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Constructing lump solutions to a generalized Kadomtsev–Petviashvili–Boussinesq equation

Xing Lü · Shou-Ting Chen · Wen-Xiu Ma

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Abstract Associated with the prime number $p = 3$, a combined model of generalized bilinear Kadomtsev–Petviashvili and Boussinesq equation (gbKPB for short) in terms of the function f is proposed, which involves four arbitrary coefficients. To guarantee the existence of lump solutions, a constraint among these four coefficients is presented firstly, and then, the lump solutions are constructed and classified via searching for positive quadratic function solutions to the gbKPB equation. Different conditions posed on lump parameters are investigated to keep the analyticity and rational localization of the resulting solutions. Finally, 3-dimensional plots, density plots and 2-dimensional curves with particular choices of the involved parameters are given to show the profile characteristics of the presented lump solutions for the potential function $u = 2(\ln f)_x$.

Keywords Lump solution · Generalized bilinear operator · Generalized Kadomtsev–Petviashvili–Boussinesq equation

Mathematics Subject Classification 35A25 · 37K10

X. Lü (✉)
Department of Mathematics, Beijing Jiao Tong University, Beijing 100044, China
e-mail: XLV@bjtu.edu.cn; xinglv655@aliyun.com

S.-T. Chen
School of Mathematics and Physical Science, Xuzhou Institute of Technology, Jiangsu 221111, China

W.-X. Ma
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

W.-X. Ma
Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa

1 Introduction

Soliton solutions, as a kind of special solutions to integrable nonlinear evolution equations (NLEEs) [1–13], are exponentially localized in certain directions, while lump solutions are rationally localized in all directions in the space [14–19]. The Hirota bilinear representation of NLEEs plays an important role in searching for soliton solutions as well as lump solutions [18–21]. In recent years, by involving different prime numbers, Hirota bilinear operators have been generalized to generate diverse nonlinear differential equations possessing potential applications [22–24]. Soliton and lump solutions have both been studied for the Hirota bilinear equations [18–20,25–28], and resonant N-wave solution and rational solutions have been solved for some generalized Hirota bilinear equations [22–24,29–31]. Therefore, it is naturally interesting to investigate lump solutions for NLEEs which possess generalized bilinear forms.

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For example, the Boussinesq equation [21]

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0, \tag{1}$$

and the Kadomtsev–Petviashvili (KP) equation [20, 32–34]

$$(u_t + 6\,u\,u_x + u_{xxx})_x + u_{yy} = 0, \tag{2}$$

enjoy the following Hirota bilinear representation as

$$(D_t^2 + D_x^4) f \cdot f = 2(f_{tt} f - f_t^2 + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) = 0, \tag{3}$$

and

$$(D_x D_t + D_x^4 + D_y^2) f \cdot f = 2(f_{xt} f - f_x f_t + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{yy} f - f_y^2) = 0, \tag{4}$$

through the transformations $u = 6\,[\ln f(x,t)]_{xx}$ and $u = 2\,[\ln f(x,y,t)]_{xx}$, respectively, where the Hirota bilinear derivatives D_t^2, D_x^4, $D_x D_t$ and D_y^2 [20] are defined by

$$D_x^{\alpha} D_y^{\beta} D_t^{\gamma} (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{\alpha} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{\beta} \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{\gamma} f(x,y,t) g(x',y',t')\Big|_{x'=x,y'=y,t'=t}.$$

Hereby, a combination version of the bilinear KP Eq. (4) and the bilinear Boussinesq Eq. (3) in terms of the function f reads

$$\begin{aligned} \text{bKPB} := &\left(c_1 D_x D_t + c_2 D_t^2 + c_3 D_x^4 + c_4 D_y^2\right) f \cdot f \\ &= 2\Big[c_1(f_{xt} f - f_x f_t) + c_2(f_{tt} f - f_t^2) \\ &\quad + c_3(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) + c_4(f_{yy} f - f_y^2)\Big] = 0, \end{aligned} \tag{5}$$

with c_i $(1 \le i \le 4)$ as arbitrary real constants, which can be regarded as the Hirota bilinear form of a combined KP–Boussinesq equation

$$\text{cKPB} := c_1 u_{xt} + c_2 u_{tt} + c_3(6\,u_x\,u_{xx} + u_{xxxx}) + c_4 u_{yy} = 0, \tag{6}$$

through the transformation[1] $u = 2\big[\ln f(x,y,t)\big]_x$.

Based on a prime number p, a kind of generalized bilinear operators has been introduced [22–24] as

$$D_{p,x_1}^{n_1} \cdots D_{p,x_M}^{n_M} (f \cdot g) = \prod_{i=1}^{M} \left(\frac{\partial}{\partial x_i} + \alpha \frac{\partial}{\partial x_i'}\right)^{n_i} \times f(x_1,\ldots,x_M) g(x_1',\ldots,x_M')\Big|_{x_1'=x_1,\ldots,x_M'=x_M},$$

where $n_1, \cdots, n_M$ are arbitrary nonnegative integers and for an integer m, the mth power of α is computed as follows:

$$\alpha^m = (-1)^{r(m)}, \text{ if } m \equiv r(m) \bmod \mathrm{p} \text{ with } 0 \le \mathrm{r(m)} < \mathrm{p}. \tag{7}$$

Under the rule given by Eq. (7), which indicates a way to take the signs $+1$ or -1, we find that if $p = 2k$ $(k \in \mathbb{N})$, all the bilinear operators defined above turn out to be the Hirota bilinear operators, since $D_{2k,x} = D_x$ [22–24]. If $p = 3$, we particularly have

$$\alpha_3 = -1, \quad \alpha_3^2 = \alpha_3^3 = 1, \quad \alpha_3^4 = -1, \quad \alpha_3^5 = \alpha_3^6 = 1, \ldots.$$

With $p = 3$, we can generalize the Hirota bilinear Boussinesq Eq. (3) and the Hirota bilinear KP Eq. (4), respectively, into

$$(D_{3,t}^2 + D_{3,x}^4) f \cdot f = 2\left(f_{tt} f - f_t^2 + 3 f_{xx}^2\right) = 0, \tag{8}$$

and

$$\left(D_{3,x} D_{3,t} + D_{3,x}^4 + D_{3,y}^2\right) f \cdot f = 2\big(f_{xt} f - f_x f_t + 3 f_{xx}^2 + f_{yy} f - f_y^2\big) = 0, \tag{9}$$

and the combined bilinear KP–Boussinesq Eq. (5) can be generalized as

[1] The transformation employed here is motivated by the Bell polynomial theories (see, e.g., [22–24,35–37]), and actually, we have $\left[\frac{\text{bKPB}}{f^2}\right]_x = \text{cKPB}$.

$$\begin{aligned}\text{gbKPB} &:= (c_1 D_{3,x} D_{3,t} + c_2 D_{3,t}^2 + c_3 D_{3,x}^4 + c_4 D_{3,y}^2) f \cdot f \\ &= 2\Big[c_1(f_{xt} f - f_x f_t) + c_2(f_{tt} f - f_t^2) \\ &\quad + 3 c_3 f_{xx}^2 + c_4(f_{yy} f - f_y^2)\Big] = 0. \qquad (10)\end{aligned}$$

Eq. (10) is a generalized bilinear KP–Boussinesq (gbKPB) equation, which is connected with the following scalar nonlinear differential equation in terms of potential function u as

$$\begin{aligned}&c_1 u_{xt} + c_2 u_{tt} + \frac{3}{2} c_3 \left(u^3 u_x + 2 u u_x^2 + u^2 u_{xx} + 2 u_x u_{xx}\right) \\ &+ c_4 u_{yy} = 0, \qquad (11)\end{aligned}$$

through the transformation $u = 2\big[\ln f(x, y, t)\big]_x$. Actually, the equality between f and u

$$\begin{aligned}\left[\frac{\text{gbKPB}}{f^2}\right]_x &= c_1 u_{xt} + c_2 u_{tt} \\ &+ \frac{3}{2} c_3 \left(u^3 u_x + 2 u u_x^2 + u^2 u_{xx} + 2 u_x u_{xx}\right) \\ &+ c_4 u_{yy},\end{aligned}$$

shows that if f is a solution to Eq. (10), then $u = 2(\ln f)_x$ solves Eq. (11).

In this paper, we will be devoted to the gbKPB equation, i.e., Eq. (10), which is generated with bilinear differential operator extension method and involves four arbitrary coefficients, c_1, c_2, c_3 and c_4. By checking the existence of lump solutions, a constraint among these four coefficients will be presented firstly, and then, the lump solutions will be constructed to Eq. (11) via searching for positive quadratic function solutions to Eq. (10). Four classes of lump solutions will be presented, and different conditions posed on lump parameters will be investigated to keep the analyticity and rational localization of the resulting solutions. Finally, a few concluding remarks will be given at the end of the paper.

2 Lump solutions to the gbKPB equation

To find the lump solutions to potential function u in Eq. (11), we search for quadratic function solutions to Eq. (10) with the assumption

$$f = g^2 + h^2 + a_9, \qquad (12)$$

with

$$\begin{aligned}g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8,\end{aligned}$$

where a_i $(1 \le i \le 9)$ are all real parameters to be determined. Symbolic computation on a direct substitution of Eq. (12) into Eq. (10) generates the following set of constraining equations on the parameters:

$$\left\{\begin{aligned} a_1 &= \frac{c_4\left(a_3 a_6^2 - 2 a_2 a_6 a_7 - a_3 a_2^2\right) - c_2 a_3 \left(a_3^2 + a_7^2\right)}{c_1\left(a_3^2 + a_7^2\right)}, \quad a_2 = a_2,\, a_3 = a_3,\, a_4 = a_4, \\ a_5 &= \frac{c_4\left(a_7 a_2^2 - 2 a_2 a_3 a_6 - a_7 a_6^2\right) - c_2 a_7 \left(a_3^2 + a_7^2\right)}{c_1\left(a_3^2 + a_7^2\right)}, \quad a_6 = a_6, \quad a_7 = a_7,\, a_8 = a_8, \\ a_9 &= -3\,\frac{c_3}{c_4}\,\frac{\left[(c_4 a_2^2 + c_2 a_3^2)^2 + (c_4 a_6^2 + c_2 a_7^2)^2 + 2\,(c_4 a_2 a_6 + c_2 a_3 a_7)^2 - 2\, c_2 c_4 (a_2 a_7 - a_3 a_6)^2\right]^2}{(a_3^2 + a_7^2)(a_2 a_7 - a_3 a_6)^2} \end{aligned}\right\},$$

which needs to satisfy both

$$\text{I}: \ \left\{c_1 c_4 \neq 0,\ c_2 = c_2,\ c_3 c_4 < 0\right\}, \qquad (13)$$

and

$$\begin{aligned}\text{II}: \ &\left\{a_2 a_7 - a_3 a_6 \neq 0,\ (c_4 a_2^2 + c_2 a_3^2)^2 + (c_4 a_6^2 + c_2 a_7^2)^2 \right. \\ &\left. + 2\,(c_4 a_2 a_6 + c_2 a_3 a_7)^2 - 2\, c_2 c_4 (a_2 a_7 - a_3 a_6)^2 \neq 0\right\}. \qquad (14)\end{aligned}$$

It can be seen that the set of Condition **I** constrains purely on the equation coefficients to realize the existence of lump solutions. Without loss of generality, we will consider two cases as $\{c_1 \neq 0, c_2 = c_2, c_3 = 1, c_4 = -1\}$ and $\{c_1 \neq 0, c_2 = c_2, c_3 = -1, c_4 = 1\}$.

2.1 $\{c_1 \neq 0,\ c_2 = c_2,\ c_3 = 1,\ c_4 = -1\}$

In this case, we can obtain two sets of constraining equations on the parameters.

The first set of constraining equations on the parameters in this case is

$$\left\{\begin{aligned} a_1 &= \frac{a_3a_2^2 + 2\,a_2a_6a_7 - a_3a_6^2 - c_2a_3(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}, \quad a_2 = a_2, a_3 = a_3, a_4 = a_4, \\ a_5 &= \frac{a_7a_6^2 + 2\,a_2a_3a_6 - a_7a_2^2 - c_2a_7(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}, \quad a_6 = a_6, a_7 = a_7, a_8 = a_8, \\ a_9 &= \frac{3\left[c_2^2(a_3^2 + a_7^2)^2 + 2\,c_2(a_2a_7 - a_3a_6)^2 - 2\,c_2(a_2a_3 + a_6a_7)^2 + (a_2^2 + a_6^2)^2\right]^2}{c_1^4(a_3^2 + a_7^2)(a_2a_7 - a_3a_6)^2} \end{aligned}\right\}, \tag{15}$$

which needs to satisfy the condition

$$a_2a_7 - a_3a_6 \neq 0, \tag{16}$$

to make the corresponding solutions f be well defined, the condition

$$a_2a_3 + a_6a_7 \neq 0, \tag{17}$$

to guarantee the positiveness of f and the condition

$$c_2(a_3^2 + a_7^2) + a_2^2 + a_6^2 \neq 0, \tag{18}$$

to realize the localization of u in all directions in the (x, y)-plane. The parameters in the set (15) yield the first class of positive quadratic function solutions to Eq. (10) as

$$\begin{aligned} f =& \left(\frac{a_3a_2^2 + 2\,a_2a_6a_7 - a_3a_6^2 - c_2a_3(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}x + a_2y + a_3t + a_4\right)^2 \\ &+ \left(\frac{a_7a_6^2 + 2\,a_2a_3a_6 - a_7a_2^2 - c_2a_7(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}x + a_6y + a_7t + a_8\right)^2 \\ &+ \frac{3\left[c_2^2(a_3^2 + a_7^2)^2 + 2\,c_2(a_2a_7 - a_3a_6)^2 - 2\,c_2(a_2a_3 + a_6a_7)^2 + (a_2^2 + a_6^2)^2\right]^2}{c_1^4(a_3^2 + a_7^2)(a_2a_7 - a_3a_6)^2}, \end{aligned} \tag{19}$$

which can be used to generate the first class of lump solutions to Eq. (11) through the transformation

$$u^{(\mathrm{I})} = \frac{4(a_1g + a_5h)}{f}, \tag{20}$$

where the function f is defined by Eq. (19), and the functions g and h are given as follows:

$$\begin{aligned} g =& \frac{a_3a_2^2 + 2\,a_2a_6a_7 - a_3a_6^2 - c_2a_3(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}x \\ &+ a_2y + a_3t + a_4, \\ h =& \frac{a_7a_6^2 + 2\,a_2a_3a_6 - a_7a_2^2 - c_2a_7(a_3^2 + a_7^2)}{c_1(a_3^2 + a_7^2)}x \\ &+ a_6y + a_7t + a_8. \end{aligned}$$

Note here that *eight* parameters c_1, c_2, a_2, a_3, a_4, a_6, a_7 and a_8 are involved in the solution $u^{(\mathrm{I})}$, and they are demanded to satisfy conditions (16), (17) and (18) to guarantee $u^{(I)}$ to be lump solutions.

The second set of constraining equations on the parameters in this case is

$$\left\{\begin{aligned} &a_1 = \frac{a_2^2-a_6^2-c_2a_3^2}{c_1a_3}, a_2=a_2, a_3=a_3, a_4=a_4,\\ &a_5=\frac{-2c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}, a_6=a_6, a_7=\frac{2a_2a_3a_6}{a_2^2-a_6^2},\\ &a_8=a_8,\\ &a_9=\frac{3\left[c_2^2a_3^4(a_2^2+a_6^2)^2-2c_2a_3^2(a_2^2-a_6^2)^3+(a_2^2-a_6^2)^4\right]^2}{c_1^4a_3^4a_6^2(a_2^2-a_6^2)^4}\end{aligned}\right\}, \tag{21}$$

which needs to satisfy the conditions

$$a_3a_6(a_2^2-a_6^2)\neq 0, \tag{22}$$

$$a_2\neq 0, \tag{23}$$

and

$$c_2a_3^2(a_2^2+a_6^2)+(a_2^2-a_6^2)^2\neq 0, \tag{24}$$

to guarantee the well-definedness of f, the positiveness of f and the localization of u in all directions in the space, respectively. The parameters in the set (21) yield the second class of positive quadratic function solutions to Eq. (10) as

$$\begin{aligned} f=&\left(\frac{a_2^2-a_6^2-c_2a_3^2}{c_1a_3}x+a_2y+a_3t+a_4\right)^2\\ &+\left(-\frac{2c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}x+a_6y+\frac{2a_2a_3a_6}{a_2^2-a_6^2}t+a_8\right)^2\\ &+\frac{3\left[c_2^2a_3^4(a_2^2+a_6^2)^2-2c_2a_3^2(a_2^2-a_6^2)^3+(a_2^2-a_6^2)^2\right]^2}{c_1^4a_3^4a_6^2(a_2^2-a_6^2)^4}, \end{aligned} \tag{25}$$

which leads to the second class of lump solutions to Eq. (11) through the transformation

$$u^{(II)}=\frac{4(a_1g+a_5h)}{f}, \tag{26}$$

where the function f is defined by Eq. (25), and the functions g and h are given as follows:

$$\begin{aligned} g&=\frac{a_2^2-a_6^2-c_2a_3^2}{c_1a_3}x+a_2y+a_3t+a_4,\\ h&=-\frac{2c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}x+a_6y+\frac{2a_2a_3a_6}{a_2^2-a_6^2}t+a_8. \end{aligned}$$

Note here that *seven* parameters c_1, c_2, a_2, a_3, a_4, a_6 and a_8 are involved in the solution $u^{(II)}$, and they are demanded to satisfy conditions (22), (23) and (24) to guarantee $u^{(II)}$ to be lump solutions.

2.2 $\{c_1\neq 0,\ c_2=c_2,\ c_3=-1,\ c_4=1\}$

In this case, we can obtain two sets of constraining equations on the parameters as well.

The first set of constraining equations on the parameters is

$$\left\{\begin{aligned} &a_1=\frac{a_3a_6^2-2a_2a_6a_7-a_3a_2^2-c_2a_3\left(a_3^2+a_7^2\right)}{c_1\left(a_3^2+a_7^2\right)}, \quad a_2=a_2, a_3=a_3, a_4=a_4,\\ &a_5=\frac{a_7a_2^2-2a_2a_3a_6-a_7a_6^2-c_2a_7\left(a_3^2+a_7^2\right)}{c_1\left(a_3^2+a_7^2\right)}, \quad a_6=a_6, a_7=a_7, a_8=a_8,\\ &a_9=\frac{3\left[c_2^2(a_3^2+a_7^2)^2-2c_2(a_2a_7-a_3a_6)^2+2c_2(a_2a_3+a_6a_7)^2+(a_2^2+a_6^2)^2\right]^2}{c_1^4(a_3^2+a_7^2)(a_2a_7-a_3a_6)^2}\end{aligned}\right\}, \tag{27}$$

which needs to satisfy the condition

$$a_2a_7 - a_3a_6 \neq 0, \tag{28}$$

to make the corresponding solutions f be well defined, the condition

$$a_2a_3 + a_6a_7 \neq 0, \tag{29}$$

to guarantee the positiveness of f and the condition

$$a_2^2 + a_6^2 - c_2\left(a_3^2 + a_7^2\right) \neq 0, \tag{30}$$

to realize the localization of u in all directions in the (x, y)-plane. The parameters in the set (27) yield the third class of positive quadratic function solutions to Eq. (10) as

$$f = \left(\frac{a_3a_6^2 - 2\,a_2a_6a_7 - a_3a_2^2 - c_2a_3(a_3^2+a_7^2)}{c_1(a_3^2+a_7^2)}x + a_2y + a_3t + a_4\right)^2 + \left(\frac{a_7a_2^2 - 2\,a_2a_3a_6 - a_7a_6^2 - c_2a_7(a_3^2+a_7^2)}{c_1(a_3^2+a_7^2)}x + a_6y + a_7t + a_8\right)^2 + \frac{3\left[c_2^2(a_3^2+a_7^2)^2 - 2\,c_2(a_2a_7-a_3a_6)^2 + 2\,c_2(a_2a_3+a_6a_7)^2 + (a_2^2+a_6^2)^2\right]^2}{c_1^4(a_3^2+a_7^2)(a_2a_7-a_3a_6)^2}, \tag{31}$$

which can be used to generate the third class of lump solutions to Eq. (11) through the transformation

$$u^{(\mathrm{III})} = \frac{4(a_1g + a_5h)}{f}, \tag{32}$$

where the function f is defined by Eq. (31), and the functions g and h are given as follows:

$$g = \frac{a_3a_6^2 - 2\,a_2a_6a_7 - a_3a_2^2 - c_2a_3(a_3^2+a_7^2)}{c_1(a_3^2+a_7^2)}x + a_2y + a_3t + a_4,$$

$$h = \frac{a_7a_2^2 - 2\,a_2a_3a_6 - a_7a_6^2 - c_2a_7(a_3^2+a_7^2)}{c_1(a_3^2+a_7^2)}x + a_6y + a_7t + a_8.$$

Note here that *eight* parameters c_1, c_2, a_2, a_3, a_4, a_6, a_7 and a_8 are involved in the solution $u^{(\mathrm{III})}$, and they are demanded to satisfy conditions (28), (29) and (30) to guarantee $u^{(\mathrm{III})}$ to be lump solutions.

The second set of constraining equations on the parameters in this case is

$$\left\{ a_1 = -\frac{a_2^2 - a_6^2 + c_2a_3^2}{c_1a_3}, a_2 = a_2, a_3 = a_3, a_4 = a_4, \quad a_5 = \frac{-2\,c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}, a_6 = a_6, a_7 = \frac{2\,a_2a_3a_6}{a_2^2-a_6^2}, a_8 = a_8, a_9 = \frac{3\left[c_2^2a_3^4(a_2^2+a_6^2)^2 + 2\,c_2a_3^2(a_2^2-a_6^2)^3 + (a_2^2-a_6^2)^4\right]^2}{c_1^4a_3^4a_6^2(a_2^2-a_6^2)^4} \right\}, \tag{33}$$

which needs to satisfy the conditions

$$a_3a_6(a_2^2 - a_6^2) \neq 0, \tag{34}$$

$$a_2 \neq 0, \tag{35}$$

and

$$(a_2^2 - a_6^2)^2 - c_2a_3^2(a_2^2 + a_6^2) \neq 0, \tag{36}$$

to guarantee the well-definedness of f, the positiveness of f and the localization of u in all directions in the space, respectively. The parameters in the set (33) yield

Table 1 Summary of the lump solutions

Cases	($\mathcal{A}$): $c_1 \neq 0,\ c_2 = c_2,\ c_3 = 1,\ c_4 = -1$		($\mathcal{B}$): $c_1 \neq 0,\ c_2 = c_2,\ c_3 = -1,\ c_4 = 1$	
Lump solution u	$u^{(\mathrm{I})}$: Eq. (20)	$u^{(\mathrm{II})}$: Eq. (26)	$u^{(\mathrm{III})}$: Eq. (32)	$u^{(\mathrm{IV})}$: Eq. (38)
Quadratic function f	Eq. (19)	Eq. (25)	Eq. (31)	Eq. (37)
Well-definedness condition	$a_2a_7 - a_3a_6 \neq 0$	$a_3a_6(a_2^2 - a_6^2) \neq 0$	$a_2a_7 - a_3a_6 \neq 0$	$a_3a_6(a_2^2 - a_6^2) \neq 0$
Positiveness condition	$a_2a_3 + a_6a_7 \neq 0$	$a_2 \neq 0$	$a_2a_3 + a_6a_7 \neq 0$	$a_2 \neq 0$
Localization condition	Eq. (18)	Eq. (24)	Eq. (30)	Eq. (36)

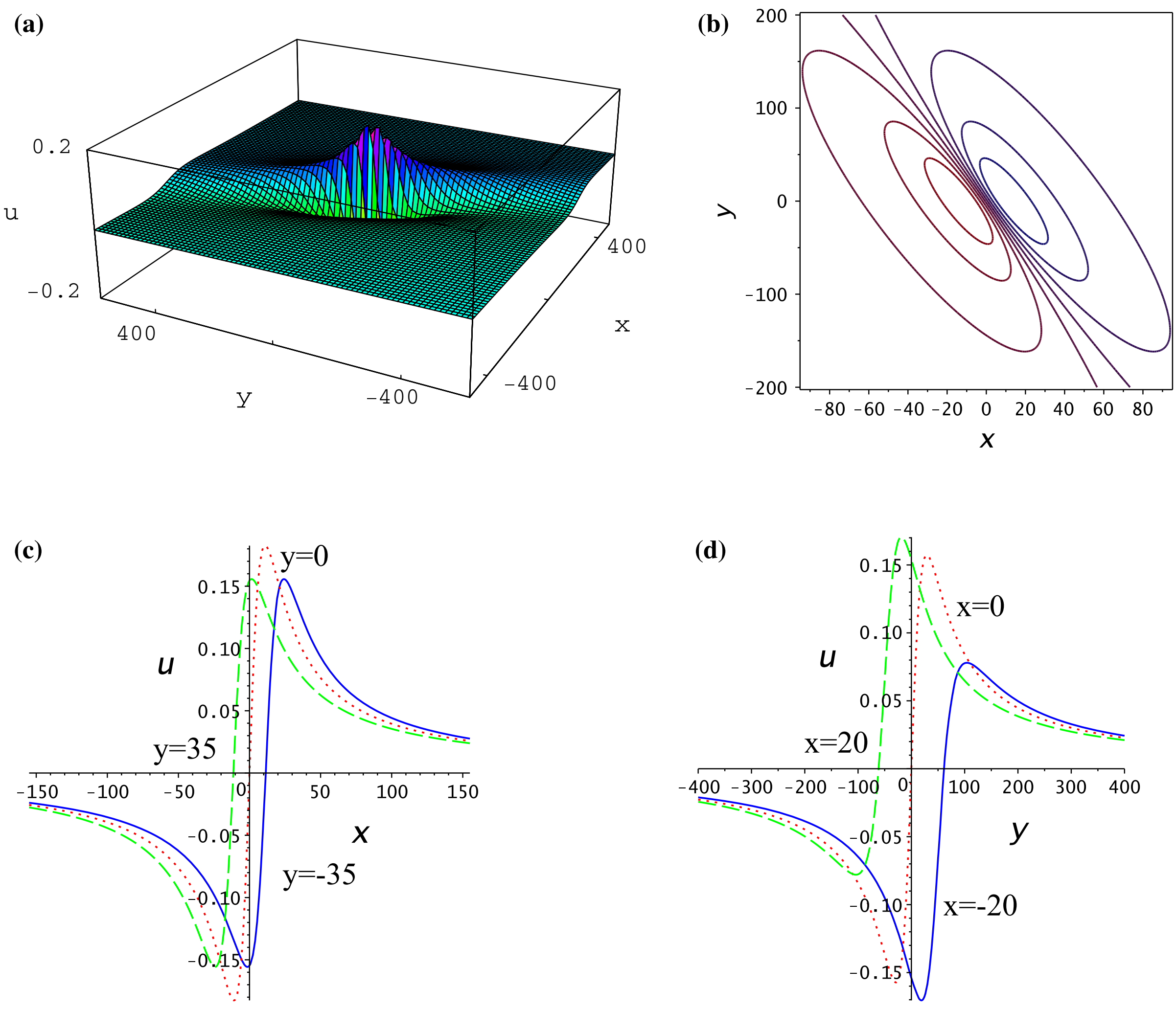

Fig. 1 Lump dynamic characteristics of $u^{(\mathrm{I})}$ via Eq. (20) with $t = 0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

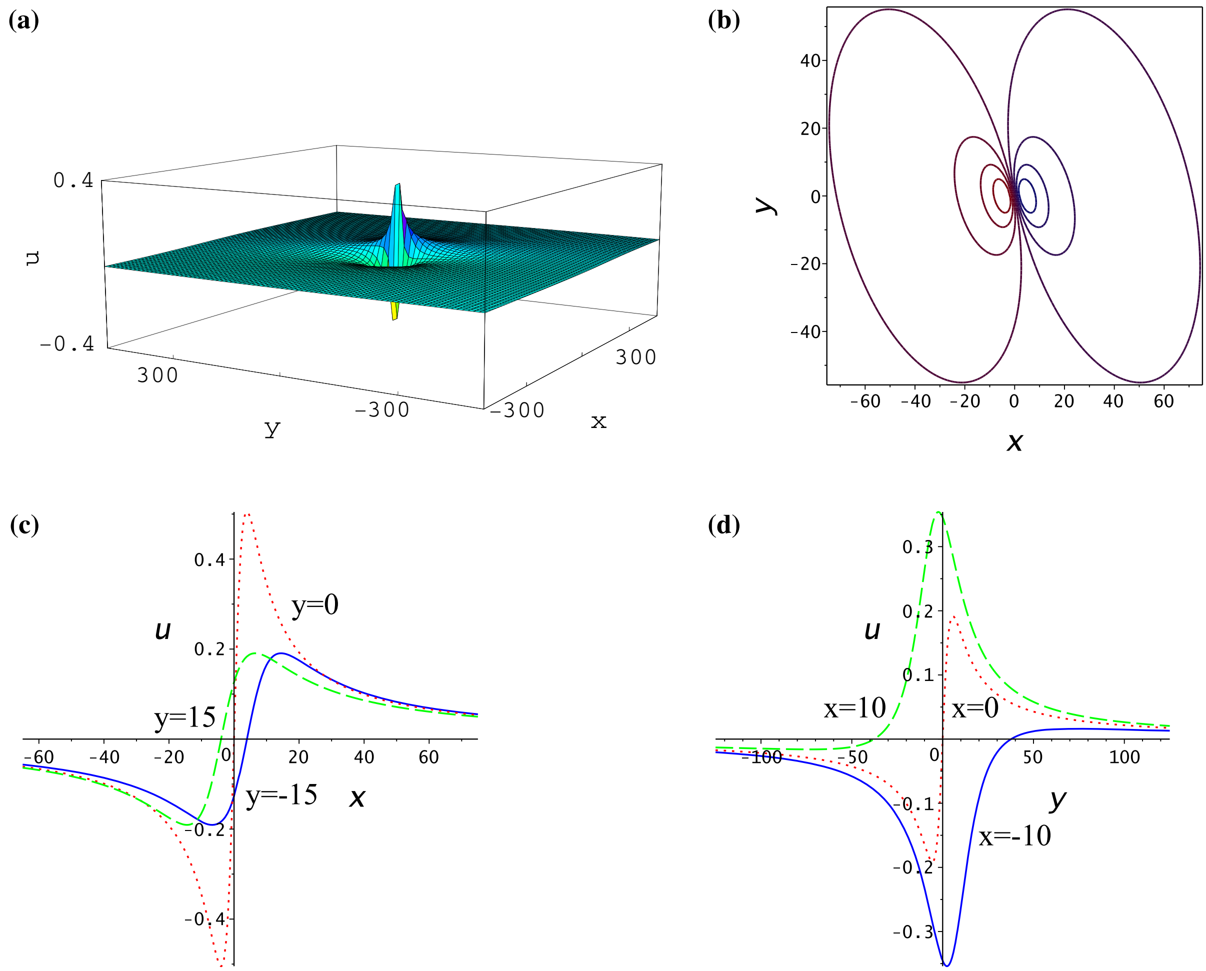

Fig. 2 Lump dynamic characteristics of $u^{(\mathrm{II})}$ via Eq. (26) with $t=0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

the fourth class of positive quadratic function solutions to Eq. (10) as

$$f=\left(-\frac{a_2^2-a_6^2+c_2a_3^2}{c_1a_3}x+a_2y+a_3t+a_4\right)^2+\left(\frac{-2\,c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}x+a_6y+\frac{2\,a_2a_3a_6}{a_2^2-a_6^2}t+a_8\right)^2+\frac{3\left[c_2^2a_3^4(a_2^2+a_6^2)^2+2\,c_2a_3^2(a_2^2-a_6^2)^3+(a_2^2-a_6^2)^4\right]^2}{c_1^4a_3^4a_6^2(a_2^2-a_6^2)^4},\tag{37}$$

which leads to the fourth class of lump solutions to Eq. (11) through the transformation

$$u^{(\mathrm{IV})}=\frac{4(a_1g+a_5h)}{f},\tag{38}$$

where the function f is defined by Eq. (37), and the functions g and h are given as follows:

$$g=-\frac{a_2^2-a_6^2+c_2a_3^2}{c_1a_3}x+a_2y+a_3t+a_4,$$
$$h=\frac{-2\,c_2a_2a_3a_6}{c_1(a_2^2-a_6^2)}x+a_6y+\frac{2\,a_2a_3a_6}{a_2^2-a_6^2}t+a_8.$$

Note here that *seven* parameters c_1, c_2, a_2, a_3, a_4, a_6 and a_8 are involved in the solution $u^{(\mathrm{IV})}$, and they are demanded to satisfy conditions (34), (35) and (36) to guarantee $u^{(\mathrm{IV})}$ to be lump solutions.

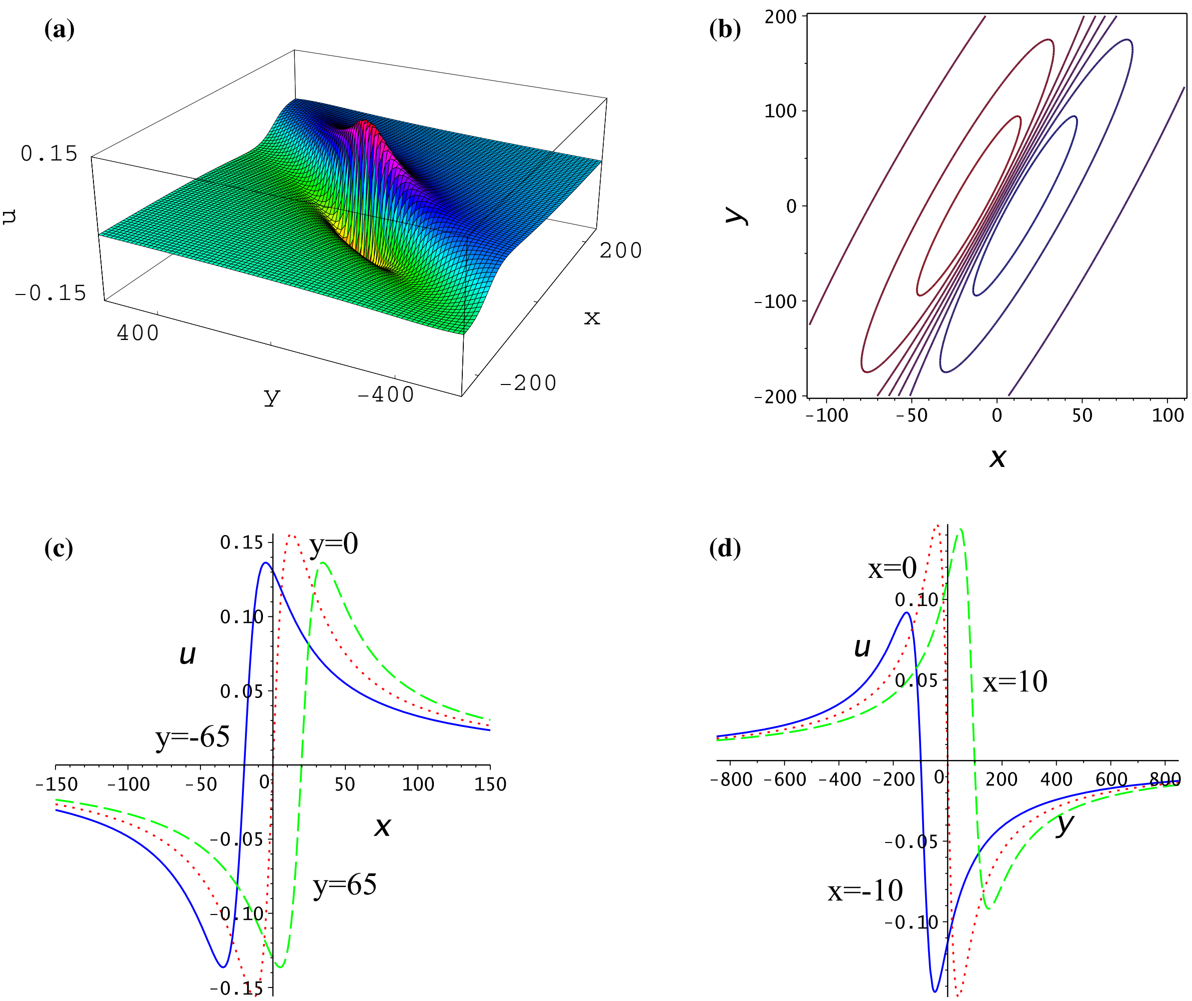

Fig. 3 Lump dynamic characteristics of $u^{(\mathrm{III})}$ via Eq. (32) with $t = 0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

By constructing four classes of positive quadratic function solutions to Eq. (10), we have found four classes of lump solutions to Eq. (11). It is necessary to summarize and compare those conditions on parameters associated with different lump solutions, which can be seen in Table 1. Different from soliton solutions exponentially localized in certain directions, lump solutions, as a type of rational solutions, are rationally localized in all directions in the space. To show the localized characteristics of the presented lump solutions clearly, 3-dimensional plots, density plots and 2-dimensional curves with particular choices of the involved parameters in the potential function u are plotted, which can be seen in Figs. 1, 2, 3 and 4. The involved parameters adopted in this paper are $c_1 = 1$, $c_2 = 1$, $a_2 = 5$, $a_3 = 1$, $a_4 = 0$, $a_6 = 4$ and $a_8 = 0$, while $a_7 = 2$ in Figs. 1 and 3.

3 Concluding remarks

Based on the bilinear differential operator extension method by taking the prime number $p = 3$ in the generalized bilinear operators, a combined model of generalized bilinear Kadomtsev–Petviashvili and Boussinesq equations in terms of the function f has been proposed and studied, which possesses four arbitrary coefficients c_1, c_2, c_3 and c_4, as seen in Eq. (10). Through the transformation $u = 2\big[\ln f(x, y, t)\big]_x$, Eq. (10) can be linked with the nonlinear differential Eq. (11). A constraint

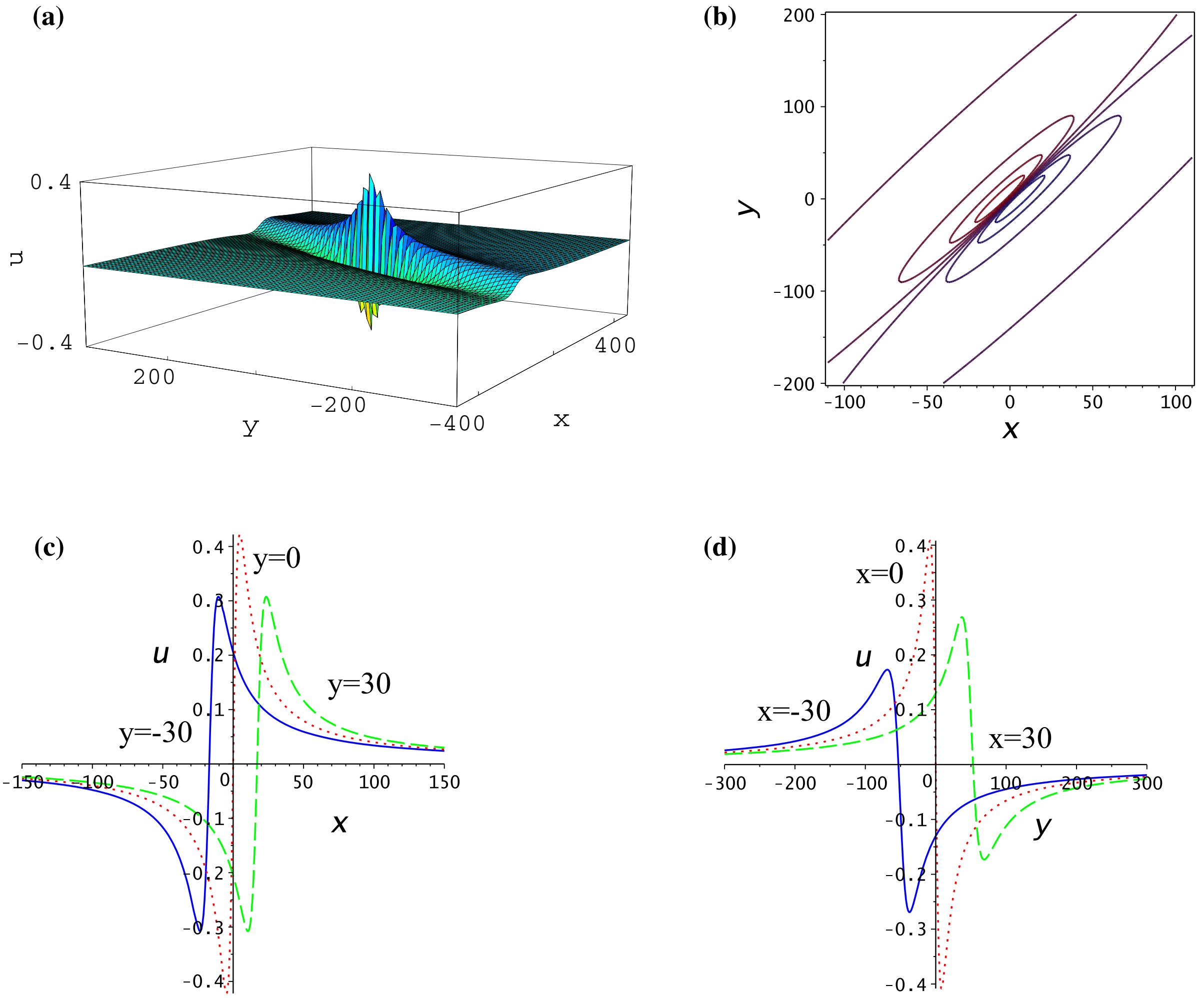

Fig. 4 Lump dynamic characteristics of $u^{(\mathrm{IV})}$ via Eq. (38) with $t = 0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

among these four coefficients [see Eq. (13)] has been presented to guarantee the existence of lump solutions and used to solve and classify the lump solutions to Eq. (11) via searching for positive quadratic function solutions to Eq. (10). Four classes of lump solutions with corresponding conditions posed on lump parameters have been constructed and plotted.

We point out that conditions posed on lump parameters, i.e., Eqs. (16)– (18), Eqs. (22)– (24), Eqs. (28)–(30) and Eqs. (34)– (36), must be satisfied to guarantee the well-definedness, the positiveness and the localization of the solutions. Otherwise, quadratic function solutions f to Eq. (10) may exist and yield rational solutions to Eq. (11), but they cannot be mapped into lump solutions u (e.g., when $a_9 \leq 0$).

Attention should also be paid to the difference between the Eqs. (5) and (10), which are both generated by a combination. However, Eq. (5) is a combined version of the Hirota-type bilinear equation, and Eq. (10) is a combined version of the generalized bilinear equation, and the former equation contains additionally two terms, $T(f) = c_3 f_{xxxx} f - 4c_3 f_{xxx} f_x$, more than the latter one. It is easy to know that the solutions derived here to Eq. (10), that is, quadratic function solutions

f in Eqs. (19), (25), (31) and (37), are solutions to Eq. (5) as well, since $T(f) = 0$ is satisfied automatically. Therefore, we can claim that we have constructed lump solutions to both Eqs. (6) and (11). Then, it is natural to ask how to construct distinct or novel lump solutions to Eqs. (6) and (11). Within the framework of this paper, we can consider sums of higher-order even function solutions or multiple sums of quadratic function solutions (more than two quadratic functions) f to search for other lump solutions, which can be written as

$$f = \sum_{i\geq 1}^{M_1} g_i^m + \sum_{j\geq 1}^{M_2} h_j^2 + c_{4(M_1+M_2)+1},$$

where $g_i = a_{1i}x + a_{2i}y + a_{3i}t + a_{4i}$, $h_j = b_{1j}x + b_{2j}y + b_{3j}t + b_{4j}$, while $a_{1i}, a_{2i}, a_{3i}, a_{4i}, b_{1j}, b_{2j}, b_{3j}, b_{4j}$ and $c_{4(M_1+M_2)+1}$ are all arbitrary real constants, $m\ (\geq 4) \in 2\mathbb{Z}^+$, M_1 and M_2 are integers; or

$$f = \sum_{i\geq 3}^{N} h_i^2 + a_{4N+1},$$

where $h_i = a_{1i}x + a_{2i}y + a_{3i}t + a_{4i}$, N is an integer, and a_{1i}, a_{2i}, a_{3i}, a_{4i} and a_{4N+1} are all arbitrary real constants.

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