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Nonlinear continuous integrable Hamiltonian couplings

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ABSTRACT

Based on a kind of special non-semisimple Lie algebras, a scheme is presented for constructing nonlinear continuous integrable couplings. Variational identities over the corresponding loop algebras are used to furnish Hamiltonian structures for the resulting continuous integrable couplings. The application of the scheme is illustrated by an example of nonlinear continuous integrable Hamiltonian couplings of the AKNS hierarchy of soliton equations.

1. Introduction

Integrable couplings [1,2] are associated with non-semisimple Lie algebras [3,4] and variational identities provide tools to generate Hamiltonian structures of integrable couplings, in both continuous and discrete cases [5,6]. Most of the presented integrable couplings are linear with respect to the supplementary variables (see, e.g., [1,7–13]). For example, the spectral matrices of the form

$$\overline{U} = \begin{bmatrix} U(u) & U'[v] \\ 0 & U(u) \end{bmatrix},$$

where the sub-spectral matrix U is associated with a given integrable equation $u_t = K(u)$ and U' denotes its Gateaux derivative, lead to integrable couplings of the perturbation type. In such resulting integrable couplings, the equation for the supplementary variable v is linear with respect to v. If the second equation of an integrable coupling

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v), \end{cases}$$

defines a nonlinear equation for v, then the whole system is called a nonlinear integrable coupling. The two variables u and v above can be either scalars or vectors.

Linear integrable couplings contain extensions of symmetry equations [1,7] and are important in classifying integrable equations, but definitely, nonlinear ones have much richer structures. There are a few systematical ways to construct linear integrable couplings, starting from the perturbed spectral matrices [2,7], defined as before, and the amended spectral matrices [8,10]:

$$\overline{U} = \begin{bmatrix} U(u) & U_a(v) \\ 0 & 0 \end{bmatrix},$$

where U_a may not be a square matrix. However, there is no feasible way which allows us to construct nonlinear integrable couplings.

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In this paper, we focus on integrable partial differential equations and are concerned with a natural question: How can one construct nonlinear continuous integrable couplings? We would like to show that the following choice of spectral matrices:

$$\overline{U} = \begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u) + U_a(v) \end{bmatrix},$$

can engender nonlinear continuous integrable couplings. The set of all matrices above is closed under the matrix product, and so it constitutes a matrix Lie algebra, which is non-semisimple. The variational identities (see [5,14,15]) over this kind of Lie algebras can be used to furnish Hamiltonian structures for the corresponding continuous integrable couplings. We will illustrate such an idea of generating nonlinear continuous integrable Hamiltonian couplings by the AKNS hierarchy of soliton equations. All these will amend the existing theories of linear integrable couplings. The resulting theory also provides an approach to another interesting mathematical question: How can one generate an infinite hierarchy of vector fields which commute with each other?

2. Constructing nonlinear integrable couplings

2.1. General scheme

Assume that an integrable equation

$$u_t = K(u) \tag{2.1}$$

has a zero curvature representation

$$U_t - V_x + [U, V] = 0, \tag{2.2}$$

where two square Lax matrices U and V usually belong to semisimple matrix Lie algebras (see, e.g., [16]). Let us then introduce an enlarged spectral matrix

$$\overline{U} = \overline{U}(\bar{u}) = \begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u) + U_a(v) \end{bmatrix}, \tag{2.3}$$

where $\bar{u}$ consists of u and v. Now, an enlarged zero curvature equation

$$\overline{U}_t - \overline{V}_x + [\overline{U}, \overline{V}] = 0 \tag{2.4}$$

with

$$\overline{V} = \overline{V}(\bar{u}) = \begin{bmatrix} V(u) & V_a(\bar{u}) \\ 0 & V(u) + V_a(\bar{u}) \end{bmatrix} \tag{2.5}$$

gives rise to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{a,t} - V_{a,x} + [U, V_a] + [U_a, V] + [U_a, V_a] = 0. \end{cases} \tag{2.6}$$

This is an integrable coupling of Eq. (2.1), due to Eq. (2.2), and it is normally a nonlinear integrable coupling because the commutator $[U_a, V_a]$ can generate nonlinear terms.

We further take a solution $\overline{W}$ to the enlarged stationary zero curvature equation

$$\overline{W}_x = [\overline{U}, \overline{W}]. \tag{2.7}$$

Then, the associated variational identity written in vector form [5]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \overline{W}, \overline{U}_\lambda \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \overline{W}, \overline{U}_{\bar{u}} \rangle, \tag{2.8}$$

with the constant γ being determined by

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle \overline{W}, \overline{W} \rangle|, \tag{2.9}$$

can be used to furnish Hamiltonian structures for those integrable couplings described above. In the variational identity (2.8), the expression on the left-hand side is the vector of variational derivatives with respect to all elements of $\bar{u}$, $\overline{U}_\lambda$ denotes the partial derivative of $\overline{U}$ with respect to λ, $\overline{U}_{\bar{u}}$ denotes the vector of partial derivatives of $\overline{U}$ with respect to all elements of $\bar{u}$, and $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form over the Lie algebra consisting of square matrices of the form (2.3) (see [5,14,15] for general discussion). In what follows, we will make an application to the AKNS hierarchy to shed light on this generating scheme.

2.2. An application

2.2.1. AKNS hierarchy

Let us consider the spectral matrix

$$U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \lambda\text{-spectral parameter}, \tag{2.10}$$

which generates the AKNS hierarchy of soliton equations [17] (see also [18]). There are other integrable equations associated with $gl(2)$ (see, e.g., [19]). Upon setting

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geqslant 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{2.11}$$

the stationary zero curvature equation $W_x = [U, W]$ gives

$$b_{i+1} = -\frac{1}{2} b_{i,x} - p a_i, \quad c_{i+1} = \frac{1}{2} c_{i,x} - q a_i, \quad a_{i+1,x} = p c_{i+1} - q b_{i+1}, \quad i \geqslant 0. \tag{2.12}$$

Choosing the initial data as

$$a_0 = -1, \quad b_0 = c_0 = 0 \tag{2.13}$$

and assuming $a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0$, $i \geqslant 1$ (equivalently selecting constants of integration to be zero), the recursion relation (2.12) uniquely defines all differential polynomial functions a_i, b_i and c_i, $i \geqslant 1$. The first few sets are listed as follows:

$$b_1 = p, \quad c_1 = q, \quad a_1 = 0;$$

$$b_2 = -\frac{1}{2} p_x, \quad c_2 = \frac{1}{2} q_x, \quad a_2 = \frac{1}{2} pq;$$

$$b_3 = \frac{1}{4} p_{xx} - \frac{1}{2} p^2 q, \quad c_3 = \frac{1}{4} q_{xx} - \frac{1}{2} p q^2, \quad a_3 = \frac{1}{4}(p q_x - p_x q);$$

$$b_4 = -\frac{1}{8} p_{xxx} + \frac{3}{4} p_x p q, \quad c_4 = \frac{1}{8} q_{xxx} - \frac{3}{4} p q_x q,$$

$$a_4 = \frac{1}{8} p_{xx} q - \frac{1}{8} p_x q_x + \frac{1}{8} p q_{xx} - \frac{3}{8} p^2 q^2.$$

The zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad V^{[m]} = (\lambda^m W)_+, \quad m \geqslant 0, \tag{2.14}$$

where $(P)_+$ denotes the polynomial part of P, generate the AKNS hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = (L^\dagger)^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geqslant 0 \tag{2.15}$$

with the Hamiltonian operator J, the hereditary recursion operator $L^\dagger$ and the Hamiltonian functions:

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} \frac{1}{2}\partial - q\partial^{-1} p & q \partial^{-1} q \\ -p\partial^{-1} p & -\frac{1}{2}\partial + p\partial^{-1} q \end{bmatrix}, \quad \mathcal{H}_m = \int \frac{2a_{m+2}}{m+1} dx, \tag{2.16}$$

where $L^\dagger$ is the adjoint operator of L, $\partial = \frac{\partial}{\partial x}$ and $m \geqslant 0$.

2.2.2. Integrable couplings

Let us now start from an enlarged spectral matrix

$$\overline{U} = \overline{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ 0 & U + U_a \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}, \tag{2.17}$$

where U is defined as in (2.10) and the supplementary matrix U_a is taken as

$$U_a = U_a(v) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \quad v = \begin{bmatrix} r \\ s \end{bmatrix}. \tag{2.18}$$

For the enlarged stationary zero curvature equation (2.7), we look for a solution

$$\overline{W}=\begin{bmatrix} W & W_a \\ 0 & W+W_a \end{bmatrix},\quad W_a=W_a(\bar{u},\lambda)=\begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \tag{2.19}$$

where W, defined by (2.11), solves $W_x=[U,W]$. Then, Eq. (2.7) requires

$$W_{a,x}=[U,W_a]+[U_a,W]+[U_a,W_a], \tag{2.20}$$

which equivalently generates

$$\begin{cases} e_x=pg-qf+rc-sb+rg-sf, \\ f_x=-2\lambda f-2pe-2ra-2re, \\ g_x=2qe+2\lambda g+2sa+2se. \end{cases} \tag{2.21}$$

Trying a formal series solution

$$e=\sum_{i=0}^{\infty}e_i\lambda^{-i},\quad f=\sum_{i=0}^{\infty}f_i\lambda^{-i},\quad g=\sum_{i=0}^{\infty}g_i\lambda^{-i}, \tag{2.22}$$

we obtain

$$\begin{cases} e_{i+1,x}=pg_{i+1}-qf_{i+1}+rc_{i+1}-sb_{i+1}+rg_{i+1}-sf_{i+1}, \\ f_{i+1}=-\frac{1}{2}f_{i,x}-pe_i-ra_i-re_i, \\ g_{i+1}=\frac{1}{2}g_{i,x}-qe_i-sa_i-se_i, \end{cases} \tag{2.23}$$

where $i\geqslant 0$. We choose the initial data as

$$e_0=-1,\quad f_0=g_0=0 \tag{2.24}$$

and assume that $e_i|_{\bar{u}=0}=f_i|_{\bar{u}=0}=g_i|_{\bar{u}=0}=0,\ i\geqslant 1$. Then the recursion relation (2.23) uniquely determines the sequence of sets of e_i, f_i and g_i, $i\geqslant 1$. The first few sets are computed as follows:

$$f_1=p+2r,\quad g_1=q+2s,\quad e_1=0;$$

$$f_2=-\frac{1}{2}p_x-r_x,\quad g_2=\frac{1}{2}q_x+s_x,\quad e_2=\frac{1}{2}pq+ps+qr+rs;$$

$$f_3=\frac{1}{4}p_{xx}+\frac{1}{2}r_{xx}-\frac{1}{2}p^2q-p^2s-2pqr-2prs-qr^2-r^2s,$$

$$g_3=\frac{1}{4}q_{xx}+\frac{1}{2}s_{xx}-\frac{1}{2}pq^2-ps^2-2pqs-2qrs-q^2r-rs^2,$$

$$e_3=-\frac{1}{4}p_xq+\frac{1}{4}pq_x-\frac{1}{2}p_xs+\frac{1}{2}ps_x+\frac{1}{2}q_xr-\frac{1}{2}qr_x-\frac{1}{2}r_xs+\frac{1}{2}rs_x;$$

$$f_4=-\frac{1}{8}p_{xxx}-\frac{1}{4}r_{xxx}+\frac{3}{4}p_xpq+\frac{3}{2}p_xps+\frac{3}{2}p_xqr+\frac{3}{2}p_xrs+\frac{3}{2}pqr_x+\frac{3}{2}pr_xs+\frac{3}{2}qr_xr+\frac{3}{2}r_xrs,$$

$$g_4=\frac{1}{8}q_{xxx}+\frac{1}{4}s_{xxx}-\frac{3}{4}pq_xq-\frac{3}{2}pq_xs-\frac{3}{2}pqs_x-\frac{3}{2}ps_xs-\frac{3}{2}q_xqr-\frac{3}{2}q_xrs-\frac{3}{2}qrs_x-\frac{3}{2}rs_xs,$$

$$e_4=\frac{1}{8}p_{xx}q-\frac{1}{8}p_xq_x+\frac{1}{8}pq_{xx}+\frac{1}{4}p_{xx}s-\frac{1}{4}p_xs_x+\frac{1}{4}ps_{xx}+\frac{1}{4}qr_{xx}-\frac{1}{4}q_xr_x+\frac{1}{4}q_{xx}r+\frac{1}{4}r_{xx}s-\frac{1}{4}r_xs_x+\frac{1}{4}rs_{xx}-\frac{3}{8}p^2q^2-\frac{3}{4}p^2s^2$$
$$-\frac{3}{4}q^2r^2-\frac{3}{2}r^2s^2-\frac{3}{2}p^2qs-\frac{3}{2}pq^2r-\frac{3}{2}qr^2s-\frac{3}{2}prs^2-3pqrs.$$

For each integer $m\geqslant 0$, take

$$\overline{V}^{[m]}=\left(\lambda^m\overline{W}\right)_+=\begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix},\quad V_a^{[m]}=(\lambda^mW_a)_+ \tag{2.25}$$

and then, the enlarged zero curvature equation

$$\overline{U}_{t_m}-\left(\overline{V}^{[m]}\right)_x+\left[\overline{U},\overline{V}^{[m]}\right]=0, \tag{2.26}$$

yields

$$U_{a,t_m} - (\overline{V}_a^{[m]})_x + \left[U, V_a^{[m]}\right] + [U_a, V^{[m]}] + [U_a, V_a^{[m]}] = 0,$$

together with the mth AKNS system in (2.15). This tells

$$v_{t_m} = S_m = S_m(u,v) = \begin{bmatrix} -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \quad m \geqslant 0, \tag{2.27}$$

where $v=(r,s)^T$ defined as in (2.18). This way, the hierarchy of enlarged zero curvature equations presents a hierarchy of integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \overline{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_m(u,v) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \quad m \geqslant 0 \tag{2.28}$$

for the AKNS hierarchy (2.15). Except the first two, all integrable couplings above are nonlinear, since the supplementary systems (2.27) of r and s with $m \geqslant 2$ are nonlinear. The third one reads

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2q, \quad q_{t_2} = \frac{1}{2}q_{xx} - pq^2, \\ r_{t_2} = -\frac{1}{2}p_{xx} - r_{xx} + p^2q + 2p^2s + 4pqr + 4prs + 2qr^2 + 2r^2s, \\ s_{t_2} = \frac{1}{2}q_{xx} + s_{xx} - pq^2 - 2ps^2 - 4pqs - 4qrs - 2q^2r - 2rs^2. \end{cases} \tag{2.29}$$

Therefore, the systems in (2.28) with $m \geqslant 2$ provide a hierarchy of nonlinear integrable couplings for the AKNS hierarchy of soliton equations.

2.2.3. Hamiltonian structures

To construct Hamiltonian structures of the obtained integrable couplings, we need to compute non-degenerate, symmetric and ad-invariant bilinear forms on the Lie algebra considered before:

$$\bar{g} = \left\{ \begin{bmatrix} A & B \\ 0 & A+B \end{bmatrix} \middle| A, B \in \mathrm{sl}(2) \right\}. \tag{2.30}$$

For brevity, let us transform the Lie algebra $\bar{g}$ into a vector form through the mapping

$$\delta : \bar{g} \to \mathbb{R}^6, \quad A \mapsto (a_1, a_2, a_3, a_4, a_5, a_6)^T, \quad A = \begin{bmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 + a_4 & a_2 + a_5 \\ 0 & 0 & a_3 + a_6 & -a_1 - a_4 \end{bmatrix} \in \bar{g}. \tag{2.31}$$

This mapping δ induces a Lie algebraic structure on $\mathbb{R}^6$, isomorphic to the matrix Lie algebra $\bar{g}$. The corresponding commutator $[\cdot,\cdot]$ on $\mathbb{R}^6$ is given by

$$[a,b]^T = a^T R(b), \quad a = (a_1, \ldots, a_6)^T, \quad b = (b_1, \ldots, b_6)^T \in \mathbb{R}^6, \tag{2.32}$$

where

$$R(b) = \begin{bmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 + 2b_5 & -2b_3 - 2b_6 \\ 0 & 0 & 0 & b_3 + b_6 & -2b_1 - 2b_4 & 0 \\ 0 & 0 & 0 & -b_2 - b_5 & 0 & 2b_1 + 2b_4 \end{bmatrix}.$$

A bilinear form on $\mathbb{R}^6$ is determined as follows

$$\langle a, b \rangle = a^T F b, \tag{2.33}$$

where F is a constant matrix. The symmetric property $\langle a,b \rangle = \langle b,a \rangle$ and the ad-invariance property

$$\langle a, [b,c] \rangle = \langle [a,b], c \rangle \tag{2.34}$$

requires that $F^T = F$ and

$$(R(b)F)^T = -R(b)F \quad \text{for all} \quad b \in \mathbb{R}^6. \tag{2.35}$$

This matrix equation gives a system of linear equations on the elements of F. Solving the resulting system, we obtain

$$F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & \eta_2 \\ 0 & \eta_2 & 0 & 0 & \eta_2 & 0 \end{bmatrix}, \tag{2.36}$$

where η_1 and η_2 are arbitrary constants.

Therefore, a bilinear form on the underlying Lie algebra $\bar{g}$ is defined by

$$\begin{aligned}\langle A, B\rangle_{\bar{g}} &= \langle \delta^{-1}(A), \delta^{-1}(B)\rangle_{\mathbb{R}^6} = (a_1, \ldots, a_6)F(b_1, \ldots, b_6)^T \\ &= \eta_1(2a_1b_1 + a_2b_3 + a_3b_2) + \eta_2[2a_1b_4 + a_2b_6 + a_3b_5 + 2a_4(b_1 + b_4) + a_5(b_3 + b_6) + a_6(b_2 + b_5)],\end{aligned} \tag{2.37}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 + a_4 & a_2 + a_5 \\ 0 & 0 & a_3 + a_6 & -a_1 - a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_4 & b_5 \\ b_3 & -b_1 & b_6 & -b_4 \\ 0 & 0 & b_1 + b_4 & b_2 + b_5 \\ 0 & 0 & b_3 + b_6 & -b_1 - b_4 \end{bmatrix}.$$

This bilinear form (2.37) is symmetric and ad-invariant:

$$\langle A, B\rangle = \langle B, A\rangle, \quad \langle A, [B, C]\rangle = \langle [A, B], C\rangle, \quad A, B, C \in \bar{g} \tag{2.38}$$

and it is non-degenerate if and only if

$$(\eta_1 - \eta_2)\eta_2 \neq 0. \tag{2.39}$$

Now, based on (2.37), we can easily compute that

$$\langle \overline{W}, \overline{U}_\lambda\rangle = -2\eta_1 a - 2\eta_2 e,$$

$$\langle \overline{W}, \overline{U}_{\bar{u}}\rangle = (\eta_1 c + \eta_2 g, \eta_1 b + \eta_2 f, \eta_2(c + g), \eta_2(b + f))^T,$$

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \overline{W}, \overline{W}\rangle| = 0,$$

where $\overline{W}$ is defined by (2.19). Thus the variational identity (2.8) with $\gamma = 0$ yields

$$\frac{\delta}{\delta\bar{u}}\int \frac{2\eta_1 a_{m+1} + 2\eta_2 e_{m+1}}{m}dx = (\eta_1 c_m + \eta_2 g_m, \eta_1 b_m + \eta_2 f_m, \eta_2(c_m + g_m), \eta_2(b_m + f_m))^T, \quad m \geqslant 1. \tag{2.40}$$

It follows now that the AKNS integrable couplings in (2.28) possess the following Hamiltonian structures:

$$\bar{u}_{t_m} = \overline{K}_m(\bar{u}) = \bar{J}\frac{\delta\overline{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geqslant 0, \tag{2.41}$$

where the Hamiltonian operator is given by

$$\bar{J} = \frac{1}{\eta_2 - \eta_1}\begin{bmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & \frac{2\eta_1}{\eta_2} \\ 2 & 0 & -\frac{2\eta_1}{\eta_2} & 0 \end{bmatrix} \tag{2.42}$$

and the Hamiltonian functionals are given by

$$\overline{\mathcal{H}}_m = \int \frac{2\eta_1 a_{m+2} + 2\eta_2 e_{m+2}}{m+1}dx, \quad m \geqslant 0. \tag{2.43}$$

It is direct to see a recursion relation

$$\overline{L}\frac{\delta\overline{\mathcal{H}}_m}{\delta\bar{u}} = \frac{\delta\overline{\mathcal{H}}_{m+1}}{\delta\bar{u}}, \quad m \geqslant 0$$

with

$$\overline{L} = \begin{bmatrix} L & L_a \\ 0 & L + L_a \end{bmatrix}, \tag{2.44}$$

where L is given by (2.16) and

$$L_a = \begin{bmatrix} -(q+s)\partial^{-1}r - s\partial^{-1}p & (q+s)\partial^{-1}s + s\partial^{-1}q \\ -(p+r)\partial^{-1}r - r\partial^{-1}p & (p+r)\partial^{-1}s + r\partial^{-1}q \end{bmatrix}. \tag{2.45}$$

Moreover, we have $\bar{J}\,\bar{L} = \bar{L}^\dagger \bar{J}$, where $\bar{L}^\dagger$ is the adjoint operator of $\bar{L}$. Based on the Tu scheme [20], it follows then that all integrable couplings in (2.28) commute with each other and so do all conserved functionals in (2.43). It is also not difficult to show that $\bar{J}$ and $\bar{J}\,\bar{L}$ is a Hamiltonian pair [21] and $\bar{L}^\dagger$ is a hereditary recursion operator [22] for the hierarchy of integrable couplings (2.28), which similarly implies that the hierarchy (2.28) commutes.

3. Concluding remarks

We proposed a kind of specific Lie algebras which allows us to generate nonlinear continuous integrable couplings and applied the variational identities on the suggested Lie algebras to the construction of Hamiltonian structures of the resulting continuous integrable couplings. An application to the AKNS hierarchy gave a hierarchy of nonlinear continuous integrable Hamiltonian couplings. The obtained results supplement the existing theories on the perturbation equations and linear integrable couplings [1,5,8].

It is clear that using the block type matrix algebras, we will be able to generate larger classes of integrable couplings. Combining the considered form of spectral matrices with the other forms in the literature (see, e.g., [10,23]) will lead to more diverse integrable couplings. The presented integrable couplings can also possess other integrable properties such as Hirota bilinear forms [24] and τ-symmetry algebras [25]. All such analyses will enrich multi-component integrable equations (see, e.g., [15,26]) and help understand them better to work towards classification of integrable equations based on loop algebras.

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