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THE ALGEBRAIC STRUCTURE OF ZERO CURVATURE REPRESENTATIONS ASSOCIATED WITH INTEGRABLE COUPLINGS

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The commutator of enlarged vector fields was explicitly computed for integrable coupling systems associated with semidirect sums of Lie algebras. An algebraic structure of zero curvature representations is then established for such integrable coupling systems. As an application example of this algebraic structure, the commutation relations of Lax operators corresponding to the enlarged isospectral and nonisospectral AKNS flows are worked out, and thus a τ-symmetry algebra for the AKNS integrable couplings is engendered from this theory.

Keywords: Zero curvature representation; integrable couplings; τ-symmetry algebra.

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1. Introduction

The theories of integrable couplings have been receiving increasing attention in recent decades,[1,2] among which is an algebraic approach to integrable couplings recently presented on base of the concept of semidirect sums of Lie algebra.[3,4] We notice that an arbitrary Lie algebra can be decomposed into a semidirect sum of a solvable Lie algebra and a semisimple Lie algebra.[5] There exist plenty of examples of both continuous and discrete integrable couplings belonging to such a class of integrable equations.[1–4,6–13] The corresponding results show various mathematical structures that integrable equations possess, such as Lax representations, infinitely many symmetries, conserved quantities and bi-Hamiltonian structures, etc., which also provide powerful tools to analyze integrable equations.

Let G be a matrix loop algebra and we assume that a pair of matrix spectral problems

$$\varphi_x = U\varphi = U(u,\lambda)\varphi\,, \quad \varphi_t = V\varphi = V(u,\lambda)\varphi\,, \quad \lambda_t = f(\lambda)\,, \tag{1.1}$$

where $x,\ t \in \mathbb{R}$, $u = u(x,t) = (u_1(x,t),\ u_2(x,t), \ldots, u_q(x,t))^T$ are the potentials, $U,\ V \in G$ are of same order square matrices, which are called a Lax pair and $\lambda \in \mathbb{C}$ is a spectral parameter and $f(\lambda) \in C^\infty(\mathbb{C})$, determines an integrable equation

$$u_t = K = K(x,t,u)\,, \tag{1.2}$$

through the zero curvature equation

$$U_t - V_x + [U,V] = 0\,. \tag{1.3}$$

This means that a triple (U,V,K) satisfies

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U,V] = 0\,, \tag{1.4}$$

where $U_\lambda = \partial U/\partial\lambda$ and $U'(u)[K]$ denotes the Gateaux derivative

$$U'(u)[K] = \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} U(u+\epsilon K)\,. \tag{1.5}$$

The Lie algebraic structure for such triple (U,V,K) has been discussed[14] and applied to nonisospectral flows.[15]

To generate integrable couplings of Eq. (1.2), take a semidirect sum of G with another matrix loop algebra G_c as introduced in Refs. 3 and 4:

$$\bar{G} = G \Subset G_c\,. \tag{1.6}$$

Note that the conception of semidirect sums implies that G and G_c satisfy

$$[G, G_c] \subseteq G_c\,,$$

where $[G,G_c] = \{[A,B] | A \in G, B \in G_c\}$. Obviously, G_c is an ideal Lie subalgebra of $\bar{G}$. Here and hereafter, the subscript c indicates a contribution to the construction of integrable couplings. Then, choose a pair of enlarged matrix spectral problems of initial matrix spectral problems (1.1) as follows:

$$\bar{\varphi}_x = \bar{U}\varphi = \bar{U}(\bar{u},\lambda)\varphi\,, \quad \bar{\varphi}_t = \bar{V}\varphi = \bar{V}(\bar{u},\lambda)\varphi\,, \quad \lambda_t = f(\lambda)\,, \tag{1.7}$$

where

$$\bar{u} = (u^T, v^T)^T\,, \quad u = u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_q(x,t))^T\,,$$
$$v = v(x,t) = (v_1(x,t), v_2(x,t), \ldots, v_{q_c}(x,t))^T\,,$$

and

$$\bar{U} = \begin{pmatrix} U & U_c \\ 0 & U \end{pmatrix}, \qquad \bar{V} = \begin{pmatrix} V & V_c \\ 0 & V \end{pmatrix}, \tag{1.8}$$

with $U_c = U_c(v,\lambda)$ and $V_c = V_c(u,v,\lambda)$ have the same size as U and V.

Thus, the corresponding enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U},\bar{V}] = 0\,, \tag{1.9}$$

determines an integrable coupling system for Eq. (1.2):

$$\bar{u}_t = \bar{K} = \bar{K}(x,t,\bar{u}) = \begin{pmatrix} K \\ K_c \end{pmatrix} = \begin{pmatrix} K(x,t,u) \\ K_c(x,t,u,v) \end{pmatrix}. \tag{1.10}$$

This means that an enlarged triple $(\bar{U},\bar{V},\bar{K})$ satisfies

$$\bar{U}'(\bar{u})[\bar{K}] + f(\lambda)\bar{U}_\lambda - \bar{V}_x + [\bar{U},\bar{V}] = 0\,, \tag{1.11}$$

where

$$\begin{aligned} \bar{U}'(\bar{u})[\bar{K}] &= \begin{pmatrix} \left.\dfrac{\partial}{\partial\epsilon}U(u+\epsilon K)\right|_{\epsilon=0} & \left.\dfrac{\partial}{\partial\epsilon_c}U_c(v+\epsilon_c K_c)\right|_{\epsilon_c=0} \\ 0 & \left.\dfrac{\partial}{\partial\epsilon}U(u+\epsilon K)\right|_{\epsilon=0} \end{pmatrix} \\ &= \begin{pmatrix} U'(u)[K] & U_c'(v)[K_c] \\ 0 & U'(u)[K] \end{pmatrix}, \end{aligned}$$

and precisely presents

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U,V] = 0\,, \tag{1.12a}$$

$$U_c'(v)[K_c] + f(\lambda)U_{c\lambda} - V_{cx} + [U,V_c] + [U_c,V] = 0\,. \tag{1.12b}$$

Stimulated by Refs. 14 and 15, in the present paper, we will be interested in the algebraic structure of integrable couplings of Eq. (1.2), i.e. the enlarged triple $(\bar{U},\bar{V},\bar{K})$ satisfying (1.11). This paper is organized as follows. In Sec. 2, we will show that the enlarged vector fields constitute a Lie algebra under the enlarged commutator. In Sec. 3, we will establish the algebraic structure of zero curvature representations associated with integrable coupling systems. Finally, we will illustrate our approach by an application example for the AKNS soliton hierarchy and a τ-symmetry algebra will be engendered for the enlarged AKNS soliton hierarchy presented in Ref. 11.

2. The Lie Algebra of Enlarged Vector Fields

For convenience, let us first fix the notation as follows: we denote by $\mathcal{B}$ all complex (or real) functions $P = P(x,t,u,v)$ which are C^∞-differentiable with respect to x,t and C^∞-Gateaux differentiable with respect to u and v, and set

$\mathcal{B}^r = \{(P_1, \ldots, P_r)^T | P_i \in \mathcal{B}\}$. Moreover, by $\mathcal{V}^r$, we denote all $r \times r$ matrix integra-differential operators:

$$\mathcal{V}^r = \bigcup_{n=-\infty}^{\infty} \sum_{k=0}^{n} \mathcal{V}^r_{(k)}\,, \quad \mathcal{V}^r_{(k)} = \left\{(P_{ij}\partial^k)_{r\times r}\,|\, P_{ij} = P_{ij}(x,t,u,v) \in \mathcal{B}\right\}.$$

Define

$$\tilde{\mathcal{V}}^r = \mathcal{V}^r \otimes [\lambda, \lambda^{-1}]\,, \qquad \tilde{\mathcal{V}}^r_{(0)} = \mathcal{V}^r_{(0)} \otimes [\lambda, \lambda^{-1}]\,.$$

We now set

$$\bar{K} = \begin{pmatrix} K \\ K_c \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} S \\ S_c \end{pmatrix} \in \mathcal{B}^{q+q_c}\,, \tag{2.1}$$

where K, $S \in \mathcal{B}^q$ and K_c, $S_c \in \mathcal{B}^{q_c}$. The Gateaux derivative is defined as follows:

$$R'[\bar{K}] = \frac{\partial}{\partial \delta} R(u + \delta K, v + \delta K_c)\bigg|_{\delta=0}\,, \qquad R \in \tilde{\mathcal{V}}^r \quad \text{or} \quad \mathcal{B}^r\,, \tag{2.2}$$

and in particular, we have

$$K'[S] = \frac{\partial}{\partial \delta} K(u + \delta S)\bigg|_{\delta=0}\,, \quad S'[K] = \frac{\partial}{\partial \epsilon} S(u + \epsilon K)\bigg|_{\epsilon=0}\,, \tag{2.3a}$$

$$\begin{aligned} K_c'[\bar{S}] &= K_c'[S] + K_c'[S_c] \\ &= \frac{\partial}{\partial \delta} K_c(u + \delta S)\bigg|_{\delta=0} + \frac{\partial}{\partial \delta_c} K_c(v + \delta_c S_c)\bigg|_{\delta_c=0}\,, \end{aligned} \tag{2.3b}$$

$$\begin{aligned} S_c'[\bar{K}] &= S_c'[K] + S_c'[K_c] \\ &= \frac{\partial}{\partial \epsilon} S_c(u + \epsilon K)\bigg|_{\epsilon=0} + \frac{\partial}{\partial \epsilon_c} S_c(v + \epsilon_c K_c)\bigg|_{\epsilon_c=0}\,. \end{aligned} \tag{2.3c}$$

Lemma 2.1. *Let $P = P(x,t,u,v) \in \mathcal{B}$, $\bar{K} = \left(\begin{smallmatrix} K \\ K_c \end{smallmatrix}\right)$, $\bar{S} = \left(\begin{smallmatrix} S \\ S_c \end{smallmatrix}\right) \in \mathcal{B}^{q+q_c}$. Then we have the relation*

$$\begin{aligned} &(P'[\bar{K}])'[\bar{S}] - (P'[\bar{S}])'[\bar{K}] \\ &\quad = P'(u)[K'[S] - S'[K]] + P'(v)[K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c]]\,. \end{aligned} \tag{2.4}$$

Proof. By the definition of the Gateaux derivative, we have

$$\begin{aligned} (P'[\bar{K}])'[\bar{S}] &= \left(\frac{\partial}{\partial \epsilon} P(u + \epsilon K)\bigg|_{\epsilon=0} + \frac{\partial}{\partial \epsilon_c} P(v + \epsilon_c K_c)\bigg|_{\epsilon_c=0}\right)'[\bar{S}] \\ &= \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon} P(u + \delta S + \epsilon K(u + \delta S))\bigg|_{\delta=\epsilon=0} \\ &\quad + \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon_c} P(v + \epsilon_c K_c(u + \delta S))\bigg|_{\delta=\epsilon_c=0} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial}{\partial\delta_c}\frac{\partial}{\partial\epsilon_c}P(v+\delta_cS_c+\epsilon_cK_c(v+\delta_cS_c))\Big|_{\delta_c=\epsilon_c=0}\\
=&\ \frac{\partial}{\partial\delta}\,\frac{\partial}{\partial\epsilon}P(u+\delta S+\epsilon K+\epsilon\delta K'[S])\Big|_{\delta=\epsilon=0}\\
&+\frac{\partial}{\partial\delta}\,\frac{\partial}{\partial\epsilon_c}P(v+\epsilon_cK_c+\epsilon_c\delta K_c'[S])\Big|_{\delta=\epsilon_c=0}\\
&+\frac{\partial}{\partial\delta_c}\,\frac{\partial}{\partial\epsilon_c}P(v+\delta_cS_c+\epsilon_cK_c+\epsilon_c\delta_cK_c'[S_c])\Big|_{\delta_c=\epsilon_c=0}\\
=&\ \frac{\partial}{\partial\delta}\,\frac{\partial}{\partial\epsilon}P(u+\delta S+\epsilon K)\Big|_{\delta=\epsilon=0}+\frac{\partial}{\partial\delta_c}\,\frac{\partial}{\partial\epsilon_c}P(v+\delta_cS_c+\epsilon_cK_c)\Big|_{\delta_c=\epsilon_c=0}\\
&+\frac{\partial}{\partial\mu}P(u+\mu K'[S])\Big|_{\mu=0}+\frac{\partial}{\partial\mu}P(v+\mu K_c'[S])\Big|_{\mu=0}\\
&+\frac{\partial}{\partial\mu}P(v+\mu K_c'[S_c])\Big|_{\mu=0}.
\end{aligned}
$$

At the same time, we similarly have

$$
\begin{aligned}
(P'[\bar S])'[\bar K]=&\ \frac{\partial}{\partial\delta}\,\frac{\partial}{\partial\epsilon}P(u+\delta S+\epsilon K)\Big|_{\delta=\epsilon=0}\\
&+\frac{\partial}{\partial\delta_c}\,\frac{\partial}{\partial\epsilon_c}P(v+\delta_cS_c+\epsilon_cK_c)\Big|_{\delta_c=\epsilon_c=0}\\
&+\frac{\partial}{\partial\mu}P(u+\mu S'[K])\Big|_{\mu=0}+\frac{\partial}{\partial\mu}P(v+\mu S_c'[K])\Big|_{\mu=0}\\
&+\frac{\partial}{\partial\mu}P(v+\mu S_c'[K_c])\Big|_{\mu=0}.
\end{aligned}
$$

These two equalities engender our required equality (2.4). The proof is finished. □

From the above Lemma, we can easily deduce the following corollary.

Corollary 2.1. *Let* $\Phi(u,v,\lambda)\in\tilde{\mathcal{V}}^r$ *and* $\bar K$, $\bar S\in\mathcal{B}^{q+q_c}$. *Then we have*

$$
\begin{aligned}
&(\Phi'[\bar K])'[\bar S]-(\Phi'[\bar S])'[\bar K]\\
&\quad=\Phi'(u)[K'[S]-S'[K]]+\Phi'(v)[K_c'[S]-S_c'[K]+K_c'[S_c]-S_c'[K_c]]\,. \qquad (2.5)
\end{aligned}
$$

Thus, for $U_c=U_c(v,\lambda)\in\tilde{\mathcal{V}}^r_{(0)}$, *we obtain*

$$
(U_c'[K_c])'[\bar S]-(U_c'[S_c])'[\bar K]=U_c'[K_c'[S]-S_c'[K]+K_c'[S_c]-S_c'[K_c]]\,. \qquad (2.6)
$$

Here, we have noted that $U_c=U_c(v,\lambda)$ *has nothing to do with* u.

Evidently, we can also compute the commutator of two enlarged vector fields $\bar{K}$, $\bar{S} \in \mathcal{B}^{q+q_c}$ as follows:

$$[\bar{K}, \bar{S}] \triangleq \bar{K}'[\bar{S}] - \bar{S}'[\bar{K}] = \begin{pmatrix} [K, S] \\ [K, S]_c \end{pmatrix}, \tag{2.7}$$

where

$$[K, S] = K'[S] - S'[K], \ [K, S]_c = K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c].$$

In the following, we directly verify that this is a Lie bracket for the space $\mathcal{B}^{q+q_c}$.

Theorem 2.1. *The product* (2.7) *defines a Lie algebra structure over vector fields in* $\mathcal{B}^{q+q_c}$.

Proof. Let $\bar{K} = \binom{K}{K_c}$, $\bar{S} = \binom{S}{S_c}$, $\bar{L} = \binom{L}{L_c} \in \mathcal{B}^{q+q_c}$. From the definition of the product (2.7), we have

$$\begin{aligned}
[[\bar{K}, \bar{S}], \bar{L}] &= \left[\begin{pmatrix} [K, S] \\ [K, S]_c \end{pmatrix}, \begin{pmatrix} L \\ L_c \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} K'[S] - S'[K] \\ K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c] \end{pmatrix}, \begin{pmatrix} L \\ L_c \end{pmatrix}\right] \\
&\triangleq \begin{pmatrix} [[K, S], L] \\ [[K, S], L]_c \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
[[K, S], L] &= \frac{\partial}{\partial \eta} \frac{\partial}{\partial \delta} K(u + \eta L + \delta S)\Big|_{\delta=\eta=0} \\
&\quad - \frac{\partial}{\partial \eta} \frac{\partial}{\partial \epsilon} S(u + \eta L + \epsilon K)\Big|_{\epsilon=\eta=0} \\
&\quad + K'[S'[L]] - S'[K'[L]] - L'[K'[S]] + L'[S'[K]], \\
[[K, S], L]_c &= (K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c])'[L] - L_c'[K'[S] - S'[K]] \\
&\quad + (K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c])'[L_c] \\
&\quad - L_c'[K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c]] \\
&= \frac{\partial}{\partial \eta} \frac{\partial}{\partial \delta} K_c(u + \delta S + \eta L)\Big|_{\delta=\eta=0} \\
&\quad + \frac{\partial}{\partial \eta_c} \frac{\partial}{\partial \delta_c} K_c(v + \delta_c S_c + \eta_c L_c)\Big|_{\delta_c=\eta_c=0} \\
&\quad - \frac{\partial}{\partial \eta} \frac{\partial}{\partial \epsilon} S_c(u + \epsilon K + \eta L)\Big|_{\epsilon=\eta=0}
\end{aligned}$$

$$- \frac{\partial}{\partial \eta_c} \frac{\partial}{\partial \epsilon_c} S_c(v + \epsilon_c K_c + \eta_c L_c)\bigg|_{\epsilon_c=\eta_c=0}$$
$$+ K'_c[S'[L]] - S'_c[K'[L]] + L'_c[S'[K]] - L'_c[K'[S]]$$
$$+ K'_c[S'_c[L]] - S'_c[K'_c[L]] + L'_c[S'_c[K]] - L'_c[K'_c[S]]$$
$$+ K'_c[S'_c[L_c]] - S'_c[K'_c[L_c]] + L'_c[S'_c[K_c]] - L'_c[K'_c[S_c]] \,.$$

So by a simple calculation, we have

$$[[K, S], L] + \text{cycle}(K, S, L) = 0 \,,$$
$$[[K, S], L]_c + \text{cycle}(K, S, L) = 0 \,,$$

that is

$$[[\bar{K}, \bar{S}], \bar{L}] + \text{cycle}(\bar{K}, \bar{S}, \bar{L}) = 0 \,.$$

This implies that (2.7) defines a Lie algebra structure over vector fields in $\mathcal{B}^{q+q_c}$, indeed. The proof is completed. □

3. Algebraic Structure of Lax Operators

In this section, we aim to discuss the algebraic structure of zero curvature representations for integrable coupling systems.

First, the commutator of two smooth functions f, $g \in C^\infty(\mathbb{C})$ (as vector fields over $\mathbb{C}$) is defined as

$$[\![f, g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) \,, \quad \lambda \in \mathbb{C} \,. \tag{3.1}$$

The bracket (3.1) defines a Lie algebra structure over $\in C^\infty(\mathbb{C})$, indeed (see Ref. 15).

In what follows, we always assume that the enlarged spectral operator $\bar{U} \in \tilde{\mathcal{V}}^{2r}_{(0)}$ has an injective Gateaux derivative operator $\bar{U}' : \mathcal{B}^{q+q_c} \to \tilde{\mathcal{V}}^{2r}_{(0)}$.

Definition 3.1. Let $\bar{V} \in \tilde{\mathcal{V}}^{2r}_{(0)}$. If there exist $\bar{K} \in \mathcal{B}^{q+q_c}$ and $f \in C^\infty(\mathbb{C})$ such that (1.11) holds, then $\bar{V}$ is called an enlarged Lax pair operator corresponding to $f(\lambda)$, and $\bar{K}$ is called an enlarged eigenvector field of $\bar{V}$ corresponding to $f(\lambda)$.

Assume that $P(\bar{U})$ denotes all triple $(\bar{V}, \bar{K}, f) \in \tilde{\mathcal{V}}^{2r}_{(0)} \times \mathcal{B}^{q+q_c} \times C^\infty(\mathbb{C})$ satisfying (1.11) and for $f(\lambda) \in C^\infty(\mathbb{C})$, we set

$$M(\bar{U}, f) = \left\{\bar{V} \in \tilde{\mathcal{V}}^{2r}_{(0)} | \exists \bar{K} \in \mathcal{B}^{q+q_c} \text{ so that } (\bar{V}, \bar{K}, f) \in P(\bar{U})\right\} , \tag{3.2}$$

i.e. all Lax operators corresponding to f, and

$$\begin{aligned} EM(\bar{U}, f) &= E_f M(\bar{U}, f) \\ &= \{\bar{K} \in \mathcal{B}^{q+q_c} | \exists \bar{V} \in M(\bar{U}, F) \text{ so that } (\bar{V}, \bar{K}, f) \in P(\bar{U})\} \,, \end{aligned} \tag{3.3}$$

i.e. all eigenvector fields of $M(\bar{U}, f)$ corresponding to f.

For $(\bar{V},\bar{K},f)$, $(\bar{W},\bar{S},g)\in P(\bar{U})$, the product $[\![\bar{V},\bar{W}]\!]\in\tilde{\mathcal{V}}^{2r}_{(0)}$ (see Ref. 15 for definition) can be computed as follows

$$\begin{aligned}[\![\bar{V},\bar{W}]\!] &= \bar{V}'[\bar{S}]-\bar{W}'[\bar{K}]+[\bar{V},\bar{W}]+g\bar{V}_\lambda-f\bar{W}_\lambda\\ &=\begin{pmatrix}[\![V,W]\!] & [\![V_c,W_c]\!]\\ 0 & [\![V,W]\!]\end{pmatrix},\end{aligned}\tag{3.4a}$$

where

$$[\![V,W]\!]=V'[S]-W'[K]+[V,W]+gV_\lambda-fW_\lambda\,,\tag{3.4b}$$

$$[\![V_c,W_c]\!]=V_c'[\bar{S}]-W_c'[\bar{K}]+[V,W_c]+[V_c,W]+gV_{c\lambda}-fW_{c\lambda}\,.\tag{3.4c}$$

This shows a special structure of the commutator of enlarged Lax operators and play a crucial role in our computation.

In what follows, we would like to show that the product $[\![\bar{V},\bar{W}]\!]$ is associated with the commutator of enlarged vector fields $[\bar{K},\bar{S}]$.

Theorem 3.1. *Let $(\bar{V},\bar{K},f)$, $(\bar{W},\bar{S},g)\in P(\bar{U})$. Then $([\![\bar{V},\bar{W}]\!],[\bar{K},\bar{S}],[\![f,g]\!])$ belongs to $P(\bar{U})$, too. That is to say*

$$\bar{U}'[[\bar{K},\bar{S}]]+[\![f,g]\!](\lambda)\bar{U}_\lambda-[\![\bar{V},\bar{W}]\!]_x+[\bar{U},[\![\bar{V},\bar{W}]\!]]=0\,,\tag{3.5}$$

which is equivalent to the following two equations:

$$U'[[K,S]]+[\![f,g]\!](\lambda)U_\lambda-[\![V,W]\!]_x+[U,[\![V,W]\!]]=0\,,\tag{3.6a}$$

$$U_c'[[K,S]_c]+[\![f,g]\!](\lambda)U_{c\lambda}-[\![V_c,W_c]\!]_x+[U,[\![V_c,W_c]\!]]+[U_c,[\![V,W]\!]]=0\,.\tag{3.6b}$$

Proof. Since $(\bar{V},\bar{K},f)$, $(\bar{W},\bar{S},g)\in P(\bar{U})$, we have

$$\begin{aligned}V_x'[S]&=(U'[K])'[S]+fU_\lambda'[S]+[U,V]'[S]\,,\\ W_x'[K]&=(U'[S])'[K]+gU_\lambda'[K]+[U,W]'[K]\,,\end{aligned}\tag{3.7a}$$

$$\begin{aligned}U_\lambda'[K]&=V_{x\lambda}-[U,V]_\lambda-f_\lambda U_\lambda-fU_{\lambda\lambda}\,,\\ U_\lambda'[S]&=W_{x\lambda}-[U,W]_\lambda-g_\lambda U_\lambda-gU_{\lambda\lambda}\,,\end{aligned}\tag{3.7b}$$

$$\begin{aligned}V_{cx}'[\bar{S}]&=(U_c'[K_c])'[\bar{S}]+fU_{c\lambda}'[\bar{S}]+[U,V_c]'[\bar{S}]+[U_c,V]'[\bar{S}]\,,\\ W_{cx}'[\bar{K}]&=(U_c'[S_c])'[\bar{K}]+gU_{c\lambda}'[\bar{K}]+[U,W_c]'[\bar{K}]+[U_c,W]'[\bar{K}]\,,\end{aligned}\tag{3.8a}$$

$$\begin{aligned}U_{c\lambda}'[\bar{K}]&=V_{cx\lambda}-[U,V_c]_\lambda-[U_c,V]_\lambda-f_\lambda U_{c\lambda}-fU_{c\lambda\lambda}\,,\\ U_{c\lambda}'[\bar{S}]&=W_{cx\lambda}-[U,W_c]_\lambda-[U_c,W]_\lambda-g_\lambda U_{c\lambda}-gU_{c\lambda\lambda}\,.\end{aligned}\tag{3.8b}$$

Let us define

$$\begin{aligned}\Theta&=\bar{V}'[\bar{S}]-\bar{W}'[\bar{K}]+[\bar{V},\bar{W}]\\ &=\begin{pmatrix}V'[S]-W'[K]+[V,W] & V_c'[\bar{S}]-W_c'[\bar{K}]+[V,W_c]+[V_c,W]\\ 0 & V'[S]-W'[K]+[V,W]\end{pmatrix}.\end{aligned}\tag{3.9}$$

Then, we have

$$\begin{aligned}\Theta_x - [\bar{U},\Theta] &= \bar{V}_x'[\bar{S}] - \bar{W}_x'[\bar{K}] + [\bar{V},\bar{W}]_x \\ &\quad - [\bar{U},\bar{V}'[\bar{S}] - \bar{W}'[\bar{K}] + [\bar{V},\bar{W}]] \\ &\triangleq \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22}\end{pmatrix}, \end{aligned} \tag{3.10a}$$

where

$$\Omega_{11} = \Omega_{22} = U'[[K,S]] + fW_{x\lambda} - f[U,W_\lambda] - gV_{x\lambda} + g[U,V_\lambda] + [\![f,g]\!]U_\lambda\,, \tag{3.10b}$$

$$\begin{aligned}\Omega_{12} &= V_{cx}'[\bar{S}] - W_{cx}'[\bar{K}] + [V,W_c]_x + [V_c,W]_x - [U,V_c[\bar{S}] - W_c'[\bar{K}] \\ &\quad + [V,W_c] + [V_c,W]] - [U_c,V'[S] - W'[K] + [V,W]]\,. \end{aligned} \tag{3.10c}$$

Making full use of (3.7) and (3.8), we can compute that

$$\begin{aligned}\Omega_{12} &= (U_c'[K_c])'[\bar{S}] - (U_c'[S_c])'[\bar{K}] + fU_{c\lambda}'[\bar{S}] \\ &\quad - gU_{c\lambda}'[\bar{K}] + [U,V_c]'[\bar{S}] + [U_c,V]'[\bar{S}] \\ &\quad + [U_c'[K_c] + fU_{c\lambda} + [U,V_c] + [U_c,V],W] \\ &\quad + [V_c,[U,W] + U'[S] + gU_\lambda] \\ &\quad - [U,V_c'[\bar{S}] - W_c'[\bar{K}] + [V,W_c] + [V_c,W]] \\ &\quad - [U_c,V'[S] - W'[K] + [V,W]] \\ &\quad - [U,W_c]'[\bar{K}] - [U_c,W]'[\bar{K}] + [[U,V] + U'[K] + fU_\lambda,W_c] \\ &\quad + [V,U_c'[S_c] + gU_{c\lambda} + [U,W_c] + [U_c,W]] \\ &= (U_c'[K_c])'[\bar{S}] - (U_c'[S_c])'[\bar{K}] + fW_{cx\lambda} - f[U,W_{c\lambda}] \\ &\quad - f[U_c,W_\lambda] - gV_{cx\lambda} + g[U,V_{c\lambda}] + g[U_c,V_\lambda] + [\![f,g]\!]U_{c\lambda} \\ &= U_c'[K_c'[S] - S_c'[K] + K_c'[S_c] - S_c'[K_c]] \\ &\quad + fW_{cx\lambda} - f[U,W_{c\lambda}] - f[U_c,W_\lambda] \\ &\quad - gV_{cx\lambda} + g[U,V_{c\lambda}] + g[U_c,V_\lambda] + [\![f,g]\!]U_{c\lambda}\,. \end{aligned}$$

On the other hand, according to (3.4b), (3.4c) and (3.9), we have

$$\Theta = \begin{pmatrix} [\![V,W]\!] + fW_\lambda - gV_\lambda & [\![V_c,W_c]\!] + fW_{c\lambda} - gV_{c\lambda} \\ 0 & [\![V,W]\!] + fW_\lambda - gV_\lambda \end{pmatrix}, \tag{3.11}$$

and thus

$$\Theta_x - [\bar{U}, \Theta] \triangleq \begin{pmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} \\ 0 & \tilde{\Omega}_{22} \end{pmatrix}, \tag{3.12a}$$

where

$$\tilde{\Omega}_{11} = \tilde{\Omega}_{22} = [\![V, W]\!]_x + fW_{x\lambda} - gV_{x\lambda} - [U, [\![V, W]\!] + fW_\lambda - gV_\lambda], \tag{3.12b}$$

$$\begin{aligned} \tilde{\Omega}_{12} = {} & [\![V_c, W_c]\!]_x - [U, [\![V_c, W_c]\!]] - [U_c, [\![V, W]\!]] + fW_{cx\lambda} \\ & - gV_{cx\lambda} - f[U, W_{c\lambda}] + g[U, V_{c\lambda}] - f[U_c, W_\lambda] + g[U_c, V_\lambda]. \end{aligned} \tag{3.12c}$$

Comparing (3.10) and (3.12), we immediately obtain (3.6). Thus, (3.5) holds. This means that $([\![\bar{V}, \bar{W}]\!], [\bar{K}, \bar{S}], [\![f, g]\!])$ belongs to $P(\bar{U})$. The proof is completed. □

We remark that (3.6a) is exactly the result presented in Ref. 15. Conversely, the whole equality (3.6) is also a consequence of the general result of Ref. 15.

It follows from the above theorem that if two enlarged evolution equations $\bar{u}_t = \bar{K}$, $\bar{u}_t = \bar{S}$, $(\bar{K}, \bar{S} \in \mathcal{B}^{q+q_c})$ are the compatibility conditions of the following spectral problems:

$$\bar{\varphi}_x = \bar{U}\bar{\varphi}, \quad \bar{\varphi}_t = \bar{V}\bar{\varphi}, \qquad \bar{V} \in \tilde{\mathcal{V}}^{2r}_{(0)}, \quad \lambda_t = a\lambda^m,$$

$$\bar{\varphi}_x = \bar{U}\bar{\varphi}, \quad \bar{\varphi}_t = \bar{W}\bar{\varphi}, \qquad \bar{W} \in \tilde{\mathcal{V}}^{2r}_{(0)}, \quad \lambda_t = b\lambda^n,$$

where a, b are constants and $m, n \geq 0$, respectively, then the product equation $\bar{u}_t = [\bar{K}, \bar{S}]$ is the compatibility condition of the following spectral problems:

$$\bar{\varphi}_x = \bar{U}\bar{\varphi}, \quad \bar{\varphi}_t = [\![\bar{V}, \bar{W}]\!]\bar{\varphi}, \quad \lambda_t = ab(m-n)\lambda^{m+n-1},$$

where

$$\begin{aligned} [\![\bar{V}, \bar{W}]\!] &= \bar{V}'[\bar{S}] - \bar{W}'[\bar{K}] + [\bar{V}, \bar{W}] + g\bar{V}_\lambda - f\bar{W}_\lambda \\ &= \begin{pmatrix} [\![V, W]\!] & [\![V_c, W_c]\!] \\ 0 & [\![V, W]\!] \end{pmatrix}. \end{aligned}$$

Therefore, we see that

$$[\![M(\bar{U}, 0), M(\bar{U}, \lambda^m)]\!] \subseteq M(\bar{U}, 0), \quad m \geq 0, \tag{3.13}$$

and further, we have

$$[\![M(\bar{U}, 0), [\![M(\bar{U}, 0), M(\bar{U}, \lambda^m)]\!]]\!] = 0, \quad m \geq 0. \tag{3.14}$$

This implies that

$$[\![EM(\bar{U}, 0), [\![EM(\bar{U}, 0), EM(\bar{U}, \lambda^m)]\!]]\!] = 0, \quad m \geq 0, \tag{3.15}$$

provided that $M(\bar{U}, 0)$ is commutative.

Now by using Theorem 3.1, we can prove the following two results.

Corollary 3.1. *Let* $(\bar{V}_i, \bar{K}_i, f_i) \in P(\bar{U})$, $i = 1, 2, 3$. *If* $[\![\bar{V}_1, \bar{V}_2]\!] = \bar{V}_3$ *and* $[\![f_1, f_2]\!] = f_3$, *then* $[\bar{K}_1, \bar{K}_2] = \bar{K}_3$.

Proof. By Theorem 3.1, namely (3.6), and the assumption, we have

$$\begin{aligned} U'[[K_1, K_2]] &= -[\![f_1, f_2]\!]U_\lambda - [\![V_1, V_2]\!]_x + [U, [\![V_1, V_2]\!]] \\ &= -f_3 U_\lambda - V_{3x} + [U, V_3] = U'[K_3]\,, \\ U'_c[[K_1, K_2]_c] &= -[\![f_1, f_2]\!]U_{c\lambda} + [\![V_{1c}, V_{2c}]\!]_x - [U, [\![V_{1c}, V_{2c}]\!]] - [U_c, [\![V_1, V_2]\!]] \\ &= -f_3 U_{c\lambda} + V_{3ax} - [U, V_{3c}] - [U_c, V_3] = U'_c[K_{3c}]\,, \end{aligned}$$

where we have noticed that $[\![\bar{V}_1, \bar{V}_2]\!] = \bar{V}_3$ implies $[\![V_1, V_2]\!] = V_3$ and $[\![V_{1c}, V_{2c}]\!] = V_{3c}$.

It now follows that $[K_1, K_2] = K_3$ and $[K_1, K_2]_c = K_{3c}$, i.e. $[\bar{K}_1, \bar{K}_2] = \bar{K}_3$, since $\bar{U}'$ is injective. The proof is completed. □

Because we assume that $\bar{U}'$ is injective, a Lax operator in $M(\bar{U}, 0)$ has only an eigenvector field corresponding to $f = 0$. Suppose that

$$\bar{U}'[\bar{K}] - \bar{V}_x + [\bar{U}, \bar{V}] = 0\,, \quad \bar{U}'[\bar{S}] - \bar{W}_x + [\bar{U}, \bar{W}] = 0\,,$$

we define

$$[\![\bar{V}, \bar{W}]\!]_0 = \bar{V}'[\bar{S}] - \bar{W}'[\bar{K}] + [\bar{V}, \bar{W}]\,,$$

which is well defined. This moment $M(\bar{U}, 0)$ constitutes an algebra with regard to $[\![\bar{V}, \bar{W}]\!]_0$ and thus $\langle EM(\bar{U}, 0), [\,\cdot\,,\cdot\,]\rangle$ is a Lie algebra. Set

$$\bar{K}(\bar{U}) = \left\{\bar{V} \in \tilde{\mathcal{V}}^r_{(0)} \,|\, \bar{V}_x = [\bar{U}, \bar{V}]\right\}.$$

Obviously, $\bar{K}(\bar{U})$ is a subspace of $M(\bar{U}, 0)$ and the bracket $[\![\,\cdot\,,\cdot\,]\!]_0$ over $\bar{K}(\bar{U})$ reduces to the matrix commutator $[\,\cdot\,,\cdot\,]$. Moreover by Theorem 3.1, we may see that

$$[\![\bar{K}(\bar{U}), M(\bar{U}, 0)]\!]_0, [\![M(\bar{U}, 0), \bar{K}(\bar{U})]\!]_0 \subseteq \bar{K}(\bar{U})\,.$$

Therefore, $\bar{K}(\bar{U})$ is an ideal subalgebra of $\langle M(\bar{U}, 0), [\![,]\!]_0\rangle$. In this way, we can generate a quotient algebra $\langle M(\bar{U}, 0)/\bar{K}(\bar{U}), [\![\,\cdot\,,\cdot\,]\!]_0\rangle$. By using Theorem 3.1, we can acquire the following result.

Theorem 3.2. *The quotient algebra* $\langle CL(M(\bar{U}, 0)) \triangleq M(\bar{U}, 0)/\bar{K}(\bar{U}), [\![\,\cdot\,,\cdot\,]\!]_0\rangle$ *is a Lie algebra and isomorphic to the Lie algebra* $\langle EM(\bar{U}, 0), [\,\cdot\,,\cdot\,]\rangle$ *under the mapping*

$$\begin{aligned} \rho : CL(M(\bar{U}, 0)) = M(\bar{U}, 0)/\bar{K}(\bar{U}) \to EM(\bar{U}, 0)\,, \\ CL(\bar{V}) \triangleq \{\bar{W} \in M(\bar{U}, 0) \,|\, \bar{W} - \bar{V} \in \bar{K}(\bar{U})\} \to \bar{K}\,, \end{aligned}$$

where $\bar{V} \in M(\bar{U}, 0), \bar{K} \in EM(\bar{U}, 0)$ *satisfy* $\bar{U}'[\bar{K}] - \bar{V}_x + [\bar{U}, \bar{V}] = 0$.

4. Application

In this section, we shall in detail illustrate our construction process by an concrete example in the AKNS case and establish the corresponding τ-symmetry algebra.

4.1. *The isospectral and nonisospectral AKNS hierarchies*

The AKNS spectral problem is given by

$$\varphi_x = U\varphi\,, \quad U = U(u,\lambda) = \begin{pmatrix} -\lambda & u_1 \\ u_2 & \lambda \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{4.1}$$

where $u_i = u_i(x,t)$, $i = 1, 2$ are two dependent variables.

Suppose that the associated temporal spectral problem is as follows:

$$\varphi_t = V\varphi\,, \quad V = V(u,\lambda) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \tilde{\mathcal{V}}^2_{(0)}\,. \tag{4.2}$$

Clearly the compatibility condition $U_t - V_x + [U, V] = 0$ in this case gives equivalently

$$\lambda_t = -a_x + u_1 c - u_2 b\,, \tag{4.3a}$$

$$u_{1t} = b_x + 2\lambda b + 2u_1 a\,, \tag{4.3b}$$

$$u_{2t} = c_x - 2\lambda c - 2u_2 a\,. \tag{4.3c}$$

From (4.3a), we have

$$a = -\lambda_t x + \partial^{-1}(u_1 c - u_2 b)\,.$$

Therefore, (4.3b) and (4.3c) become

$$\begin{aligned} u_{1t} &= b_x + 2\lambda b + 2u_1\partial^{-1}(u_1 c - u_2 b) - 2u_1\lambda_t x\,, \\ u_{2t} &= c_x - 2\lambda c - 2u_2\partial^{-1}(u_1 c - u_2 b) + 2u_2\lambda_t x\,. \end{aligned} \tag{4.4}$$

***Case 1*: $\boldsymbol{\lambda_t = 0}$**

Substituting the following expansions:

$$b = \sum_{j=0}^{m} b_j \lambda^{m-j}\,, \qquad c = \sum_{j=0}^{m} c_j \lambda^{m-j} \tag{4.5}$$

into (4.4), and comparing the same powers of both sides in λ, we arrive at

$$b_0 = c_0 = 0\,, \quad b_1 = u_1\,, \quad c_1 = u_2\,,$$

and

$$u_t = K_m = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \end{pmatrix} = \Phi(u)^m K_0\,, \quad m \geq 0\,, \tag{4.6a}$$

where

$$\Phi(u) = \begin{pmatrix} -\frac{1}{2}\partial + u_1\partial^{-1}u_2 & u_1\partial^{-1}u_1 \\ -u_2\partial^{-1}u_2 & \frac{1}{2}\partial - u_2\partial^{-1}u_1 \end{pmatrix}, \quad K_0 = \begin{pmatrix} -2u_1 \\ 2u_2 \end{pmatrix}. \tag{4.6b}$$

This is the isospectral ($\lambda_t = 0$) AKNS hierarchy.

Case 2: $\boldsymbol{\lambda_t = \lambda^{m+1}}$

Similarly, substituting (4.5) into (4.4), we arrive at

$$b_0 = u_1 x\,, \quad c_0 = u_2 x\,, \quad b_1 = -\frac{1}{2}(u_1 x)_x\,, \quad c_1 = \frac{1}{2}(u_2 x)_x\,,$$

and

$$u_t = \rho_m = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \end{pmatrix} = \Phi(u)^m \rho_0\,, \quad \rho_0 = \begin{pmatrix} (u_1 x)_x \\ (u_2 x)_x \end{pmatrix}, \quad m \geq 0\,, \tag{4.7}$$

which is the nonisospectral ($\lambda_t = \lambda^{m+1}$) AKNS hierarchy.

4.2. *The enlarged isospectral and nonisospectral AKNS hierarchies*

We define the corresponding enlarged AKNS spectral problem by semidirect sums of Lie algebras as follows:[11]

$$\begin{gathered} \bar{\varphi}_x = \bar{U}\bar{\varphi}\,, \quad \bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{pmatrix} U & U_c \\ 0 & U \end{pmatrix} \in G \Subset G_c\,, \\ U_c = U_c(v) = \begin{pmatrix} 0 & v_1 \\ v_2 & 0 \end{pmatrix}, \end{gathered} \tag{4.8}$$

where $v_i = v_i(x,t)$, $i = 1, 2$ are two new dependent variables and

$$v = (v_1, v_2)^T\,, \quad \bar{u} = (u^T, v^T)^T = (u_1, u_2, v_1, v_2)^T\,.$$

The associated enlarged temporal spectral problem is assumed to be

$$\begin{gathered} \bar{\varphi}_t = \bar{V}\bar{\varphi}\,, \quad \bar{V} = \bar{V}(\bar{u}, \lambda) = \begin{pmatrix} V & V_c \\ 0 & V \end{pmatrix} \in \tilde{\mathcal{V}}^4_{(0)}\,, \\ V_c = V_c(u, v) = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}. \end{gathered} \tag{4.9}$$

Then the corresponding enlarged zero curvature equation $\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0$ becomes

$$\begin{gathered} U_t - V_x + [U, V] = 0\,, \\ U_{ct} - V_{cx} + [U, V_c] + [U_c, V] = 0\,, \end{gathered} \tag{4.10}$$

which are equivalent to

$$\begin{aligned} u_{1t} &= b_x + 2\lambda b + 2u_1\partial^{-1}(u_1c - u_2b) - 2u_1\lambda_t x\,, \\ u_{2t} &= c_x - 2\lambda c - 2u_2\partial^{-1}(u_1c - u_2b) + 2u_2\lambda_t x\,, \\ v_{1t} &= f_x + 2\lambda f + 2u_1e + 2b_1[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,, \\ v_{2t} &= g_x - 2u_2e - 2\lambda e - 2\lambda g - 2v_2[-\lambda_t x + \partial^{-1}(u_1c - u_2b)]\,. \end{aligned} \tag{4.11}$$

Set

$$e = \sum_{j=0}^{m} e_j\lambda^{m-j}\,, \quad f = \sum_{j=0}^{m} f_j\lambda^{m-j}\,, \quad g = \sum_{j=0}^{m} g_j\lambda^{m-j}\,, \tag{4.12}$$

and thus, we can derive the enlarged isospectral and nonisospectral AKNS hierarchies as follows.

Case 1: $\lambda_t = 0$

Substituting the expansions of (4.5) and (4.12) into (4.11) leads to

$$f_0 = g_0 = 0\,, \quad f_1 = u_1 + v_1\,, \quad g_1 = u_2 + v_2$$

and

$$\bar{u}_t = \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}_t = \bar{K}_m = \begin{pmatrix} K_m \\ K_{mc} \end{pmatrix} = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{pmatrix} = \bar{\Phi}^m \bar{K}_0\,, \quad m \geq 0\,, \tag{4.13a}$$

where

$$\begin{aligned} \bar{\Phi} &\triangleq \begin{pmatrix} \Phi(u) & 0 \\ \Phi_c(u,v) & \Phi(u) \end{pmatrix}, \\ \Phi_c(u,v) &= \begin{pmatrix} v_1\partial^{-1}u_2 + u_1\partial^{-1}v_2 & v_1\partial^{-1}u_1 + u_1\partial^{-1}v_1 \\ -v_2\partial^{-1}u_2 - u_2\partial^{-1}v_2 & -v_2\partial^{-1}u_1 - u_2\partial^{-1}v_1 \end{pmatrix}, \\ \bar{K}_0 &= \begin{pmatrix} K_0 \\ K_{0c} \end{pmatrix} = \begin{pmatrix} -2u_1 \\ 2u_2 \\ -2u_1 - 2v_1 \\ 2u_2 + 2v_2 \end{pmatrix}. \end{aligned} \tag{4.13b}$$

Thus, this enlarged isospectral ($\lambda_t = 0$) AKNS hierarchy (4.13a) (see Ref. 11 for its Hamiltonian structure) has the recursion relation

$$\bar{u}_t = \bar{K}_m = \begin{pmatrix} K_m \\ K_{mc} \end{pmatrix} = \begin{pmatrix} \Phi(u)K_{m-1} \\ \Phi_c(u,v)K_{m-1} + \Phi(u)K_{m-1,c} \end{pmatrix}, \quad m \geq 0\,. \tag{4.14}$$

***Case 2:* $\boldsymbol{\lambda_t = \lambda^{m+1}}$**

In this case, substituting the expansions of (4.5) and (4.12) into (4.11) leads to

$$f_0 = v_1 x\,, \quad g_0 = v_2 x\,, \quad f_1 = -\frac{1}{2}(v_1 x)_x\,, \quad g_1 = \frac{1}{2}(v_2 x)_x\,,$$

and

$$\bar{u}_t = \bar{\rho}_m = \begin{pmatrix} \rho_m \\ \rho_{mc} \end{pmatrix} = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{pmatrix} = \bar{\Phi}^m \bar{\rho}_0\,, \quad m \geq 0\,,$$

where

$$\bar{\rho}_0 = \begin{pmatrix} \rho_0 \\ \rho_{0c} \end{pmatrix} = \begin{pmatrix} (u_1 x)_x \\ (u_2 x)_x \\ (v_1 x)_x \\ (v_2 x)_x \end{pmatrix}, \tag{4.15}$$

and $\Phi_c(u, v)$ is defined as in (4.13). Therefore, this enlarged nonisospectral ($\lambda_t = \lambda^{m+1}$) AKNS hierarchy has a recursion relation

$$\bar{u}_t = \bar{\rho}_m = \begin{pmatrix} \rho_m \\ \rho_{mc} \end{pmatrix} = \begin{pmatrix} \Phi(u)\rho_{m-1} \\ \Phi_c(u,v)\rho_{m-1} + \Phi(u)\rho_{m-1,c} \end{pmatrix}, \quad m \geq 1\,. \tag{4.16}$$

Next let us consider how to compute the corresponding τ-symmetry algebra for the obtained integrable coupling systems. The procedure below is an application of the idea in Ref. 16 and can be applied to other cases. As in Ref. 16, we first make the following computation at $\bar{u} = 0$:

$$\begin{aligned} \bar{K}_m|_{\bar{u}=0} &= \begin{pmatrix} K_m \\ K_{mc} \end{pmatrix}\Bigg|_{\bar{u}=0} = \bar{\Phi}^m \bar{K}_0|_{\bar{u}=0} = 0\,, \\ \bar{\rho}_n|_{\bar{u}=0} &= \begin{pmatrix} \rho_n \\ \rho_{nc} \end{pmatrix}\Bigg|_{\bar{u}=0} = \bar{\Phi}^n \bar{\rho}_0|_{\bar{u}=0} = 0\,, \end{aligned} \tag{4.17}$$

where m, $n \geq 0$. We denote by $\bar{V}_m$ and $\bar{W}_n$ the Lax operators corresponding to the vector fields $\bar{K}_m$ and $\bar{\rho}_n$, respectively (see Ref. 17 for nonisospectral Lax operators). Then as in Ref. 16, we compute that

$$\bar{V}_m|_{\bar{u}=0} = \begin{pmatrix} V_m & V_{mc} \\ 0 & V_m \end{pmatrix}\Bigg|_{\bar{u}=0} = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \lambda^m\,,$$

$$\bar{V}_{m\lambda}|_{\bar{u}=0} = (\bar{V}_m|_{\bar{u}=0})_\lambda = m \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \lambda^{m-1}\,,$$

$$\bar{W}_n|_{\bar{u}=0} = \begin{pmatrix} W_n & W_{nc} \\ 0 & W_n \end{pmatrix}\Bigg|_{\bar{u}=0} = \begin{pmatrix} -\lambda x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -\lambda x & 0 \\ 0 & 0 & 0 & \lambda x \end{pmatrix} \lambda^n ,$$

$$\bar{W}_{n\lambda}|_{\bar{u}=0} = (\bar{W}_n|_{\bar{u}=0})_\lambda = (n+1) \begin{pmatrix} -\lambda x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -\lambda x & 0 \\ 0 & 0 & 0 & \lambda x \end{pmatrix} \lambda^{n-1} ,$$

where m, $n \geq 0$. Now we can find by the definition (3.4) of the product of Lax operators that

$$\begin{aligned} [\![\bar{V}_m, \bar{V}_n]\!]|_{\bar{u}=0} &= 0\,, && m,n \geq 0\,, \\ [\![\bar{V}_m, \bar{W}_n]\!]|_{\bar{u}=0} &= m\bar{V}_{m+n}|_{\bar{u}=0}\,, && m,n \geq 0\,, \\ [\![\bar{W}_m, \bar{W}_n]\!]|_{\bar{u}=0} &= (m-n)\bar{W}_{m+n}|_{\bar{u}=0}\,, && m,n \geq 0\,. \end{aligned} \tag{4.18}$$

For example, we can compute that

$$[\![\bar{V}_m, \bar{W}_n]\!]|_{\bar{u}=0} = [\bar{V}_m, \bar{W}_n] + \lambda^{n+1}\bar{V}_{m\lambda}|_{\bar{u}=0} = m\bar{V}_{m+n}|_{\bar{u}=0}\,, \quad m,n \geq 0\,.$$

Because $[\![\bar{V}_m, \bar{V}_n]\!]$, $[\![\bar{V}_m, \bar{W}_n]\!] - m\bar{V}_{m+n}$, $[\![\bar{W}_m, \bar{W}_n]\!] - (m-n)\bar{V}_{m+n}$, m, $n \geq 0$, are all isospectral ($\lambda_t = 0$) Lax operators belonging to $\widetilde{\mathcal{V}}^4_{(0)}$ by Theorem 3.1, based upon (4.18), we obtain a Lax operator algebra by the uniqueness property of the enlarged spectral problem (4.8):

$$\begin{aligned} [\![\bar{V}_m, \bar{V}_n]\!] &= 0\,, && m,n \geq 0\,, \\ [\![\bar{V}_m, \bar{W}_n]\!] &= m\bar{V}_{m+n}\,, && m,n \geq 0\,, \\ [\![\bar{W}_m, \bar{W}_n]\!] &= (m-n)\bar{W}_{m+n}\,, && m,n \geq 0\,. \end{aligned} \tag{4.19}$$

Further, due to the injective property of $\bar{U}'$, we finally obtain a vector field algebra of the enlarged isospectral and nonisospectral AKNS hierarchies:

$$\begin{aligned} [\bar{K}_m, \bar{K}_n] &= 0\,, && m,n \geq 0\,, \\ [\bar{K}_m, \bar{\rho}_n] &= m\bar{K}_{m+n}\,, && m,n \geq 0\,, \\ [\bar{\rho}_m, \bar{\rho}_n] &= (m-n)\bar{\rho}_{m+n}\,, && m,n \geq 0\,, \end{aligned} \tag{4.20}$$

which automatically give rise to the τ-symmetry algebra[18] for the enlarged isospectral AKNS hierarchy (4.13a).

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