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A direct bilinear Bäcklund transformation of a (2+1)-dimensional Korteweg–de Vries-like model

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ABSTRACT

We directly construct a bilinear Bäcklund transformation (BT) of a (2+1)-dimensional Korteweg–de Vries-like model. The construction is based on a so-called quadrilinear representation. The resulting bilinear BT is in accordance with the auxiliary-independent-variable-involved one derived with the Bell-polynomial scheme. Moreover, by applying the gauge transformation and the Hirota perturbation technique, multisoliton solutions are iteratively computed.

1. Introduction

In the study of nonlinear partial differential equations of mathematical physics, the Bäcklund transformation (BT) provides a means to construct new solutions from known ones [1–20]. BTs can connect solutions of different equations or the same equation [6,11]. In the latter case, BTs are generally called auto-BTs. In soliton theory, BTs are also closely related to integrable properties such as Lax pairs, nonlinear superposition formulas, and infinitely many conservation laws [3–5,12].

Among others, the Hirota bilinear method [2,3,5,12,13,16] is a powerful approach to solving soliton equations and deriving bilinear BTs. Some generalization of bilinear forms has been given, including a natural extension: multilinear forms [17]. A class of generalized Hirota derivatives has been introduced by assigning specific signs to derivatives [18], and as a result, some generalized nonlinear differential equations have been built and studied, for which the linear superposition principle [19] can be applied to the construction of

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sub-spaces of solutions [20]. For example, the Korteweg–de Vries (KdV) model [3,4,11],

$$u_t + 6\,u\,u_x + u_{xxx} = 0 \tag{1}$$

enjoys the bilinear representation, via the dependent variable transformation $u(x,t) = 2\big[\ln f(x,t)\big]_{xx}$, as follows

$$(D_x D_t + D_x^4) f \cdot f = 0, \tag{2}$$

where the Hirota derivatives D_x, D_t and D_x^4 [3] are bilinear operators defined by

$$D_x^\alpha D_y^\beta D_t^\gamma (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^\beta \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\gamma f(x,y,t) g(x',y',t')\Big|_{x'=x, y'=y, t'=t.}$$

Based on the bilinear Eq. (2), it is easy to derive the following bilinear BT for the KdV model [3],

$$(D_x^2 + \lambda) g \cdot f = 0, \tag{3a}$$

$$(D_x^3 + D_t - 3\lambda D_x) g \cdot f = 0, \tag{3b}$$

where λ is an arbitrary constant, and $g = g(x,t)$ is another solution of Eq. (2).

Generally speaking, there exist some difficulties in applying the Hirota bilinear method to construct bilinear BTs for nonlinear equations. In some cases, employing the Bell-polynomial technique and introducing the auxiliary independent variable(s) are helpful [7,9,10].

The following (2+1)-dimensional KdV-like model,

$$u_t + 6\,u\,u_x + u_{xxx} + 4\,u\,u_y + u_{xxy} + 2\,u_x \int u_y \,\mathrm{d}x = 0,$$

which is firstly proposed with the Lax pair generating technique [14], can be converted into

$$u_t + 6\,u\,u_x + u_{xxx} + 4\,u\,u_y + u_{xxy} + 2v\,u_x = 0, \tag{4}$$

via $u_y = v_x$. The auxiliary-independent-variable-involved bilinear form of Eq. (4) has been derived with the Bell-polynomial scheme [15]:

$$(D_x^4 - m\,D_x D_s)\tau \cdot \tau = 0, \tag{5a}$$

$$(D_x D_t + \frac{2}{3}\,D_x^3 D_y + \frac{m}{3}\,D_y D_s + m\,D_x D_s)\tau \cdot \tau = 0, \tag{5b}$$

where $u = 2\big[\ln \tau(x,y,t)\big]_{xx}$, $v = 2\left[\ln \tau(x,y,t)\right]_{xy}$, s is an auxiliary independent variable, while $m \neq 0$ is an arbitrary constant. Through symbolic computation, the auxiliary-independent-variable-involved bilinear BT, with $\tau' = \tau'(x,y,t)$ being another solution of Eq. (5), can be determined by [15]:

$$\Big[D_x^2 - \lambda\,D_x - k\Big]\tau' \cdot \tau = 0, \tag{6a}$$

$$\Big[m\,D_s - D_x^3 - 3\,k\,D_x\Big]\tau' \cdot \tau = 0, \tag{6b}$$

$$\Big[D_t + D_x^2 D_y + D_x^3 - \lambda\,D_x D_y + (\lambda^2 + 3\,k)\,D_y + 3\,k\,D_x - \nu(y,t)\Big]\tau' \cdot \tau = 0, \tag{6c}$$

where λ and k are two arbitrary constants and $\nu(y,t)$ is an arbitrary function.

In this Letter, without employing the Bell-polynomial technique and introducing the auxiliary independent variable, we will directly construct a bilinear BT for Eq. (4). We will firstly give a so-called quadrilinear representation in Section 2, and directly derive a bilinear BT in Section 3. In Section 4, we will apply a gauge transformation (GT) and the Hirota perturbation technique to compute multisoliton solutions iteratively. Section 5 will be our concluding remarks.

2. Quadrilinear representation

Substitution of the dependent variable transformation, $u = 2\left[\ln \tau(x,y,t)\right]_{xx}$ and $v = 2\left[\ln \tau(x,y,t)\right]_{xy}$, into Eq. (4) generates

$$\begin{aligned} u_t + 6\,u\,u_x + u_{xxx} + 4\,u\,u_y + u_{xxy} + 2v\,u_x &= \Big\{[\tau^2(D_xD_t\tau\cdot\tau) + \tau^2(D_x^4\tau\cdot\tau) \\ &\quad + (D_x^2\tau\cdot\tau)(D_xD_y\tau\cdot\tau)]/\tau^4\Big\}_x + \Big\{[\tau^2(D_x^4\tau\cdot\tau) - 2(D_x^2\tau\cdot\tau)^2]/\tau^4\Big\}_y \\ &= \frac{1}{\tau^4}\left\{D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau\cdot\tau\Big]\cdot\tau^2 + \frac{1}{3}D_y\big(D_x^4\tau\cdot\tau\big)\cdot\tau^2\right\} \\ &= 0. \end{aligned}$$

Hereby, we can take

$$D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau\cdot\tau\Big]\cdot\tau^2 + \frac{1}{3}D_y\big(D_x^4\tau\cdot\tau\big)\cdot\tau^2 = 0 \tag{7}$$

as a quadrilinear representation of Eq. (4). If we introduce the auxiliary independent variable s via $(D_x^4 - m\,D_xD_s)\tau\cdot\tau = 0$, then the auxiliary-independent-variable-involved bilinear form (5) can be decoupled from Eq. (7). In the following, starting from Eq. (7), we will directly construct a bilinear BT of Eq. (4) without any decoupling.

3. Bilinear BT

Assuming $u'(x,y,t) = 2\left[\ln \tau'(x,y,t)\right]_{xx}$ and $v'(x,y,t) = 2\left[\ln \tau'(x,y,t)\right]_{xy}$ to be another solution of Eq. (4), we consider

$$\begin{aligned} 0 = \mathcal{P} &\equiv \tau'^4\Big\{D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau\cdot\tau\Big]\cdot\tau^2 + \frac{1}{3}D_y\big(D_x^4\tau\cdot\tau\big)\cdot\tau^2\Big\} \\ &\quad - \tau^4\Big\{D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau'\cdot\tau'\Big]\cdot\tau'^2 + \frac{1}{3}D_y\big(D_x^4\tau'\cdot\tau'\big)\cdot\tau'^2\Big\} \\ &= (\tau'^2)^2\Big\{D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau\cdot\tau\Big]\cdot\tau^2\Big\} + (\tau'^2)^2\Big\{\frac{1}{3}D_y\big(D_x^4\tau\cdot\tau\big)\cdot\tau^2\Big\} \\ &\quad - (\tau^2)^2\Big\{D_x\Big[\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau'\cdot\tau'\Big]\cdot\tau'^2\Big\} - (\tau^2)^2\Big\{\frac{1}{3}D_y\big(D_x^4\tau'\cdot\tau'\big)\cdot\tau'^2\Big\} \\ &= D_x\Big[\tau'^2\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau\cdot\tau - \tau^2\big(D_xD_t + D_x^4 + \frac{2}{3}D_x^3D_y\big)\tau'\cdot\tau'\Big]\cdot\tau^2\tau'^2 \\ &\quad + \frac{1}{3}D_y\Big[\tau'^2\big(D_x^4\tau\cdot\tau\big) - \tau^2\big(D_x^4\tau'\cdot\tau'\big)\Big]\cdot\tau^2\tau'^2 \\ &= D_x\Big[2D_t\big(D_x\tau\cdot\tau'\big)\cdot\big(\tau\tau'\big) + 2D_x\big(D_x^3\tau\cdot\tau'\big)\cdot\big(\tau\tau'\big) - 6D_x\big(D_x^2\tau\cdot\tau'\big)\cdot\big(D_x\tau\cdot\tau'\big) \\ &\quad + 2D_x\big(D_x^2D_y\tau\cdot\tau'\big)\cdot\big(\tau\tau'\big) - 2D_x\big(D_x^2\tau\cdot\tau'\big)\cdot\big(D_y\tau\cdot\tau'\big) \\ &\quad - \frac{2}{3}D_y\big(D_x^3\tau\cdot\tau'\big)\cdot\big(\tau\tau'\big) - 2D_y\big(D_x^2\tau\cdot\tau'\big)\cdot\big(D_x\tau\cdot\tau'\big)\Big]\cdot\tau^2\tau'^2 \\ &\quad - \frac{2}{3}D_y\Big[D_x\big(D_x^3\tau\cdot\tau'\big)\cdot\big(\tau\tau'\big) - 3D_x\big(D_x^2\tau\cdot\tau'\big)\cdot\big(D_x\tau\cdot\tau'\big)\Big]\cdot\tau^2\tau'^2. \end{aligned} \tag{8}$$

At this stage, if we take

$$D_x^2(\tau\cdot\tau') = \lambda D_x(\tau\cdot\tau') + k\,\tau\tau' \tag{9a}$$

with λ and k as two arbitrary constants, Eq. (8) can be simplified into

$$\mathcal{P} = D_x\Big\{D_x\Big[\big(2D_t + 2D_x^3 + 2D_x^2D_y + 6kD_x - 2\lambda D_xD_y + (6k + 2\lambda^2)D_y\big)\tau\cdot\tau'\Big]\cdot\big(\tau\tau'\big)\Big\}\cdot\tau^2\tau'^2,$$

from which, we can assume

$$\Big[D_t + D_x^3 + D_x^2 D_y + 3kD_x - \lambda D_x D_y + (3k+\lambda^2)D_y\Big]\tau \cdot \tau' = \nu(y,t)\tau\tau', \tag{9b}$$

where $\nu(y,t)$ is an arbitrary function, and then, the set of Eqs. (9) constitutes a bilinear BT for Eq. (4). We remark that Eqs. (9) are as the same as Eqs. (6a) and (6c).

Solving Eqs. (9) with $\tau = 1$ (corresponding to the vacuum solution $u = 0$), $\lambda = 0$ and $\nu(y,t) = 0$, we can obtain a solitary wave solution of Eq. (4) as follows:

$$u = 2\left[\ln \tau'(x,y,t)\right]_{xx} = 2\,k\,\mathrm{sech}^2\Big(\sqrt{k}\,x + \alpha\,y - 4\,k\,(\sqrt{k}+\alpha)\,t + \beta\Big),$$

where α, β and $k > 0$ are arbitrary constants. Hereby, the existence of the above new solution u shows that the BT, Eqs. (9), is powerful in solving Eq. (4). Based on the BT, Eqs. (9), multisoliton solutions can be constructed iteratively.

4. Gauge transformation and multisoliton solutions

Noticing the arbitrariness of λ, k and $\nu(y,t)$ in the BT, Eqs. (9), we apply the following GT to Eqs. (9):

$$\tau \longrightarrow e^{\xi}\tau, \qquad \tau' \longrightarrow e^{\eta}\tau', \qquad \xi = \omega_1\,t + m_1\,x + n_1\,y + \xi^{(0)}, \qquad \eta = \sigma_1\,t + l_1\,x + p_1\,y + \eta^{(0)},$$

where ω_1, m_1, n_1, $\xi^{(0)}$, σ_1, l_1, p_1 and $\eta^{(0)}$ are all arbitrary constants. With symbolic computation, another form of the bilinear BT (9) can be given by

$$(D_x^2 + 2\kappa D_x)\tau \cdot \tau' = 0, \tag{10a}$$

$$(D_t + D_x^3 + D_x^2 D_y + 2\kappa D_x D_y + 4\kappa^2 D_y)\tau \cdot \tau' = 0, \tag{10b}$$

upon taking

$$\begin{cases} k = (m_1 - l_1)^2, \\ \nu(y,t) = (m_1-l_1)^2(n_1-p_1) + 3k(m_1-l_1) - \lambda(m_1-l_1)(n_1-p_1) + (3k+\lambda^2)(n_1-p_1), \end{cases}$$

setting $\lambda = 0$ and denoting $m_1 - l_1 = \kappa$.

Substituting the following expansions with a formal parameter ϵ

$$\tau = 1 + \epsilon\tau^{(1)} + \epsilon^2\tau^{(2)} + \epsilon^3\tau^{(3)} + \cdots,$$
$$\tau' = 1 + \epsilon\,\tau'^{(1)} + \epsilon^2\tau'^{(2)} + \epsilon^3\tau'^{(3)} + \cdots,$$

into Eqs. (10), and comparing the like powers of the coefficients of ϵ, we can derive multisoliton solutions of Eq. (4) as follows.

• One-soliton solution: Taking $\tau' = 1$ and $\kappa = \kappa_1 \neq 0$, and solving Eqs. (10), we can obtain the one-soliton solution:

$$u = 2\left[\ln(1+e^{\xi_1})\right]_{xx} = 2\,\kappa_1^2\,\mathrm{sech}^2\Big(\frac{\xi_1}{2}\Big),$$

where $\xi_1 = -2\kappa_1 x + \mu_1 y + (8\kappa_1^3 - 4\kappa_1^2\mu_1)t + \varphi_1$, while κ_1, μ_1 and φ_1 are all arbitrary constants.

• Two-soliton solution: Setting $\kappa = \kappa_2 \neq 0$ and $\tau' = 1 + e^{\xi_1}$ corresponding to the one-soliton solution, we can solve Eqs. (10) to obtain the two-soliton solution:

$$u = 2\left[\ln\Big(1 + \mathscr{A}e^{\xi_1} + \mathscr{B}e^{\xi_2} + \mathscr{Q}e^{\xi_1+\xi_2}\Big)\right]_{xx},$$

with $\mathscr{A} = \frac{\kappa_1+\kappa_2}{\kappa_2-\kappa_1}$, $\mathscr{B} = \frac{\kappa_1+\kappa_2}{\kappa_2-\kappa_1}\mathscr{Q}$, where $\xi_2 = -2\kappa_2 x + \mu_2 y + (8\kappa_2^3 - 4\kappa_2^2\mu_2)t + \varphi_2$, while κ_2, μ_2, φ_2 and $\mathscr{Q} \neq 0$ are all arbitrary constants.

• Three-soliton solution: Setting $\kappa = \kappa_3 \neq 0$ and $\tau' = 1 + \mathscr{A} e^{\xi_1} + \mathscr{B} e^{\xi_2} + \mathscr{Q} e^{\xi_1+\xi_2}$ corresponding to the two-soliton solution, we can solve Eqs. (10) to obtain the three-soliton solution:

$$u = 2\left[\ln\left(1 + \mathscr{A}_1 e^{\xi_1} + \mathscr{A}_2 e^{\xi_2} + \mathscr{A}_3 e^{\xi_3} + \mathscr{A}_{12} e^{\xi_1+\xi_2} + \mathscr{A}_{13} e^{\xi_1+\xi_3} + \mathscr{A}_{23} e^{\xi_2+\xi_3} + \mathscr{R} e^{\xi_1+\xi_2+\xi_3}\right)\right]_{xx},$$

with

$$\mathscr{A}_1 = \frac{(\kappa_1+\kappa_2)(\kappa_1+\kappa_3)}{(\kappa_1-\kappa_2)(\kappa_1-\kappa_3)}, \qquad \mathscr{A}_2 = \frac{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)}{(\kappa_1-\kappa_2)(\kappa_2-\kappa_3)}\mathscr{Q}, \qquad \mathscr{A}_3 = \frac{(\kappa_1+\kappa_3)(\kappa_2+\kappa_3)\mathscr{R}}{(\kappa_1-\kappa_3)(\kappa_2-\kappa_3)\mathscr{Q}},$$
$$\mathscr{A}_{12} = \frac{(\kappa_1+\kappa_3)(\kappa_2+\kappa_3)}{(\kappa_1-\kappa_3)(\kappa_2-\kappa_3)}, \qquad \mathscr{A}_{13} = \frac{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)\mathscr{R}}{(\kappa_1-\kappa_2)(\kappa_2-\kappa_3)\mathscr{Q}}, \qquad \mathscr{A}_{23} = \frac{(\kappa_1+\kappa_2)(\kappa_1+\kappa_3)}{(\kappa_1-\kappa_2)(\kappa_1-\kappa_3)}\mathscr{R},$$

where $\xi_3 = -2\kappa_3 x + \mu_3 y + (8\kappa_3^3 - 4\kappa_3^2\mu_3)t + \varphi_3$, while κ_3, μ_3, φ_3 and $\mathscr{R} \neq 0$ are all arbitrary constants.

5. Concluding remarks

Based on the quadrilinear representation [Eq. (7)], we have directly constructed the bilinear BT [Eqs. (9)] for the (2+1)-dimensional Korteweg–de Vries-like model [Eq. (4)]. Without employing the Bell-polynomial scheme, the construction of the bilinear BT is based on the quadrilinear representation, but not the bilinear equations like normal bilinear BTs in the literature (see, e.g., [2,3,16]); and no auxiliary independent variable has been introduced in our construction. Finally, we have applied the gauge transformation and the Hirota perturbation technique on the bilinear BT to iteratively compute the specific multisoliton solutions. We hope that the construction manner of the BT in this Letter would be useful in studying other soliton problems.

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