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Integrable couplings and matrix loop algebras

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Abstract. We will discuss how to generate integrable couplings from zero curvature equations associated with matrix spectral problems. The key elements are matrix loop algebras consisting of block matrices with blocks of the same size or different sizes. Hamiltonian structures are furnished by applying the variational identity defined over semi-direct sums of Lie algebras. Illustrative examples include integrable couplings of the AKNS hierarchy by using the irreducible representations V_2 and V_3 of the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$.

Keywords: Zero curvature equation, Recursion operator, Variational identity, Hamiltonian structure, Symmetry, Conservation law
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1. INTRODUCTION

A system of differential equations or differential-difference equations is said to be integrable by quadratures if its solutions to well-possed Cauchy problems can be obtained after a finite number of steps involving algebraic operations and integration of given functions [1]. Given a Hamiltonian system of ordinary differential equations (ODEs) on a symplectic space, when is it integrable by quadratures? A first answer is the Liouville theorem using action-angle variables [2, 3]. The modern formulation of the Liouville theorem is called the Liouville-Arnold theorem. It states that for a Hamiltonian system of ODEs associated with a Hamiltonian function H defined on a symplectic manifold (M^{2N},ω^2):

$$p_t = \{p,H\} = \omega^2(IdH,Idp),\ q_t = \{q,H\} = \omega^2(IdH,Idq), \tag{1}$$

the following conditions guarantee integrability by quadratures:

(i) Existence of N integrals of motion $\{F_i(p,q)\}_{i=1}^N$;

(ii) $F_1,F_2,\cdots,F_N$ commute with each other under the associated Poisson bracket: $\{F_i,F_j\}=0,\ 1\le i,j\le N$;

(iii) $F_1,F_2,\cdots,F_N$ are functionally independent on the level surface $\{F_i=a_i|1\le i\le N\}$ containing the initial data.

Those three conditions are nowadays called the Liouville conditions, and Liouville integrable systems of ODEs mean Hamiltonian systems possessing the above Liouville conditions.

For partial differential equations (PDEs) or differential-difference equations (DDEs), no similar Liouville conditions are found to guarantee their integrability by quadratures. There are, however, many different but related integrable criteria for PDEs or DDEs widely adopted in the soliton community [4], a few of which are listed as follows (see, e.g., [5, 6, 7] for details):

(i) Existence of Lax pair and solvable by the inverse scattering transform (S-integrable case);

(ii) Transforming into linear equations (C-integrable case);

(iii) Passing the Painlevé test and the singularity confinement;

(iv) Existence of Darboux transformations;

(v) Algebraic complete integrability;

(vi) Existence of three-soliton solutions;

(vii) Existence of infinitely many symmetries;

(viii) Existence of infinitely many conservation laws;

(ix) Bi-Hamiltonian formulations (which often imply the existence of infinitely many symmetries and conservation laws); etc.

In what follows, we say a system of PDEs or DDEs is integrable if it has infinitely many symmetries which are functionally independent on the jet spaces, and Liouville integrable if it is Hamiltonian and possesses infinitely many conserved densities being functionally independent on the jet spaces. The Liouville integrability is stronger since Hamiltonian systems generate symmetries from conserved densities associated with conservation laws. The

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fundamental question in the integrability theory of PDEs and DDEs is how to explore Hamiltonian structures and infinitely many commuting functionally independent conserved densities.

A continuous (or discrete) Hamiltonian system of PDEs or DDEs reads

$$u_t = K(u) = K(u,u_x,\cdots) \ \ [\text{or } K(u,Eu,E^{-1}u,\cdots)] = J\frac{\delta \mathscr{H}}{\delta u}, \tag{2}$$

where u denotes a column vector of dependent variables, $\frac{\delta}{\delta u}$ is the variational derivative with respect to u, J is a Hamiltonian operator and $\mathscr{H} = \int H\,dx$ [or $\mathscr{H} = \sum_{n\in\mathbb{Z}} H$] is called a Hamiltonian functional associated with a Hamiltonian function H. For a Hamiltonian system, there exists a relationship chain from conserved densities through adjoint symmetries to symmetries:

$$I \mapsto \frac{\delta}{\delta u}\int I\,dx \mapsto J\frac{\delta}{\delta u}\int I\,dx \ \ [\text{or } I \mapsto \frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} I \mapsto J\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} I] \tag{3}$$

and a Lie algebra homomorphism $J\frac{\delta}{\delta u}$:

$$J\frac{\delta}{\delta u}\{\mathscr{I}_1,\mathscr{I}_2\} = [J\frac{\delta \mathscr{I}_1}{\delta u}, J\frac{\delta \mathscr{I}_2}{\delta u}], \tag{4}$$

with the Poisson bracket of functionals being defined by

$$\{\mathscr{I}_1,\mathscr{I}_2\} = \int \Big(\frac{\delta \mathscr{I}_1}{\delta u}\Big)^T J\frac{\delta \mathscr{I}_2}{\delta u}\,dx \ \ [\text{or } \{\mathscr{I}_1,\mathscr{I}_2\} = \sum_{n\in\mathbb{Z}} \Big(\frac{\delta \mathscr{I}_1}{\delta u}\Big)^T J\frac{\delta \mathscr{I}_2}{\delta u}], \tag{5}$$

and the Lie bracket of vector fields being defined by

$$[K,S] = K'(u)[S] - S'(u)[K]. \tag{6}$$

Here and in what follows, $P'(u)[v]$ denotes the Gateaux derivative of an object $P = P(u,u_x,\cdots)$ with respect to u along a direction v:

$$P'(u)[v] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P(u+\varepsilon v, u_x+\varepsilon v_x,\cdots). \tag{7}$$

We say that two conserved densities of a Hamiltonian system (2) commute, if their corresponding functionals commute under the Poisson bracket (5) (see, e.g., [8] for details).

Lax proposed an equivalent operator formulation, now called the Lax pair approach, for studying the KdV equation [9]. A Lax pair formulation is generally equivalent to a zero curvature equation formulation [10] and it also provides a novel type of separability [11]. An integrable system of PDEs or DDEs is said to possess a zero curvature equation representation, if it is generated from a continuous zero curvature equation

$$U_t - V_x + [U,V] = 0 \tag{8}$$

or a discrete zero curvature equation

$$U_t + UV - (EV)U = 0, \tag{9}$$

where the two square matrices, U and V, are called a Lax pair, often belonging to simple matrix loop algebras [12]. One of our primary tasks in the field of integrable systems is how to construct, from the continuous and discrete zero curvature equation formulations, a Hamiltonian structure with a recursion operator [13] or a bi-Hamiltonian structure [14]:

$$u_t = K(u) = J\frac{\delta \mathscr{H}_1}{\delta u} = M\frac{\delta \mathscr{H}_2}{\delta u}, \tag{10}$$

which naturally engenders infinitely many commuting conserved densities, for a given system of evolution equations. The existence of bi-Hamiltonian structures often guarantee recursion operators [14]-[17], which can be hereditary [18].

A Lax pair of $U = U(u,\lambda)$ and $V = V(u,\lambda)$ involving a spectral parameter λ, from a matrix Lie algebra with a free parameter, called a matrix loop algebra, is a starting point in formulating integrable systems which possess bi-Hamiltonian structures. Simple matrix loop algebras yield typical integrable systems (see, e.g., [19, 20] for examples associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$). Semisimple matrix loop algebras lead to separated integrable systems, i.e., collections of typical integrable systems, each of which corresponds to a simple matrix loop algebra. Non-semisimple matrix loop algebras generate integrable couplings [21, 22].

Specially, an integrable coupling of an integrable system (2) is a triangular integrable system of the following form [23, 24]:

$$\begin{cases} u_t = K(u), \\ v_t = S(u,v). \end{cases} \tag{11}$$

If S is nonlinear with respect to the second sub-vector v of dependent variables, the integrable coupling (11) is called nonlinear. An example of integrable couplings is the first-order perturbation system [23]:

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \end{cases} \tag{12}$$

where K' is the Gateaux derivative as defined earlier. A bi-integrable coupling of a given integrable system (2) is an enlarged triangular integrable system of the following form [25]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2); \end{cases} \tag{13}$$

and similarly, by a tri-integrable coupling, we mean an enlarged triangular integrable system of the following form [26, 27]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2), \\ u_{3,t} = S_3(u,u_1,u_2,u_3). \end{cases} \tag{14}$$

Non-semisimple Lie algebras possess semi-direct sum decompositions [28]:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c,\ \mathfrak{g} \text{ - semisimple},\ \mathfrak{g}_c \text{ - solvable}, \tag{15}$$

and so, semi-direct sums of Lie algebras lay a foundation for constructing integrable couplings. The notion of semi-direct sums

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c \tag{16}$$

means that the two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}_c$ satisfy

$$[\mathfrak{g},\mathfrak{g}_c] \subseteq \mathfrak{g}_c, \tag{17}$$

where $[\mathfrak{g},\mathfrak{g}_c] = \{[A,B]\,|\,A \in \mathfrak{g}, B \in \mathfrak{g}_c\}$, with $[\cdot,\cdot]$ denoting the Lie bracket of $\bar{\mathfrak{g}}$. Obviously, $\mathfrak{g}_c$ is an ideal Lie subalgebra of $\bar{\mathfrak{g}}$. The subscript c indicates a contribution to the construction of coupling systems. We also require the closure property between $\mathfrak{g}$ and $\mathfrak{g}_c$ under the matrix multiplication

$$\mathfrak{g}\mathfrak{g}_c, \mathfrak{g}_c\mathfrak{g} \subseteq \mathfrak{g}_c, \tag{18}$$

where $\mathfrak{g}_1\mathfrak{g}_2 = \{AB\,|\,A \in \mathfrak{g}_1, B \in \mathfrak{g}_2\}$, while we use discrete zero curvature equations over semi-direct sums of matrix Lie algebras to generate discrete integrable couplings [22].

Integrable couplings provide inspiration and insights into classifying integrable systems with multi-components [23, 24]. Continuous and discrete zero curvature equation formulations over non-semisimple matrix loop algebras are the basis for generating integrable couplings, and the associated (classical and super) variational identities offer fundamental tools to build their Hamiltonian structures [29]-[32].

In this report, we would like to explore non-semisimple loop algebras consisting of block matrices to generate integrable couplings. Hamiltonian structures of the resulting coupling systems will be furnished through the variational identity on general matrix loop algebras. In particular, we will present matrix loop algebras consisting of block matrices by using two irreducible representations V_2 and V_3 of the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and apply them to the construction of integrable couplings, based on enlarged zero curvature equations. As illustrative examples, two applications will be made for the AKNS soliton hierarchy, and Hamiltonian structures will be generated for one of the two resulting hierarchies of integrable couplings. The presented matrix loop algebras provide key elements for constructing integrable couplings.

2. SOLITON HIERARCHIES

2.1. General procedure

Let us recall a standard procedure for generating soliton hierarchies from Lax pairs (see [33, 34, 35] for details).

Assume that $\mathfrak{g}$ is a given matrix Lie algebra, which can be either semisimple or non-semisimple, and its corresponding matrix loop algebra is formulated as follows:

$$\tilde{\mathfrak{g}} = \{M \in \mathfrak{g} \,|\, \text{entries of } M \text{ - Laurent series of } \lambda\}. \tag{19}$$

The beginning is to introduce a spatial spectral problem

$$\phi_x = U\phi,\ U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{20}$$

where u is a column vector of dependent variables and λ is a spectral parameter. We then take a solution of the form

$$W = W(u,\lambda) = \sum_{i\ge 0} W_{0,i}\lambda^{-i},\ W_{0,i} \in \mathfrak{g},\ i \ge 0, \tag{21}$$

to the stationary zero curvature equation

$$W_x = [U,W] \ \ [\text{or } (EW)U - UW = 0]. \tag{22}$$

Further, introduce the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi,\ m \ge 0, \tag{23}$$

with the Lax matrices being chosen as

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}},\ m \ge 0, \tag{24}$$

where P_+ denotes the polynomial part of P in λ. The introduction of the modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ aims to guarantee that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U,V^{[m]}] = 0 \ \ [\text{or } U_{t_m} - (EV^{[m]})U + UV^{[m]} = 0],\ m \ge 0, \tag{25}$$

engender a hierarchy of soliton equations with Hamiltonian structures:

$$u_{t_m} = K_m(u) = J\frac{\delta \mathscr{H}_m}{\delta u},\ m \ge 0. \tag{26}$$

Those Hamiltonian functionals are usually generated by applying the trace identity:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(\frac{\partial U}{\partial \lambda}W)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(\frac{\partial U}{\partial u}W),\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{27}$$

or

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \mathrm{tr}(\frac{\partial U}{\partial \lambda}W) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(\frac{\partial U}{\partial u}W),\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}((WU)^2)|, \tag{28}$$

and more generally, the variational identity:

$$\frac{\delta}{\delta u}\int \langle\frac{\partial U}{\partial \lambda},W\rangle\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle\frac{\partial U}{\partial u},W\rangle,\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W,W\rangle|, \tag{29}$$

or

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \langle\frac{\partial U}{\partial \lambda},W\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle\frac{\partial U}{\partial u},W\rangle,\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle WU,WU\rangle|, \tag{30}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the matrix loop algebra $\tilde{\mathfrak{g}}$ [31, 32].

The non-degenerate property means that if $\langle A,B\rangle = 0$ for all $A \in \tilde{\mathfrak{g}}$, then $B=0$; and if $\langle A,B\rangle = 0$ for all $B \in \tilde{\mathfrak{g}}$, then $A=0$. The symmetric and ad-invariant properties are

$$\langle A,B\rangle = \langle B,A\rangle$$

and

$$\langle A,[B,C]\rangle = \langle [A,B],C\rangle,$$

where $[\cdot,\cdot]$ denotes the Lie bracket of $\tilde{\mathfrak{g}}$, respectively.

When the underlying Lie algebra $\tilde{\mathfrak{g}}$ is semisimple, the trace identity will most likely work. When $\tilde{\mathfrak{g}}$ is non-semisimple, we have to use the variational identity since the trace identity is not valid, and the bilinear form $\langle\cdot,\cdot\rangle$ in the variational identity (29) must not be of the Killing type. If $\langle A,A\rangle$ are positive for non-zero matrices $A \in \tilde{\mathfrak{g}}$, then the Lie algebra $(\tilde{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ becomes quadratic.

Matrix loop algebras provide key elements to study soliton hierarchies, including hierarchies of integrable couplings, and both the trace identity and the variational identity are basic tools for generating Hamiltonian structures which usually guarantee the Liouville integrability.

2.2. The AKNS soliton hierarchy

Let us recall the AKNS soliton hierarchy associated with $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})$ [19]. The traditional spatial spectral problem for the AKNS hierarchy is given by

$$\phi_x = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{31}$$

The stationary continuous zero curvature equation in (22), i.e., $W_x = [U,W]$, is equivalent to

$$\begin{cases} a_x = pc - qb, \\ b_x = -2\lambda\, b - 2\, pa, \\ c_x = 2qa + 2\lambda\, c, \end{cases} \tag{32}$$

if we assume that W is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} W_{0,i}\lambda^{-i},\ W_{0,i} = \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix},\ i \geq 0. \tag{33}$$

Equivalently, the system (32) leads to the recursion relations:

$$\begin{cases} b_{i+1} = -\frac{1}{2} b_{i,x} - pa_i, \\ c_{i+1} = \frac{1}{2} c_{i,x} - qa_i, \qquad i \geq 0, \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \tag{34}$$

upon taking the initial values

$$a_0 = -1,\ b_0 = c_0 = 0. \tag{35}$$

We impose the conditions on constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{36}$$

such that the recursion relations in (34) will uniquely determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$. This way, the first three sets can be worked out as follows:

$$\begin{aligned} &b_1 = p,\ c_1 = q,\ a_1 = 0; \\ &b_2 = -\tfrac{1}{2} p_x,\ c_2 = \tfrac{1}{2} q_x,\ a_2 = \tfrac{1}{2} pq; \\ &b_3 = \tfrac{1}{4} p_{xx} - \tfrac{1}{2} p^2 q,\ c_3 = \tfrac{1}{4} q_{xx} - \tfrac{1}{2} pq^2,\ a_3 = \tfrac{1}{4}(pq_x - p_x q). \end{aligned}$$

Now, the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0 \ \text{ with } \ V^{[m]} = (\lambda^m W)_+, \ \ m \geq 0, \tag{37}$$

where all modification terms, $\Delta_m, \ m \geq 0$, are taken as zero, generate the AKNS soliton hierarchy:

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J \frac{\delta \mathscr{H}_m}{\delta u}, \ \ m \geq 0, \tag{38}$$

where the Hamiltonian operator, the hereditary recursion operator and the Hamiltonian functionals are given by

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \ \ \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \ \ \mathscr{H}_m = \int \frac{2a_{m+2}}{m+1}\, dx, \ m \geq 0, \tag{39}$$

respectively. The above Hamiltonian structures are generated by applying the trace identity [33, 35].

3. MATRIX LOOP ALGEBRAS

To generate integrable couplings, one needs to create non-semisimple Lie algebras, also called enlarged Lie algebras (see, e.g., [36]-[45] for typical examples). We will present a few classes of matrix Lie algebras consisting of block matrices with blocks of the same size or different sizes. Larger numbers of blocks show diversity of integrable couplings, though they bring complexity in theoretical verification.

3.1. Lie algebras consisting of block matrices with blocks of the same size

In this subsection, we would particularly like to show matrix Lie algebras consisting of 2×2, 3×3 or 4×4 block matrices, which can be used to generate dark equations and bi- and tri-integrable couplings [26].

Class 1 - Matrix Lie algebras consisting of 2×2 block matrices:

All 2×2 block matrices of the type:

$$M_1(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}, \tag{40}$$

where A_1 and A_2 are square matrices of the same size, define an enlarged matrix Lie algebra with the following semi-direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \in \mathfrak{g}_c, \ \text{ where } \ \mathfrak{g} = \{M(A_1, 0)\} \text{ and } \mathfrak{g}_c = \{M(0, A_2)\}.$$

The corresponding matrix product reads

$$M(A_1, A_2)M(B_1, B_2) = M(C_1, C_2), \tag{41}$$

where C_1 and C_2 are given by

$$\begin{cases} C_1 = A_1 B_1, \\ C_2 = A_1 B_2 + A_2 B_1. \end{cases} \tag{42}$$

This kind of enlarged matrix Lie algebras can be used to generate dark equations [32, 46, 47], and the variational identity in this case reduces to the bi-trace identity [48].

Class 2 - Matrix Lie algebras consisting of 3×3 block matrices:

Let α and β be two arbitrarily given constants, which can be zero. All 3×3 block matrices of the type:

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{43}$$

where A_1, A_2 and A_3 be square matrices of the same size, define an enlarged matrix Lie algebra with the following semi-direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \text{ where } \mathfrak{g} = \{M(A_1,0,0)\} \text{ and } \mathfrak{g}_c = \{M(0,A_2,A_3)\}.$$

The corresponding matrix product reads

$$M(A_1,A_2,A_3)M(B_1,B_2,B_3) = M(C_1,C_2,C_3), \tag{44}$$

where C_1, C_2 and C_3 are given by

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + A_2B_1 + \alpha A_2B_2, \\ C_3 = A_1B_3 + A_3B_1 + \beta A_2B_2 + \alpha A_2B_3 + \alpha A_3B_2. \end{cases} \tag{45}$$

This kind of enlarged matrix Lie algebras can be used to generate bi-integrable couplings [25].

Class 3 - Matrix Lie algebras consisting of 4×4 block matrices:

Let α, β, μ and ν be four arbitrarily given constants, which can be zero. All following 4×4 block matrices:

$$M_3(A_1,A_2,A_3,A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1+\alpha A_2 & \alpha A_3 & \beta A_2 + \alpha A_4 \\ 0 & 0 & A_1 + \alpha A_2 + \mu A_3 & \nu A_3 \\ 0 & 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{46}$$

where A_i, $1 \le i \le 4$, be square matrices of the same size, define an enlarged matrix Lie algebra with the semi-direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \text{ where } \mathfrak{g} = \{M(A_1,0,0,0)\} \text{ and } \mathfrak{g}_c = \{M(0,A_2,A_3,A_4)\}.$$

The corresponding matrix product reads

$$M(A_1,A_2,A_3,A_4)M(B_1,B_2,B_3,B_4) = M(C_1,C_2,C_3,C_4), \tag{47}$$

where C_1, C_2, C_3 and C_4 are given by

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + A_2B_1 + \alpha A_2B_2, \\ C_3 = A_1B_3 + A_3B_1 + \alpha A_2B_3 + \alpha A_3B_2 + \mu A_3B_3, \\ C_4 = A_1B_4 + A_4B_1 + \alpha A_2B_4 + \alpha A_4B_2 + \beta A_2B_2 + \nu A_3B_3. \end{cases} \tag{48}$$

This kind of enlarged matrix Lie algebras can be used to generate tri-integrable couplings [27].

3.2. Lie algebras consisting of block matrices with blocks of different sizes

We will use the finite-dimensional irreducible representations V_d of the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ (see, e.g., [49]), to create enlarged matrix Lie algebras.

Class 1 - Matrix Lie algebra by using the irreducible representation V_2:

Let $M(V_2, A_a) = \bar{\mathfrak{g}}$ be a Lie algebra of enlarged matrices of the type:

$$M(a,b,c,f,g) = \left[\begin{array}{ccc|cc} 2a & b & 0 & & \\ 2c & 0 & 2b & & A_a \\ 0 & c & -2a & & \\ \hline 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & -a \end{array}\right] \text{ with } A_a = \begin{bmatrix} f & 0 \\ g & f \\ 0 & g \end{bmatrix}. \tag{49}$$

Here the (1,1)-block is generated from the irreducible representation V_2 of $\mathrm{sl}(2,\mathbb{R})$. This Lie algebra has the semi-direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \text{ where } \mathfrak{g} = \{M(a,b,c,0,0)\} \text{ and } \mathfrak{g}_c = \{M(0,0,0,f,g\}.$$

The corresponding matrix commutator reads

$$[M(a,b,c,f,g), M(a',b',c',f',g')] = M(a'',b'',c'',f'',g''), \tag{50}$$

where

$$\begin{cases} a'' = bc' - cb', \\ b'' = 2ab' - 2ba', \\ c'' = 2ca' - 2ac', \\ f'' = af' - fa' + bg' - gb', \\ g'' = cf' - fc' + ga' - ag'. \end{cases} \tag{51}$$

This enlarged matrix Lie algebra will be used to generate a class of integrable couplings of different type from the previous ones.

Class 2 - Matrix Lie algebra by using the irreducible representation V_3:

Let $M(V_3, A_a) = \bar{\mathfrak{g}}$ be a Lie algebra of enlarged matrices of the type:

$$M(a,b,c,e,f,g) = \left[\begin{array}{cccc|cc} 3a & b & 0 & 0 & & \\ 3c & a & 2b & 0 & & \\ 0 & 2c & -a & 3b & \multicolumn{2}{c}{A_a} \\ 0 & 0 & c & -3a & & \\ \hline 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & -a \end{array}\right] \text{ with } A_a = \begin{bmatrix} f & 0 \\ e & f \\ g & e \\ 0 & g \end{bmatrix}. \tag{52}$$

Here the (1,1)-block is generated from the irreducible representation V_3 of $\mathrm{sl}(2,\mathbb{R})$. This enlarged matrix Lie algebra has the semi-direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \text{where } \mathfrak{g} = \{M(a,b,c,0,0,0)\} \text{ and } \mathfrak{g}_c = \{M(0,0,0,e,f,g\}.$$

The corresponding matrix commutator reads

$$[M(a,b,c,e,f,g), M(a',b',c',e',f',g')] = M(a'',b'',c'',e'',f'',g''), \tag{53}$$

where

$$\begin{cases} a'' = bc' - cb', \\ b'' = 2ab' - 2ba', \\ c'' = 2ca' - 2ac', \\ e'' = 2cf' - 2fc' + 2bg' - 2gb', \\ f'' = 2af' - 2fa' + be' - eb', \\ g'' = ce' - ec' - 2ag' + 2ga'. \end{cases} \tag{54}$$

This enlarge matrix Lie algebra will be used to generate another class of integrable couplings of different type from the previous ones.

4. INTEGRABLE COUPLINGS BY USING IRREDUCIBLE REPRESENTATIONS

4.1. Using the irreducible representation V_2 of $\mathbf{sl}(2,\mathbb{R})$

Based on $M(V_2,A_a)$, let us introduce a spatial spectral problem

$$\phi_x = \bar{U}\phi = \bar{U}(\bar{u},\lambda)\phi, \tag{55}$$

where λ is a spectral parameter, $\phi = (\phi_1,\cdots,\phi_5)^T$ is an eigenfunction, and the enlarged spatial matrix $\bar{U}$ is given by

$$\bar{U} = \bar{U}(\bar{u},\lambda) = \begin{bmatrix} -2\lambda & p & 0 & r & 0 \\ 2q & 0 & 2p & s & r \\ 0 & q & 2\lambda & 0 & s \\ 0 & 0 & 0 & -\lambda & p \\ 0 & 0 & 0 & q & \lambda \end{bmatrix}, \ \bar{u} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}. \tag{56}$$

With $\bar{W} = M(V_2,W_a)$ being chosen from the enlarged matrix loop algebra $\tilde{M}(V_2,A_a)$:

$$\bar{W} = M(V_2,W_a) = \begin{bmatrix} 2a & b & 0 & f & 0 \\ 2c & 0 & 2b & g & f \\ 0 & c & -2a & 0 & g \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & -a \end{bmatrix}, \ W_a = \begin{bmatrix} f & 0 \\ g & f \\ 0 & g \end{bmatrix}, \tag{57}$$

the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U},\bar{W}] \tag{58}$$

leads equivalently to both (32) and

$$\begin{cases} f_x = -\lambda f + pg - ra - sb, \\ g_x = qf + sa - rc + \lambda g. \end{cases} \tag{59}$$

Assume that W_a is of the following form

$$W_a = \sum_{i\geq 0} W_{a,i}\lambda^{-i}, \ W_{a,i} = \begin{bmatrix} f_i & 0 \\ g_i & f_i \\ 0 & g_i \end{bmatrix}, \ i \geq 0. \tag{60}$$

Then, upon taking the initial values

$$f_0 = g_0 = 0, \tag{61}$$

the system (59) equivalently yields the recursion relations:

$$\begin{cases} f_{i+1} = -f_{i,x} + pg_i - ra_i - sb_i, \\ g_{i+1} = g_{i,x} - qf_i - sa_i + rc_i, \end{cases} \quad i \geq 0. \tag{62}$$

The first three sets then can be computed as follows:

$$b_1 = r, \ g_1 = s;$$
$$f_2 = -r_x, \ g_2 = s_x;$$
$$f_3 = r_{xx} + ps_x - \tfrac{1}{2}pqr + \tfrac{1}{2}p_x s, \ g_3 = s_{xx} + qr_x - \tfrac{1}{2}pqs + \tfrac{1}{2}q_x r.$$

Further, taking

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = \sum_{i=0}^{m} \bar{W}_i \lambda^{m-i}, \ m \geq 0, \tag{63}$$

where $\bar{W}_i \in M(V_2, A_a),\ i \geq 0$, we see that the zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{64}$$

generate a hierarchy of integrable couplings for the AKNS hierarchy (38):

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \bar{K}_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -f_{m+1} \\ g_{m+1} \end{bmatrix} = \bar{\Phi}^m \begin{bmatrix} -2p \\ 2q \\ -r \\ s \end{bmatrix},\ m \geq 0, \tag{65}$$

where the operator $\bar{\Phi}$ is determined by using the recursion relations in (34) and (62):

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 & \Phi_2 \end{bmatrix},\ \Phi_1 = \begin{bmatrix} \frac{1}{2} r\partial^{-1} q - \frac{1}{2} s & \frac{1}{2} r\partial^{-1} p \\ -\frac{1}{2} s\partial^{-1} q & -\frac{1}{2} s\partial^{-1} p + \frac{1}{2} r \end{bmatrix},\ \Phi_2 = \begin{bmatrix} -\partial & -p \\ q & \partial \end{bmatrix}. \tag{66}$$

This recursion operator is of different type from the previous ones (see, e.g., [26, 31, 32] for previous examples).

We point out that any symmetric and ad-invariant bilinear form on the enlarged matrix loop algebra $\tilde{M}(V_2, A_a)$ is degenerate. Therefore, the variational identity does not work for $\tilde{M}(V_2, A_a)$, and we do not know how to explore Hamiltonian structures, if any, for the hierarchy of integrable couplings (65).

4.2. Using the irreducible representation V_3 of $\mathbf{sl}(2, \mathbb{R})$

4.2.1. Soliton hierarchy of integrable couplings

Based on $M(V_3, A_a)$, let us introduce a spatial spectral problem

$$\phi_x = \bar{U}\phi = \bar{U}(\bar{u}, \lambda)\phi, \tag{67}$$

where λ is a spectral parameter, $\phi = (\phi_1, \cdots, \phi_6)^T$ is an eigenfunction, and the enlarged spatial matrix $\bar{U}$ is given by

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} -3\lambda & p & 0 & 0 & r & 0 \\ 3q & -\lambda & 2p & 0 & 0 & r \\ 0 & 2q & \lambda & 3p & s & 0 \\ 0 & 0 & q & 3\lambda & 0 & s \\ 0 & 0 & 0 & 0 & -\lambda & p \\ 0 & 0 & 0 & 0 & q & \lambda \end{bmatrix},\ \bar{u} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}.$$

With $\bar{W} = M(V_3, W_a)$ being chosen from the enlarge matrix loop algebra $\tilde{M}(V_3, A_a)$:

$$\bar{W} = M(V_3, W_a) = \begin{bmatrix} 3a & b & 0 & 0 & f & 0 \\ 3c & a & 2b & 0 & e & f \\ 0 & 2c & -a & 3b & g & e \\ 0 & 0 & c & -3a & 0 & g \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & -a \end{bmatrix},\ W_a = \begin{bmatrix} f & 0 \\ e & f \\ g & e \\ 0 & g \end{bmatrix}, \tag{68}$$

the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}] \tag{69}$$

leads equivalently to both (32) and

$$\begin{cases} f_x = -2\lambda f + pe - 2ra, \\ g_x = qe + 2sa + 2\lambda g, \\ e_x = 2qf + 2pg - 2rc - 2sb. \end{cases} \tag{70}$$

Assume that W_a is of the following form

$$W_a = \sum_{i\geq 0} W_{a,i}\lambda^{-i},\ W_{a,i} = \begin{bmatrix} f_i & 0 \\ e_i & f_i \\ g_i & e_i \\ 0 & g_i \end{bmatrix},\ i \geq 0. \tag{71}$$

Then, upon taking the initial values

$$e_0 = -1,\ f_0 = g_0 = 0, \tag{72}$$

the system (70) equivalently yields the recursion relations:

$$\begin{cases} f_{i+1} = -\frac{1}{2} f_{i,x} + \frac{1}{2} pe_i - ra_i, \\ g_{i+1} = \frac{1}{2} g_{i,x} - \frac{1}{2} qe_i - sa_i, \\ e_{i+1,x} = 2qe_{i+1} + 2pg_{i+1} - 2rc_{i+1} - 2sb_{i+1}, \end{cases} \quad i \geq 0. \tag{73}$$

We impose the conditions on constants of integration

$$e_i|_{\bar{u}=0} = f_i|_{\bar{u}=0} = g_i|_{\bar{u}=0} = 0,\ i \geq 1, \tag{74}$$

to determine the sequence of $\{e_i, f_i, g_i \,|\, i \geq 1\}$ uniquely. The first three sets then can be computed as follows:

$$\begin{aligned} & f_1 = -\tfrac{1}{2}p + r,\ g_1 = \tfrac{1}{2}q + s,\ e_1 = 0; \\ & f_2 = \tfrac{1}{4}p_x - \tfrac{1}{2}r_x,\ g_2 = \tfrac{1}{4}q_x + \tfrac{1}{2}s_x,\ e_2 = \tfrac{1}{2}pq + ps - qr; \\ & f_3 = -\tfrac{1}{8}p_{xx} + \tfrac{1}{4}r_{xx} + \tfrac{1}{4}p^2 q - \tfrac{1}{2}p^2 s - pqr, \\ & g_3 = \tfrac{1}{8}q_{xx} + \tfrac{1}{4}s_{xx} - \tfrac{1}{4}pq^2 - pqs - \tfrac{1}{2}q^2 r, \\ & e_3 = -\tfrac{1}{4}(p_x q - pq_x) - \tfrac{1}{2}(p_x s - ps_x) - \tfrac{1}{2}(q_x r - qr_x). \end{aligned}$$

Further, taking

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = \sum_{i=0}^{m} \bar{W}_i \lambda^{m-i},\ m \geq 0, \tag{75}$$

where $\bar{W}_i \in M(V_3, A_a),\ i \geq 0$, we see that the zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{76}$$

generate a hierarchy of integrable couplings for the AKNS hierarchy (38):

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \bar{K}_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{bmatrix} = \bar{\Phi}^m \begin{bmatrix} -2p \\ 2q \\ p-2r \\ q+2s \end{bmatrix},\ m \geq 0, \tag{77}$$

where the operator $\bar{\Phi}$ is determined by using the recursion relations in (34) and (73):

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 & \Phi_2 \end{bmatrix}, \tag{78}$$

with the sub-matrix operators Φ_1 and Φ_2 being given by

$$\Phi_1 = \begin{bmatrix} -p\partial^{-1}s + r\partial^{-1}q & p\partial^{-1}r + r\partial^{-1}p \\ -q\partial^{-1}s - s\partial^{-1}q & q\partial^{-1}r - s\partial^{-1}p \end{bmatrix}, \ \Phi_2 = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}. \tag{79}$$

This recursion operator is also of different type from the previous ones (see, e.g, [26, 31, 32] for previous examples).

4.2.2. Hamiltonian structures

It can be directly shown that any symmetric and ad-invariant bilinear form on the enlarged matrix loop algebra $\tilde{M}(V_3, A_a)$ is given by

$$\begin{aligned} &\langle M(a,b,c,e,f,g), M(a',b',c',e',f',g')\rangle \\ &= (a,b,c,e,f,g)F(a',b',c',e',f',g')^T \\ &= \eta_1(aa' + \frac{1}{2}bc' + \frac{1}{2}cb') + \eta_2(ae' + bg' - cf' + ea' - fc' + gb'), \end{aligned} \tag{80}$$

where the square matrix F is defined by

$$F = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\eta_1 & 0 & 0 & \eta_2 \\ 0 & \frac{1}{2}\eta_1 0 & & 0 & -\eta_2 & 0 \\ \eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{81}$$

with η_1 and η_2 being arbitrary constants. Note that the determinant of F is $-\eta_2^6$. Thus, the bilinear form $\langle\cdot,\cdot\rangle$ is non-degenerate iff the constant η_2 is non-zero.

Now a simple application of the variational identity [29, 31, 32]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{W}, \frac{\partial \bar{U}}{\partial \lambda} \rangle \, dx = \lambda^{-\gamma} \frac{\lambda}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \frac{\partial \bar{U}}{\partial \bar{u}} \rangle, \tag{82}$$

where γ is a constant, engenders

$$\frac{\delta}{\delta \bar{u}} \int (-\eta_1 a - \eta_2 e) \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{bmatrix} \frac{1}{2}\eta_1 c + \eta_2 g \\ \frac{1}{2}\eta_1 b - \eta_2 f \\ -\eta_2 c \\ \eta_2 b \end{bmatrix}.$$

Balancing coefficients of each power of λ in the above equality leads to

$$\frac{\delta}{\delta \bar{u}} \int (-\eta_1 a_{m+1} - \eta_2 e_{m+1}) \, dx = (\gamma - m) \begin{bmatrix} \frac{1}{2}\eta_1 c_m + \eta_2 g_m \\ \frac{1}{2}\eta_1 b_m - \eta_2 f_m \\ -\eta_2 c_m \\ \eta_2 b_m \end{bmatrix}, \ m \geq 0.$$

The case of $m = 1$ tells that $\gamma = 0$, and thus we have

$$\frac{\delta}{\delta u} \int \frac{\eta_1 a_{m+2} + \eta_2 e_{m+2}}{m+1} \, dx = \begin{bmatrix} \frac{1}{2}\eta_1 c_{m+1} + \eta_2 g_{m+1} \\ \frac{1}{2}\eta_1 b_{m+1} - \eta_2 f_{m+1} \\ -\eta_2 c_{m+1} \\ \eta_2 b_{m+1} \end{bmatrix}, \ m \geq 0. \tag{83}$$

Consequently, we obtain the Hamiltonian structures for the soliton hierarchy of integrable couplings (77):

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \bar{K}_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{bmatrix} = \bar{J}\frac{\delta \bar{\mathscr{H}}_m}{\delta \bar{u}},\ m \geq 0, \tag{84}$$

with the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & -\frac{2}{\eta_2} \\ 0 & 0 & -\frac{2}{\eta_2} & 0 \\ 0 & \frac{2}{\eta_2} & 0 & -\frac{\eta_1}{\eta_2^2} \\ \frac{2}{\eta_2} & 0 & \frac{\eta_1}{\eta_2^2} & 0 \end{bmatrix}, \tag{85}$$

and the Hamiltonian functionals

$$\bar{\mathscr{H}}_m = \int \frac{\eta_1 a_{m+2} + \eta_2 e_{m+2}}{m+1}\, dx,\ m \geq 0. \tag{86}$$

4.2.3. Liouville integrability

It is direct but lengthy to show by compte algebra systems that $\bar{J}$ defined by (85) and

$$\bar{M} = \bar{\Phi}\bar{J} = \left[\begin{array}{c|c} 0 & -\frac{2}{\eta_2}\Phi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \hline \frac{2}{\eta_2}\Phi_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -\frac{2}{\eta_2}\Phi_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{\eta_1}{\eta_2^2}\Phi_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array}\right] \tag{87}$$

with Φ being defined as in (39) and Φ_1 and Φ_2 being defined by (79), constitute a Hamiltonian pair. It means that any linear combination $\bar{N}$ of $\bar{J}$ and $\bar{M}$ satisfies

$$\int \bar{K}^T \bar{N}'(\bar{u})[\bar{N}\bar{S}]\bar{T}\, dx + \text{cycle}(\bar{K}, \bar{S}, \bar{T}) = 0 \tag{88}$$

for all vector fields $\bar{K}$, $\bar{S}$ and $\bar{T}$. This implies that $\bar{\Phi}$ is hereditary [18], i.e., it satisfies

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{K}]\bar{S} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{K}]\bar{S} = \bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{S}]\bar{K} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{S}]\bar{K} \tag{89}$$

for all vector fields $\bar{K}$ and $\bar{S}$. The hereditary property (89) is equivalent to

$$L_{\bar{\Phi}\bar{K}}\bar{\Phi} = \bar{\Phi}L_{\bar{K}}\bar{\Phi} \tag{90}$$

for an arbitrary vector field $\bar{K}$, where the Lie derivative $L_{\bar{K}}\bar{\Phi}$ is defined by

$$(L_{\bar{K}}\bar{\Phi})\bar{S} = \bar{\Phi}[\bar{K}, \bar{S}] - [\bar{K}, \bar{\Phi}\bar{S}]. \tag{91}$$

It is known (see, e.g., [50]) that an autonomous operator $\bar{\Phi} = \bar{\Phi}(\bar{u}, \bar{u}_x, \cdots)$ is a recursion operator of an evolution equation $\bar{u}_t = \bar{K}$ iff the operator $\bar{\Phi}$ needs to satisfy

$$L_{\bar{K}}\bar{\Phi} = 0. \tag{92}$$

Obviously, we have $L_{\bar{K}_0}\bar{\Phi} = 0$, and thus

$$L_{\bar{K}_m}\bar{\Phi} = \bar{\Phi}L_{\bar{K}_{m-1}}\bar{\Phi} = \cdots = \bar{\Phi}^m L_{\bar{K}_0}\bar{\Phi} = 0,\ m \geq 1. \tag{93}$$

This shows that the operator $\bar{\Phi}$ is a common hereditary recursion operator for the soliton hierarchy of integrable couplings (77).

Now, it follows that the soliton hierarchy of integrable couplings (77) is bi-Hamiltonian and Liouville integrable, and thus, it possesses infinitely many commuting symmetries and conserved densities. In particular, we have the Abelian symmetry algebra:

$$[\bar{K}_k, \bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l] - \bar{K}_l'(\bar{u})[\bar{K}_k] = 0,\ k,l \geq 0, \tag{94}$$

and the two Abelian algebras of conserved Hamiltonian functionals:

$$\{\bar{\mathscr{H}}_k, \bar{\mathscr{H}}_l\}_{\bar{J}} = \int \Big(\frac{\delta \bar{\mathscr{H}}_k}{\delta \bar{u}}\Big)^T \bar{J} \frac{\delta \bar{\mathscr{H}}_l}{\delta \bar{u}}\, dx = 0,\ k,l \geq 0, \tag{95}$$

and

$$\{\bar{\mathscr{H}}_k, \bar{\mathscr{H}}_l\}_{\bar{M}} = \int \Big(\frac{\delta \bar{\mathscr{H}}_k}{\delta \bar{u}}\Big)^T \bar{M} \frac{\delta \bar{\mathscr{H}}_l}{\delta \bar{u}}\, dx = 0,\ k,l \geq 0. \tag{96}$$

These conserved Hamiltonian functionals generate infinitely many conservation laws for each system of integrable couplings in the hierarchy (77). Such differential polynomial type conservation laws can also be computed systematically by computer algebra systems (see, e.g., [51]) or generated from a Riccati equation from the associated spatial spectral problem (see, e.g., [52, 53]).

5. SUPER VARIATIONAL IDENTITIES

If the underlying Lie algebra $\bar{\mathfrak{g}}$ is superalgebra, we have similar super variational identities [26]. Let $\bar{\mathfrak{g}}(\lambda)$ be a loop superalgebra over a supercommutative ring. Then the continuous super variational identity on $\bar{\mathfrak{g}}(\lambda)$ holds:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial \lambda}\rangle\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{97}$$

and the discrete super variational identity on $\bar{\mathfrak{g}}(\lambda)$ holds:

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \langle W, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{98}$$

where γ is a fixed constant, $U, W \in \bar{\mathfrak{g}}(\lambda)$ satisfy $W_x = [U, W]$ or $(EW)U = UW$, and $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the loop superalgebra $\bar{\mathfrak{g}}(\lambda)$.

When the supertrace str is non-degenerate, the above super variational identities reduce to the supertrace identities [54]:

$$\frac{\delta}{\delta u}\int \mathrm{str}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial \lambda}})\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial u}}), \tag{99}$$

and

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \mathrm{str}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial \lambda}}) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial u}}), \tag{100}$$

where $\mathrm{ad}_a b = [a, b]$. A second class of the powerful super variational identities are bi-supertrace identities for constructing super Hamiltonian dark equations:

$$\frac{\delta}{\delta u}\int \big[\mathrm{str}\,\big(W_0\frac{\partial U_1}{\partial \lambda}\big) + \mathrm{str}\,\big(W_1\frac{\partial U_0}{\partial \lambda}\big)\big]\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\big[\mathrm{str}\,\big(W_0\frac{\partial U_1}{\partial u}\big) + \mathrm{str}\,\big(W_1\frac{\partial U_0}{\partial u}\big)\big] \tag{101}$$

in the continuous case, and

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \big[\mathrm{str}\,\big(W_0\frac{\partial U_1}{\partial \lambda}\big) + \mathrm{str}\,\big(W_1\frac{\partial U_0}{\partial \lambda}\big)\big] = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\big[\mathrm{str}\,\big(W_0\frac{\partial U_1}{\partial u}\big) + \mathrm{str}\,\big(W_1\frac{\partial U_0}{\partial u}\big)\big] \tag{102}$$

in the discrete case. More generally, the super variational identities can reduce to the last-component-trace identities:

$$\frac{\delta}{\delta u}\int \sum_{i=0}^{N} \mathrm{str}\,\big(W_i\frac{\partial U_{N-j}}{\partial \lambda}\big)\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\sum_{i=0}^{N} \mathrm{str}\,\big(W_i\frac{\partial U_{N-i}}{\partial u}\big), \tag{103}$$

and

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\sum_{i=0}^{N}\operatorname{str}\left(W_i\frac{\partial U_{N-i}}{\partial\lambda}\right)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\sum_{i=0}^{N}\operatorname{str}\left(W_0\frac{\partial U_{N-i}}{\partial u}\right), \tag{104}$$

which help generate Hamiltonian structures of super multi-integrable couplings.

If a spectral matrix $U=U(u,\lambda)$ is of order 2, we can make a super generalization to construct a super soliton hierarchy:

$$\bar{U}=U(u,\lambda)+\alpha E_3+\beta E_4=\left[\begin{array}{cc|c} \multicolumn{2}{c|}{U(u,\lambda)} & \begin{array}{c}\alpha\\ \beta\end{array} \\ \hline \beta & -\alpha & 0\end{array}\right], \tag{105}$$

where E_3 and E_4 are odd generators of the super sl(2), u is a vector of commuting variables, and α and β are anticommuting variables. Applications of the super-trace identities lead to super integrable systems and super-symmetric integrable systems (see, e.g., [54]-[57]).

We can also form semi-direct sums of Lie superalgebras and take new enlarged spectral matrices from the resulting semi-direct sums of Lie superalgebras, to construct super integrable couplings. More specifically, we can make

$$\bar{\mathfrak{g}}(\lambda)=\tilde{\mathfrak{g}}\Subset\tilde{\mathfrak{g}}_c \tag{106}$$

with the Lie product:

$$\bar{W}=[\bar{U},\bar{V}]=[U+U_c,V+V_c]=W+W_c,\ \bar{U},\bar{V}\in\bar{\mathfrak{g}}(\lambda), \tag{107}$$

where

$$W=[U,V]\in\tilde{\mathfrak{g}},\ W_c=[U,V_c]+[U_c,V]+[U_c,V_c]\in\tilde{\mathfrak{g}}_c. \tag{108}$$

Applications of the super variational identities such as the bi-supertrace identities will yield super Hamiltonian structures for super integrable couplings. The procedure for constructing super integrable couplings is similar to the one in the classical case, and one only needs to pay special attention to the anticommuting fermionic variables in deriving integrable couplings.

6. CONCLUDING REMARKS

We proposed a few classes of matrix Lie algebras consisting of block matrices to generate integrable couplings, and successfully constructed two new type hierarchies of integrable couplings for the AKNS equations. The presented enlarged matrix loop algebras provide key elements to construct integrable couplings, and we point out that the construction idea by using irreducible representations can also be applied to the other existing soliton hierarchies like the Dirac hierarchy (see, e.g., [58]) and the Kaup-Newell hierarchy (see [59]).

Integrable couplings bring inspiration and insights into the general structure of integrable systems with multi-components. It will be very helpful in building an exhaustive list of integrable systems to collect particular examples of integrable couplings, both linear and nonlinear. The theory of integrable couplings generates various and complicated recursion operators in block matrix form, and the mathematics behind integrable couplings is deep and interesting. It is extremely important to explore more specific examples of integrable couplings in establishing mathematical structures that integrable systems, particularly multi-component ones, possess.

There are many further questions on integrable couplings and their solution theories. We list some of them for interested readers as follows.

Para-Grassmann zero curvature equations:

It is not clear how to generalize a procedure for constructing integrable systems by zero curvature equations to the para-Grassmann case. Even in the special super-symmetric case of $D=1$ and $N=1$, it is not known how to solve

$$D_xW=[U,W],\ D_x=\partial_\theta+\theta\partial_x \text{ - super derivative,}$$

for W such that the super-symmetric zero curvature equation

$$U_t-D_xV+[U,V]=0$$

generates super-symmetric integrable systems. The general para-Grassmann case definitely brings more difficulties.

Open question on linear ordinary differential equations:

It was shown recently that for a general linear ordinary differential equation (ODE) with continuous coefficients

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0,$$

where $y^{(i)}$ denotes the i-th order derivative of $y=y(x)$ with respect to x, there is a solution $y(x)$ such that

$$y(x),\ y'(x),\ \cdots,y^{(n-1)}(x)$$

are linearly independent over the interval where solutions exist, and this helps recognize solution structures of linear ODEs and generate subspaces of solutions invariant under the flows of given systems of evolution equations [60].

There is, however, an unsolved problem about the representation of the general solutions of linear systems of ODEs with continuous coefficients. Let $n\geq 2$, $I=(a,b)\subseteq\mathbb{R}$ and $t_0\in I$. Consider a Cauchy problem

$$\begin{cases} x'(t)=A(t)x(t), \\ x(t_0)=x_0\in\mathbb{R}^n, \end{cases}$$

where $A(t)$ is an $n\times n$ matrix whose entries are continuous on I. It is known that the general solution is given by

$$x(t)=e^{\int_{t_0}^t A(s)\,ds}x_0,$$

if the coefficient matrix commutes with its integral, i.e., we have

$$[A(t),\int_{t_0}^t A(s)\,ds]=0,\ t\in I.$$

An open question is whether this commutativity condition is necessary to guarantee that the vector function $x(t)$ defined above solves the given Cauchy problem [61].

Criterion for multivariate polynomials with one zero:

Hirota bilinear equations have recently been generalized by a technique of assigning signs to derivatives and the linear superposition principle was used to classify bilinear equations and determine their linear subspaces of solutions [62]. The related analysis presents an open question on multivariate polynomials [63]: how to determine if a real multivariate polynomial has one and only one zero? There are many such polynomials, for example,

$$\begin{aligned} &x^2+y^2-2x+2y+2,\ \text{zero}\ (x,y)=(1,-1); \\ &5x^2+2xy+2y^2-12x-6y+9,\ \text{zero}\ (x,y)=(1,1). \end{aligned}$$

Note that all such multivariate polynomials satisfy the requirement in Hilbert's 17th problem [64]. So, the above problem is more general than Hilbert's 17th problem.

Conjecture on integrability of commuting soliton equations:

There are infinitely many functionally independent symmetries generated from a recursion operator in a hierarchy of commuting soliton equations. Those symmetries generate an infinite number of one-parameter Lie groups of solutions to each member in the given soliton hierarchy. We conjecture that those infinitely many one-parameter Lie groups of solutions form a dense subset of solutions in the whole solution set to each member in the soliton hierarchy.

More specifically, let us denote a hierarchy of commuting soliton equations by $u_{t_m}=K_m(u),\ m\geq 0$. For a given member $u_{t_n}=K_n(u)$, we assume that

$$\text{symmetry}\ K_m\ \Rightarrow\ \text{Lie group of solutions}\ S_m(\varepsilon_m),\ \varepsilon_m\in I_m=(a_m,b_m)\subseteq\mathbb{R}.$$

Let T_n be the set of solutions to the n-th member $u_{t_n}=K_n$, and make a metric space $(T_n(\mathscr{D}),d)$ with a bounded domain $\mathscr{D}$:

$$T_n(\mathscr{D})=\{f|_{\mathscr{D}}\,|\,f\in T_n\},\ d(f,g)=sup_{(t,x)\in\mathscr{D}}|f(t,x)-g(t,x)|.$$

Is the union $\cup_{m=0}^{\infty}S_m(\varepsilon_m)$ dense in the metric space $(T_n(\mathscr{D}),d)$ with any bounded domain $\mathscr{D}$ for each member $u_{t_n}=K_n$ in the soliton hierarchy?

If the answer is yes, the solution to a Cauchy problem can be approximated by the solutions generated from those Lie symmetries. Thus, soliton hierarchies present good models of integrable nonlinear partial differential equations from a computational point of view, indeed.

Criterion for existence of Hamiltonian structures:

We know that Hamiltonian structures exist for the perturbation systems (see, e.g., [23], [65]-[68]). But some enlarged matrix loop algebras do not possess any non-degenerate, symmetric and ad-invariant bilinear forms required in the variational identities and a specific such example was given in subsection 4.1 (see also [69, 70] for other examples). Is there any concrete criterion which tells when there exist Hamiltonian structures for integrable couplings, even bi- and tri-integrable couplings? A concrete example is the following bi-integrable coupling

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \\ w_t = K'(u)[w], \end{cases}$$

where K' denotes the Gateaux derivative as before. Is there any Hamiltonian structure for this specific bi-integrable coupling?

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