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A soliton hierarchy associated with $so(3,\mathbb{R})$

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ABSTRACT

We generate a hierarchy of soliton equations from zero curvature equations associated with the real Lie algebra $so(3,\mathbb{R})$ and show that each equation in the resulting hierarchy has a bi-Hamiltonian structure and thus integrable in the Liouville sense.

1. Introduction

Zero curvature equations on simple Lie algebras lay the foundation for constructing soliton hierarchies of evolution equations. Among the well-known soliton hierarchies are the KdV hierarchy, the AKNS hierarchy and the Kaup–Newell hierarchy [1]. The trace identity is used to construct Hamiltonian structures of soliton equations [2].

More generally, zero curvature equations associated with non-simple Lie algebras generate integrable couplings [3,4], and the variational identity provides the basic tool for finding the corresponding Hamiltonian structures, which often lead to hereditary recursion operators [5,6].

Let us first recall the standard procedure for constructing soliton hierarchies (see, e.g., [2,7]). Usually, one starts from a spectral problem

$$\phi_x = U\phi, \quad U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{1.1}$$

where λ is the spectral parameter and $\tilde{\mathfrak{g}}$ is a matrix loop algebra associated with a given matrix Lie algebra $\mathfrak{g}$, often being simple. We take a solution of the form

$$W = W(u,\lambda) = \sum_{i\geqslant 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} \in \mathfrak{g},\ i \geqslant 0, \tag{1.2}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{1.3}$$

Then, introduce the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi, \quad m \geqslant 0, \tag{1.4}$$

with the Lax matrices being defined by

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geqslant 0, \tag{1.5}$$

where P_+ denotes the polynomial part of P in λ. The introduction of the modification terms Δ_m aims to guarantee that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geqslant 0, \tag{1.6}$$

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generate a hierarchy of soliton equations with Hamiltonian structures:

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geqslant 0. \tag{1.7}$$

The Hamiltonian functionals $\mathcal{H}_m$ in (1.7) are generally presented by using the trace identity [2,7]:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right), \quad \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{tr}(W^2)|, \tag{1.8}$$

or the variational identity [6]:

$$\frac{\delta}{\delta u}\int \left\langle\frac{\partial U}{\partial \lambda}, W\right\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle\frac{\partial U}{\partial u}, W\right\rangle, \quad \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{1.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the loop algebra $\tilde{\mathfrak{g}}$ [6]. If $\tilde{\mathfrak{g}}$ is non-semisimple, the bilinear form $\langle\cdot,\cdot\rangle$ in the variational identity (1.9) must not be of the Killing type. If $\langle A, A\rangle$ is positive for every non-zero matrix $A \in \tilde{\mathfrak{g}}$, then the Lie algebra $(\tilde{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ becomes quadratic.

In this paper, we would like to form a spectral problem, based on the real Lie algebra $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$, to generate a hierarchy of soliton equations from the associated zero curvature equations. The Hamiltonian structures will be furnished by applying the corresponding trace identity and so all equations in the resulting soliton hierarchy are Liouville integrable, i.e., they possess infinitely many commuting symmetries and conservation laws. Two concrete examples will be computed, together with their bi-Hamiltonian structures.

2. Matrix loop algebra and soliton equations

2.1. Matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$

Let us consider the simple real Lie algebra of the special orthogonal group, $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$, the Lie algebra of 3×3 trace-free, skew-symmetric real matrices. It has a basis

$$e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

with which, the structure equations of $\mathrm{so}(3,\mathbb{R})$ are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{2.2}$$

The derived algebra $[\mathrm{so}(3,\mathbb{R}), \mathrm{so}(3,\mathbb{R})]$ is $\mathrm{so}(3,\mathbb{R})$ itself. This is one of the only two three-dimensional real Lie algebras with a three-dimensional derived algebra. The other one is $\mathrm{sl}(2,\mathbb{R})$, which has been frequently used to analyze soliton equations.

The matrix loop algebra we will adopt in our construction is

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3,\mathbb{R}) = \{M \in \mathrm{so}(3,\mathbb{R}) | \text{entries of } M \ - \ \text{Laurent series in } \lambda\}, \tag{2.3}$$

where λ is a spectral parameter. The algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ contains matrices of the form $\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$ with arbitrary integers m, n, l. This matrix loop algebra gives a basis for us to generate soliton equations. Based on the perturbation-type loop algebras of $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we can also consider the problem of integrable couplings such as bi-integrable couplings and tri-integrable couplings.

2.2. Soliton hierarchy

We are going to construct a soliton hierarchy from the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$. Let us introduce a spectral problem

$$\phi_x = U\phi, \quad U = U(u,\lambda) = \lambda e_1 + p e_2 + q e_3 = \begin{bmatrix} 0 & q & \lambda \\ -q & 0 & -p \\ -\lambda & p & 0 \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}. \tag{2.4}$$

Then, the stationary zero curvature equation

$$W_x = [U, W], \tag{2.5}$$

becomes

$$\begin{cases} a_x = pc - qb, \\ b_x = -\lambda c + qa, \\ c_x = \lambda b - pa, \end{cases} \tag{2.6}$$

if we assume that W is of the form

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & c & a \\ -c & 0 & -b \\ -a & b & 0 \end{bmatrix} = \sum_{i\geqslant 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} = \begin{bmatrix} 0 & c_i & a_i \\ -c_i & 0 & -b_i \\ -a_i & b_i & 0 \end{bmatrix}, \quad i \geqslant 0. \tag{2.7}$$

Upon taking the initial values

$$a_0 = -1, \quad b_0 = c_0 = 0,$$

the system (2.6) equivalently yields

$$\begin{cases} b_{i+1} = c_{i,x} + pa_i, \\ c_{i+1} = -b_{i,x} + qa_i, \qquad i \geqslant 0. \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \tag{2.8}$$

We impose the integration conditions

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geqslant 1,$$

to determine the sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$ uniquely. Therefore, the first few sets can be computed as follows:

$$b_1 = -p, \quad c_1 = -q, \quad a_1 = 0;$$
$$b_2 = -q_x, \quad c_2 = p_x, \quad a_2 = \tfrac{1}{2}(p^2 + q^2);$$
$$b_3 = p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2, \quad c_3 = q_{xx} + \tfrac{1}{2}p^2 q + \tfrac{1}{2}q^3, \quad a_3 = pq_x - p_x q;$$
$$b_4 = q_{xxx} + \tfrac{3}{2}p^2 q_x + \tfrac{3}{2}q^2 q_x, \quad c_4 = -p_{xxx} - \tfrac{3}{2}p^2 p_x - \tfrac{3}{2}p_x q^2,$$
$$a_4 = -pp_{xx} - qq_{xx} + \tfrac{1}{2}p_x^2 + \tfrac{1}{2}q_x^2 - \tfrac{3}{8}p^4 - \tfrac{3}{4}p^2q^2 - \tfrac{3}{8}q^4;$$
$$b_5 = -p_{xxxx} - \tfrac{5}{2}p^2 p_{xx} - \tfrac{5}{2}pp_x^2 - \tfrac{3}{2}p_{xx}q^2 - 3p_x q q_x - pqq_{xx} + \tfrac{1}{2}pq_x^2 - \tfrac{3}{8}p^5 - \tfrac{3}{4}p^3q^2 - \tfrac{3}{8}pq^4,$$
$$c_5 = -q_{xxxx} - \tfrac{5}{2}q^2 q_{xx} - \tfrac{5}{2}qq_x^2 - \tfrac{3}{2}p^2 q_{xx} - 3pp_x q_x - pp_{xx}q + \tfrac{1}{2}p_x^2 q - \tfrac{3}{8}q^5 - \tfrac{3}{4}p^2q^3 - \tfrac{3}{8}p^4 q,$$
$$a_5 = p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_x q_{xx} + \tfrac{3}{2}p^2 p_x q - \tfrac{3}{2}pq^2 q_x - \tfrac{3}{2}p^3 q_x + \tfrac{3}{2}p_x q^3.$$

Now, taking

$$V^{[m]} = (\lambda^m W)_+ = \sum_{i=0}^{m} W_{0,i}\lambda^{m-i}, \quad m \geqslant 0, \tag{2.9}$$

the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geqslant 0, \tag{2.10}$$

generate a hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} q \\ -p \end{bmatrix}, \quad m \geqslant 0, \tag{2.11}$$

where the operator Φ can be determined by the recursion relation (2.8):

$$\Phi = \begin{bmatrix} q\partial^{-1}p & \partial + q\partial^{-1}q \\ -\partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix}, \quad \partial = \frac{\partial}{\partial x}. \tag{2.12}$$

2.3. Hamiltonian structures

We use the trace identity [2] (or more generally the variational identity [6]):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right), \tag{2.13}$$

where the constant γ is determined as in (1.8).

It is direct to compute that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so we obtain

$$\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -2a, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -2b, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = -2c.$$

Now the corresponding trace identity (1.8) becomes

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} b \\ c \end{bmatrix}.$$

Balancing coefficients of each power of λ in the above equality gives rise to

$$\frac{\delta}{\delta u}\int a_{m+1}\,dx = (\gamma - m)\begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geqslant 0.$$

The case of $m = 1$ tells $\gamma = 0$, and thus we have

$$\frac{\delta}{\delta u}\int \left(-\frac{a_{m+2}}{m+1}\right)dx = \begin{bmatrix} b_{m+1} \\ c_{m+1} \end{bmatrix}, \quad m \geqslant 0. \tag{2.14}$$

Consequently, we obtain the following Hamiltonian structures for the soliton hierarchy (2.11):

$$u_{t_m} = K_m = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geqslant 0, \tag{2.15}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.16}$$

and the Hamiltonian functionals

$$\mathcal{H}_m = \int \left(-\frac{a_{m+2}}{m+1}\right)dx, \quad m \geqslant 0. \tag{2.17}$$

These correspond to infinitely many conservation laws of each system in the soliton hierarchy (2.11), which can also be computed directly by computer algebra systems (see, e.g., [8]).

2.4. Liouville integrability

It is direct but lengthy to show by computer algebra systems that the operator Φ defined by (2.12) is hereditary (see [9] for definition), i.e., it satisfies

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K \tag{2.18}$$

for all vector fields K and S; and that J and

$$M = \Phi J = \begin{bmatrix} \partial + q\partial^{-1}q & -q\partial^{-1}p \\ -p\partial^{-1}q & \partial + p\partial^{-1}p \end{bmatrix}, \tag{2.19}$$

where J is defined by (2.16), constitute a Hamiltonian pair (see [10] for details), i.e., any linear combination N of J and M satisfies

$$\int K^T N'(u)[NS]T\,dx + \mathrm{cycle}(K,S,T) = 0 \tag{2.20}$$

for all vector fields K, S and T.

The hereditary property (2.18) is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi, \tag{2.21}$$

where K is an arbitrary vector field. The Lie derivative $L_K\Phi$ above is defined by

$$(L_K\Phi)S = \Phi[K,S] - [K,\Phi S],$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields. It is known that an autonomous operator $\Phi = \Phi(u, u_x, \ldots)$ is a recursion operator of an evolution equation $u_t = K$ iff the operator Φ needs to satisfy

$$L_K\Phi = 0. \tag{2.22}$$

Obviously, for the operator Φ defined by (2.12), we have

$$L_{K_0}\Phi = 0, \quad K_0 = \begin{bmatrix} q \\ -p \end{bmatrix}$$

and thus

$$L_{K_m}\Phi = L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi = 0, \quad m \geqslant 1, \tag{2.23}$$

where the K_m's are given by (2.11). This implies that the operator Φ defined by (2.12) is a common hereditary recursion operator for the soliton hierarchy (2.11). We point out that there are also direct symbolic algorithms for computing recursion operators of nonlinear partial differential equations by computer algebra systems (see, e.g., [11]).

Now, the soliton hierarchy (2.11) is bi-Hamiltonian (see, e.g., [10,12]):

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geqslant 1, \tag{2.24}$$

where J, M and $\mathcal{H}_m$ are defined by (2.16), (2.19) and (2.17) respectively, and so, the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k, l \geqslant 0 \tag{2.25}$$

and the Abelian algebras of conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J\frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \quad k, l \geqslant 0 \tag{2.26}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T M\frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \quad k, l \geqslant 0. \tag{2.27}$$

2.5. Two nonlinear examples

The first two nonlinear integrable systems in the hierarchy (2.11) read

$$u_{t_2} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_2} = K_2 = \begin{bmatrix} -q_{xx} - \frac{1}{2}p^2q - \frac{1}{2}q^3 \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \end{bmatrix} \tag{2.28}$$

and

$$u_{t_3} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_3} = K_3 = \begin{bmatrix} p_{xxx} + \frac{3}{2}p^2p_x + \frac{3}{2}p_xq^2 \\ q_{xxx} + \frac{3}{2}p^2q_x + \frac{3}{2}q^2q_x \end{bmatrix}. \tag{2.29}$$

They possess the following bi-Hamiltonian structures

$$u_{t_2} = K_2 = J\frac{\delta \mathcal{H}_2}{\delta u} = M\frac{\delta \mathcal{H}_1}{\delta u} \tag{2.30}$$

and

$$u_{t_3} = K_3 = J\frac{\delta \mathcal{H}_3}{\delta u} = M\frac{\delta \mathcal{H}_2}{\delta u}, \tag{2.31}$$

where the Hamiltonian pair $\{J, M\}$ is defined by (2.16) and (2.19), and the Hamiltonian functionals, $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, are given by

$$\begin{cases} \mathcal{H}_1 = -\frac{1}{2}\int (pq_x - p_xq)\, dx, \\ \mathcal{H}_2 = -\frac{1}{3}\int \left(-pp_{xx} - qq_{xx} + \frac{1}{2}p_x^2 + \frac{1}{2}q_x^2 - \frac{3}{8}p^4 - \frac{3}{4}p^2q^2 - \frac{3}{8}q^4\right) dx, \\ \mathcal{H}_3 = -\frac{1}{4}\int \left(p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_xq_{xx} + \frac{3}{2}p^2p_xq - \frac{3}{2}pq^2q_x - \frac{3}{2}p^3q_x + \frac{3}{2}p_xq^3\right) dx. \end{cases} \tag{2.32}$$

3. Conclusions and remarks

We presented a hierarchy of soliton equations from zero curvature equations associated with the real loop algebra $\widetilde{\text{so}}(3, \mathbb{R})$. The Liouville integrability of the resulting soliton equations has been shown by establishing a bi-Hamiltonian formulation.

The real Lie algebra of the special orthogonal group, so$(3, \mathbb{R})$, is not isomorphic to the real Lie algebra sl$(2, \mathbb{R})$ over the real field $\mathbb{R}$, and thus the soliton hierarchy (2.11) is not gauge equivalent to the AKNS soliton hierarchy [13] over the real field $\mathbb{R}$. But the two Lie algebras are isomorphic to each other over the complex filed $\mathbb{C}$, which implies that the hierarchy (2.11) is gauge equivalent to the AKNS hierarchy over the complex filed $\mathbb{C}$.

There is a growing interest in soliton equations generating from zero curvature equations associated with non-semisimple Lie algebras. Bi-integrable couplings and tri-integrable couplings provide us with insightful thoughts about general structures of integrable systems with multi-components [14]. Multi-integrable couplings generate diverse recursion operators in block matrix form. The mathematical structures behind integrable couplings are rich and interesting [6,14]. It is known that Hamiltonian structures exist for the perturbation equations [15,16], but some non-semisimple matrix Lie algebras generating spectral matrices do not possess any non-degenerate bilinear forms required in the variational identities [17,18]. It is an open question to us how to guarantee the existence of Hamiltonian structures for bi- or tri-integrable couplings, based on zero curvature equations.

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