Instruction: There are five questions and each is worth 20 points. Please show all work necessary to produce your solutions.

1. (20 points) Note that \( y = x \) would be a solution of

\[
(1 - x)y'' + xy' - y = 2(x - 1)^2
\]

if the right side were zero. Use this fact to find the general solution of the equation as given.

2. (20 points) Solve the integral equation for \( f(x) \):

\[
f(x) = e^{-|x|} + 2 \int_0^\infty f(y) \cos xy \, dy.
\]

3. (20 points) Use the separation of variables to solve the vibrating beam problem:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} &= 0, \quad 0 < x < 1, \quad t > 0, \\
u(x = 0, t) &= u(x = 1, t) = \frac{\partial^2 u}{\partial x^2}(x = 0, t) = \frac{\partial^2 u}{\partial x^2}(x = 1, t) = 0, \quad t > 0, \\
u(x, t = 0) &= f(x), \quad \frac{\partial u}{\partial t}(x, t = 0) = g(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

4. (20 points) Show that for the potential

\[
u(x) = \begin{cases} U_0, & -1 < x < 0, \\ 0, & x \leq -1, \quad x \geq 0,
\end{cases}
\]

where \( U_0 \) is a positive constant, the Sturm-Liouville problem

\[\psi'' + \{\lambda - u(x)\} \psi = 0\]

has no discrete eigenvalue.

5. (20 points) Solve the Cauchy problem of the Korteweg-de Vries equation on the whole \( x \)-axis by the inverse scattering transform:

\[
\begin{align*}
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= 0, \quad t > 0, \\
u(x, t = 0) &= -\frac{3}{2} \operatorname{sech}^2 \frac{x}{2}.
\end{align*}
\]

- End -