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ABSTRACT

Variable-coefficient equations can be used to describe certain phenomena when inhomogeneous media and nonuniform boundaries are taken into consideration. Describing the fluid dynamics of shallow-water wave in an open ocean, a $(2+1)$-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equation is investigated in this paper. The integrability is first examined by the Painlevé analysis method. Secondly, the one-soliton and two-soliton solutions and lump solutions of the $(2+1)$-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equations are derived by virtue of the Hirota bilinear method. In the exact solutions, parameter values and variable-coefficient functions are chosen and analyzed for different effects on the shallow-water waves.

I. INTRODUCTION

Nonlinear partial differential equations (NLPDEs) have important applications in physics, aerodynamics, fluid mechanics, atmospheric physics, ocean physics, explosion physics, chemistry, physiology, biology, ecology, and other fields.[1–3] In order to understand and explore the nature of these nonlinear physical phenomena, it is necessary to deeply study the exact solutions and qualitative properties of these mathematical physics equations.[4,5] The $(2+1)$-dimensional Hirota–Satsuma–Ito equation finds significant application in fluid physics, allowing for the description of phenomena such as solitons and wave patterns.[6–9] This implies its capability to explain the propagation of solitary waves in fluids, including phenomena observed in oceans, lakes, and other behaviors of solitary waves in fluid dynamics. Furthermore, the equation describes nonlinear wave dynamics, enabling it to explain wave phenomena with complex nonlinear characteristics, which is essential for understanding vortices, shock waves, and other irregular wave dynamics phenomena. Additionally, certain fluid phenomena may result in wave breaking or compressibility effects, and this equation provides a framework to study these phenomena, aiding our understanding of these complex fluid behaviors. Finally, the equation can also be utilized to explore complex dynamics phenomena within fluid systems, such as chaos and nonlinear resonance.[10,11] Therefore, the $(2+1)$-dimensional Hirota–Satsuma–Ito equation serves as a powerful tool in fluid physics, allowing us to explain and explore a wide range of complex wave and nonlinear phenomena, thereby deepening our understanding of fluid behavior. Integrability plays a crucial role in finding analytical solutions. The Painlevé analysis proposed in 1983 by Weiss, Tabor, and Carnevale provides an accurate description of the integrability of NLPDEs.[12–14] The Painlevé analysis approach is one of the most reliable methods to characterize the integrability of NLPDEs. It is well known that the Hirota bilinear method is a direct and powerful technique for generating solitonic solutions. The availability of computer symbolic systems such as Maple and Mathematica enables us to perform more complex and tedious algebraic computations in the investigation of exact closed-form solutions of NLPDEs.[15]

Research of solutions to NLPDEs plays an important role in the field of nonlinearity. Therefore, search for efficient solution methods has gained increasing attention. To date, there are several effective methods that have been commonly used by mathematicians, such as Darboux transformation, Bäcklund transformation, Hirota bilinear method, and hyperbolic method.[16–21] Among them, the Hirota bilinear method proposed by Hirota is a straightforward method for finding exact solutions of NLPDEs. The exact solutions to NLPDEs are of great value in many areas, such as resonant soliton solutions under Bell polynomials and Hirota bilinear method, N-rational solutions by the long-wave limit method, M-lump solution and the Wronskian solutions by the long-wave limit method,[22–24] lump and lump-multi-kink solutions via the test function method.[25]

In 1981, Hirota and Satsuma presented a Hirota–Satsuma (HS) shallow-water wave equation based on a Bäcklund transformation of the Boussinesq equation.[26–28] It described the propagation of unidirectional shallow-water waves and interactions of two long waves with different dispersion relations. Among the NLPDEs, $(2+1)$-dimensional HSI equation can be written in the following form:

$$w_t = u_{xxt} + 3uu_t - 3u_x v_t + u_x, \quad w_x = -u_y, \quad v_x = -u, \tag{1}$$

where $u = u(x, y, t)$ is the physical field, while the functions v and w are the potentials of physical field derivatives. u_{xxt} signifies the spatial and temporal evolution of u; $3uu_t$ indicates a non-linear relationship in the temporal development of a wave packet; $-3u_x v_t$ describes the interaction between two fields; and u_x denotes the rate of change of u with respect to x. Equation (1) describes the propagation of small amplitude surface waves in a channel or a large channel where the depth and width change slowly. Liu *et al.*[29] analyzed the HSI equation with constant coefficient using the Hirota bilinear method and obtained some interactions of different types of solutions. Kumar *et al.*[30] studied the HSI equation by Lie symmetry analysis and obtained multiple soliton solutions and interactions of soliton solutions.

However, to describe the phenomena in the realistic world, the variable-coefficient NLPDEs have been constructed and studied. Variable-coefficient equations can be used to describe certain phenomena when the inhomogeneous media and nonuniform boundaries are taken into consideration.[31–33] In this paper, we investigate a variable-coefficient extension of Eq. (1), i.e.,

$$\begin{gathered} w_t = a(t)u_{xxt} + b(t)uu_t - b(t)u_x v_t + c(t)u_x, \\ w_x = -u_y, \quad v_x = -u, \end{gathered} \tag{2}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$, and $\omega = \omega(x, y, t)$ are analytic functions of the space variables x, y, and time variables t, and where $a(t)$, $b(t)$, and $c(t)$ are some differentiable functions of t. The dispersive term is $a(t)u_{xxt}$, and $a(t)u_{xxt}$ denotes the time-dependent coefficient $a(t)$ multiplied by the second derivative of u with respect to x and its derivative with respect to time, describing the spatial and temporal evolution of u; $b(t)uu_t$ indicates the time-dependent coefficient $b(t)$ multiplied by the product of u and u_t, describing the relationship between the temporal development of a wave packet and itself; $-b(t)u_x v_t$ describes the interaction between two fields; $c(t)$ is the linear phase speed and dependent on time t. Equation (2) can be seen as an extension of Eq. (1) with variable coefficients. In this case, the equations consider time-dependent coefficients, reflecting the characteristics of the physical system as it changes over time. When simulating and exploring complex nonlinear phenomena in various physical backgrounds, the nonlinear evolution equation with variable coefficient[34–37] is more necessary to study than the nonlinear evolution equation with constant coefficient, and can reflect the essence of the problem. Therefore, in recent years, the study of the nonlinear evolution equation with the variable coefficient has received further attention.[38–45]

The outline of the paper is as follows. In Sec. II, Eq. (2) is tested for integrability through Painlevé analysis. The criterion of Painlevé property is that the solutions of the NLPDEs should have no singularities other than poles. The result gives the condition that Eq. (2) can be Painlevé integrable. In Sec. III, the one-soliton and two-soliton solutions will be derived and discussed. The process of motion of the soliton solution will be indicated by a set of diagrams at different times t. In Sec. IV, the lump solutions will be determined through symbolic computations with Maple. For a special presented lump solution, three-dimensional plots and contour plots will be made via the Maple plot tool to shed light on the characteristics of the presented lump solutions. A few concluding remarks will be given in Sec. V. The variable-coefficient Hirota–Satsuma–Ito equation provides a closer representation of real systems by accommodating varying system characteristics over time or space, offering more flexibility in modeling as parameters change and demonstrating broader applicability to a wide range of phenomena. As a result, it delivers more accurate results when describing nonlinear wave phenomena and fluid behaviors.

II. PAINLEVÉ ANALYSIS

A NLPDE has the Painlevé property when its solutions are single-valued about all the movable singularity manifold.[46] We will discuss whether Eq. (2) is Painlevé-integrable by virtue of the Weiss–Tabor–Carnevale method. Assuming that the singularity manifold is

$$\varphi(x, y, t) = 0. \tag{3}$$

We seek a solution to Eq. (2) in the following form:

$$u = \sum_{j=0}^{\infty} u_j(x, y, t)\varphi(x, y, t)^{j+\alpha}, \tag{4}$$

where α is a negative integer, $u_j(x, y, t)$ and $\varphi(x, y, t)$ are analytic functions in the neighborhood of Eq. (3).

Substituting $u \sim u_0\varphi^{(\alpha)}$ into Eq. (2) and balancing the dominant term, we find that $\alpha = -1$ and $u_0 = \frac{2a(t)}{b(t)}\varphi_x$ by a leading order analysis. Substituting $u \sim u_0\varphi^{-1} + \sum_{j=1}^{\infty} u_j\varphi^{j-1}$ into Eq. (2) and making the coefficient of φ^{j-6} to zero lead to the resonance values $j = -1, 1, 4, 6$. The resonance $j = -1$ means that φ is arbitrary. Thinking about u_1, u_4 and u_6 are free functions, we find the integrable conditions that $a(t)$ and $b(t)$ reduce into constants.

III. SOLITON SOLUTIONS

A. One-soliton solution

When the constraint $b(t) = 3a(t)$ is satisfied, Eq. (2) is converted into

$$w_t = a(t)u_{xxt} + 3a(t)uu_t - 3a(t)u_x v_t + c(t)u_x,$$
$$w_x = -u_y, \quad v_x = -u. \tag{5}$$

Through the dependent variable transformation,

$$u = 2[\ln f(x, y, t)]_{xx}. \tag{6}$$

Equation (5) enjoys the bilinear form

$$(D_y D_t + a(t)D_x^3 D_t + c(t)D_x^2) f \cdot f = 0, \tag{7}$$

where D is the Hirota bilinear operator[47] defined by

$$D_x^\alpha D_y^\beta D_t^\gamma (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^\beta \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\gamma \times f(x, y, t) g(x', y', t')|_{x'=x, y'=y, t'=t}. \tag{8}$$

One basic property of the Hirota bilinear derivatives is that

$$D_x D_y f \cdot g = D_x D_y g \cdot f. \tag{9}$$

Expanding f into a power series of the small parameter ε using the standard perturbation method,

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots. \tag{10}$$

Using the properties of the D-operator, effective truncation can be applied to the function f.

The Hirota bilinear method derives solitary wave solutions by solving the equation's bilinear form, allowing for the representation of localized waves or solitons that maintain their shape during propagation and enables the determination of precise two-soliton solutions by manipulating the equation's bilinear form, capturing the interaction between two solitons as they move through a medium or space. The Hirota bilinear method also involves discovering lump solutions, representing nonsingular localized structures within integrable systems and often arising from nonlinear wave equations, crucial for understanding complex system behaviors. The innovative concept of bilinear derivatives is the key in the basic theory of exact solutions. The Hirota bilinear method can be considered as the simplest and effective technique to investigate integrability aspect of a nonlinear equation. The Hirota bilinear method concisely converts a nonlinear equation into a bilinear form using a dependent variable transformation and produces quasi-periodic wave solutions, rational solutions, lump solutions, multi-soliton solutions, and other exact solutions via the bilinear form.

To find the one-soliton solution, f is given by

$$f = 1 + e^{\eta_1}, \tag{11}$$

where

$$\eta_1 = p_1 x + q_1 y + \Omega_1(t), \tag{12}$$

$$\Omega_1(t) = -\int \frac{c(t)p_1^2}{a(t)p_1^3 + q_1} dt, \tag{13}$$

where p_1 and q_1 are constants, and $\Omega_1(t)$ is the function of t; thus, the one-soliton solutions for Eq. (2) under constraint Eq. (12) can be written as

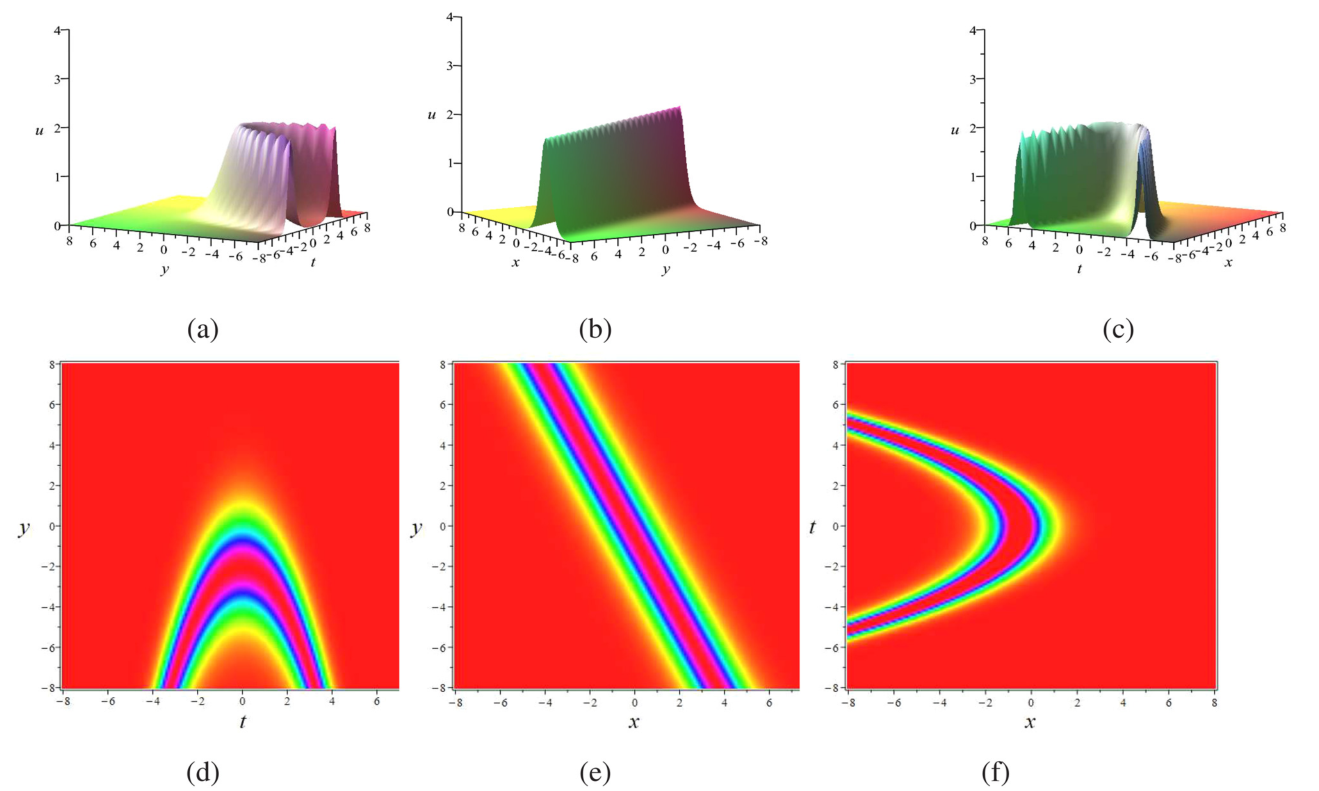

FIG. 1. One-soliton solutions via Eq. (14) with $c(t) = -2t$, $a(t) = 1$, $p_1 = 2$, $q_1 = 1$, $\Omega_1(t) = \frac{4}{7}t^2$. (a) $x = 1$, (b) $t = 1$, (c) $y = 1$, and (d)–(f) are the density figures of (a)–(c).

$$u=\frac{2p_1^2e^{p_1x+q_1y-\int\frac{c(t)p_1^2}{a(t)p_1^3+q_1}dt}}{1+e^{p_1x+q_1y-\int\frac{c(t)p_1^2}{a(t)p_1^3+q_1}dt}}-\frac{2p_1^2e^{2\left(p_1x+q_1y-\int\frac{c(t)p_1^2}{a(t)p_1^3+q_1}dt\right)}}{\left(1+e^{p_1x+q_1y-\int\frac{c(t)p_1^2}{a(t)p_1^3+q_1}dt}\right)^2}. \tag{14}$$

Figures 1 and 2 show the effect of parameters $a(t)$ and $c(t)$ on nonlinear wave depicted by Eq. (14). Their physical structures are described in some 3D plots. A parabolic-shape rational solution is presented in Fig. 1, and the solitary wave has an amplitude of 2, with the velocity in the x and y directions being $\left(v_x=-\frac{8}{7}t, v_y=-\frac{8}{7}t\right)$. A periodic-shape rational solution is plotted in Fig. 2. The solitary wave solution has an amplitude of $\frac{1}{2}$, with the velocities in the x and y directions being $\left(v_x=\frac{\sin(t)}{\cos(t)+1}, v_y=\frac{\sin(t)}{\cos(t)+1}\right)$. These solitary waves maintain their shape and size unchanged during the process of movement. Due to the presence of variable coefficients, the speed of solitary wave movement will depend on the time t. The one-soliton solution to Eq. (7) is related to the variable coefficient, and when different parameters are selected, the one-soliton solution to Eq. (11) will adjust accordingly.

B. Two-soliton solutions

To find the two-soliton solution, f is given by

$$f=1+e^{p_1x+q_1y+\Omega_1(t)}+e^{p_2x+q_2y+\Omega_2(t)} + A_{12}e^{p_1x+q_1y+\Omega_1(t)+p_2x+q_2y+\Omega_2(t)}, \tag{15}$$

where

$$\Omega_i(t)=-\int\frac{c(t)p_i^2}{a(t)p_i^3+q_i}dt, \quad i=1,2, \tag{16}$$

$$A_{12}=-\frac{(\Omega_1'(t)-\Omega_2'(t))((q_1-q_2)+a(t)(p_1-p_2)^3))+b(t)(p_1-p_2)^2}{(\Omega_1'(t)+\Omega_2'(t))((q_1+q_2)+a(t)(p_1+p_2)^3))+b(t)(p_1+p_2)^2}. \tag{17}$$

We obtain the two-soliton solutions under constraint Eq. (15) as

$$u=\frac{2(p_1^2e^{\eta_1}+p_2^2e^{\eta_2}+A_{12}(p_1+p_2)^2e^{\eta_1+\eta_2})}{1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2}} - \frac{2(p_1e^{\eta_1}+p_2e^{\eta_2}+A_{12}(p_1+p_2)e^{\eta_1+\eta_2})^2}{(1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2})^2}. \tag{18}$$

Figures 3 and 4 show the effect of parameters $a(t)$ and $c(t)$ on nonlinear wave depicted by Eq. (18). Figure 4 shows the dynamical behavior of the two-solitons in (x, y) plane. It is clear that the position of the two-solitons changes along the x axis with time from

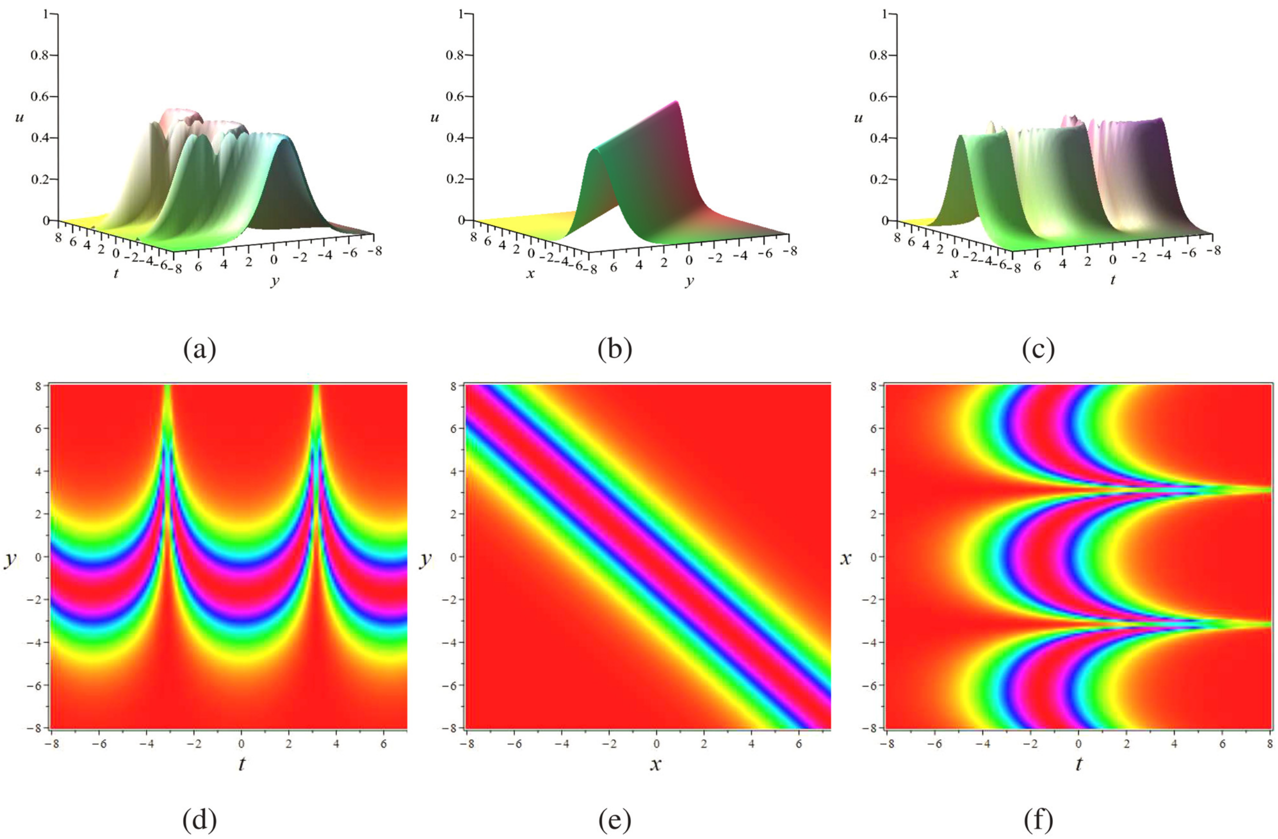

FIG. 2. One-soliton solutions via Eq. (14) with $c(t)=\sin(t)$, $a(t)=\cos(t)$, $p_1=1$, $q_1=1$, $\Omega_1(t)=ln(\cos(t)+1)$. (a) $x=1$, (b) $t=1$, (c) $y=1$, and (d)–(f) are the density figures of (a)–(c).

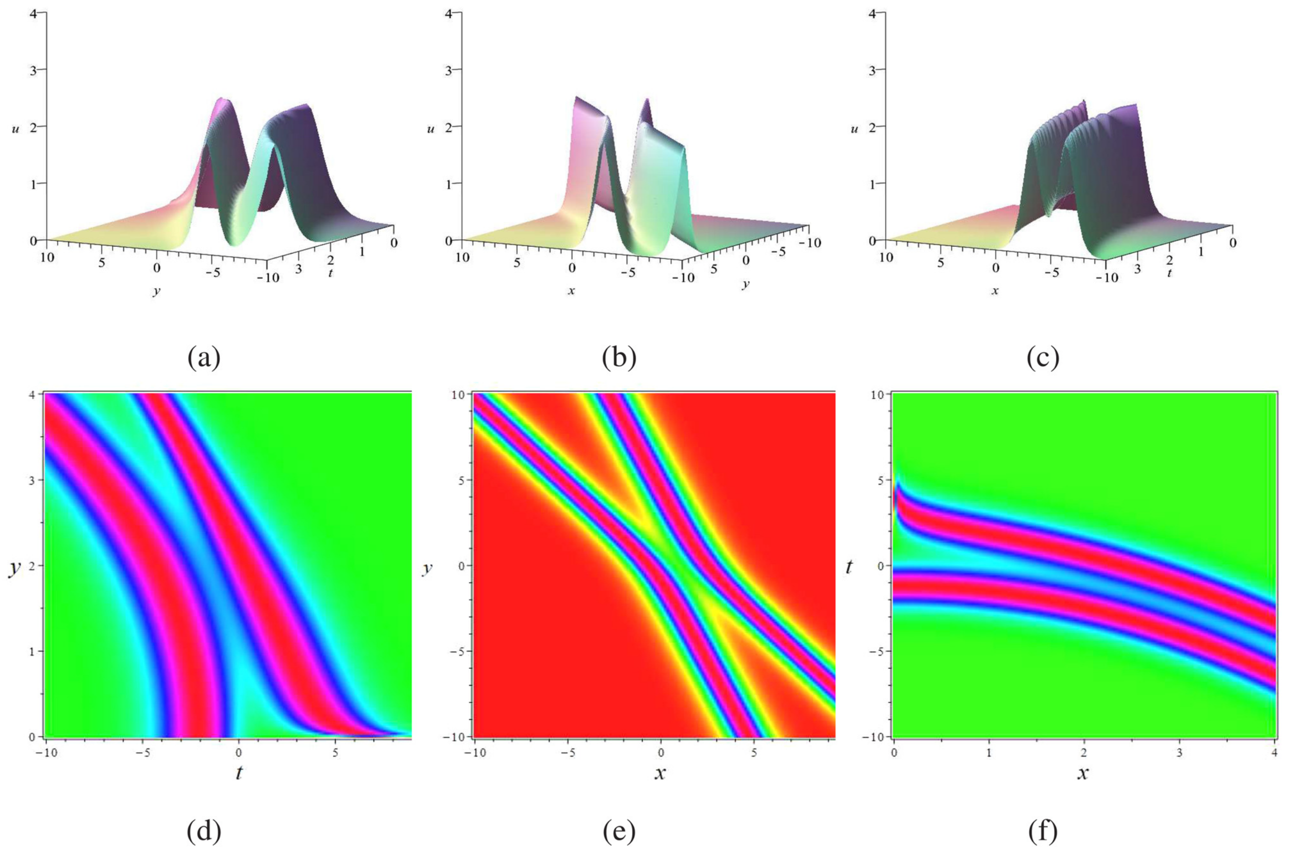

FIG. 3. Two-soliton solutions via Eq. (18) with $a(t)=1$, $c(t)=-2t$, $p_1=2$, $p_2=2$, $q_1=1$, $q_2=2$, $\Omega_1(t)=\frac{4}{7}t^2$, $\Omega_2(t)=\frac{2}{3}t^2$. (a) $x=1$, (b) $t=1$, (c) $y=1$, and (d)–(f) are the density figures of (a)–(c).

Figs. 4(a)–4(c). The speeds of two solitary waves in the x and y directions are $\left(v_{1_x}=-\frac{4}{7}t, v_{1_y}=-\frac{8}{7}t\right)$ and $\left(v_{2_x}=-\frac{2}{3}t, v_{2_y}=-\frac{2}{3}t\right)$, respectively. The movement velocities of the two solitary waves are dependent on the values of the variable coefficients. However, the amplitude and the waveform of the two-solitons remain the same. Figures 4(d)–4(f) are their corresponding density plots. Figure 5 illustrates the interaction between the two solitons. Moreover, the amplitudes of the two solitary waves are 2, and the interaction between the two solitons is seen to be elastic in the sense that the velocities and amplitudes of the two solitons remain constant after interaction except for a phase shift.

IV. LUMP SOLUTIONS

A lump solution is a rational function solution, which is real analytic and decays in all directions of space variables. Lump functions provide appropriate prototypes to model rogue wave dynamics in both oceanography[48] and nonlinear optics.[49] The lump solutions presented in the context of the shallow-water wave equation exhibit nonsingular localized structures. Their significance lies in their ability to represent specific concentrated features within the broader wave system, offering a means to depict phenomena such as wave interactions, energy concentration, and localized disturbances. To search for quadratic function solutions to the $(2+1)$-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equations in Eq. (7), we begin with

$$f=g^2+h^2+a_7(t),\quad g=a_1(t)x+a_2(t)y+a_3(t),\\ h=a_4(t)x+a_5(t)y+a_6(t), \tag{19}$$

where $a_i(t)$, $1\le i\le 7$ are real parameters to be determined. A direct Maple symbolic computation with f generates the following set of constraining equations for the parameters:

$$\begin{gathered} a_1(t)=a_1(t),\quad a_2(t)=c_3a_1(t), a_3(t)=a_3(t),\\ a_4(t)=c_4a_1(t),\quad a_5(t)=-\frac{2c_3c_4a_1(t)}{c_4^2-1},\\ a_6(t)=c_2a_1(t),\quad a_7(t)=c_1a_1(t)^2,\\ a(t)=-\frac{c_1c_3c_4^2}{3c_4^4-3},\quad c(t)=\frac{c_3\frac{da_3(t)}{dt}a_1(t)-\frac{da_1(t)}{dt}a_3(t)}{(c_4^2-1)a_1(t)^2}, \end{gathered} \tag{20}$$

where $(c_4^2-1)a_1(t)^2\neq 0$, c_1, c_2, c_3 and c_4 are constants, $c_1>0$.

This set leads to a class of positive quadratic function solutions to the $(2+1)$-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equations in Eq. (6),

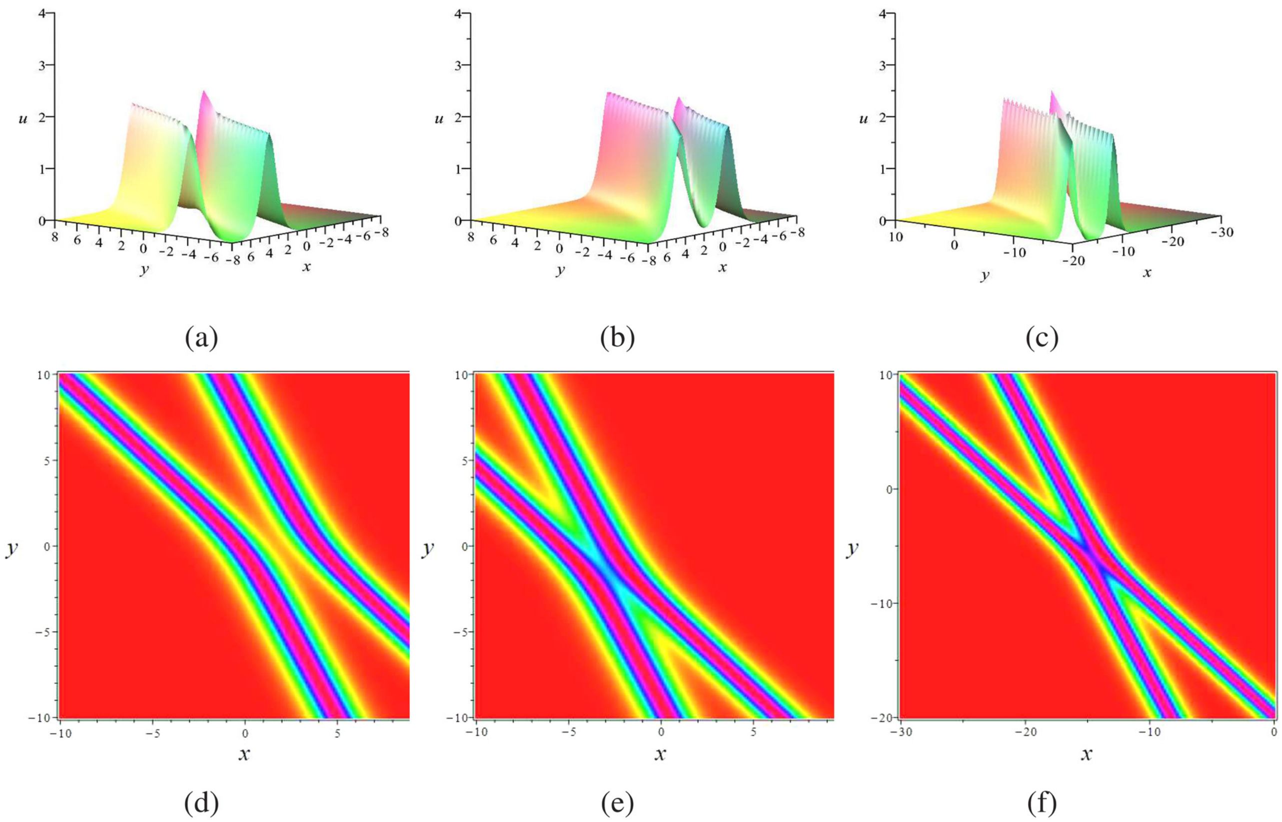

FIG. 4. Two-soliton solutions via Eq. (18) with $a(t) = 1$, $c(t) = -2t$, $p_1 = 2$, $p_2 = 2$, $q_1 = 1$, $q_2 = 2$, $\Omega_1(t) = \frac{4}{7}t^2$, $\Omega_2(t) = \frac{2}{3}t^2$. (a) $t = 1$, (b) $t = 4$, (c) $t = 8$, and (d)–(f) are the density figures of (a)–(c).

$$f = (a_1(t)x + c_3 a_1(t)y + a_3(t))^2 + \left(c_4 a_1(t)x - \frac{2c_3 c_4 a_1(t)}{c_4^2 - 1}y + c_2 a_1(t)\right)^2 + c_1 a_1(t)^2, \tag{21}$$

and the resulting class of quadratic function solutions, in turn, yields a class of lump solutions to the (2 + 1)-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equations in Eq. (2) through the transformation $u = 2[\ln f(x, y, t)]_{xx}$,

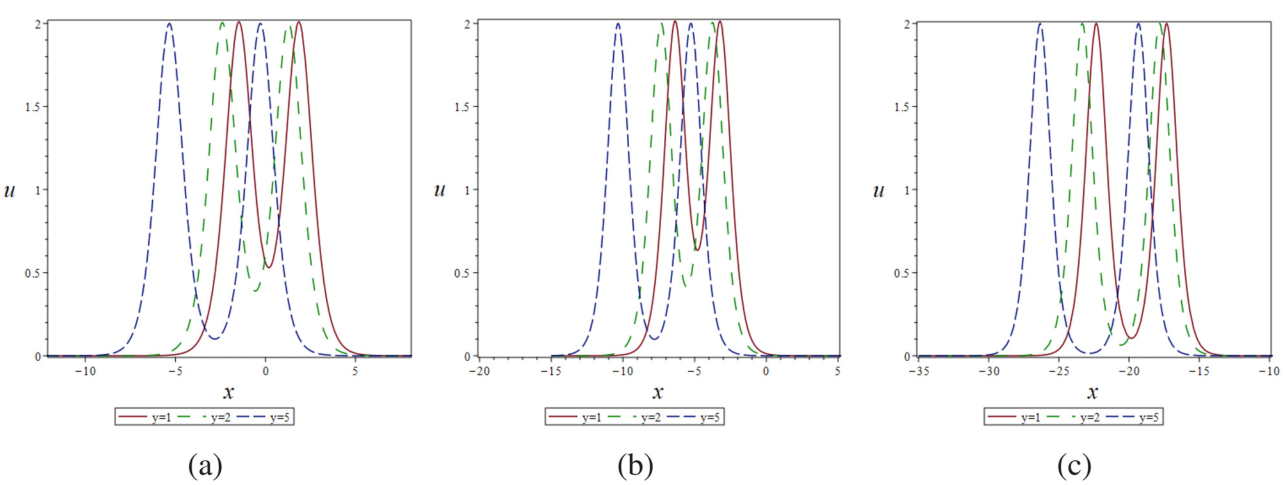

FIG. 5. Spatial structure of the two-soliton solution with (a) $t = 1$, (b) $t = 4$, and (c) $t = 8$.

$$u = \frac{4(a_1(t) + a_4(t))f - 8(a_1(t)g + a_4(t)h)^2}{f^2}, \tag{22}$$

where the function f is defined by Eq. (19), and the functions of g and h are given as follows:

$$g = a_1(t)x + c_3 a_1(t)y + a_3(t), \tag{23}$$

$$h = c_4 a_1(t)x - \frac{2c_3 c_4 a_1(t)}{c_4^2 - 1}y + c_2 a_1(t). \tag{24}$$

In this set of lump solutions, all two involved parameters of $a_1(t)$ and $a_3(t)$ are arbitrary provided that the solutions are well defined, i.e., if the determinant condition $(c_4^2 - 1)a_1(t)^2 \neq 0$ is satisfied. That determinant condition precisely means that two directions $(a_1(t), a_2(t))$ and $(a_4(t), a_5(t))$ in the (x, y)-plane are not parallel.

Note that the solutions in Eq. (21) are analytic in the (x, y)-plane if and only if the parameter $a_7(t) > 0$. The analyticity of the solutions in Eq. (21) is guaranteed if $(c_4^2 - 1)a_1(t)^2 \neq 0$ and $c_1 > 0$ hold.

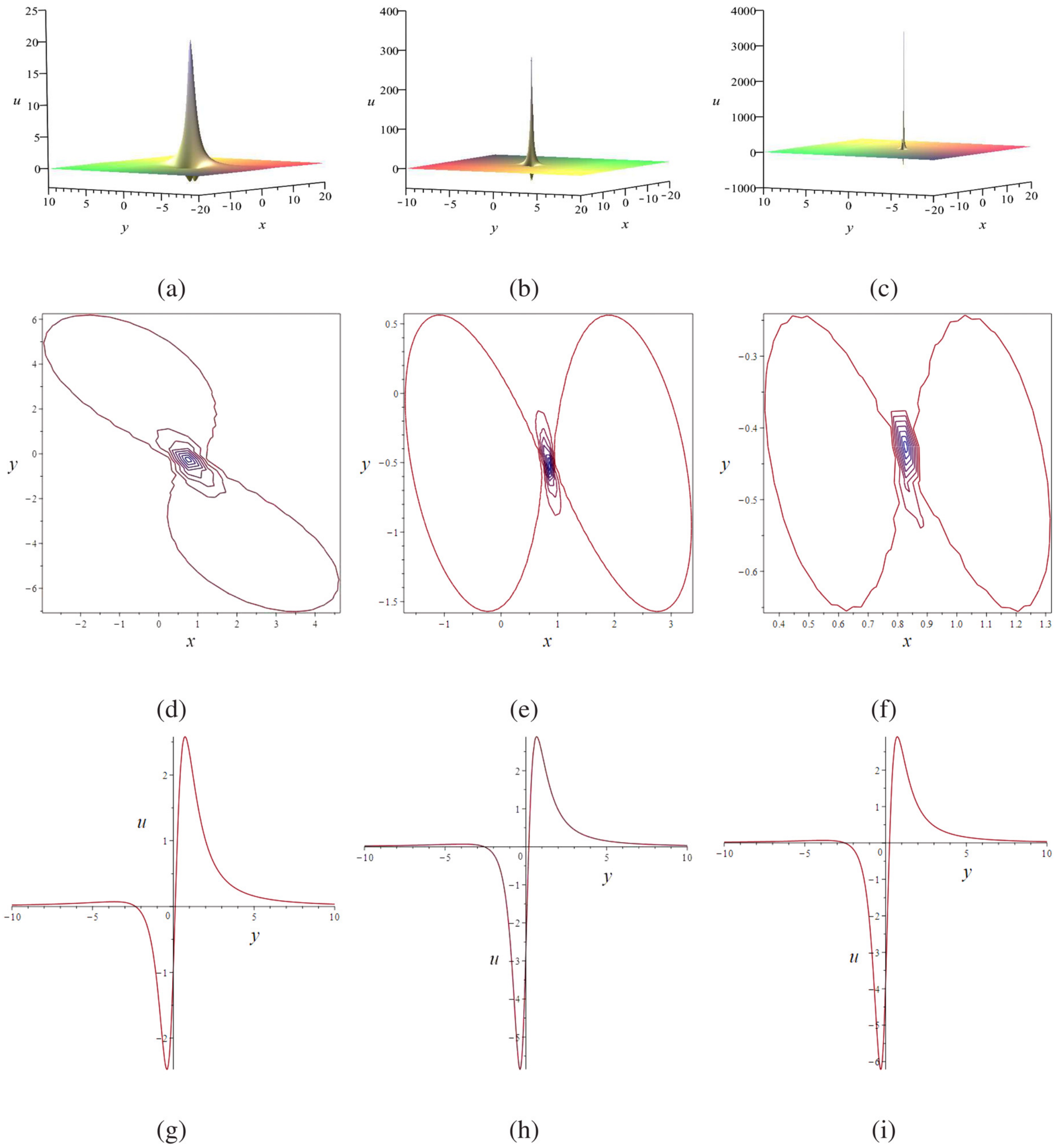

FIG. 6. Lump solutions via Eq. (21) with $a_1(t) = t^2$, $a_2(t) = 2t^2$, $a_3(t) = t^2$, $a_4(t) = t^2$, $a_5(t) = 0$, $a_6(t) = -\frac{3}{2}ln(t)t^2 - \frac{1}{8} + t^2$, $a_7(t) = 4t^2$. (a) $t = 1$, (b) $t = 2$, (c) $t = 5$, and (d)–(f) are the contour figures of (a)–(c), and (g)–(i) are the 2D-plots for $x = 0$ of (a)–(c).

The dynamics of the solution u show that one lump wave is initially moving. Its amplitude and velocity are dependent on the values of the variable coefficients. The plots of function u at $t = 1$, 2, and 5 are shown in Fig. 6.

The coordinates of the center points are obtained by means of the extreme points found by Maple. In general, the maximum point coordinates can be used to study and represent the properties of lump wave in the process of motion, such as the change of velocity and waveform. After software calculation, the extreme values and extreme points are obtained as follows:

$$(u_{max}, u_{min}) = \left(20t^4, -\frac{5}{2}t^4\right), \tag{25}$$

$$(x, y) = \left(\frac{5}{6}, -\frac{5t^2 + 6ln(t)}{12t^2}\right),$$

$$(x, y) = \left(\frac{5t^2 - \frac{6\sqrt{15}}{5}}{6t^2}, -\frac{5t^2 + 6ln(t)}{12t^2}\right), \tag{26}$$

$$(x, y) = \left(\frac{5t^2 - \frac{6\sqrt{15}}{5}}{6t^2}, -\frac{5t^2 + 6ln(t)}{12t^2}\right).$$

The peak height of lump wave is represented by u_{max}, and the trough height of lump wave is represented by u_{min}. From the above-mentioned expression, it can be concluded that the amplitude of u and the coordinates of extreme value points are related to t, which means that different shapes of the lump wave can be obtained when different values are assigned to t.

From the coordinates of maximum points, the lump wave motion in the x and y axis directions is considered as $x = x(t), y = y(t)$. The velocity of lump wave at specific time are $x_t = 0, y_t = -\frac{10t+\frac{6}{t}}{12t^2} + \frac{5t^2+6ln(t)}{6t^3}$. At the time t, lump wave moves along the y axis with the velocity of y_t, which is reflected in the image that lump wave always moves along the y axis. The minimum velocity of the lump wave is $x_t = \frac{5}{3t} - \frac{5t^2+\frac{6\sqrt{15}}{5}}{3t^3}, y_t = -\frac{10t+\frac{6}{t}}{12t^2} + \frac{5t^2+6ln(t)}{6t^3}$.

By comparing the velocity of the maximum lump wave with the velocity of the minimum lump wave, it is found that the motion velocity is different. Therefore, it can be concluded that the moving speed of the lump wave is related to the variable coefficient, and as $t = 1$, $t = 2$, $t = 5$, the amplitude of the lump wave increases. The lump solutions play a crucial role in understanding the impact of varying coefficients on the formation and behavior of localized structures within the equations, shedding light on how the coefficient variations affect the dynamics and stability of such solutions.

V. CONCLUSIONS

Investigated in this paper is a $(2+1)$-dimensional generalized variable-coefficient Hirota–Satsuma–Ito equations, i.e., Eq. (2). We have systematically checked its Painlevé-integrability by performing the Painlevé test. It has been shown that Eq. (2) is Painlevé-integrable when $a(t)$ and $b(t)$ reduce into constants are satisfied. By virtue of the Hirota bilinear method, one-soliton solutions [Eq. (14)], two-soliton solutions [Eq. (18)], and lump solutions [Eq. (21)] have been constructed. The variable-coefficient Hirota–Satsuma–Ito equations introduce several innovations in fluid physics compared to its constant coefficient counterpart. It offers improved simulation of real fluid systems by accommodating their time and space-varying properties, providing a broader applicability that explains a wider range of fluid phenomena, and better representation of nonlinear effects. As a result, these equations provide a more accurate and flexible framework for understanding and modeling intricate fluid dynamics, thereby fostering innovation in the field of fluid physics. Due to the diverse forms of the variable-coefficient parameters, the resulting solutions are naturally different when various values are assigned, which can produce more intriguing and peculiar interaction phenomena. It is hoped that the exploration of the variable-coefficient equations presented in this paper can be used as a basis for the subsequent study of some complex physical phenomena.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Xing Lü: Supervision (lead). **Liang Li Zhang:** Writing – original draft (equal). **Wen-Xiu Ma:** Software (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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