution law:

$$S_t = i\lambda^2 [\Omega, S]. \tag{3.21}$$

This yields the time evolution of the time-dependent scattering coefficients:

$$\begin{cases} s_{12} = s_{12}(0,\lambda) e^{i\beta\lambda^2 t}, \ s_{13} = s_{13}(0,\lambda) e^{i\beta\lambda^2 t}, \ \cdots, \ s_{1,n+1} = s_{1,n+1}(0,\lambda) e^{i\beta\lambda^2 t}, \\ s_{21} = s_{21}(0,\lambda) e^{-i\beta\lambda^2 t}, \ s_{31} = s_{31}(0,\lambda) e^{-i\beta\lambda^2 t}, \ \cdots, \ s_{n+1,1} = s_{n+1,1}(0,\lambda) e^{-i\beta\lambda^2 t}, \end{cases}$$

but all other scattering coefficients are independent of the time variable t.

**Closing the systems:** To close the systems for the matrix Jost solutions, we transform the Riemann–Hilbert problems in (3.18) into

$$\begin{cases} G^+ - G^- = G^- v, \ v = G_0 - I_{n+1}, \ \text{on } \mathbb{R}, \\ G \to I_{n+1} \ \text{as } \lambda \to \infty. \end{cases}$$
(3.22)

Suppose that G has simple poles off  $\mathbb{R}$ :  $\{\lambda_j\}_{j=1}^R$ , where R is an arbitrary integer. Define

$$\tilde{G}(\lambda) = G(\lambda) - \sum_{j=1}^{R} \frac{G_j}{\lambda - \lambda_j}, \ \lambda \in \mathbb{C} \setminus \mathbb{R},$$
(3.23)

where  $G_j$  is the residue of G at  $\lambda = \lambda_j$ , i.e.,  $G_j = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j) G(\lambda)$ . Then, we have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v, \text{ on } \mathbb{R}, \\ \tilde{G} \to I_{n+1} \text{ as } \lambda \to \infty. \end{cases}$$
(3.24)

By the Sokhotski–Plemelj formula [11], we obtain the solutions

$$\tilde{G}(\lambda) = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G^- v(\xi)}{\xi - \lambda} d\xi.$$
(3.25)

Now, taking the limit as  $\lambda \to \lambda_l$  yields the closed systems

$$I_{n+1} + \sum_{j \neq l}^{R} \frac{G_j}{\lambda_l - \lambda_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda_l} d\xi = 0, \ 1 \le l \le R,$$
(3.26)

where the integral takes the Cauchy principle value. These determine solutions to the Riemann–Hilbert problems and thus the Jost solutions.

**Recovery of the potential:** To recover the potential matrix P from the Jost solutions, we make an asymptotic expansion

$$P^{+}(x,t,\lambda) = I_{n+1} + \frac{1}{\lambda} P_{1}^{+}(x,t) + O(\frac{1}{\lambda^{2}}), \ \lambda \to \infty.$$
(3.27)

Plugging this into the matrix spectral problem (3.1) and comparing O(1) terms yields

$$P = \lim_{\lambda \to \infty} \lambda[P^+(\lambda), \Lambda] = -[\Lambda, P_1^+].$$
(3.28)

One needs to check an involution property for  $P_1^+$ :

$$(P_1^+)^T(x, -t) = CP_1^+(x, t)C^{-1}.$$
(3.29)

Then, solutions to the nonlocal reverse-time NLS equations (2.10) are determined by

$$p_j = -\alpha(P_1^+)_{1,j+1}, \ P_1^+ = ((P_1^+)_{jl})_{(n+1)\times(n+1)}, \ 1 \le j \le n.$$
(3.30)

This completes the inverse scattering procedure from the scattering matrix  $S(\lambda)$  through the jump matrix  $G_0(\lambda)$  to the potential matrix P. The final potential P defines solutions to the nonlocal reverse-time NLS equations (2.10).

#### 4. Soliton solutions

Let  $N \in \mathbb{N}$  be arbitrary. Assume that det  $P^+(x,\lambda) = s_{11}$  has N zeros  $\{\lambda_k \in \mathbb{C}^+, 1 \leq k \leq N\}$ , and det  $P^-(x,\lambda) = \hat{s}_{11}$  has N zeros  $\{\hat{\lambda}_k \in \mathbb{C}^-, 1 \leq k \leq N\}$ . We also assume that all these zeros are simple. Then, each ker  $P^+(\lambda_k)$  contains only a single basis column vector, denoted by  $v_k$ ; and each ker  $P^-(\hat{\lambda}_k)$ , a single basis row vector, denoted by  $\hat{v}_k$ .

Soliton solutions are associated with  $G_0 = I_{n+1}$  in the Riemann-Hilbert problems, achieved under zero reflection coefficients:  $s_{i1} = \hat{s}_{1i} = 0$ ,  $2 \le i \le n+1$ . Solutions to this kind of special Riemann-Hilbert problems can be formulated as follows (see, e.g., [8,12]):

$$P^{+}(\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \ P^{-}(\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k},$$
(4.1)

where  $M = (m_{kl})_{N \times N}$  is a square matrix whose entries are determined by

$$m_{kl} = \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, \ 1 \le k, l \le N.$$

$$(4.2)$$

To satisfy the involution property (3.29), we take zeros of det  $P^+(\lambda)$  and det  $P^-(\lambda)$  as follows:

$$\lambda_k \in \mathbb{C}^+, \ \hat{\lambda}_k = -\lambda_k \in \mathbb{C}^-, \ 1 \le k \le N.$$

$$(4.3)$$

Then, ker  $P^+(\lambda_k)$  and ker  $P^-(\lambda_k)$ ,  $1 \le k \le N$ , are determined by

$$v_k(x,t) = v_k(x,t,\lambda_k) = e^{i\lambda_k\Lambda x + i\lambda_k^2\Omega t} w_k, \ 1 \le k \le N,$$
(4.4)

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^T(x,-t,-\lambda_k)C = w_k^T e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t}C, \ 1 \le k \le N,$$
(4.5)

respectively. Here  $w_k$ ,  $1 \le k \le N$ , are arbitrary constant column vectors.

Finally, we see that the solutions to the specific Riemann–Hilbert problems, determined by (4.1) and (4.2), satisfy  $(P^+)^T(x, -t, -\lambda) = CP^-(x, t, \lambda)C^{-1}$ , which implies that  $P_1^+$  satisfies (3.29). Thus, N-soliton solutions to the nonlocal reverse-time NLS equations (2.10) read

$$p_j = \alpha \sum_{k,l=1}^N v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1}, \ 1 \le j \le n,$$
(4.6)

where M is defined by (4.2), and  $v_k = (v_{k,1}, v_{k,2}, \ldots, v_{k,n+1})^T$  and  $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \ldots, \hat{v}_{k,n+1})$  are determined by (4.4) and (4.5), respectively. When n = N = 1, (4.6) presents

$$p(x,t) = \frac{2(\alpha_1 - \alpha_2)\lambda_1 w_{1,1} w_{1,2} \Sigma e^{i\lambda_1(\alpha_1 + \alpha_2)x + i\lambda_1^2(\beta_1 - \beta_2)t}}{w_{1,1}^2 e^{2i\lambda_1\alpha_1 x} + w_{1,2}^2 \Sigma e^{2i\lambda_1\alpha_2 x}},$$
(4.7)

where  $w_1 = (w_{1,1}, w_{1,2})^T \in \mathbb{C}^2$  and  $\Sigma, \lambda_1 \in \mathbb{C}$  are arbitrary. This solution is analytic on the real line at any time when the denominator is nonzero but has time-independent singularity otherwise.

We remark that it would also be interesting to see how to construct different kinds of exact solutions in nonlinear dispersive waves, for example, lump solutions [13], Rossby wave solutions [14], solitonless solutions [15] and algebro-geometric solutions [16], through the Riemann–Hilbert technique.

#### **CRediT** authorship contribution statement

Wen-Xiu Ma: Conceptualization, Methodology, Validation, Investigation, Data curation, Writing.

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