Note: The first four questions are compulsory and the last question is optional. All four questions carry equal weight, and the total points are 100. Students may earn a maximum of 5 bonus points by attempting the last question.

1. (25 points) Let $A(t)$ be an $n \times n$ matrix of functions continuous for $t \in (-\infty, \infty)$. Suppose that $A(t)$ is $T$-periodic [i.e., $A(t + T) = A(t)$] for some constant $T > 0$. Show that $x'(t) = A(t)x(t)$ has a non-zero $T$-periodic solution if and only if 1 is an eigenvalue of $e^{TC} = U(t_0 + T, t_0)$.

2. (25 points) Show that the zero solution of the linear system $x'(t) = A(t)x(t)$ with

$$A(t) = \begin{bmatrix} -2 & e^{2t} \\ 0 & -1 \end{bmatrix}$$

on $[0, \infty)$ is unstable.

3. (25 points) Construct a real function $a(t)$ continuous on $[0, \infty)$ such that the zero solution of the scalar differential equation $x'(t) = a(t)x(t)$ on $[0, \infty)$ is stable but not uniformly stable.

4. (25 points) Consider the differential equation

$$x'' + 2x' + x = \frac{m}{1 + t^4}x,$$

where $m$ is a real constant. Show that (a) the zero solution on $[0, \infty)$ is uniformly stable for each $m$, and (b) there is a constant $m_0 > 0$ such that the zero solution on $[0, \infty)$ is uniformly asymptotically stable for each $|m| < m_0$.

5. (5 bonus points) Rewrite the scalar differential equation $x'' + ax' + bx = 0$, with two arbitrary real constant coefficients $a$ and $b$, as a system of linear differential equations, and then determine the stability properties of the zero solution on $[0, \infty)$.

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