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Reduced AKNS Spectral Problems and Associated Complex Matrix Integrable Models

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Abstract

The aim of this paper is to conduct two group reductions for matrix spectral problems simultaneously. We formulate reduced Ablowitz-Kaup-Newell-Segur matrix spectral problems under two local group reductions, and construct associated hierarchies of matrix integrable models, which keep the corresponding zero curvature equations invariant. In this way, various integrable models can be generated via zero curvature equations.

Keywords AKNS matrix spectral problem · Integrable hierarchy · Zero curvature equation

Mathematics Subject Classification 37K15 · 35Q55 · 37K40

1 Introduction

Matrix spectral problems are key objects in formulating integrable models and their soliton solutions. Zero curvature equations are representations of integrable models, which are the compatibility conditions of spatial and temporal matrix spectral problems. The universal approach for Cauchy problems of integrable models, called the inverse scattering transform, is completely based on Lax pairs of matrix spectral problems [1].

Motivated by gauge transformations, conducting group reductions for matrix spectral problems, which keep the zero curvature equations invariant, can yield reduced zero curvature equations, and thus, reduced integrable models [2]. The nonlinear Schrödinger equations and the modified Korteweg-de Vries equation are such typical examples, which are generated by one group reduction (see, e.g., [3–6], for more examples). Based on the

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Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems, taking one group reduction leads to one kind of reduced integrable nonlinear Schrödinger type models and two kinds of reduced integrable modified Kortweweg-de Vries type models. Moreover, taking a pair of group reductions can engender more diverse integrable models. However, some additional constraint conditions are required to satisfy, which are raised by the compatibility conditions between two group reductions. Such conditions bring relations between entries of spectral matrices in matrix spectral problems and are usually hard to achieve.

The idea of taking group reductions has also been applied to construction of nonlocal integrable models [7]. Based on one nonlocal group reduction for the AKNS matrix spectral problems, one can generate three kinds of reduced integrable nonlinear Schrödinger type models, and two kinds of reduced integrable modified Kortweweg-de Vries type models [8]. The inverse scattering transform has also been successfully applied to nonlocal integrable models (see, e.g., [9–12]). Moreover, many other efficient approaches solve nonlocal integrable models, and particularly, construct soliton solutions. Among those methods are the Hirota bilinear method, Darboux transformation, Bäclund transforms and the Riemann-Hilbert technique (see, for example, [13–16]). It is also shown [17] that a nonlocal reduction on the full-line, leading to the nonlocal nonlinear Schrödinger equation from the AKNS spectral problem, can be recast as a local reduction on the half-line.

In this paper, we would like to propose a pair of specific local group reductions for the AKNS matrix spectral problems, to generate reduced integrable models. The other sections of the paper are structured as follows. In the next section, we recall the AKNS hierarchies of matrix integrable models and their matrix spectral problems to prepare the subsequent analyses. Then we consider two local group reductions for the AKNS matrix spectral problems simultaneously and compute reduced hierarchies of local integrable models, which consist of commuting flows. The whole theory of formulating reduced AKNS matrix spectral problems and reduced corresponding matrix integrable models is illustrated by a few concrete examples., which also present novel integrable models. A conclusion and some discuss ions are given in the last section.

2 The AKNS Matrix Integrable Hierarchies Revisited

In order to facilitate the subsequent analyses, let us recall the AKNS hierarchies of matrix integrable models and their corresponding matrix spectral problems.

First, let λ stand for the spectral parameter, and p and q be two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{1}$$

where $m, n \geq 1$ are two arbitrarily given natural numbers. The matrix AKNS spectral problems are defined by

$$-i\phi_x = U\phi,\ U = U(u,\lambda) = (\lambda\Lambda + P), \tag{2}$$

and

$$-i\phi_t = V^{[r]}\phi,\ V^{[r]} = V^{[r]}(u,\lambda) = (\lambda^r\Omega + Q^{[r]}),\ r \geq 0, \tag{3}$$

with $u = u(p,q)$ being the potential. In this Lax pair of matrix spectral problems, the $(m + n)$-th order square matrices, Λ and Ω, are given by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{4}$$

where I_k is the identity matrix of size k, and α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct real constants, which show the diversity of matrix spectral problems but do not affect associated integrable models very much. The other two $(m+n)$-th order square matrices, P and $Q^{[r]}$, are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{5}$$

which is called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{6}$$

where $a^{[s]}$, $b^{[s]}$, $c^{[s]}$ and $d^{[s]}$ are defined recursively through

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a^{[0]} = \beta_1 I_m,\ d^{[0]} = \beta_2 I_n, \tag{7}$$

and

$$\begin{cases} b^{[s+1]} = \dfrac{1}{\alpha}(-i b_x^{[s]} - p d^{[s]} + a^{[s]} p), \\ c^{[s+1]} = \dfrac{1}{\alpha}(i c_x^{[s]} + q a^{[s]} - d^{[s]} q), \qquad s \geq 0, \\ a_x^{[s+1]} = i(p c^{[s+1]} - b^{[s+1]} q), \\ d_x^{[s+1]} = i(q b^{[s+1]} - c^{[s+1]} p), \end{cases} \tag{8}$$

where $\alpha = \alpha_1 - \alpha_2$ and zero constants of integration are taken in the determination of $a^{[s]}$ and $d^{[s]}$. Particularly, we can have

$$Q^{[1]} = \frac{\beta}{\alpha} P,\ Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3),$$

where $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. It is easy to see from the recursive relations in (8) with (7) that

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{9}$$

determines a Laurent series solution to the stationary zero curvature equation

$$W_x = i[U, W], \tag{10}$$

where U is given in (2). Such a formal series solution is crucial in generating integrable hierarchies (see, e.g., [18, 19] for examples).

Then, it is direct to see that for each pair of $m, n \geq 1$, the compatibility conditions of the two matrix spectral problems in (2) and (3), namely, the zero curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \; r \geq 0, \tag{11}$$

present one matrix AKNS integrable hierarchy

$$p_t = i\alpha b^{[r+1]}, \; q_t = -i\alpha c^{[r+1]}, \; r \geq 0. \tag{12}$$

The case of $m = n = 1$ reduces to the typical AKNS integrable hierarchy [20]. By the trace identity [21], each member in the above matrix integrable hierarchy can be showed to possess a bi-Hamiltonian structure and infinitely many symmetries and conserved quantities (see, e.g., [22, 23] for more details).

The first and second nonlinear (corresponding to $r = 2, 3$) integrable models in (12) give us the AKNS matrix nonlinear Schrödinger equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \; q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq), \tag{13}$$

and the AKNS matrix modified Korteweg-de Vries equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \; q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{14}$$

where p and q are the two matrix potentials defined by (1). More examples could be found in the literature (see, e.g., [24, 25]).

3 Reduced AKNS Spectral Problems and Integrable Hierarchies

3.1 Reduced AKNS Matrix Spectral Problems

Assume that Σ_1 and Σ_2 are two constant invertible Hermitian matrices of orders m and n, respectively, and Δ_1 and Δ_2 are other two constant invertible matrices of orders m and n, respectively. Then, we form the two bigger invertible constant matrices of order $m+n$:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \; \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{15}$$

All such matrice form a group Γ under the matrix multiplication.

For the spectral matrix U in (2), we propose the following pair of group reductions:

$$\Sigma U(\lambda)\Sigma^{-1} = U^{\dagger}(\lambda^*) = (U(\lambda^*))^{\dagger}, \tag{16}$$

and

$$\Delta U(\lambda)\Delta^{-1} = U(\lambda), \tag{17}$$

where $\dagger$ and $*$ denotes the Hermitian transpose and the complex conjugate. These show the two invariance properties under similarity transformations, and in (16) and (17), two groups

$$\{(\Sigma, f(\lambda^*)) \in (\Gamma, C^{\infty})\} \text{ with } (\Sigma^{[1]}, f^{[1]}(\lambda^*))(\Sigma^{[2]}, f^{[2]}(\lambda^*)) = (\Sigma^{[1]}\Sigma^{[2]}, f^{[1]}(f^{[2]}(\lambda^*))), \tag{18}$$

and

$$\{(\Delta, g(\lambda)) \in (\Gamma, C^{\infty})\} \text{ with } (\Delta^{[1]}, g^{[1]}(\lambda))(\Delta^{[2]}, g^{[2]}(\lambda)) = (\Delta^{[1]}\Delta^{[2]}, g^{[1]}(g^{[2]}(\lambda))), \quad (19)$$

are taken, respectively.

Noting the characteristic form of U, we can see that these two group reductions lead equivalently to

$$\Sigma P \Sigma^{-1} = P^{\dagger}, \quad (20)$$

and

$$\Delta P \Delta^{-1} = P, \quad (21)$$

respectively. These actually require the following corresponding constraints for the two matrix potentials p and q:

$$p = \Sigma_1^{-1} q^{\dagger} \Sigma_2 \text{ or } q = \Sigma_2^{-1} p^{\dagger} \Sigma_1, \quad (22)$$

and

$$p = \Delta_1 p \Delta_2^{-1}, \; q = \Delta_2 q \Delta_1^{-1}. \quad (23)$$

As a consequence of (22) and (23), the first matrix potential p needs to satisfy

$$\Delta_1 p = p \Delta_2, \; \Sigma_1^{-1} \Delta_1^{\dagger} \Sigma_1 p = p \Sigma_2^{-1} \Delta_2^{\dagger} \Sigma_2, \quad (24)$$

or the second matrix potential q needs to satisfy

$$q \Delta_1 = \Delta_2 q, \; q \Sigma_1^{-1} \Delta_1^{\dagger} \Sigma_1 = \Sigma_2^{-1} \Delta_2^{\dagger} \Sigma_2 q. \quad (25)$$

Therefore, under both group reductions in (16) and (17), we have a class of reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \; U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ \Sigma_2^{-1} p^{\dagger} \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \quad (26)$$

where p needs to satisfy the constraints in (24), or equivalently,

$$-i\phi_x = U\phi, \; U = \begin{bmatrix} \alpha_1 \lambda I_m & \Sigma_1^{-1} q^{\dagger} \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \quad (27)$$

where q needs to satisfy the constraints in (25).

3.2 Associated Reduced Matrix Integrable Hierarchies

Following the two group reductions in (16) and (17), one can show that

$$\begin{cases} \Sigma W(\lambda) \Sigma^{-1} = W^{\dagger}(\lambda^*) = (W(\lambda^*))^{\dagger}, \\ \Delta W(\lambda) \Delta^{-1} = W(\lambda), \end{cases} \quad (28)$$

where W is given by (9). These invariance properties guarantee that for each $r \geq 0$, we have

$$\begin{cases} \Sigma V^{[r]}(\lambda)\Sigma^{-1} = V^{[r]\dagger}(\lambda^*) = (V^{[r]}(\lambda^*))^\dagger, \\ \Delta V^{[r]}(\lambda)\Delta^{-1} = V^{[r]}(\lambda), \end{cases} \tag{29}$$

and

$$\begin{cases} \Sigma Q^{[r]}(\lambda)\Sigma^{-1} = Q^{[r]\dagger}(\lambda^*) = (Q^{[r]}(\lambda^*))^\dagger, \\ \Delta Q^{[r]}(\lambda)\Delta^{-1} = Q^{[r]}(\lambda), \end{cases} \tag{30}$$

where $V^{[r]}$ and $Q^{[r]}$ are given in (3) and (6), respectively. Now, as a consequence of the potential constraints (22) and (23), we see that

$$\begin{cases} \Sigma(U_t - V_x^{[r]} + i[U, V^{[r]}])\Sigma^{-1} = U_t^\dagger - V_x^{[r]\dagger} + i[U^\dagger, V^{[r]\dagger}], \\ \Delta(U_t - V_x^{[r]} + i[U, V^{[r]}])\Delta^{-1} = U_t - V_x^{[r]} + i[U, V^{[r]}], \end{cases} \quad r \geq 0, \tag{31}$$

and thus, the matrix AKNS integrable models in (12) become a hierarchy of reduced AKNS matrix integrable models:

$$p_t = i\alpha b^{[r+1]}|_{q=\Sigma_2^{-1}p^\dagger\Sigma_1}, \ r \geq 0, \tag{32}$$

where p is a reduced $m \times n$ matrix potential satisfying (24), or equivalently,

$$q_t = -i\alpha c^{[r+1]}|_{p=\Sigma_1^{-1}q^\dagger\Sigma_2}, \ r \geq 0, \tag{33}$$

where q is a reduced $n \times m$ matrix potential satisfying (25). Moreover, every member in the reduced hierarchy (32) or (33) has a Lax pair consisting of the reduced matrix spectral problems in (2) and (3) and possesses a hierarchy of commuting symmetries and conserved densities reduced from those for the matrix integrable AKNS models in (12). The Lax pair of reduced matrix spectral problems are made of (26) and

$$-i\phi_t = V^{[r]}|_{q=\Sigma_2^{-1}p^\dagger\Sigma_1}\phi, \ r \geq 0, \tag{34}$$

or equivalently, (27) and

$$-i\phi_t = V^{[r]}|_{p=\Sigma_1^{-1}q^\dagger\Sigma_2}\phi, \ r \geq 0. \tag{35}$$

Since Σ_1 and Σ_2 are arbitrary invertible constant Hermitian matrices of orders m and n, respectively, and Δ_1 and Δ_2 are arbitrary invertible constant matrices of orders m and n, respectively, we can generate various reduced hierarchies of matrix AKNS integrable models.

4 Illustrative Examples

4.1 Case of $m = 1$ and $n = 2$

In the case of $m = 1$ and $n = 2$, we present two examples.

If we firstly take

$$\Sigma_1 = 1,\ \Sigma_2 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{36}$$

where δ and σ take on values of either 1 or -1, then we have

$$p = (p_1, \delta p_1),\ q = \sigma p^\dagger = \sigma(p_1^*, \delta p_1^*)^T, \tag{37}$$

and the reduced matrix spectral problem becomes

$$-i\phi_x = U|_{q=\Sigma_2^{-1}p^\dagger\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & p_1 & \delta p_1 \\ \sigma p_1^* & \alpha_2\lambda & 0 \\ \sigma\delta p_1^* & 0 & \alpha_2\lambda \end{bmatrix}\phi. \tag{38}$$

Upon going with the choice for p and q in (37), we can see that the 2nd-order reduced integrable model is just the nonlinear Schrödinger equation

$$ip_{1,t} = \frac{\beta}{\alpha^2}(p_{1,xx} + 4\sigma|p_1|^2 p_1), \tag{39}$$

and the 3rd-order reduced integrable equation is exactly the modified Korteweg-de Vries equation

$$p_{1,t} = -\frac{\beta}{\alpha^3}(p_{1,xxx} + 12\sigma|p_1|^2 p_{1,x}). \tag{40}$$

If we secondly take

$$\Sigma_1 = 1,\ \Sigma_2 = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{41}$$

where δ and σ take on values of either 1 or -1, then we have

$$p = (p_1, \delta p_1),\ q = \sigma\delta p^\dagger = \sigma(\delta p_1^*, p_1^*)^T, \tag{42}$$

and the reduced matrix spectral problem becomes

$$-i\phi_x = U|_{q=\Sigma_2^{-1}p^\dagger\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & p_1 & \delta p_1 \\ \sigma\delta p_1^* & \alpha_2\lambda & 0 \\ \sigma p_1^* & 0 & \alpha_2\lambda \end{bmatrix}\phi. \tag{43}$$

Now going with the choice for p and q in (42), we can see that the 2nd-order reduced integrable model is precisely the nonlinear Schrödinger equation

$$ip_{1,t} = \frac{\beta}{\alpha^2}(p_{1,xx} + 4\sigma\delta|p_1|^2 p_1), \tag{44}$$

and the 3rd-order reduced integrable model is exactly the modified Korteweg-de Vries equation

$$p_{1,t} = -\frac{\beta}{\alpha^3}(p_{1,xxx} + 12\sigma\delta|p_1|^2 p_{1,x}). \tag{45}$$

To conclude, we have shown that the nonlinear Schrödinger equation and the modified Korteweg-de Vries equation possess different 3×3 matrix Lax pairs, which amend the 2×2 matrix Lax pairs in the existing literature [1].

4.2 Case of $m = 2$ and $n = 2$

In the case of $m = n = 2$, we present a few examples below.

Let us generally take

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_4 & 0 \end{bmatrix},\ \Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} 0 & \delta_3 \\ \delta_4 & 0 \end{bmatrix}, \tag{46}$$

where each of σ_i and δ_i takes on values of either 1 or -1 and all of them satisfy

$$\delta_1\delta_2\delta_3\delta_4 = 1,\ \sigma_1\sigma_2\sigma_3\sigma_4 = 1, \tag{47}$$

which come from the two group reductins, then we have

$$p = \begin{bmatrix} p_1 & p_2 \\ \delta_2\delta_3 p_2 & \delta_2\delta_4 p_1 \end{bmatrix},\ q = \begin{bmatrix} \delta_2\delta_4\sigma_2\sigma_4 p_1^* & \sigma_1\sigma_4 p_2^* \\ \delta_2\delta_3\sigma_2\sigma_3 p_2^* & \sigma_1\sigma_3 p_1^* \end{bmatrix}, \tag{48}$$

and so the reduced matrix spectral problem takes the form

$$-i\phi_x = \begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 \\ 0 & \alpha_1\lambda & \delta_2\delta_3 p_2 & \delta_2\delta_4 p_1 \\ \delta_2\delta_4\sigma_2\sigma_4 p_1^* & \sigma_1\sigma_4 p_2^* & \alpha_2\lambda & 0 \\ \delta_2\delta_3\sigma_2\sigma_3 p_2^* & \sigma_1\sigma_3 p_1^* & 0 & \alpha_2\lambda \end{bmatrix} \phi. \tag{49}$$

Particularly, if we firstly take

$$\begin{cases} \delta_1 = -\delta_2 = \delta_3 = -\delta_4 = 1, \\ \sigma_1 = \sigma_2 = \pm\sigma_3 = \pm\sigma_4 = 1, \end{cases} \tag{50}$$

then we have

$$p = \begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix},\ q = \pm p^* = \pm \begin{bmatrix} p_1^* & p_2^* \\ -p_2^* & p_1^* \end{bmatrix}. \tag{51}$$

The second example here is the reduction analyzed in [26]. The two coupled nonlinear Schrödinger integrable models read

$$\begin{cases} i p_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} \pm 2(|p_1|^2 - 2|p_2|^2)p_1 \mp 2p_2^2 p_1^*], \\ i p_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} \mp 2(|p_2|^2 - 2|p_1|^2)p_2 \pm 2p_1^2 p_2^*], \end{cases} \tag{52}$$

and the two coupled modified Korteweg-de Vries integrable models are

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} \pm 6(|p_1|^2 - |p_2|^2)p_{1,x} \mp 6(p_1 p_2^* + p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} \pm 6(p_1 p_2^* + p_1^* p_2)p_{1,x} \pm 6(|p_1|^2 - |p_2|^2)p_{2,x}]. \end{cases} \tag{53}$$

If we secondly take

$$\begin{cases} \delta_1 = -\delta_2 = \delta_3 = -\delta_4 = 1, \\ \sigma_1 = -\sigma_2 = \pm\sigma_3 = \mp\sigma_4 = -1, \end{cases} \tag{54}$$

then we have

$$p = \begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix}, \; q = \pm p^\dagger = \pm \begin{bmatrix} p_1^* & -p_2^* \\ p_2^* & p_1^* \end{bmatrix}. \tag{55}$$

The second example above is the reduction discussed in [27, 28]. The two coupled nonlinear Schrödinger integrable models read

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} \pm 2(|p_1|^2 + 2|p_2|^2)p_1 \mp 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} \pm 2(|p_2|^2 + 2|p_1|^2)p_2 \mp 2p_1^2 p_2^*], \end{cases} \tag{56}$$

and the two coupled modified Korteweg-de Vries integrable models are

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} \pm 6(|p_1|^2 + |p_2|^2)p_{1,x} \pm 6(p_1 p_2^* - p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} \mp 6(p_1 p_2^* - p_1^* p_2)p_{1,x} \pm 6(|p_1|^2 + |p_2|^2)p_{2,x}]. \end{cases} \tag{57}$$

If we thirdly take

$$\delta_1 = -\delta_2 = -\delta_3 = \delta_4 = 1, \tag{58}$$

then we have

$$p = \begin{bmatrix} p_1 & p_2 \\ p_2 & -p_1 \end{bmatrix}. \tag{59}$$

Further, let us take

$$\begin{cases} \sigma_1 = -\sigma_2 = -\sigma_3 = \sigma_4 = 1; \\ \sigma_1 = \sigma_2 = -\sigma_3 = -\sigma_4 = 1; \\ \sigma_1 = -\sigma_2 = \sigma_3 = -\sigma_4 = 1; \end{cases} \tag{60}$$

and then we have

$$q = \begin{bmatrix} p_1^* & p_2^* \\ p_2^* & -p_1^* \end{bmatrix}, \; q = \begin{bmatrix} p_1^* & -p_2^* \\ -p_2^* & -p_1^* \end{bmatrix}, \; q = \begin{bmatrix} -p_1^* & -p_2^* \\ -p_2^* & p_1^* \end{bmatrix}, \tag{61}$$

respectively. All three corresponding coupled nonlinear Schrödinger integrable models read

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} + 2(|p_1|^2 + 2|p_2|^2)p_1 - 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} + 2(|p_2|^2 + 2|p_1|^2)p_2 - 2p_1^2 p_2^*], \end{cases} \tag{62}$$

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} + 2(|p_1|^2 - 2|p_2|^2)p_1 - 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} - 2(|p_2|^2 - 2|p_1|^2)p_2 + 2p_1^2 p_2^*], \end{cases} \tag{63}$$

and

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} - 2(|p_1|^2 + 2|p_2|^2)p_1 + 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} - 2(|p_2|^2 + 2|p_1|^2)p_2 + 2p_1^2 p_2^*], \end{cases} \tag{64}$$

All three corresponding coupled modified Korteweg-de Vries integrable models are

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6(|p_1|^2 + |p_2|^2)p_{1,x} + 6(p_1 p_2^* - p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} - 6(p_1 p_2^* - p_1^* p_2)p_{1,x} + 6(|p_1|^2 + |p_2|^2)p_{2,x}], \end{cases} \tag{65}$$

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6(|p_1|^2 - |p_2|^2)p_{1,x} - 6(p_1 p_2^* + p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} + 6(p_1 p_2^* + p_1^* p_2)p_{1,x} + 6(|p_1|^2 - |p_2|^2)p_{2,x}], \end{cases} \tag{66}$$

and

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6(|p_1|^2 + |p_2|^2)p_{1,x} - 6(p_1 p_2^* - p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} + 6(p_1 p_2^* - p_1^* p_2)p_{1,x} - 6(|p_1|^2 + |p_2|^2)p_{2,x}]. \end{cases} \tag{67}$$

These three examples of coupled nonlinear Schrödinger and coupled modified Korteweg-de Vries integrable models are covered in the previous examples.

Finally, let us take

$$\delta_1 = \delta_2 = -\delta_3 = -\delta_4 = -1, \tag{68}$$

and then we have

$$p = \begin{bmatrix} p_1 & p_2 \\ -p_2 & -p_1 \end{bmatrix}. \tag{69}$$

Further, we choose

$$\begin{cases} \sigma_1 = \sigma_2, \ \sigma_3 = \sigma_4, \ \sigma_1\sigma_3 = \pm 1; \\ \sigma_1 = -\sigma_2, \ \sigma_3 = -\sigma_4, \ \sigma_1\sigma_3 = \pm 1; \end{cases} \tag{70}$$

we get

$$q = \pm \begin{bmatrix} -p_1^* & p_2^* \\ -p_2^* & p_1^* \end{bmatrix}, \ q = \pm \begin{bmatrix} -p_1^* & -p_2^* \\ p_2^* & p_1^* \end{bmatrix}, \tag{71}$$

respectively. The two pairs of corresponding coupled nonlinear Schrödinger integrable models are as follows:

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} \mp 2(|p_1|^2 + 2|p_2|^2)p_1 \mp 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} \mp 2(|p_2|^2 + 2|p_1|^2)p_2 \mp 2p_1^2 p_2^*]; \end{cases} \tag{72}$$

and

$$\begin{cases} ip_{1,t} = \frac{\beta}{\alpha^2}[p_{1,xx} \mp 2(|p_1|^2 - 2|p_2|^2)p_1 \mp 2p_2^2 p_1^*], \\ ip_{2,t} = \frac{\beta}{\alpha^2}[p_{2,xx} \pm 2(|p_2|^2 - 2|p_1|^2)p_2 \pm 2p_1^2 p_2^*]. \end{cases} \tag{73}$$

The two pairs of corresponding coupled modified Korteweg-de Vries integrable models read

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} \mp 6(|p_1|^2 + |p_2|^2)p_{1,x} \mp 6(p_1 p_2^* + p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} \mp 6(p_1 p_2^* + p_1^* p_2)p_{1,x} \mp 6(|p_1|^2 + |p_2|^2)p_{2,x}]; \end{cases} \tag{74}$$

and

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} \mp 6(|p_1|^2 - |p_2|^2)p_{1,x} \pm 6(p_1 p_2^* - p_1^* p_2)p_{2,x}], \\ p_{2,t} = -\frac{\beta}{\alpha^3}[p_{2,xxx} \pm 6(p_1 p_2^* - p_1^* p_2)p_{1,x} \mp 6(|p_1|^2 - |p_2|^2)p_{2,x}]. \end{cases} \tag{75}$$

5 Conclusion and Remarks

Two group reductions have been discussed, which reduce the AKNS matrix spectral problems, and associated reduced AKNS matrix integrable hierarchies have been presented. Illustrative examples of reduced AKNS matrix spectral problems and associated reduced integrable models were given, which also show the diversity of Lax pairs that reduced integrable models could possess. One of the group reductions yields a constraint on the two matrix potentials, and the other leads to a constraint on one of the two matrix potentials. This is a different kind of pairs of group reductions from the ones discussed in the literature for the nonlinear Schrödinger equations [29–32] and the modified Korteweg-de Vries equations [33, 34].

It will be significantly important to construct soliton type solutions through various approaches, such as the Darboux transformation, the Hirota bilinear tool, Bäcklund transforms and the Wronskian determinant technique. Breather wave solutions [35, 36] and lump wave solutions [37–39] are particularly interesting. Moreover, Riemann-Hilbert problems are used to formulate soliton solutions to integrable models, both local and nonlocal. Reduced integrable models need to satisfy additional constraint conditions and so require special attention while formulating solitons. Applications of Riemann-Hilbert problems to reduced integrable models will be another intriguing problem worthy of further exploration.

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Data Availability All data generated or analyzed during this study are included in this published article.

Declarations

Competing Interests The author declares no competing interests.

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