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AN ALGEBRAIC STRUCTURE OF ZERO CURVATURE REPRESENTATIONS ASSOCIATED WITH COUPLED INTEGRABLE COUPLINGS AND APPLICATIONS TO τ-SYMMETRY ALGEBRAS

LIN LUO[*,§], WEN-XIU MA[†,¶] and ENGUI FAN[‡,‖]

[*] *Department of Mathematics, Shanghai Second Polytechnic University, Shanghai 201209, P. R. China*

[†] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA*

[‡] *School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. China*

[§] *luolinmath@yahoo.cn*

[¶] *mawx@cas.usf.edu*

[‖] *faneg@fudan.edu.cn*

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We establish an algebraic structure for zero curvature representations of coupled integrable couplings. The adopted zero curvature representations are associated with Lie algebras possessing two sub-Lie algebras in form of semi-direct sums of Lie algebras. By applying the presented algebraic structures to the AKNS systems, we give an approach for generating τ-symmetry algebras of coupled integrable couplings.

Keywords: Zero curvature representation; coupled integrable couplings; τ-symmetry algebra.

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1. Introduction

It is well-known that there is a close connection between the integrability properties of differential equations and their infinitely many symmetries. An extensive literature on this subject already exists, including continuous and discrete nonlinear equations.[1–21] Moreover, these symmetries form nice and interesting algebraic structures, such as: Virasoro algebras,[13,14] $W_{1+\infty}$ algebras,[20,21] master symmetry algebras[1,2] and so on.

In recent years, there has been an increasing interest in the theories of integrable couplings on the basis of the concept of semidirect sums of Lie algebras,[23–25] in particular, loop algebras.[26–28] Ones have constructed plenty of examples of both continuous and discrete integrable couplings for given classes of integrable equations.[22–29] The corresponding results show various mathematical structures that integrable equations possess, such as Lax representations, infinitely many symmetries, conserved quantities and bi-Hamiltonian structures, and also provide powerful tools to analyze integrable equations.

Very recently, Ma and Gao[31] put forward a new result on generating integrable couplings by coupled integrable couplings. Such new integrable couplings are associated with Lie algebras possessing two sub-Lie algebras in the form of semidirect sums of Lie algebras, and infinitely many commuting symmetries and recursion operators are presented for such coupled integrable couplings by Ma and Gao.[31] In this paper, we are concerned with an algebraic structure of zero curvature representations of the coupled integrable couplings and apply such a structure to the coupled integrable couplings of the AKNS systems to obtain their τ-symmetry algebras.

This paper is organized as follows. In Sec. 2, we briefly introduce coupled integrable couplings. In Sec. 3, we compute the Lie algebra of the corresponding enlarged vector fields under the enlarged commutator, and further, establish an algebraic structure of zero curvature representations associated with coupled integrable coupling systems. Finally, we apply such a structure to the coupled integrable couplings of the AKNS systems to propose an approach for generating τ-symmetry algebras of coupled integrable couplings.

2. Coupled Integrable Coupling Systems

Let us consider an integrable evolution equation

$$u_t = K = K(u) = K(x, t, u, u_x, u_{xx}, \ldots)\,, \tag{2.1}$$

where $x,\, t \in \mathbb{R}$ and $u = (u_1, u_2, \ldots, u_q)^T$ is a potential vector. Assume that it has a zero curvature representation

$$U_t - V_x + [U, V] = 0\,, \tag{2.2}$$

where the Lax matrices U and V belong to a matrix loop algebra g. This means that a triple (U, V, K) satisfies

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0\,, \tag{2.3}$$

where $\lambda_t = f(\lambda) \in C^\infty(\mathbb{C})$, $U_\lambda = \partial U/\partial\lambda$ and $U'(u)[K] = \partial/\partial\epsilon|_{\epsilon=0}U(u + \epsilon K)$.

Now let us assume Eq. (2.1) has two integrable couplings

$$\bar{u}_{1,t} = \bar{K}_1(\bar{u}_1) = \begin{pmatrix} K \\ K_1 \end{pmatrix}, \tag{2.4}$$

and

$$\bar{u}_{2,t} = \bar{K}_2(\bar{u}_2) = \begin{pmatrix} K \\ K_2 \end{pmatrix}, \tag{2.5}$$

and their corresponding zero curvature representations read as

$$\bar{U}_i'(\bar{u}_i)[\bar{K}_i] + f(\lambda)\bar{U}_{i\lambda} - \bar{V}_{ix} + [\bar{U}_i, \bar{V}_i] = 0\,, \tag{2.6}$$

with the Lax pairs being given by

$$\begin{aligned} \bar{U}_i &= \bar{U}_i(\bar{u}_i) = \begin{pmatrix} U(u) & U_i(\bar{u}_i) \\ 0 & U(u) \end{pmatrix}, \\ \bar{V}_i &= \bar{V}_i(\bar{u}_i) = \begin{pmatrix} V(u) & V_i(\bar{u}_i) \\ 0 & V(u) \end{pmatrix}, \end{aligned} \tag{2.7}$$

where $i = 1, 2$ and $\bar{u}_1 = (u^T, v^T)^T$, $\bar{u}_2 = (u^T, w^T)^T$, $v = (v_1, v_2, \ldots, v_{q_1})^T$, $w = (w_1, w_2, \ldots, w_{q_2})^T$.

The algebraic structures for zero curvature representations (2.3) and (2.6) were discussed systematically in Refs. 13, 14 and 29.

To generate coupled integrable couplings of (2.4) and (2.5) as

$$\hat{u}_t = \hat{K}(\hat{u}) = \begin{pmatrix} K \\ K_1 \\ K_2 \end{pmatrix}, \quad \hat{u}_t = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{2.8}$$

let us now form a matrix Lie algebra $\hat{g}$ consisting of square matrices of the following block form as introduced in Ref. 31:

$$\hat{P} = \begin{pmatrix} P & 0 & P_1 \\ 0 & P & P_2 \\ 0 & 0 & P \end{pmatrix}, \tag{2.9}$$

where P, P_1, P_2 are the same size square submatrices as U and V. This Lie algebra $\hat{g}$ has two sub-Lie algebras

$$\bar{g}_1 = \{\hat{P} | P_2 = 0\}, \quad \bar{g}_2 = \{\hat{P} | P_1 = 0\}, \tag{2.10}$$

which can be written as semidirect sums of sub-Lie algebras

$$\bar{g}_1 = \bar{g}_1|_{P_1=0} \Subset \bar{g}_1|_{P=0}, \quad \bar{g}_2 = \bar{g}_2|_{P_2=0} \Subset \bar{g}_2|_{P=0}, \tag{2.11}$$

and thus, the Lie algebra $\hat{g}$ is nonsemisimple.

So, the coupled integrable couplings (2.8) are determined by the following enlarged zero curvature representation

$$\hat{U}_t - \hat{V}_x + [\hat{U}, \hat{V}] = 0, \tag{2.12}$$

where

$$\hat{U} = \begin{pmatrix} U & 0 & U_1 \\ 0 & U & U_2 \\ 0 & 0 & U \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} V & 0 & V_1 \\ 0 & V & V_2 \\ 0 & 0 & V \end{pmatrix}. \tag{2.13}$$

This implies that an enlarged triple $(\hat{U}, \hat{V}, \hat{K})$ satisfies

$$\hat{U}'(\hat{u})[\hat{K}] + f(\lambda)\hat{U}_\lambda - \hat{V}_x + [\hat{U}, \hat{V}] = 0, \quad \lambda_t = f(\lambda), \tag{2.14}$$

with

$$\hat{U}'(\hat{u})[\hat{K}] = \begin{pmatrix} \frac{\partial}{\partial\epsilon}U(u+\epsilon K)\Big|_{\epsilon=0} & 0 & \frac{\partial}{\partial\epsilon}U_1(v+\epsilon K_1)\Big|_{\epsilon=0} \\ 0 & \frac{\partial}{\partial\epsilon}U(u+\epsilon K)\Big|_{\epsilon=0} & \frac{\partial}{\partial\epsilon}U_2(w+\epsilon K_2)\Big|_{\epsilon=0} \\ 0 & 0 & \frac{\partial}{\partial\epsilon}U(u+\epsilon K)\Big|_{\epsilon=0} \end{pmatrix}$$

$$= \begin{pmatrix} U'(u)[K] & 0 & U_1'(v)[K_1] \\ 0 & U'(u)[K] & U_2'(w)[K_2] \\ 0 & 0 & U'(u)[K] \end{pmatrix},$$

which precisely presents

$$\begin{cases} U'(u)[K] + f(\lambda)U_\lambda - V_x + [U,V] = 0\,, \\ U_1'(v)[K_1] + f(\lambda)U_{1\lambda} - V_{1x} + [U,V_1] + [U_1,V] = 0\,, \\ U_2'(w)[K_2] + f(\lambda)U_{2\lambda} - V_{2x} + [U,V_2] + [U_2,V] = 0\,. \end{cases} \tag{2.15}$$

3. Algebraic Structure of Zero Curvature Representation for Coupled Integrable Couplings

In this section, we aim to discuss the algebraic structure of zero curvature representations for coupled integrable coupling systems (2.8), i.e. the enlarged triple $(\hat{U}, \hat{V}, \hat{K})$ satisfying (2.14).

For the sake of convenience, let us first fix the notation as in Refs. 13 and 29. We denote by $\mathcal{B}$ all complex (or real) functions $P = P(x,t,u,v,w)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to u, v and w, and set $\mathcal{B}^r = \{(P_1,\ldots,P_r)^T | P_i \in \mathcal{B}\}$. Moreover, by $\mathcal{V}^r$, we denote all $r \times r$ matrix integrable-differential operators:

$$\mathcal{V}^r = \{(\Phi_{ij})_{r\times r} | \Phi_{ij} = \Phi_{ij}(x,t,u,v,w)\text{-integra-differential operators}, 1 \le i,\ j \le r\}\,,$$

and by $\mathcal{V}^r_{(0)}$, we denote all following $r \times r$ matrices:

$$\mathcal{V}^r_{(0)} = \{(P_{ij})_{r\times r} | P_{ij} = P_{ij}(x,t,u,v,w) \in \mathcal{B},\ 1 \le i,\ j \le r\}\,.$$

Define

$$\tilde{\mathcal{V}}^r = \mathcal{V}^r \otimes [\lambda, \lambda^{-1}]\,, \quad \tilde{\mathcal{V}}^r_{(0)} = \mathcal{V}^r_{(0)} \otimes [\lambda, \lambda^{-1}]\,.$$

We now set

$$\hat{K} = \begin{pmatrix} K \\ K_1 \\ K_2 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} S \\ S_1 \\ S_2 \end{pmatrix} \in \mathcal{B}^{q+q_1+q_2}\,, \tag{3.1}$$

where $K,\ S \in \mathcal{B}^q$, $K_1,\ S_1 \in \mathcal{B}^{q_1}$ and $K_2,\ S_2 \in \mathcal{B}^{q_2}$. The Gateaux derivative is defined as follows:

$$R'[\hat{K}] = \frac{\partial}{\partial\epsilon} R(u+\epsilon K, v+\epsilon K_1, w+\epsilon K_2)\Big|_{\epsilon=0}, \quad R \in \tilde{\mathcal{V}}^r \text{ or } \mathcal{B}^r\,, \tag{3.2}$$

and in particular, we have

$$K_1'[\hat{S}] = K_1'[S] + K_1'[S_1]\,, \quad S_1'[\hat{K}] = S_1'[K] + S_1'[K_1]\,, \tag{3.3}$$

$$K_2'[\hat{S}] = K_2'[S] + K_2'[S_2]\,, \quad S_2'[\hat{K}] = S_2'[K] + S_2'[K_2]\,, \tag{3.4}$$

By a direct computation, we have the following result:

Theorem 3.1. *Let $\Phi(u,v,\lambda) \in \tilde{\mathcal{V}}^r$ and $\hat{K},\ \hat{S} \in \mathcal{B}^{q+q_1+q_2}$. Then we have*

$$\begin{aligned}
&(\Phi'[\hat{K}])'[\hat{S}] - (\Phi'[\hat{S}])'[\hat{K}] \\
&\quad = \Phi'(u)[K'[S]-S'[K]] + \Phi'(v)[K_1'[S]-S_1'[K]+K_1'[S_1]-S_1'[K_1]] \\
&\qquad + \Phi'(w)[K_2'[S]-S_2'[K]+K_2'[S_2]-S_2'[K_2]]\,.
\end{aligned} \tag{3.5}$$

Thus, for $U_1 = U_1(v,\lambda)$, $U_2 = U_2(w,\lambda) \in \tilde{\mathcal{V}}^r_{(0)}$, we can obtain

$$(U_1'[K_1])'[\hat{S}] - (U_1'[S_1])'[\hat{K}] = U_1'[K_1'[S]-S_1'[K]+K_1'[S_1]-S_1'[K_1]], \tag{3.6}$$

and

$$(U_2'[K_2])'[\hat{S}] - (U_2'[S_2])'[\hat{K}] = U_2'[K_2'[S]-S_2'[K]+K_2'[S_2]-S_2'[K_2]]\,. \tag{3.7}$$

Here we have noted that $U_1 = U_1(v,\lambda)$ and $U_2 = U_2(w,\lambda)$ have nothing to do with the original potential vector u. Evidently, we can also compute the commutator of two enlarged vector fields $\bar{K},\ \bar{S} \in \mathcal{B}^{q+q_1+q_2}$ as follows:

$$[\hat{K},\hat{S}] \triangleq \hat{K}'[\hat{S}] - \hat{S}'[\hat{K}] = \begin{pmatrix} [K,S] \\ [K,S]_1 \\ [K,S]_2 \end{pmatrix}, \tag{3.8}$$

where

$$\begin{aligned}
[K,S] &= K'[S] - S'[K]\,, \\
[K,S]_1 &= K_1'[S] - S_1'[K] + K_1'[S_1] - S_1'[K_1]\,, \\
[K,S]_2 &= K_2'[S] - S_2'[K] + K_2'[S_2] - S_2'[K_2]\,.
\end{aligned}$$

which implies (3.8) *defines a Lie algebra structure over vector fields in $\mathcal{B}^{q+q_1+q_2}$.*

The commutator of two smooth functions $f,\ g \in C^\infty(\mathbb{C})$ (as vector fields over $\mathbb{C}$) is defined as

$$[\![f,g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda)\,, \quad \lambda \in \mathbb{C}\,, \tag{3.9}$$

which defines a Lie algebra structure over $C^\infty(\mathbb{C})$. In what follows, we always assume that the enlarged spectral operator $\hat{U} \in \tilde{\mathcal{V}}^{3r}_{(0)}$ has an injective Gateaux derivative operator $\hat{U}' : \mathcal{B}^{q+q_1+q_2} \to \tilde{\mathcal{V}}^{3r}_{(0)}$.

We further assume that $P(\hat{U})$ denotes all triple $(\hat{V}, \hat{K}, f) \in \tilde{\mathcal{V}}^{3r}_{(0)} \times \mathcal{B}^{q+q_1+q_2} \times C^\infty(\mathbb{C})$ satisfying Eq. (2.14), and for $f(\lambda) \in C^\infty(\mathbb{C})$, we set

$$M(\hat{U}, f) = \left\{\hat{V} \in \tilde{\mathcal{V}}^{3r}_{(0)} \middle| \exists \hat{K} \in \mathcal{B}^{q+q_1+q_2} \text{ so that } (\hat{V}, \hat{K}, f) \in P(\hat{U})\right\}, \tag{3.10}$$

$$EM(\hat{U}, f) = \left\{\hat{K} \in \mathcal{B}^{q+q_1+q_2} \middle| \exists \hat{V} \in M(\hat{U}, f) \text{ so that } (\hat{V}, \hat{K}, f) \in P(\hat{U})\right\}. \tag{3.11}$$

For $(\hat{V}, \hat{K}, f), (\hat{W}, \hat{S}, g) \in P(\hat{U})$, the product $[\![\hat{V}, \hat{W}]\!] \in \tilde{\mathcal{V}}^{3r}_{(0)}$ can be computed as follows (see Ref. 13):

$$\begin{aligned}[\![\hat{V}, \hat{W}]\!] &= \hat{V}'[\hat{S}] - \hat{W}'[\hat{K}] + [\hat{V}, \hat{W}] + g\hat{V}_\lambda - f\hat{W}_\lambda \\ &= \begin{pmatrix} [\![V, W]\!] & 0 & [\![V_1, W_1]\!] \\ 0 & [\![V, W]\!] & [\![V_2, W_2]\!] \\ 0 & 0 & [\![V, W]\!] \end{pmatrix},\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}[\![V, W]\!] &= V'[S] - W'[K] + [V, W] + gV_\lambda - fW_\lambda\,, \\ [\![V_1, W_1]\!] &= V_1'[\hat{S}] - W_1'[\hat{K}] + [V, W_1] + [V_1, W] + gV_{1\lambda} - fW_{1\lambda}\,, \\ [\![V_2, W_2]\!] &= V_2'[\hat{S}] - W_2'[\hat{K}] + [V, W_2] + [V_2, W] + gV_{2\lambda} - fW_{2\lambda}\,.\end{aligned} \tag{3.13}$$

This shows a special structure of the commutator of enlarged Lax operators and play a crucial role in our following computation.

Theorem 3.2. *Let $(\hat{V}, \hat{K}, f), (\hat{W}, \hat{S}, g) \in P(\hat{U})$. Then $([\![\hat{V}, \hat{W}]\!], [\bar{K}, \hat{S}], [\![f, g]\!])$ belongs to $P(\hat{U})$, too. That is to say*

$$\hat{U}'[[\hat{K}, \hat{S}]] + [\![f, g]\!](\lambda)\hat{U}_\lambda - [\![\hat{V}, \hat{W}]\!]_x + [\hat{U}, [\![\hat{V}, \hat{W}]\!]] = 0\,, \tag{3.14}$$

which is equivalent to the following three equations:

$$\begin{aligned}U'[[K, S]] + [\![f, g]\!](\lambda)U_\lambda - [\![V, W]\!]_x + [U, [\![V, W]\!]] &= 0\,, \\ U_1'[[K, S]_1] + [\![f, g]\!](\lambda)U_{1\lambda} - [\![V_1, W_1]\!]_x + [U, [\![V_1, W_1]\!]] + [U_1, [\![V, W]\!]] &= 0\,, \\ U_2'[[K, S]_2] + [\![f, g]\!](\lambda)U_{2\lambda} - [\![V_2, W_2]\!]_x + [U, [\![V_2, W_2]\!]] + [U_2, [\![V, W]\!]] &= 0\,.\end{aligned} \tag{3.15}$$

Proof. Since $(\hat{V}, \hat{K}, f), (\hat{W}, \hat{S}, g) \in P(\hat{U})$, we have

$$\begin{aligned}V_x'[S] &= (U'[K])'[S] + fU_\lambda'[S] + [U, V]'[S]\,, \\ W_x'[K] &= (U'[S])'[K] + gU_\lambda'[K] + [U, W]'[K]\,, \\ U_\lambda'[K] &= V_{x\lambda} - [U, V]_\lambda - f_\lambda U_\lambda - fU_{\lambda\lambda}\,, \\ U_\lambda'[S] &= W_{x\lambda} - [U, W]_\lambda - g_\lambda U_\lambda - gU_{\lambda\lambda}\,;\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
V'_{1x}[\hat{S}] &= (U'_1[K_1])'[\hat{S}] + fU'_{1\lambda}[\hat{S}] + [U, V_1]'[\hat{S}] + [U_1, V]'[\hat{S}]\,, \\
W'_{1x}[\hat{K}] &= (U'_1[S_1])'[\hat{K}] + gU'_{1\lambda}[\hat{K}] + [U, W_1]'[\hat{K}] + [U_1, W]'[\hat{K}]\,, \\
U'_{1\lambda}[\hat{K}] &= V_{1x\lambda} - [U, V_1]_\lambda - [U_1, V]_\lambda - f_\lambda U_{1\lambda} - fU_{1\lambda\lambda}\,, \\
U'_{1\lambda}[\hat{S}] &= W_{1x\lambda} - [U, W_1]_\lambda - [U_1, W]_\lambda - g_\lambda U_{1\lambda} - gU_{1\lambda\lambda}\,;
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
V'_{2x}[\hat{S}] &= (U'_1[K_2])'[\hat{S}] + fU'_{2\lambda}[\hat{S}] + [U, V_2]'[\hat{S}] + [U_2, V]'[\hat{S}]\,, \\
W'_{2x}[\hat{K}] &= (U'_2[S_2])'[\hat{K}] + gU'_{2\lambda}[\hat{K}] + [U, W_2]'[\hat{K}] + [U_2, W]'[\hat{K}]\,, \\
U'_{2\lambda}[\hat{K}] &= V_{2x\lambda} - [U, V_2]_\lambda - [U_2, V]_\lambda - f_\lambda U_{2\lambda} - fU_{2\lambda\lambda}\,, \\
U'_{2\lambda}[\hat{S}] &= W_{2x\lambda} - [U, W_2]_\lambda - [U_2, W]_\lambda - g_\lambda U_{2\lambda} - gU_{2\lambda\lambda}\,.
\end{aligned}
\tag{3.18}
$$

Let us define

$$
\begin{aligned}
\Theta &= \hat{V}'[\hat{S}] - \hat{W}'[\hat{K}] + [\hat{V}, \hat{W}] \\
&= \begin{pmatrix} Q & 0 & V'_1[\hat{S}] - W'_1[\hat{K}] + [V, W_1] + [V_1, W] \\ 0 & Q & V'_2[\hat{S}] - W'_2[\hat{K}] + [V, W_2] + [V_2, W] \\ 0 & 0 & Q \end{pmatrix},
\end{aligned}
\tag{3.19}
$$

where $Q = V'[S] - W'[K] + [V, W]$, then we have

$$
\begin{aligned}
\Theta_x - [\hat{U}, \Theta] &= \hat{V}'_x[\hat{S}] - \hat{W}'_x[\hat{K}] + [\hat{V}, \hat{W}]_x \\
&\quad - [\hat{U}, \hat{V}'[\hat{S}] - \hat{W}'[\hat{K}] + [\hat{V}, \hat{W}]] \\
&\triangleq \begin{pmatrix} \Omega & 0 & \Omega_1 \\ 0 & \Omega & \Omega_2 \\ 0 & 0 & \Omega \end{pmatrix},
\end{aligned}
\tag{3.20}
$$

where

$$
\Omega = U'[[K, S]] + fW_{x\lambda} - f[U, W_\lambda] - gV_{x\lambda} + g[U, V_\lambda] + [\![f, g]\!]U_\lambda\,, \tag{3.21}
$$

$$
\begin{aligned}
\Omega_1 = {} & V'_{1x}[\hat{S}] - W'_{1x}[\hat{K}] + [V, W_1]_x + [V_1, W]_x \\
& - [U, V_1[\hat{S}] - W'_1[\hat{K}] + [V, W_1] + [V_1, W]] \\
& - [U_1, V'[S] - W'[K] + [V, W]]\,,
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\Omega_2 = {} & V'_{2x}[\hat{S}] - W'_{2x}[\hat{K}] + [V, W_2]_x + [V_2, W]_x \\
& - [U, V_2[\hat{S}] - W'_2[\hat{K}] + [V, W_2] + [V_2, W]] \\
& - [U_2, V'[S] - W'[K] + [V, W]]\,.
\end{aligned}
\tag{3.23}
$$

Making full use of (3.16) and (3.17), we can compute that

$$\begin{aligned}\Omega_1 &= (U_1'[K_1])'[\hat{S}] - (U_1'[S_1])'[\hat{K}] + fU_{1\lambda}'[\hat{S}] - gU_{1\lambda}'[\hat{K}] + [U,V_1]'[\hat{S}] + [U_1,V]'[\hat{S}] \\ &\quad + [U_1'[K_1] + fU_{1\lambda} + [U,V_1] + [U_1,V], W] + [V_1, [U,W] + U'[S] + gU_\lambda] \\ &\quad - [U, V_1'[\hat{S}] - W_1'[\hat{K}] + [V,W_1] + [V_1,W]] \\ &\quad - [U_1, V'[S] - W'[K] + [V,W]] - [U,W_1]'[\hat{K}] - [U_1,W]'[\hat{K}] \\ &\quad + [[U,V] + U'[K] + fU_\lambda, W_1] + [V, U_1'[S_1] + gU_{1\lambda} + [U,W_1] + [U_1,W]] \\ &= U_1'[K_1'[S] - S_1'[K] + K_1'[S_1] - S_1'[K_1]] + fW_{1x\lambda} \\ &\quad - f[U,W_{1\lambda}] - f[U_1,W_\lambda] - gV_{1x\lambda} + g[U,V_{1\lambda}] + g[U_1,V_\lambda] + [\![f,g]\!]U_{1\lambda}\,. \end{aligned} \tag{3.24}$$

Similarly, from (3.16) and (3.18), we obtain that

$$\begin{aligned}\Omega_2 &= U_2'[K_2'[S] - S_2'[K] + K_2'[S_2] - S_2'[K_2]] + fW_{2x\lambda} - f[U,W_{2\lambda}] \\ &\quad - f[U_2,W_\lambda] - gV_{2x\lambda} + g[U,V_{2\lambda}] + g[U_2,V_\lambda] + [\![f,g]\!]U_{2\lambda}\,. \end{aligned} \tag{3.25}$$

On the other hand, according to (3.13) and (3.19), we have

$$\Theta = \begin{pmatrix} \tilde{Q} & 0 & [\![V_1,W_1]\!] + fW_{1\lambda} - gV_{1\lambda} \\ 0 & \tilde{Q} & [\![V_2,W_2]\!] + fW_{2\lambda} - gV_{2\lambda} \\ 0 & 0 & \tilde{Q} \end{pmatrix}, \tag{3.26}$$

where $\tilde{Q} = [\![V,W]\!] + fW_\lambda - gV_\lambda$. Thus, we obtain

$$\Theta_x - [\hat{U},\Theta] \triangleq \begin{pmatrix} \tilde{\Omega} & 0 & \tilde{\Omega}_1 \\ 0 & \tilde{\Omega} & \tilde{\Omega}_2 \\ 0 & 0 & \tilde{\Omega} \end{pmatrix}, \tag{3.27}$$

where

$$\tilde{\Omega} = [\![V,W]\!]_x + fW_{x\lambda} - gV_{x\lambda} - [U, [\![V,W]\!] + fW_\lambda - gV_\lambda]\,, \tag{3.28}$$

$$\begin{aligned}\tilde{\Omega}_1 &= [\![V_1,W_1]\!]_x - [U,[\![V_1,W_1]\!]] - [U_1,[\![V,W]\!]] + fW_{1x\lambda} - gV_{1x\lambda} \\ &\quad - f[U,W_{1\lambda}] + g[U,V_{1\lambda}] - f[U_1,W_\lambda] + g[U_1,V_\lambda]\,, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\tilde{\Omega}_2 &= [\![V_2,W_2]\!]_x - [U,[\![V_2,W_2]\!]] - [U_2,[\![V,W]\!]] + fW_{2x\lambda} - gV_{2x\lambda} \\ &\quad - f[U,W_{2\lambda}] + g[U,V_{2\lambda}] - f[U_2,W_\lambda] + g[U_2,V_\lambda]\,. \end{aligned} \tag{3.30}$$

Comparing Ω, Ω_1, Ω_2 with $\tilde{\Omega}$, $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, we immediately obtain (3.15) . Thus, (3.14) holds. This means that $([\![\hat{V},\hat{W}]\!], [\hat{K},\hat{S}], [\![f,g]\!])$ belongs to $P(\hat{U})$. The proof is completed. □

Remark. The first two equations in (3.15) is exactly the result presented in Refs. 13 and 29. The whole equality (3.15) is an application but also a generalization of the result in Refs. 13 and 29.

It follows from the above theorem that if two enlarged evolution equations $\hat{u}_t = \hat{K}, \hat{u}_t = \hat{S}, (\hat{K}, \hat{S} \in \mathcal{B}^{q+q_1+q_2})$ are the compatibility conditions of the spectral problems

$$\hat{\varphi}_x = \hat{U}\hat{\varphi}\,, \quad \hat{\varphi}_t = \hat{V}\hat{\varphi}\,, \qquad \hat{V} \in \tilde{\mathcal{V}}^r_{(0)}\,, \quad \lambda_t = p\lambda^m\,,$$
$$\hat{\varphi}_x = \hat{U}\hat{\varphi}\,, \quad \hat{\varphi}_t = \hat{W}\hat{\varphi}\,, \quad \hat{W} \in \tilde{\mathcal{V}}^r_{(0)}\,, \quad \lambda_t = q\lambda^n\,,$$

where p, q are constants and m, $n \geq 0$, respectively, then the product equation $\hat{u}_t = [\hat{K}, \hat{S}]$ is the compatibility condition of the following spectral problems

$$\hat{\varphi}_x = \hat{U}\hat{\varphi}\,, \quad \hat{\varphi}_t = [\![\hat{V}, \hat{W}]\!]\hat{\varphi}\,, \quad \lambda_t = ab(m-n)\lambda^{m+n-1}\,,$$

where

$$\begin{aligned}[\![\hat{V}, \hat{W}]\!] &= \hat{V}'[\hat{S}] - \hat{W}'[\hat{K}] + [\hat{V}, \hat{W}] + g\hat{V}_\lambda - f\hat{W}_\lambda \\ &= \begin{pmatrix} [\![V, W]\!] & 0 & [\![V_1, W_1]\!] \\ 0 & [\![V, W]\!] & [\![V_2, W_2]\!] \\ 0 & 0 & [\![V, W]\!] \end{pmatrix}.\end{aligned}$$

This will give us an approach for generating τ-symmetry algebras of coupled integrable couplings.

4. Application

In this section, we shall illustrate our construction process by a concrete example in the AKNS case and establish the corresponding τ-symmetry algebra.

4.1. *The isospectral and nonisospectral AKNS hierarchies*

The AKNS spectral problem is given by

$$\varphi_x = U\varphi\,, \quad U = U(u, \lambda) = \begin{pmatrix} -\lambda & u_1 \\ u_2 & \lambda \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{4.1}$$

where $u_i = u_i(x, t)$, $i = 1, 2$, are two dependent variables.

Suppose that the associated temporal spectral problem is as follows:

$$\varphi_t = V\varphi\,, \quad V = V(u, \lambda) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \tilde{\mathcal{V}}^2_{(0)}\,, \tag{4.2}$$

where $a = \sum_{j=0}^m a_j\lambda^{m-j}$, $b = \sum_{j=0}^m b_j\lambda^{m-j}$ and $c = \sum_{j=0}^m c_j\lambda^{m-j}$. Clearly, the compatibility condition $U_t - V_x + [U, V] = 0$ in this case gives equivalently

$$u_{1t} = b_x + 2\lambda b + 2u_1\partial^{-1}(u_1c - u_2b) - 2u_1\lambda_t x\,, \tag{4.3}$$

$$u_{2t} = c_x - 2\lambda c - 2u_2\partial^{-1}(u_1c - u_2b) + 2u_2\lambda_t x\,. \tag{4.4}$$

For the isospectral case ($\lambda_t = 0$), if we choose

$$b_0 = c_0 = 0\,, \quad b_1 = u_1\,, \quad c_1 = u_2\,, \tag{4.5}$$

and set integration constants to be zero, then the corresponding isospectral AKNS hierarchy reads (see Ref. 7):

$$u_t = K_m = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \end{pmatrix} = \Phi(u)^m K_0\,, \quad K_0 = \begin{pmatrix} -2u_1 \\ 2u_2 \end{pmatrix}, \quad m \geq 0\,, \tag{4.6}$$

where the hereditary operator $\Phi(u)$ is determined by

$$\Phi(u) = \begin{pmatrix} -\frac{1}{2}\partial + u_1\partial^{-1}u_2 & u_1\partial^{-1}u_1 \\ -u_2\partial^{-1}u_2 & \frac{1}{2}\partial - u_2\partial^{-1}u_1 \end{pmatrix}. \tag{4.7}$$

For the nonisospectral ($\lambda_t = \lambda^{m+1}$) case, if we choose

$$b_0 = u_1 x\,, \quad c_0 = u_2 x\,, \quad b_1 = -\frac{1}{2}(u_1 x)_x\,, \quad c_1 = \frac{1}{2}(u_2 x)_x\,, \tag{4.8}$$

then we can arrive at the nonisospectral AKNS hierarchy (see Ref. 7):

$$u_t = S_m = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \end{pmatrix} = \Phi(u)^m S_0\,, \quad S_0 = \begin{pmatrix} (u_1 x)_x \\ (u_2 x)_x \end{pmatrix}, \quad m \geq 0\,. \tag{4.9}$$

4.2. *Coupled integrable couplings of the isospectral and nonisospectral AKNS hierarchies*

Following Refs. 25 and 29, we know that each of the isospectral and nonisospectral AKNS hierarchies, (4.6) and (4.9), can have two hierarchies of integrable couplings:

$$\bar{K}_{1m} = \begin{pmatrix} K_m \\ K_{1m} \end{pmatrix}, \quad \bar{S}_{1m} = \begin{pmatrix} S_m \\ S_{1m} \end{pmatrix} \tag{4.10}$$

and

$$\bar{K}_{2m} = \begin{pmatrix} K_m \\ K_{2m} \end{pmatrix}, \quad \bar{S}_{2m} = \begin{pmatrix} S_m \\ S_{2m} \end{pmatrix}, \tag{4.11}$$

with the corresponding zero curvature equation being given by

$$\bar{U}_{it} - \bar{V}_{ix} + [\bar{U}_i, \bar{V}_i] = 0\,, \quad i = 1, 2\,, \tag{4.12}$$

where

$$\bar{U}_i = \begin{pmatrix} U & U_i \\ 0 & U \end{pmatrix}, \quad \bar{V}_i = \begin{pmatrix} V & V_i \\ 0 & V \end{pmatrix}, \quad i = 1, 2\,.$$

Now we define enlarged AKNS spectral problem as follows:[31]

$$\begin{aligned}
&\hat{\varphi}_x = \hat{U}\hat{\varphi}\,, \quad \hat{U} = \hat{U}(\hat{u},\lambda) = \begin{pmatrix} U & 0 & U_1 \\ 0 & U & U_2 \\ 0 & 0 & U \end{pmatrix}, \\
&U_1 = U_1(v) = \begin{pmatrix} 0 & v_1 \\ v_2 & 0 \end{pmatrix}, \\
&U_2 = U_2(w) = \begin{pmatrix} -1 & w_1 \\ w_2 & 1 \end{pmatrix},
\end{aligned} \tag{4.13}$$

where $v_i = v_i(x,t)$, $i = 1,2$, $w_i = w_i(x,t)$, $i = 1,2$ are new dependent variables and

$$\hat{u} = \left(u^T, v^T, w^T\right)^T = (u_1, u_2, v_1, v_2, w_1, w_2)^T\,.$$

The associated enlarged temporal spectral problem is assumed to be

$$\begin{aligned}
&\hat{\varphi}_t = \hat{V}\hat{\varphi}\,, \quad \hat{V} = \hat{V}(\hat{u},\lambda) = \begin{pmatrix} V & 0 & V_1 \\ 0 & V & V_2 \\ 0 & 0 & V \end{pmatrix} \in \tilde{\mathcal{V}}^6_{(0)}\,, \\
&V_1 = V_1(u,v) = \begin{pmatrix} e_1 & f_1 \\ g_1 & -e_1 \end{pmatrix}, \\
&V_2 = V_2(u,w) = \begin{pmatrix} e_2 & f_2 \\ g_2 & -e_2 \end{pmatrix}.
\end{aligned} \tag{4.14}$$

Then the corresponding enlarged zero curvature equation becomes

$$\begin{aligned}
U_t - V_x + [U,V] &= 0\,, \\
U_{1t} - V_{1x} + [U,V_1] + [U_1,V] &= 0\,, \\
U_{2t} - V_{2x} + [U,V_2] + [U_2,V] &= 0\,,
\end{aligned} \tag{4.15}$$

which are equivalent to

$$\begin{aligned}
u_{1t} &= b_x + 2\lambda b + 2u_1\partial^{-1}(u_1c - u_2b) - 2u_1\lambda_t x\,, \\
u_{2t} &= c_x - 2\lambda c - 2u_2\partial^{-1}(u_1c - u_2b) + 2u_2\lambda_t x\,, \\
v_{1t} &= f_{1x} + 2\lambda f_1 + 2u_1e_1 + 2b_1[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,, \\
v_{2t} &= g_{1x} - 2u_2e_1 - 2\lambda g_1 - 2v_2[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,, \\
w_{1t} &= f_{2x} + 2\lambda f_2 + 2u_1e_2 + 2w_3b + 2w_1[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,, \\
w_{2t} &= g_{2x} - 2\lambda g_2 - 2u_2e_2 - 2w_3c - 2w_2[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,.
\end{aligned} \tag{4.16}$$

Set

$$e_i = \sum_{j=0}^{m} e_{ij}\lambda^{m-j}\,, \quad f_i = \sum_{j=0}^{m} f_{ij}\lambda^{m-j}\,, \quad g_i = \sum_{j=0}^{m} g_{ij}\lambda^{m-j}\,, \quad i = 1,2\,, \tag{4.17}$$

and then, we can derive the isospectral and nonisospectral coupled integrable couplings for the isospectral and nonisospectral AKNS hierarchies, respectively.

(i) For the isospectral case ($\lambda_t = 0$), we choose

$$\begin{aligned} &f_{10} = g_{10} = 0\,, \quad f_{11} = u_1 + v_1\,, \quad g_{11} = u_2 + v_2\,, \\ &f_{20} = g_{20} = 0\,, \quad f_{21} = u_1 + w_1\,, \quad g_{21} = u_2 + w_2\,, \end{aligned} \tag{4.18}$$

then we obtain the isospectral coupled integrable couplings of the isospectral AKNS hierarchy:

$$\begin{aligned} \hat{u}_t &= \begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \hat{K}_m = \begin{pmatrix} K_m \\ K_{1m} \\ K_{2m} \end{pmatrix} \\ &= \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{1\ m+1} \\ 2g_{1\ m+1} \\ -2f_{2\ m+1} \\ 2g_{2\ m+1} \end{pmatrix} = \hat{\Phi}(\hat{u})^m \hat{K}_0\,, \quad m \geq 0\,. \end{aligned} \tag{4.19}$$

Here the hereditary operator $\hat{\Phi}(\hat{u})$ reads

$$\hat{\Phi}(\hat{u}) = \begin{pmatrix} \Phi(u) & 0 & 0 \\ \Phi_1(u,v) & \Phi(u) & 0 \\ \Phi_2(u,w) & 0 & \Phi(u) \end{pmatrix}, \tag{4.20}$$

with Φ_1 and Φ_2 being given by

$$\Phi_1(u,v) = \begin{pmatrix} v_1\partial^{-1}u_2 + u_1\partial^{-1}v_2 & v_1\partial^{-1}u_1 + u_1\partial^{-1}v_1 \\ -v_2\partial^{-1}u_2 - u_2\partial^{-1}v_2 & -v_2\partial^{-1}u_1 - u_2\partial^{-1}v_1 \end{pmatrix}, \tag{4.21}$$

and

$$\Phi_2(u,w) = \begin{pmatrix} w_1\partial^{-1}u_2 + u_1\partial^{-1}w_2 - 1 & w_1\partial^{-1}u_1 + u_1\partial^{-1}w_1 \\ -w_2\partial^{-1}u_2 - u_2\partial^{-1}w_2 & -w_2\partial^{-1}u_1 - u_2\partial^{-1}w_1 - 1 \end{pmatrix}, \tag{4.22}$$

and the initial coupled vector field is

$$\hat{K}_0 = \begin{pmatrix} K_0 \\ K_{10} \\ K_{20} \end{pmatrix} = \begin{pmatrix} -2u_1 \\ 2u_2 \\ -2u_1 - 2v_1 \\ 2u_2 + 2v_2 \\ -2u_1 - 2w_1 \\ 2u_2 + 2w_2 \end{pmatrix}. \tag{4.23}$$

(ii) For the nonisospectral case ($\lambda_t = \lambda^{m+1}$), we choose

$$\begin{aligned} &f_{10} = v_1 x\,, && g_{10} = v_2 x\,, \\ &f_{11} = -\frac{1}{2}(v_1 x)_x\,, && g_{11} = \frac{1}{2}(v_2 x)_x\,, \\ &f_{20} = w_1 x\,, && g_{20} = w_2 x\,, \\ &f_{21} = -\frac{1}{2}(w_1 x)_x, -u_1 x\,, && g_{21} = \frac{1}{2}(w_2 x)_x - u_2 x\,, \end{aligned} \tag{4.24}$$

then the nonisospectral coupled integrable couplings of the nonisospectral AKNS hierarchy reads

$$\hat{u}_t = \hat{S}_m = \begin{pmatrix} S_m \\ S_{1m} \\ S_{2m} \end{pmatrix} = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{1\,m+1} \\ 2g_{1\,m+1} \\ -2f_{2\,m+1} \\ 2g_{2\,m+1} \end{pmatrix} = \hat{\Phi}(\hat{u})^m \hat{S}_0\,, \quad m \geq 0\,, \tag{4.25}$$

where the nonisospectral ($\lambda_t = \lambda$) initial coupled vector field is

$$\hat{S}_0 = \begin{pmatrix} S_0 \\ S_{10} \\ S_{20} \end{pmatrix} = \begin{pmatrix} (u_1 x)_x \\ (u_2 x)_x \\ (v_1 x)_x \\ (v_2 x)_x \\ (w_1 x)_x + 2u_1 x \\ (w_2 x)_x - 2u_2 x \end{pmatrix}, \tag{4.26}$$

and $\hat{\Phi}(\hat{u})$ is defined as in (4.20).

Let us next consider how to compute the corresponding τ-symmetry algebra for the coupled integrable coupling systems generated above. As in Ref. 13, we first make the following computation at $\hat{u} = 0$

$$\begin{aligned} \hat{K}_m|_{\hat{u}=0} &= \left.\begin{pmatrix} K_m \\ K_{1m} \\ K_{2m} \end{pmatrix}\right|_{\hat{u}=0} = \hat{\Phi}^m \hat{K}_0|_{\hat{u}=0} = 0\,, \\ \hat{S}_n|_{\hat{u}=0} &= \left.\begin{pmatrix} S_n \\ S_{1n} \\ S_{2n} \end{pmatrix}\right|_{\hat{u}=0} = \hat{\Phi}^n \hat{S}_0|_{\hat{u}=0} = 0\,, \end{aligned} \tag{4.27}$$

where $m,\, n \geq 0$. We denote by $\hat{V}_m$ and $\hat{W}_n$ the Lax operators corresponding to the vector fields $\hat{K}_m$ and $\hat{S}_n$, respectively, we compute that

$$\hat{V}_m|_{\hat{u}=0} = \left.\begin{pmatrix} V_m & 0 & V_{1m} \\ 0 & V_m & V_{2m} \\ 0 & 0 & V_m \end{pmatrix}\right|_{\hat{u}=0} = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda^m ,$$

$$\hat{V}_{m\lambda}|_{\hat{u}=0} = (\hat{V}_m|_{\hat{u}=0})_\lambda = m \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda^{m-1} ,$$

$$\hat{W}_n|_{\hat{u}=0} = \left.\begin{pmatrix} W_n & 0 & W_{1n} \\ 0 & W_n & W_{2n} \\ 0 & 0 & W_n \end{pmatrix}\right|_{\hat{u}=0} = \begin{pmatrix} -\lambda x & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda x & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda x & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda x \end{pmatrix} \lambda^n ,$$

$$\hat{W}_{n\lambda}|_{\hat{u}=0} = (\hat{W}_n|_{\hat{u}=0})_\lambda = (n+1) \begin{pmatrix} -\lambda x & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda x & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda x & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda x \end{pmatrix} \lambda^{n-1} ,$$

where $m,\, n \geq 0$. Now we can find by the definition (3.12) of the product of Lax operators that

$$\begin{aligned} [\![\hat{V}_m, \hat{V}_n]\!]|_{\hat{u}=0} &= 0 , & m,n \geq 0 , \\ [\![\hat{V}_m, \hat{W}_n]\!]|_{\hat{u}=0} &= m\hat{V}_{m+n}|_{\hat{u}=0} , & m,n \geq 0 , \\ [\![\hat{W}_m, \hat{W}_n]\!]|_{\hat{u}=0} &= (m-n)\hat{W}_{m+n}|_{\hat{u}=0} , & m,n \geq 0 . \end{aligned} \tag{4.28}$$

Because $[\![\hat{V}_m, \hat{V}_n]\!]$, $[\![\hat{V}_m, \hat{W}_n]\!] - m\hat{V}_{m+n}$, $[\![\hat{W}_m, \hat{W}_n]\!] - (m-n)\hat{W}_{m+n}$, $m,\, n \geq 0$, are all isospectral ($\lambda_t = 0$) Lax operators belonging to $\tilde{\mathcal{V}}^6_{(0)}$ by Theorem 3.2, based upon (4.28), we obtain a Lax operator algebra by the uniqueness property of the enlarged spectral problem (4.13):

$$\begin{aligned} [\![\hat{V}_m, \hat{V}_n]\!] &= 0\,, & m, n \geq 0\,, \\ [\![\hat{V}_m, \hat{W}_n]\!] &= m\hat{V}_{m+n}\,, & m, n \geq 0\,, \\ [\![\hat{W}_m, \hat{W}_n]\!] &= (m-n)\hat{W}_{m+n}\,, & m, n \geq 0\,. \end{aligned} \tag{4.29}$$

Further, due to the injective property of $\hat{U}'$, we finally obtain a vector field algebra for the coupled integrable couplings of the isospectral and nonisospectral AKNS hierarchies:

$$\begin{aligned} [\hat{K}_m, \hat{K}_n] &= 0\,, & m, n \geq 0\,, \\ [\hat{K}_m, \hat{S}_n] &= m\hat{K}_{m+n}\,, & m, n \geq 0\,, \\ [\hat{S}_m, \hat{S}_n] &= (m-n)\hat{S}_{m+n}\,, & m, n \geq 0\,, \end{aligned} \tag{4.30}$$

which implies that $\hat{S}_n$, $n \geq 0$, are all master symmetries of each equation $\hat{u}_t = \hat{K}_l$, $l \geq 0$ in the isospectral hierarchy (4.19), and the symmetries

$$\hat{K}_m\,, \quad m \geq 0\,, \quad \text{and} \quad \hat{\tau}_n^l = t[\hat{K}_l\,, \quad \hat{S}_n] + \hat{S}_n = tl\hat{K}_{n+l} + \hat{S}_n\,, \quad n, l \geq 0\,, \tag{4.31}$$

constitute an infinite-dimensional τ-symmetry algebra, whose commutator satisfies

$$\begin{aligned} [\hat{K}_m, \hat{K}_n] &= 0\,, & m, n \geq 0\,, \\ [\hat{K}_m, \hat{\tau}_n^l] &= m\hat{K}_{m+n}\,, & m, n, l \geq 0\,, \\ [\hat{\tau}_m^l, \hat{\tau}_n^l] &= (m-n)\hat{\tau}_{m+n}^l\,, & m, n, l \geq 0\,. \end{aligned} \tag{4.32}$$

This is a union of infinitely many τ-symmetry algebras, offering a supplement to the structure of τ-algebras introduced in Ref. 9. However, it is still an open question to us if the variational identities[30] can generate Hamiltonian structures for the isospectral coupled AKNS hierarchy (4.19).

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