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A note on rational solutions to a Hirota-Satsuma-like equation

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ABSTRACT

With the generalized bilinear operators based on a prime number $p = 3$, a Hirota-Satsuma-like equation is proposed. Rational solutions are generated and graphically described by using symbolic computation software Maple.

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1. Introduction

Among the methods of solving nonlinear partial differential equations in various areas [1–6], the Hirota bilinear method is a powerful approach [1]. The shallow water wave equation studied by Hirota and Satsuma reads [7]

$$u_{xxt} + 3\,u\,u_t - 3\,u_x v_t - u_x = u_t, \tag{1}$$

where $v_x = -u$. Through the dependent variable transformation $u(x,t) = 2\big[\ln f(x,t)\big]_{xx}$, the Hirota bilinear form of Eq. (1) is

$$(D_x^3 D_t - D_x^2 - D_x D_t) f \cdot f = 0, \tag{2}$$

where $D_x^3 D_t$, D_x^2 and $D_x D_t$ are the Hirota bilinear operators [1] defined by

$$D_x^a D_t^b (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^a \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^b f(x,t) g(x',t')\Big|_{x'=x,t'=t}.$$

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Now we extend the Hirota bilinear Eq. (2) with the generalized bilinear operators, based on a prime number $p = 3$, into

$$(D_{3,x}^3 D_{3,t} - D_{3,x}^2 - D_{3,x} D_{3,t}) f \cdot f = 6 f_{xx} f_{xt} - 2 f_{xx} f + 2 f_x^2 - 2 f f_{xt} + 2 f_x f_t = 0, \tag{3}$$

where the generalized differential operators are introduced in [8]:

$$\begin{aligned} D_{p,x}^a D_{p,t}^b (f \cdot g) &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^a \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^b f(x,t) g(x',t')\Big|_{x'=x,t'=t} \\ &= \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} \alpha_p^i \alpha_p^j \frac{\partial^{a-i}}{\partial x^{a-i}} \frac{\partial^i}{\partial x'^{(i)}} \frac{\partial^{b-j}}{\partial t^{b-j}} \frac{\partial^j}{\partial t'^{(j)}} f(x,t) g(x',t')\Big|_{x'=x,t'=t}, \end{aligned}$$

while

$$\alpha_p^s = (-1)^{r_p(s)}, \quad s = r_p(s) \bmod p. \tag{4}$$

Under the rule given by Eq. (4), we particularly have

$$\alpha_3 = -1, \qquad \alpha_3^2 = \alpha_3^3 = 1, \qquad \alpha_3^4 = -1, \qquad \alpha_3^5 = \alpha_3^6 = 1.$$

Employing a dependent variable transformation

$$u(x,t) = 2\big[\ln f(x,t)\big]_x, \tag{5}$$

which is motivated by a general Bell polynomial theory [9,10], we can directly find that the generalized bilinear Eq. (3) is linked to a Hirota-Satsuma-like equation as below:

$$\frac{3}{8} v_t u^3 - u_t + \frac{3}{4} u^2 u_t - u_x + \frac{3}{4} v_t u u_x + \frac{3}{2} u_t u_x = 0 \tag{6}$$

with $u = v_x$, in virtue of the following equality:

$$\frac{3}{8} v_t u^3 - u_t + \frac{3}{4} u^2 u_t - u_x + \frac{3}{4} v_t u u_x + \frac{3}{2} u_t u_x = \frac{(D_{3,x}^3 D_{3,t} - D_{3,x}^2 - D_{3,x} D_{3,t}) f \cdot f}{f^2}.$$

2. Rational solutions by maple

How to find exact solutions to Eq. (6)? Within the framework of investigation on resonant solutions to generalized bilinear equations, we find that Eq. (6) does not satisfy the conditions for resonant solutions given in Refs. [9,11]. A conjecture on rational solutions to the Boussinesq-like equation has been given [6]. *In this Letter, we will discuss rational solutions to the Hirota-Satsuma-like Eq. (6), based on polynomial solutions to the generalized bilinear Eq. (3).*

Through submitting

$$f = \sum_{i=0}^{7} \sum_{j=0}^{7} c_{ij} x^i t^j$$

into Eqs. (3) and (5), we find (i) the degree of t is as the same as x in any such polynomial solution f, and (ii) Eq. (6) possesses only three types of rational solutions.

With the formula of a polynomial solution to the generalized bilinear Eq. (3) as

$$f = \sum_{k=0}^{n} g_k x^k = \sum_{k=0}^{n} g_k(t) x^k,$$

where n is a nonnegative integer and the g_k's are polynomials of t with $g_n \neq 0$, we can compute that

$$
\begin{aligned}
&3\, f_{xx}\, f_{xt} - f_{xx}\, f + f_x^2 - f\, f_{xt} + f_x\, f_t = 3\left[\sum_{k=0}^{n-2}(k+2)(k+1)g_{k+2}x^k\right]\left[\sum_{k=0}^{n-1}(k+1)g'_{k+1}x^k\right] \\
&\quad - \left[\sum_{k=0}^{n-2}(k+2)(k+1)g_{k+2}x^k\right]\left[\sum_{k=0}^{n} g_k x^k\right] + \left[\sum_{k=0}^{n-1}(k+1)g_{k+1}x^k\right]^2 \\
&\quad - \left[\sum_{k=0}^{n-1}(k+1)g'_{k+1}x^k\right]\left[\sum_{k=0}^{n} g_k x^k\right] + \left[\sum_{k=0}^{n-1}(k+1)g_{k+1}x^k\right]\left[\sum_{k=0}^{n} g'_k x^k\right] \\
&= 3\sum_{k=0}^{2n-3}\left[\sum_{i+j=k,\, 0\le i\le n-2,\, 0\le j\le n-1}(i+2)(i+1)(j+1)g_{i+2}\, g'_{j+1}\right]x^k \\
&\quad - \sum_{k=0}^{2n-2}\left[\sum_{i+j=k,\, 0\le i\le n-2,\, 0\le j\le n}(i+2)(i+1)g_{i+2}\, g_j\right. \\
&\quad \left. - \sum_{i+j=k,\, 0\le i,\, j\le n-1}(i+1)(j+1)g_{i+1}\, g_{j+1} + \sum_{i+j=k,\, 0\le i\le n-1,\, 0\le j\le n}(i+1)(g'_{i+1}g_j - g_{i+1}\, g'_j)\right]x^k. \qquad (7)
\end{aligned}
$$

It is clear that the right-hand side of Eq. (7) is a polynomial of x with coefficients being polynomials of t. Adopting $g_k = 0$ if $k < 0$, we can compute the coefficients of the like powers of x, e.g., the coefficients of the first three highest orders are

$$x^{2n-2}: \quad ng_n^2 - g'_n g_{n-1} + g_n g'_{n-1}, \qquad (8a)$$

$$x^{2n-3}: \quad 3(n-1)n^2 g_n g'_n + 2(n-1)g_n g_{n-1} + 2(g'_{n-2}g_n - g_{n-2}g'_n), \qquad (8b)$$

$$
\begin{aligned}
x^{2n-4}: \quad & 3n(n-1)^2 g_n g'_{n-1} + 3n(n-1)(n-2)g_{n-1}g'_n - (n^2-4n+6)g_n g_{n-2} \\
& + (n-1)g_{n-1}^2 - g'_{n-1}g_{n-2} + g_{n-1}g'_{n-2} - (n-3)(g'_{n-3}g_n - g_{n-3}g'_n). \qquad (8c)
\end{aligned}
$$

The relationship among g_k's (say, Expressions (8)) determines the specific requirement on the structure of polynomial solutions to Eq. (3). With solving the generalized bilinear Eq. (3), we can conclude the rational solutions to the Hirota-Satsuma-like Eq. (6) as follows:

Case (1): Degree $(f, x) = 1$
With the assumption

$$f = \sum_{k=0}^{1} g_k x^k = \sum_{k=0}^{1} g_k(t)x^k, \qquad (9)$$

we find it demands that

$$g_1^2 - g_0 g'_1 + g_1 g'_0 = 0, \qquad (10)$$

and then the first type of rational solutions to the Hirota-Satsuma-like Eq. (6) is derived as

$$u^I = \frac{2f_x}{f} = \frac{2}{x - t + c_1} \qquad (11)$$

with c_1 as an arbitrary constant.

Case (2): Degree $(f, x) = 2$
With the assumption

$$f = \sum_{k=0}^{2} g_k x^k = \sum_{k=0}^{2} g_k(t)x^k, \qquad (12)$$

we find it demands that

$$\begin{cases} 2g_2^2 - g_2'g_1 + g_2g_1' = 0, \\ 6g_2g_2' + g_2g_1 + g_0'g_2 - g_0g_2' = 0, \\ 6g_2g_1' - 2g_2g_0 + g_1^2 - g_1'g_0 + g_1g_0' = 0, \end{cases} \tag{13}$$

then

$$g_2 = g_1 = g_0 = 0, \quad (\textbf{trivial})$$

or

$$\begin{cases} g_2 = 0, \\ g_1^2 - g_0g_1' + g_1g_0' = 0. \quad \Big(\textbf{reduce to Case}(\mathbf{1})\Big) \end{cases}$$

Case (3): **Degree** $(f, x) = 3$
With the assumption

$$f = \sum_{k=0}^{3} g_k x^k = \sum_{k=0}^{3} g_k(t)x^k, \tag{14}$$

we find two types of new rational solutions to the Hirota-Satsuma-like Eq. (6) as

$$u^{II} = \frac{2f_x}{f} = \frac{6(x-t)^2}{(x-t)^3 + 36t + c_2}, \tag{15}$$

$$u^{III} = \frac{2f_x}{f} = \frac{6\Big[(x-t)^2 + 2c_3(x-t) + c_3^2\Big]}{(x-t)^3 + 3c_3(x-t)^2 + 3c_3^2(x-t) + 36t + c_4}, \tag{16}$$

where c_2, $c_3 \neq 0$ and c_4 are all arbitrary constants.

Case (4): **Degree** $(f, x) \geq 4$
In this case, we find in the expression

$$f = \sum_{k=0}^{n} g_k x^k = \sum_{k=0}^{n} g_k(t)x^k \tag{17}$$

that $g_k(t) = 0$ with $k \geq 4$, and therefore the **Case (4)** reduces to **Case (3)**.

We guess that the three classes of rational solutions (11), (15) and (16) exhaust all rational solutions to the Hirota-Satsuma-like Eq. (6), and make the following conjecture.

Conjecture. *If a polynomial $f = f(x, t)$ of x and t solves the generalized bilinear Eq.* (3)*, then Degree $(f, x) \leq 3$.*

3. Concluding remarks

With the generalized bilinear operators based on a prime number $p = 3$, a Hirota-Satsuma-like equation was proposed, that is, Eq. (6). Rational solutions were generated, based on polynomial solutions to the generalized bilinear Eq. (3) by using symbolic computation software Maple.

We conjecture that the three classes of rational solutions (11), (15) and (16) exhaust all rational solutions to the Hirota-Satsuma-like Eq. (6). Finally, we give some figures to describe the rational solutions (11), (15) and (16) as seen in Figs. 1, 2 and 3, respectively.

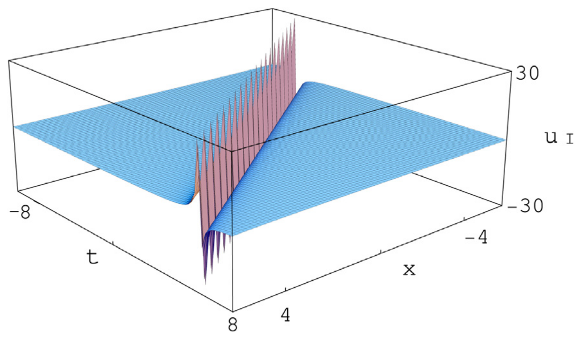

Fig. 1. Rational solution (11) with $c_1 = 0$.

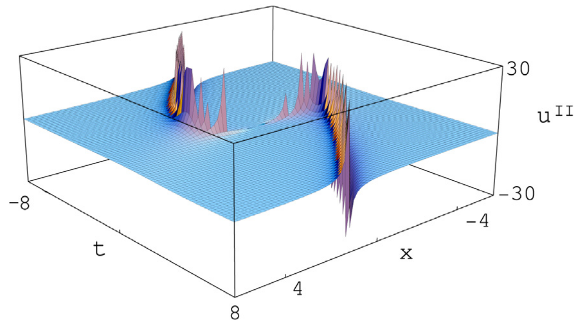

Fig. 2. Rational solution (15) with $c_2 = 0$.

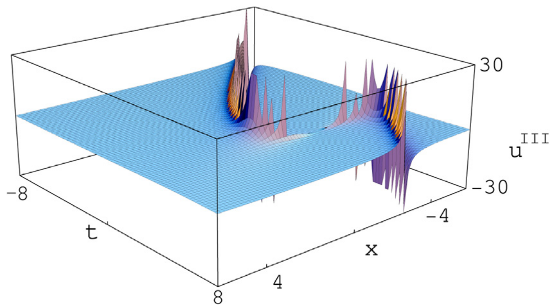

Fig. 3. Rational solution (16) with $c_3 = 5$ and $c_4 = 0$.

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