

A refined invariant subspace method and applications to evolution equations

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Abstract The invariant subspace method is refined to present more unity and more diversity of exact solutions to evolution equations. The key idea is to take subspaces of solutions to linear ordinary differential equations as invariant subspaces that evolution equations admit. A two-component nonlinear system of dissipative equations is analyzed to shed light on the resulting theory, and two concrete examples are given to find invariant subspaces associated with 2nd-order and 3rd-order linear ordinary differential equations and their corresponding exact solutions with generalized separated variables.

Keywords invariant subspace, generalized separation of variables, evolution equation

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1 Introduction

Exact solutions to differential equations are significantly important in exploring the nature of motion. Based on the classification of elementary functions, there are only three kinds of explicit elementary exact solutions, classified as soliton-like, positon-like and complexiton-like solutions [9, 12, 17, 18], to differential equations, both linear and nonlinear. Hirota direct method presents Wronskian and Pfaffian formulations of solutions to soliton equations, leading to solitons, positons and complexitons [6, 13, 14]. For general nonlinear partial differential equations, symmetry related methods (see, e.g., [3, 21, 22, 28]) provide powerful approaches to their exact solutions. Group-invariant solutions stemming from symmetries play crucial roles in studying asymptotical behavior, blow up phenomena and fractal properties of motion, and they can be used to justify numerical schemes of solving partial differential equations [1, 2, 20].

The invariant subspace method, recently proposed in [4, 27], is one of powerful approaches for constructing exact solutions to nonlinear evolution equations. Various invariant subspaces defined through linear ordinary differential equations have been presented for solving specific nonlinear evolution equations (see [5, 8, 25] and references therein). Indeed, the invariant subspace method generates many interesting exact solutions to nonlinear evolution equations in mechanics and physics and a systematical solution procedure was given by Galaktionov and Svirshchevskii in their book [5].

In particular, Galaktionov [4] utilized the invariant subspace method to generate exact solutions to nonlinear evolution equations with quadratic nonlinearities, and showed that exact positive solutions to the quasi-linear heat equations

$$u_t = (u^{-\frac{4}{3}}u_x)_x - au^{-\frac{1}{3}} + bu^{\frac{7}{3}} + cu,$$

where $a, b, c \in \mathbb{R}$ are constants, can be constructed through invariant subspaces of functions of the polynomial or trigonometric form, admitted by the spatial differential operator. Actually, evolution equations that admit invariant subspaces can be defined to be symmetries of given ordinary differential equations [25, 26]. Interestingly, the N -soliton solutions to soliton equations such as the KdV equation, the mKdV equation, the nonlinear Schrödinger equation and the sine-Gordon equation, derived by the Hirota bilinear method [6, 7], are all in a linear space of exponential functions under change of variables [5].

The invariant subspace method was also used to construct exact solutions to systems of nonlinear evolution equations. On the basis of the existence of invariant subspaces that systems of linear ordinary differential equations define, Qu and Zhu [23] classified the systems of nonlinear parabolic equations of the form

$$\begin{aligned} u_t &= [f(u, v)u_x + p(u, v)v_x]_x + r(u, v), \\ v_t &= [g(u, v)u_x + q(u, v)v_x]_x + s(u, v). \end{aligned}$$

Zhu and Qu [29] presented an estimation of maximal dimensions of invariant subspaces for two-component systems of nonlinear evolution equations, and Shen et al. [24] generalized this estimation to multi-component systems of nonlinear evolution equations, together with some classifications of the considered systems of nonlinear parabolic equations and computation of the exact solutions derived from the corresponding invariant subspaces.

In this paper, we would like to refine the invariant subspace method by taking invariant subspaces as subspaces of solution spaces to systems of linear ordinary differential equations. Note that a solution to an n -th-order ordinary differential equation may not satisfy another ordinary differential equation of order less than n . Our idea will generalize the invariant subspace method from the point of view of unity and diversity of invariant subspaces and exact solutions. A two-component nonlinear system of dissipative equations is analyzed carefully and a set of sufficient and necessary conditions is presented for the existence of invariant subspaces. Two concrete examples illustrate the effectiveness of the resulting refined theory in presenting exact solutions with generalized separated variables.

2 Refining the invariant subspace method

2.1 Scalar case

Let us consider a scalar evolution equation

$$u_t = F[u], \quad (2.1)$$

where $u = u(x, t)$ is a function of $x, t \in \mathbb{R}$ and F is a differential operator of order m :

$$F[u] = F(x, t, u, u_1, \dots, u_m), \quad u_i = \frac{\partial^i}{\partial x^i} u, \quad i \geq 0. \quad (2.2)$$

Let $n \geq 1$ be a given natural number. Take n linearly independent functions

$$f_1(x), \quad f_2(x), \quad \dots, \quad f_n(x),$$

and form an n -dimensional linear space

$$W_n = \mathcal{L}\{f_1(x), f_2(x), \dots, f_n(x)\} = \left\{ \sum_{i=1}^n C_i f_i(x) \mid C_i = \text{const}, 1 \leq i \leq n \right\}, \quad (2.3)$$

i.e., the linear span of $f_1(x), f_2(x), \dots, f_n(x)$ over \mathbb{R} or \mathbb{C} .

Definition 2.1. A finite-dimensional linear space W_n is said to be invariant with respect to a differential operator F , if $F[W_n] \subseteq W_n$, i.e., $F[u] \in W_n, \forall u \in W_n$.

Suppose that W_n is invariant with respect to a given differential operator F . Then there exist n functions $\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_n$ such that

$$F\left[\sum_{i=1}^n C_i f_i(x)\right] = \sum_{i=1}^n \tilde{F}_i(C_1, C_2, \dots, C_n) f_i(x) \quad (2.4)$$

for whatever constants C_1, C_2, \dots, C_n . It follows that the evolution equation (2.1) possesses a solution of the form

$$u(x, t) = \sum_{i=1}^n \phi_i(t) f_i(x), \quad (2.5)$$

if and only if $\phi_1, \phi_2, \dots, \phi_n$ satisfy a system of ordinary differential equations:

$$\frac{d\phi_i}{dt} = \tilde{F}_i(\phi_1, \phi_2, \dots, \phi_n), \quad 1 \leq i \leq n. \quad (2.6)$$

We usually take an invariant subspace W_n as the space of solutions to a given n -th-order linear ordinary differential equation

$$L[y] = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = 0, \quad y^{(i)} = D^i y, \quad D = \frac{d}{dx}, \quad i \geq 0, \quad (2.7)$$

where a_0, a_1, \dots, a_{n-1} are given continuous functions. The linearity of the above equation brings a good possibility to generate exact solutions to nonlinear evolution equations.

The above approach for constructing exact solutions of the form (2.5) is called the invariant subspace method [4, 27]. It is also called a generalized separation of variables [25], whose resulting solutions of the form (2.5) we call solutions with generalized separated variables.

To refine the invariant subspace method discussed above, let us consider a k -dimensional subspace W_k of the n -dimensional linear space W_n , and without loss of generality, we set

$$W_k = \mathcal{L}\{f_1(x), f_2(x), \dots, f_k(x)\} = \left\{ \sum_{i=1}^k C_i f_i(x) \mid C_i = \text{const}, 1 \leq i \leq k \right\}, \quad (2.8)$$

where $k \leq n$. The invariance condition $F[W_k] \subseteq W_k$ means that there exist k functions $\bar{F}_1, \bar{F}_2, \dots, \bar{F}_k$ such that

$$F\left[\sum_{i=1}^k C_i f_i(x)\right] = \sum_{i=1}^k \bar{F}_i(C_1, C_2, \dots, C_k) f_i(x) \quad (2.9)$$

for whatever constants C_1, C_2, \dots, C_k . This way, beginning with a similar system of ordinary differential equations

$$\frac{d\psi_i}{dt} = \bar{F}_i(\psi_1, \psi_2, \dots, \psi_k), \quad 1 \leq i \leq k, \quad (2.10)$$

we can engender a set of exact solutions to the evolution equation (2.1):

$$u = \sum_{i=1}^k \psi_i(t) f_i(x). \quad (2.11)$$

We call this approach a refined invariant subspace method. One of its advantages is that when $k < n$, the invariance condition $F[W_k] \subseteq W_k$ requires much less conditions on the evolution equation (2.1).

The following results are helpful in searching for conditions to guarantee the existence of invariant subspaces that nonlinear evolution equations admit.

Theorem 2.1. *Let $I = (a, b)$ be an open interval, $x_0 \in I$ be an arbitrary point and $m \geq 0$ be an integer. If a real function f defined on I satisfies*

$$f(x_0) = 0, \quad f'(x_0) = 0, \quad \dots, \quad f^{(m-1)}(x_0) = 0, \quad f^{(m)}(x_0) = d, \quad (2.12)$$

where d is nonzero, then $f, f', \dots, f^{(m)}$ are linearly independent over I .

Proof. If $m = 0$, the theorem is true, since based on (2.12), f is not a zero function. In what follows, let $m > 0$. Suppose that the theorem is not true. That is to say, there are an integer $0 \leq k \leq m$ and real constants c_i , $0 \leq i \leq m - k$, with $c_{m-k} \neq 0$ such that

$$c_0 f(x) + c_1 f'(x) + \cdots + c_{m-k} f^{(m-k)}(x) = 0, \quad x \in I. \quad (2.13)$$

This is an ordinary differential equation with constant coefficients. Thus, the solution f is analytic in I , and so, using the conditions for derivatives in (2.12), we have

$$f(x) = \frac{d}{m!} (x - x_0)^m + \sum_{i=m+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i. \quad (2.14)$$

Now, the coefficient of $(x - x_0)^k$ in the Taylor series expansion of the function on the left-hand side of (2.13) is $c_{m-k} d/k!$, which is not zero. This contradicts the linear dependence equation (2.13). Therefore, $f, f', \dots, f^{(m)}$ are linearly independent over I . \square

Theorem 2.2. Let $a_i(x)$, $0 \leq i \leq n - 1$, be real continuous functions on an open interval $I = (a, b)$. Then there exists a solution y to the linear ordinary differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

such that $y, y', \dots, y^{(n-1)}$ are linearly independent over I .

Proof. For a fixed $x_0 \in I$, let us consider a Cauchy problem of the linear ordinary differential equation (2.7) with the initial data

$$y(x_0) = 0, \quad y'(x_0) = 0, \quad \dots, \quad y^{(n-2)}(x_0) = 0, \quad y^{(n-1)}(x_0) = d, \quad (2.15)$$

where the real constant d is nonzero. The theory of ordinary differential equations tells that there exists a unique solution y defined on I to this Cauchy problem. It now follows from Theorem 2.1 with $m = n - 1$ that this solution y is the desired solution. \square

Theorem 2.2 tells that for an n -th-order linear ordinary differential equation, there always exists a solution y such that $y, y', \dots, y^{(n-1)}$ are linearly independent. Thus, in principle, for a given differential operator F , we can get necessary and sufficient conditions to guarantee the existence of an invariant subspace W_k , by collecting all coefficients of linearly independent terms, generated from the linearly independent functions $y, y', \dots, y^{(n-1)}$, in the invariance condition

$$D^n F[y] + a_{n-1}(x)D^{n-1}F[y] + \cdots + a_0(x)F[y] = 0, \quad y \in W_k, \quad (2.16)$$

and setting them to be zero. The process may be difficult and frustrating. However, from analyzing different terms involving $y, y', \dots, y^{(n-1)}$ in (2.16), we can always get sufficient conditions for the existence of invariant subspaces that F admits.

It should be interesting to note that W_k may not possibly be generated by a k -th-order linear ordinary differential equation. An example is given as follows:

$$W_1 = \mathcal{L}\{y\}, \quad y = e^{\lambda_1 x} + e^{\lambda_2 x}, \quad \lambda_{1,2} = \text{consts},$$

where y is a solution to the 2nd-order linear differential equation

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1 \lambda_2 y = 0,$$

but does not solve any 1st-order linear differential equation when $\lambda_1 \neq \lambda_2$. Therefore, our refinement does make sense in generalizing the invariant subspace method.

2.2 Multi-component case

In what follows, we adopt the notations

$$u_0^i = u^i(x, t), \quad u_j^i = \frac{\partial^j u^i(x, t)}{\partial x^j}, \quad 1 \leq i \leq q, \quad j \geq 1, \quad (2.17)$$

such that the discussion can be easily extended to cases of multiple spatial variables. A system of evolution equations is assumed to take the form

$$u_t = F[u] = (F^1[u], F^2[u], \dots, F^q[u])^T, \quad u = (u^1, u^2, \dots, u^q)^T, \quad (2.18)$$

where

$$F^i[u] = F^i(x, t, u^1, \dots, u^q, \dots, u_{m_i}^1, \dots, u_{m_i}^q), \quad 1 \leq i \leq q, \quad (2.19)$$

are given sufficiently smooth functions in the indicated variables. Therefore, for each $1 \leq i \leq q$, F^i can be viewed as a differential operator of order m_i .

Let W_{k_1, \dots, k_q} denote a linear space $W_{k_1}^1 \times \dots \times W_{k_q}^q$, with $W_{k_i}^i$ being defined by

$$W_{k_i}^i = \mathcal{L}\{f_1^i(x), \dots, f_{k_i}^i(x)\} = \left\{ \sum_{j=1}^{k_i} C_j^i f_j^i(x) \mid C_j^i = \text{const}, 1 \leq j \leq k_i \right\}, \quad 1 \leq i \leq q, \quad (2.20)$$

where for each $1 \leq i \leq q$, $f_1^i(x), \dots, f_{k_i}^i(x)$ are linearly independent. If the above vector differential operator F satisfies the invariance condition

$$F[u] \in W_{k_1, \dots, k_q}, \quad \forall u \in W_{k_1, \dots, k_q},$$

i.e.,

$$F^i[u] \in W_{k_i}^i, \quad \forall u \in W_{k_1, \dots, k_q}, \quad 1 \leq i \leq q, \quad (2.21)$$

then the vector differential operator F (or the system of evolution equations (2.18)) is said to admit an invariant subspace W_{k_1, \dots, k_q} , or W_{k_1, \dots, k_q} is said to be invariant under the given differential operator F . The above invariance condition (2.21) means that there exist functions \tilde{F}_j^i , $1 \leq j \leq k_i$, $1 \leq i \leq q$, such that

$$F^i \left[\sum_{j=1}^{k_1} C_j^1 f_j^1(x), \dots, \sum_{j=1}^{k_q} C_j^q f_j^q(x) \right] = \sum_{j=1}^{k_i} \tilde{F}_j^i(C_1^1, \dots, C_{k_1}^1, \dots, C_1^q, \dots, C_{k_q}^q) f_j^i(x), \quad (2.22)$$

where $1 \leq i \leq q$.

Now if a space W_{k_1, \dots, k_q} is admitted by the vector differential operator F , then the system of evolution equations (2.18) possesses an exact solution of the form

$$u^i = \sum_{j=1}^{k_i} C_j^i(t) f_j^i(x), \quad 1 \leq i \leq q, \quad (2.23)$$

if and only if the $C_j^i(t)$'s satisfy a system of ordinary differential equations:

$$\frac{dC_j^i}{dt} = \tilde{F}_j^i(C_1^1, \dots, C_{k_1}^1, \dots, C_1^q, \dots, C_{k_q}^q), \quad 1 \leq j \leq k_i, \quad 1 \leq i \leq q. \quad (2.24)$$

The last step is that for each $1 \leq i \leq q$, we take the space $W_{k_i}^i = \mathcal{L}\{f_1^i(x), \dots, f_{k_i}^i(x)\}$ as a subspace of solutions to an n_i -th-order linear ordinary differential equation:

$$L_i[y_i] = y_i^{(n_i)} + a_{n_i-1}^i(x) y_i^{(n_i-1)} + \dots + a_1^i(x) y_i' + a_0^i(x) y_i = 0, \quad (2.25)$$

where $n_i \geq k_i$. The invariance conditions for the subspace $W_{k_1, \dots, k_q} = W_{k_1}^1 \times \dots \times W_{k_q}^q$ with respect to $F = (F^1, \dots, F^q)^T$ read

$$D^{n_i} F^i[u] + a_{n_i-1}^i(x) D^{n_i-1} F^i[u] + \dots + a_0^i(x) F^i[u] = 0, \quad u \in W_{k_1, \dots, k_q}, \quad 1 \leq i \leq q. \quad (2.26)$$

This set of equations is our starting point to construct exact solutions to systems of evolution equations by looking for their invariant subspaces.

Note that the orders of linear ordinary differential equations defining invariant subspaces can not be arbitrary, and they are subject to the differential orders of the nonlinear operators F^i , $1 \leq i \leq q$. Once the maximal orders of the required linear ordinary differential equations are determined, we will be able to classify systems of evolution equations under consideration, and compute exact solutions from the associated invariant subspaces.

The problem of maximal orders of linear ordinary differential equations defining invariant subspaces was firstly posed and solved for the scalar case in [5]. For the scalar case, the maximal order of a linear ordinary differential equation defining an invariant subspace is not greater than $2m + 1$, where m is the order of the differential operator F in (2.1). For a q -component nonlinear really-coupled system defined by (2.18) and (2.19) with $m_1 \geq m_2 \geq \dots \geq m_q \geq 0$, the orders $\{n_1, \dots, n_q\}$ of linear ordinary differential equations defining invariant subspaces with $n_1 \geq n_2 \geq \dots \geq n_q > 0$ must satisfy [24]

$$n_{i-1} - n_i \leq m_i, \quad 2 \leq i \leq q, \quad n_1 \leq 2 \sum_{i=1}^q m_i + 1.$$

That the system defined by (2.18) and (2.19) is really-coupled means that for each pair $1 \leq i \neq j \leq q$, there exists an integer $0 \leq k \leq m_i$ such that $\frac{\partial F^i}{\partial u_k^j}$ is not a zero function. If F^i , $1 \leq i \leq q$, are all real, then the really-coupled condition can be concisely written as

$$\sum_{k=0}^{m_i} \left(\frac{\partial F^i}{\partial u_k^j} \right)^2 \neq 0, \quad 1 \leq i \neq j \leq q.$$

3 Invariant subspaces and exact solutions

In this section, we analyze a (1+1)-dimensional nonlinear system of dissipative equations to illustrate how to generate invariant subspaces and the corresponding exact solutions. We consider the following nonlinear system of dissipative equations:

$$u_t = F = (u_{xx} + \alpha_1 v v_x)_x + \alpha_2 v^2, \quad (3.1)$$

$$v_t = G = u_{xx} + \beta_1 u + \beta_2 v, \quad (3.2)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants, α_1, α_2 are not simultaneously equal to zero, and we have used the traditional notation

$$u_x = \frac{\partial u}{\partial x}, \quad v_x = \frac{\partial v}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad v_{xx} = \frac{\partial^2 v}{\partial x^2}, \quad \dots$$

Let us take an invariant subspace $W_{2,2} = W_2^1 \times W_2^2$ defined by

$$L_1[y] = y'' + a_1 y' + a_0 y = 0, \quad L_2[z] = z'' + b_1 z' + b_0 z = 0, \quad (3.3)$$

where a_0, a_1, b_0, b_1 are constants to be determined. The corresponding invariance conditions read

$$(D^2 F + a_1 D F + a_0 F)|_{u \in W_2^1, v \in W_2^2} = 0, \quad (3.4)$$

$$(D^2 G + b_1 D G + b_0 G)|_{u \in W_2^1, v \in W_2^2} = 0. \quad (3.5)$$

Substitute the expressions for F and G into the above equations, and replace u_{xx} and v_{xx} by $-a_1 u_x - a_0 u$ and $-b_1 v_x - b_0 v$ a few times, respectively. Then we collect the coefficients of $(v_x)^2, v v_x$ and v^2 in the first simplified equation and the coefficients of u_x and u in the second simplified equation, and set them to be zero, to obtain the sufficient conditions:

$$(v_x)^2 : 7\alpha_1 b_1^2 + \alpha_1 a_0 + 2\alpha_2 - 4\alpha_1 b_0 - 3\alpha_1 a_1 b_1 = 0, \quad (3.6)$$

$$vv_x : 12\alpha_1 b_1 b_0 - \alpha_1 a_0 b_1 - 4\alpha_1 a_1 b_0 + 2\alpha_2 a_1 - \alpha_1 b_1^3 - 2\alpha_2 b_1 + \alpha_1 a_1 b_1^2 = 0, \quad (3.7)$$

$$v^2 : 4\alpha_1 b_0^2 + \alpha_2 a_0 - \alpha_1 b_1^2 b_0 + \alpha_1 a_1 b_0 b_1 - \alpha_1 a_0 b_0 - 2\alpha_2 b_0 = 0, \quad (3.8)$$

$$u_x : -a_1^3 + 2a_0 a_1 - \beta_1 a_1 + a_1^2 b_1 - a_0 b_1 + \beta_1 b_1 - a_1 b_0 = 0, \quad (3.9)$$

$$u : -a_0 a_1^2 + a_0^2 - \beta_1 a_0 + a_0 a_1 b_1 - a_0 b_0 + \beta_1 b_0 = 0, \quad (3.10)$$

which guarantees the invariance conditions (3.4) and (3.5). We began with two second order differential equations, and so definitely there exist linearly dependent terms in $(v_x)^2$, vv_x and v^2 for whatever solution v , but u and u_x could be linearly independent (see Theorem 2.2). Therefore, the conditions (3.6)–(3.8) are sufficient but not necessary to guarantee the first invariance condition (3.4), but the conditions (3.9) and (3.10) are both necessary and sufficient to guarantee the second invariance condition (3.5).

Under the first condition (3.6), the second and third conditions, (3.7) and (3.8), are equivalent to

$$-\alpha_1 a_1 b_1^3 - 2\alpha_1 b_0 b_1^2 + 3\alpha_1 b_1^4 - \alpha_2 a_0 + \alpha_2 a_1 b_1 = 0, \quad (3.11)$$

$$-3\alpha_1 b_1^3 - \alpha_2 a_1 + \alpha_1 a_1 b_1^2 + 2\alpha_1 a_1 b_0 - 4\alpha_1 b_0 b_1 = 0. \quad (3.12)$$

This can be shown directly by using Maple and we will see later where they come from.

Let us now assume $\Delta_2 = b_1^2 - 4b_0 > 0$. Then

$$W_2^2 = \mathcal{L}\{e^{\lambda_+ x}, e^{\lambda_- x}\}, \quad (3.13)$$

where

$$\lambda_{\pm} = \frac{-b_1 \pm \sqrt{\Delta_2}}{2}.$$

Collecting the coefficients of three linearly independent terms $e^{(\lambda_+ + \lambda_-)x}$, $e^{2\lambda_+ x}$ and $e^{2\lambda_- x}$ in the first invariance condition (3.4) and setting them to be zero gives rise to

$$\gamma_1 = 0, \quad \gamma_2 \pm \sqrt{\Delta_2} \gamma_3 = 0, \quad (3.14)$$

respectively, where

$$\gamma_1 = \alpha_1 a_0 b_1^2 - \alpha_1 a_1 b_1^3 - 2\alpha_2 a_1 b_1 + \alpha_1 b_1^4 + 2\alpha_2 b_1^2 + 2\alpha_2 a_0, \quad (3.15)$$

$$\gamma_2 = -4\alpha_2 b_0 + \alpha_2 a_0 + 8\alpha_1 b_0^2 - 16\alpha_1 b_0 b_1^2 - 2\alpha_1 a_0 b_0 + 4\alpha_1 b_1^4 + 2\alpha_2 b_1^2 + 6\alpha_1 a_1 b_0 b_1 - 2\alpha_1 a_1 b_1^3 - \alpha_2 a_1 b_1 + \alpha_1 a_0 b_1^2, \quad (3.16)$$

$$\gamma_3 = -4\alpha_1 b_1^3 - 2\alpha_2 b_1 + \alpha_2 a_1 + 2\alpha_1 a_1 b_1^2 - 2\alpha_1 a_1 b_0 - \alpha_1 a_0 b_1 + 8\alpha_1 b_0 b_1. \quad (3.17)$$

The conditions in (3.14) equivalently lead to

$$\gamma_1 = 0, \quad \gamma_2 = 0, \quad \gamma_3 = 0. \quad (3.18)$$

These are necessary and sufficient conditions for guaranteeing the first invariance condition (3.4).

Under the first condition (3.6), $\gamma_2 = 0$ and $\gamma_3 = 0$ become the equations (3.11) and (3.12), respectively. When the three equations, (3.6), $\gamma_2 = 0$ and $\gamma_3 = 0$, hold, the condition $\gamma_1 = 0$ is automatically satisfied. Therefore, the conditions (3.6)–(3.8) yield (3.18). But conversely, it is not true. This is because the system (3.18) has a solution $\alpha_2 = b_0 = b_1 = 0$ with the other variables being arbitrary, but the left-hand side of (3.6) under $\alpha_2 = b_0 = b_1 = 0$ is $\alpha_1 a_0$, not always zero.

More specifically, under $\Delta_2 = b_1^2 - 4b_0 > 0$, let us take a smaller invariant subspace

$$W_{2,1} = W_2^1 \times W_1^2, \quad W_1^2 = \mathcal{L}\{e^{\mu x}\}, \quad (3.19)$$

where $\mu = \lambda_+$ (or $\mu = \lambda_-$). Then the invariance conditions

$$(D^2 F + a_1 D F + a_0 F)|_{u \in W_2^1, v \in W_1^2} = 0, \quad (3.20)$$

$$(D^2 G + b_1 D G + b_0 G)|_{u \in W_2^1, v \in W_1^2} = 0, \quad (3.21)$$

only require

$$\gamma_2 + \sqrt{\Delta_2} \gamma_3 = 0 \quad (\text{or } \gamma_2 - \sqrt{\Delta_2} \gamma_3 = 0) \quad (3.22)$$

plus (3.9) and (3.10). Any of these two conditions in (3.22) is much weaker than the conditions in (3.14), i.e., (3.18). Therefore, we can have a more general system of dissipative equations which still possesses exact solutions with generalized separated variables.

In what follows, we give two concrete examples of getting exact solutions with generalized separated variables.

Example 3.1. Let us consider a system

$$u_t = (u_{xx} + \alpha_1 v v_x)_x + \left(3\alpha_1 a_1 b_1 - \frac{9}{2} \alpha_1 b_1^2 - \frac{1}{2} \alpha_1 a_1^2 \right) v^2, \quad (3.23)$$

$$v_t = u_{xx} + \beta_1 u + \beta_2 v, \quad (3.24)$$

which admits an invariant subspace $W_{2,2}$ defined by

$$L_1[y] = y'' + a_1 y' + (a_1 b_1 - b_1^2) y = 0, \quad (3.25)$$

$$L_2[z] = z'' + b_1 z' + \left(a_1 b_1 - \frac{3}{4} b_1^2 - \frac{1}{4} a_1^2 \right) z = 0, \quad (3.26)$$

where a_1 can take any of the following three choices:

$$a_1 = b_1 \quad \text{or} \quad a_1 = \frac{4b_1^2 - \beta_1 \pm \sqrt{4b_1^4 - 20\beta_1 b_1^2 + \beta_1^2}}{6b_1}. \quad (3.27)$$

We analyze the case of $a_1 = b_1$, for which we have

$$u_t = (u_{xx} + \alpha_1 v v_x)_x - 2\alpha_1 a_1^2 v^2, \quad (3.28)$$

$$v_t = u_{xx} + \beta_1 u + \beta_2 v, \quad (3.29)$$

and

$$L_1[y] = y'' + a_1 y' = 0, \quad (3.30)$$

$$L_2[z] = z'' + a_1 z' = 0. \quad (3.31)$$

From these two equations $L_1[y] = 0$ and $L_2[z] = 0$, we obtain an invariant subspace

$$W_2^1 \times W_2^2 = \mathcal{L}\{1, e^{-a_1 x}\} \times \mathcal{L}\{1, e^{-a_1 x}\} \quad (3.32)$$

that the system of (3.28) and (3.29) admits. It then follows that an exact solution takes the form

$$u = C_1(t) + C_2(t)e^{-a_1 x}, \quad v = D_1(t) + D_2(t)e^{-a_1 x}. \quad (3.33)$$

Substituting this solution into the system of (3.28) and (3.29), we get the following system of ordinary differential equations:

$$C_1' = -2\alpha_1 a_1^2 D_1^2, \quad C_2' = -a_1^3 C_2 - 3\alpha_1 a_1^2 D_1 D_2, \quad (3.34)$$

$$D_1' = \beta_1 C_1 + \beta_2 D_1, \quad D_2' = a_1^2 C_2 + \beta_1 C_2 + \beta_2 D_2. \quad (3.35)$$

Based on the refined theory, let us further focus on a smaller invariant subspace

$$W_{1,1} = W_1^1 \times W_1^2 = \mathcal{L}\{e^{-a_1 x}\} \times \mathcal{L}\{e^{-a_1 x}\}, \quad (3.36)$$

to present exact solutions. Solving the system of (3.34) and (3.35) with $C_1 = D_1 = 0$, we arrive at

$$C_2 = c e^{-a_1^3 t}, \quad D_2 = d e^{\beta_2 t} - \frac{c(a_1^2 + \beta_1)}{a_1^3 + \beta_2} e^{-a_1^3 t}, \quad (3.37)$$

and then according to (3.33), we obtain an exact solution to the system of (3.28) and (3.29):

$$u = ce^{-a_1x-a_1^3t}, \quad v = de^{-a_1x+\beta_2t} - \frac{c(a_1^2 + \beta_1)}{a_1^3 + \beta_2} e^{-a_1x-a_1^3t}, \quad (3.38)$$

where c and d are arbitrary constants.

Example 3.2. Let us finally consider a system of the following evolution equations:

$$u_t = (u_x + \alpha_1 vv_x)_x - a_1^2 \alpha_1 v^2 - 3\alpha_1 vv_{xx}, \quad (3.39)$$

$$v_t = u_{xx} + a_1^2 u + \beta_2 v, \quad (3.40)$$

where $a_1 \neq 0, \alpha_1, \beta_2$ are arbitrary constants. This system admits an invariant subspace $W_{1,1}$ defined by

$$W_{1,1} = W_1^1 \times W_1^2 = \mathcal{L}\{\cos(a_1x)\} \times \mathcal{L}\{1 + \sin(a_1x)\}. \quad (3.41)$$

These two basis solutions $y = \cos(a_1x)$ and $z = 1 + \sin(a_1x)$ satisfy

$$L_1[y] = y'' + a_1^2 y = 0, \quad (3.42)$$

$$L_2[z] = z''' + a_1^2 z' = 0, \quad (3.43)$$

but they cannot satisfy any lower-order linear ordinary differential equations with constant coefficients.

Now, assuming a solution with the form

$$u = C(t) \cos(a_1x), \quad v = D(t)[1 + \sin(a_1x)], \quad (3.44)$$

and substituting back into the system of (3.39) and (3.40), we find

$$C'(t) = -a_1^2 C(t), \quad D'(t) = \beta_2 D(t). \quad (3.45)$$

The general solution of this system yields the following exact solution to the system of (3.39) and (3.40):

$$u = ce^{-a_1^2t} \cos(a_1x), \quad v = de^{\beta_2t}[1 + \sin(a_1x)], \quad (3.46)$$

where c and d are two arbitrary constants.

This gives us a concrete example which uses the refined invariant subspace method to construct exact solutions to nonlinear systems of evolution equations.

4 Concluding remarks

The invariant subspace method was refined by taking invariant subspaces as subspaces of solutions to linear ordinary differential equations. Our discussions were concentrated on how to identify necessary and sufficient conditions for the existence of invariant subspaces that nonlinear evolution equations admit. Two concrete examples illustrated the effectiveness of the refined approach for exploring solution structures of systems of nonlinear differential equations, which are notoriously more difficult to solve than scalar ones.

The invariant subspace method is also called a generalized separation of variables for nonlinear differential equations by Svirshchevskii [25], presenting a kind of complexiton-like solutions [10, 12]. It is interesting to see that the linear superposition principle takes on key role in constructing exact solutions to either evolution equations [5] or Hirota bilinear equations [15, 19]. All related theories furnish linear combination solutions of functions with separated variables, which shows a sort of integrability of nonlinear differential equations [11].

Motivated by the multiple exp-function method [16], we can also characterize invariant subspaces through linear partial differential equations. This will lead to more diverse situations of solutions with generalized separated variables and open a much larger research area.

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