East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.101120.180221

Vol. **11**, No. 3, pp. 594-603
August 2021

Study on Lump Behavior for a New $(3+1)$-Dimensional Generalised Kadomtsev-Petviashvili Equation

Xing Lü[1,2,*], Si-Jia Chen[2], Guo-Zhu Liu[1] and Wen-Xiu Ma[3,4,5,6,*]

[1] *State Key Laboratory of Complex Electromagnetic Environment Effects on Electronics and Information System (CEMEE), Luoyang 471003, China.*
[2] *Department of Mathematics, Beijing Jiao Tong University, Beijing 100044, China.*
[3] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.*
[4] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.*
[5] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA.*
[6] *International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa.*

Received 10 November 2020; Accepted (in revised version) 18 February 2021.

Abstract. In this paper, we investigate two dimensionally reduced cases of a new (3+1)-dimensional generalised Kadomtsev-Petviashvili equation. With symbolic computation, lump solutions are derived via searching for positive quadratic function solutions to the associated bilinear equations. Localised characteristics and lump motion are analysed and illustrated as well.

AMS subject classifications: 35A25, 35G50, 35Q35, 37K10

Key words: Bilinear method, lump dynamics, dimensionally reduced equation, symbolic computation.

1. Introduction

In the nonlinear science, more and more attention has been paid to the two-dimensional or three-dimensional nonlinear models [1, 2, 7, 10, 11, 13, 15, 17–21, 23, 24, 26, 28]. In contrast with the (1+1)-dimensional equations (one for space and the other one for time), multi-dimensional ones are more realistic in describing the nonlinear phenomena in science and engineering [4–6, 9, 27, 29]. For example, the Kadomtsev-Petviashvili (KP) equation, modeling water waves of long wavelengths with weakly non-linear restoring forces and

*Corresponding author. *Email addresses:* `mawx@cas.usf.edu` (W.-X. Ma), `xinglv655@163.com, XLV@bjtu.edu.cn` (X. Lü)

frequency dispersion [10], is usually written as

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \quad \sigma = \pm 1,$$

which is classified as the KPI equation when $\sigma = 1$ and the KPII equation when $\sigma = -1$. The KP equation is completely integrable, and its soliton solutions and lump solutions have been solved [1,18].

The KP equation is a two-dimensional generalisation of the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

where the spatial variable is generalised into two dimensions with x and y. Actually, more and more generalised KdV or KP equations are proposed, which maybe integrable or non-integrable — cf. Refs. [11,17,19–21,24] and references therein.

The KP-like equation has attracted more attention recently. The generalised perturbation Darboux transformations have been reported for the $(2+1)$-dimensional KP equation and its extension by using the Taylor expansion of the Darboux matrix [25]. Fission and fusion interaction phenomena of mixed lump kink solutions for a generalised $(3+1)$-dimensional B-type KP equation has been studied by using the Hirota bilinear method [12].

In recent, a new $(3+1)$-dimensional generalised KP equation [24] has been introduced as

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tz} - u_{zz} = 0. \tag{1.1}$$

Via the simplified Hirota bilinear method, multiple soliton solutions to the Eq. (1.1) have been derived with the coefficients of the spatial variables left free, and the phase shifts depending on all these coefficients. It has also been proved that the Eq. (1.1) fails to pass the Painlevé integrability test although it enjoys multiple soliton solutions. Moreover, the resonant multiple wave solutions to the Eq. (1.1) have been constructed by using linear superposition principle [11].

As well known, soliton solutions are exponentially localised in certain directions, while lump solutions are a kind of rational function solutions, localised in all directions in the space [3,14,16]. Based on bilinear forms, one can derive both soliton solutions and lump solutions [1,18]. The dynamics of lump, lumpoff and rogue wave solutions of $(2+1)$-dimensional Hirota-Satsuma-Ito equations has been studied through bilinear method [30]. For a fourth-order nonlinear generalised Boussinesq water wave equation, symmetry reductions and twelve families of soliton wave solutions have been derived by employing Lie symmetry method [22]. The Riemann-Hilbert approach has also been used to solve N-soliton solutions of a four-component nonlinear Schrödinger equation associated with a 5×5 Lax pair [31].

In this paper, we will focus on the dimensionally reduced cases of Eq. (1.1) and present two classes of lump solutions with symbolic computation. It is clear that the Eq. (1.1) is a $(3+1)$-dimensional model with the spatial variables (x, y, z) and the time variable t. Through a dependent variable transformation

$$u = 2\big[\ln f(x,y,z,t)\big]_x = 2\frac{f_x(x,y,z,t)}{f(x,y,z,t)}, \tag{1.2}$$

Eq. (1.1) can be mapped into its Hirota bilinear form as

$$\left(D_x^3 D_y + D_t D_x + D_t D_y + D_t D_z - D_z^2\right) f \cdot f = 0, \tag{1.3}$$

where the derivatives $D_x^3 D_y$, $D_t D_x$, $D_t D_y$, $D_t D_z$ and D_z^2 are bilinear operators [8] defined by

$$D_x^\alpha D_y^\beta D_z^\gamma D_t^\delta (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^\beta \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^\gamma \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\delta \times f(x,y,z,t) g(x',y',z',t')\Big|_{x'=x,y'=y,z'=z,t'=t.}$$

In details, we will search for positive quadratic function solutions to the dimensionally reduced forms of the Eq. (1.3) via taking $z = x$ or $z = y$, and begin with

$$\begin{aligned} f &= g^2 + h^2 + a_9, \\ g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \end{aligned} \tag{1.4}$$

where a_i $(1 \le i \le 9)$ are all real parameters to be determined. To obtain the lump solutions, we note that the conditions guaranteeing the well-definedness of f, positiveness of f and localisation of u in all directions in the space need to be satisfied. With a selection of the parameters in the solutions, the localised structure and lump motion will be displayed. Finally, a few concluding remarks will be given at the end of the paper.

2. Lump Solutions to the Reduction with $z = x$

With $z = x$, the dimensionally reduced form of the Eq. (1.3) turns out to be

$$\left(D_x^3 D_y + 2 D_t D_x + D_t D_y - D_x^2\right) f \cdot f = 0, \tag{2.1}$$

which is transformed into

$$u_{xxxy} + 3(u_x u_y)_x + 2u_{tx} + u_{ty} - u_{xx} = 0, \tag{2.2}$$

through the link between f and u

$$u = 2\left[\ln f(x,y,t)\right]_x = 2\frac{f_x(x,y,t)}{f(x,y,t)}. \tag{2.3}$$

A direct substitution of f in the Eq. (1.4) into Eq. (2.1) leads to the following set of constraining equations on the parameters:

$$\left\{ a_1 = a_1, \quad a_2 = \frac{a_1^2 a_3 - 2a_1 a_3^2 + 2a_1 a_5 a_7 - 2a_1 a_7^2 - a_3 a_5^2}{a_3^2 + a_7^2}, \quad a_3 = a_3, \quad a_4 = a_4, \right.$$

$$a_5 = a_5, \quad a_6 = -\frac{a_1^2 a_7 - 2a_1 a_3 a_5 + 2a_3^2 a_5 - a_5^2 a_7 + 2a_5 a_7^2}{a_3^2 + a_7^2}, \quad a_7 = a_7,$$
$$\left. a_8 = a_8, \quad a_9 = \frac{3\left(a_1^2 + a_5^2\right)^2 \left(a_1 a_3 - 2a_3^2 + a_5 a_7 - 2a_7^2\right)}{(a_1 a_7 - a_3 a_5)^2} \right\}, \tag{2.4}$$

which needs to satisfy the conditions

$$a_1 a_7 - a_3 a_5 \neq 0, \tag{2.5}$$
$$a_1 a_3 - 2a_3^2 + a_5 a_7 - 2a_7^2 > 0 \tag{2.6}$$

to guarantee the well-definedness of f, the positiveness of f and the localisation of u in all directions in the space. The parameters in the set (2.4) yield a class of positive quadratic function solution to the Eq. (2.1) as

$$\begin{aligned} f = &\left(a_1 x + \frac{a_1^2 a_3 - 2a_1 a_3^2 + 2a_1 a_5 a_7 - 2a_1 a_7^2 - a_3 a_5^2}{a_3^2 + a_7^2} y + a_3 t + a_4 \right)^2 \\ &+ \left(a_5 x - \frac{a_1^2 a_7 - 2a_1 a_3 a_5 + 2a_3^2 a_5 - a_5^2 a_7 + 2a_5 a_7^2}{a_3^2 + a_7^2} y + a_7 t + a_8 \right)^2 \\ &+ \frac{3\left(a_1^2 + a_5^2\right)\left(a_1 a_3 - 2a_3^2 + a_5 a_7 - 2a_7^2\right)}{(a_1 a_7 - a_3 a_5)^2}, \end{aligned} \tag{2.7}$$

which, in turn, generates a class of lump solutions to the dimensionally reduced the Eq. (2.2) through transformation (2.3) as

$$u^{(\mathrm{I})} = \frac{4(a_1 g + a_5 h)}{f}, \tag{2.8}$$

where the function f is defined by the Eq. (2.7), and the functions g and h are given as follows:

$$g = a_1 x + \frac{a_1^2 a_3 - 2a_1 a_3^2 + 2a_1 a_5 a_7 - 2a_1 a_7^2 - a_3 a_5^2}{a_3^2 + a_7^2} y + a_3 t + a_4,$$
$$h = a_5 x - \frac{a_1^2 a_7 - 2a_1 a_3 a_5 + 2a_3^2 a_5 - a_5^2 a_7 + 2a_5 a_7^2}{a_3^2 + a_7^2} y + a_7 t + a_8.$$

3. Lump Solutions to the Reduction with $z = y$

With $z = y$, the dimensionally reduced form of the Eq. (1.3) reads

$$\left(D_x^3 D_y + D_t D_x + 2D_t D_y - D_y^2\right) f \cdot f = 0, \tag{3.1}$$

which is cast into

$$u_{xxxy} + 3\left(u_x u_y\right)_x + u_{tx} + 2u_{ty} - u_{yy} = 0 \tag{3.2}$$

through the link between f and u, that is, transformation (2.3).

A direct substitution of f into the Eq. (3.1) gives rise to the following set of constraining equations for the parameters:

$$\left\{a_1 = \frac{a_2^2a_3 - 2a_2a_3^2 + 2a_2a_6a_7 - 2a_2a_7^2 - a_3a_6^2}{a_3^2+a_7^2}, \quad a_2=a_2, \quad a_3=a_3, \quad a_4=a_4, \right. \tag{3.3}$$
$$a_5 = -\frac{a_2^2a_7 - 2a_2a_3a_6 + 2a_3^2a_6 - a_6^2a_7 + 2a_6a_7^2}{a_3^2+a_7^2}, \quad a_6=a_6, \quad a_7=a_7, \quad a_8=a_8,$$
$$\left. a_9 = \frac{3(a_2^2+a_6^2)^2(a_2a_3-2a_3^2+a_6a_7-2a_7^2)(a_2^2-4a_2a_3+4a_3^2+a_6^2-4a_6a_7+4a_7^2)}{(a_2a_7-a_3a_6)^2(a_3^2+a_7^2)}\right\},$$

which needs to satisfy the conditions

$$\begin{aligned} &a_2a_7 - a_3a_6 \neq 0, \\ &a_2a_3 - 2a_3^2 + a_6a_7 - 2a_7^2 > 0 \end{aligned} \tag{3.4}$$

to guarantee the well-definedness of f, the positiveness of f and the localisation of u in all directions in the space. The parameters in the set (3.3) yield a class of positive quadratic function solution to the Eq. (3.1) as

$$\begin{aligned} f =& \left(\frac{a_2^2a_3 - 2a_2a_3^2 + 2a_2a_6a_7 - 2a_2a_7^2 - a_3a_6^2}{a_3^2+a_7^2}x + a_2y + a_3t + a_4\right)^2 \\ &+\left(-\frac{a_2^2a_7 - 2a_2a_3a_6 + 2a_3^2a_6 - a_6^2a_7 + 2a_6a_7^2}{a_3^2+a_7^2}x + a_6y + a_7t + a_8\right)^2 \\ &+\frac{3\left(a_2^2+a_6^2\right)^2\left(a_2a_3-2a_3^2+a_6a_7-2a_7^2\right)\left(a_2^2-4a_2a_3+4a_3^2+a_6^2-4a_6a_7+4a_7^2\right)}{(a_2a_7-a_3a_6)^2\left(a_3^2+a_7^2\right)}, \end{aligned} \tag{3.5}$$

which, in turn, generates a class of lump solutions to the dimensionally reduced the Eq. (3.2) through transformation (2.3) as

$$u^{(\mathrm{II})} = \frac{4(a_1g + a_5h)}{f}, \tag{3.6}$$

where the function f is defined by the Eq. (3.5), and the functions g and h are given as follows:

$$\begin{aligned} g &= \frac{a_2^2a_3 - 2a_2a_3^2 + 2a_2a_6a_7 - 2a_2a_7^2 - a_3a_6^2}{a_3^2+a_7^2}x + a_2y + a_3t + a_4, \\ h &= -\frac{a_2^2a_7 - 2a_2a_3a_6 + 2a_3^2a_6 - a_6^2a_7 + 2a_6a_7^2}{a_3^2+a_7^2}x + a_6y + a_7t + a_8. \end{aligned}$$

4. Localised Characteristics and Lump Motion

The transformations (1.2) and (2.3) [with the Eq. (1.4)] clearly denote that lump solution is a type of rational solution. By virtue of the following property

$$\lim_{x^2+y^2\to\infty} f(x,y,t)=\infty \quad \forall t\in\mathbb{R}$$

it is easy to have

$$\lim_{x^2+y^2\to\infty} u^{(\mathrm{I})}(x,y,t)=\lim_{x^2+y^2\to\infty} u^{(\mathrm{II})}(x,y,t)=0 \quad \forall t\in\mathbb{R},$$

therefore, all the solutions derived in this paper ($u^{(\mathrm{I})}$ and $u^{(\mathrm{II})}$) are rationally localised in all directions in the space.

The amplitude of a lump solution u is defined as $\max|u|$, and the location of a lump solution is then defined as the place where the $\max|u|$ is attained. To study the lump motion, we firstly derive all the critical points of the lump solutions at a fixed time t as

$$\begin{aligned}
x^*_\pm(t)&=\frac{\left(a_1^2+a_5^2\right)\left[(a_2a_7-a_3a_6)t+a_2a8-a_4a_6\right]\pm\sqrt{a_9\left(a_1^2+a_5^2\right)(a_1a_6-a_2a_5)^2}}{\left(a_1^2+a_5^2\right)(a_1a_6-a_2a_5)},\\
y^*(t)&=\frac{(a_3a_5-a_1a_7)t+a_4a5-a_1a_8}{a_1a_6-a_2a_5}.
\end{aligned}$$

For the lump $u^{(\mathrm{I})}$ with parameter constraining conditions (2.4)-(2.6), the second partial derivative test

$$\begin{aligned}
&u_{xx}\big(x^*_+(t),y^*(t)\big)=-2\frac{\left(a_1^2+a_5^2\right)^{3/2}}{a_9^{3/2}}<0,\\
&u_{xx}\big(x^*_+(t),y^*(t)\big)u_{yy}\big(x^*_+(t),y^*(t)\big)-u_{xy}^2\big(x^*_+(t),y^*(t)\big)\\
=&\frac{4\left(a_1^2+a_5^2\right)^2(a_1a_6-a_2a_5)^2}{a_9^3}>0
\end{aligned}$$

tells us that $(x^*_+(t),\ y^*(t))$ is the maximum point, and

$$u_{\max}=\frac{2|a_3a_5-a_1a_7|}{\sqrt{3\left(a_1^2+a_5^2\right)\left(a_1a_3-2a_3^2+a_5a_7-2a_7^2\right)}}.$$

Meanwhile, we can find

$$\begin{aligned}
&u_{xx}\big(x^*_-(t),y^*(t)\big)=2\frac{\left(a_1^2+a_5^2\right)^{3/2}}{a_9^{3/2}}>0,\\
&u_{xx}\big(x^*_-(t),y^*(t)\big)u_{yy}\big(x^*_-(t),y^*(t)\big)-u_{xy}^2\big(x^*_-(t),y^*(t)\big)\\
=&\frac{4\left(a_1^2+a_5^2\right)^2(a_1a_6-a_2a_5)^2}{a_9^3}>0,
\end{aligned}$$

so that $(x_-^*(t),\ y^*(t))$ is the minimum point, and

$$u_{\min} = \frac{-2|a_3a_5 - a_1a_7|}{\sqrt{3\left(a_1^2+a_5^2\right)\left(a_1a_3-2a_3^2+a_5a_7-2a_7^2\right)}}.$$

Hereby, the amplitude of the lump

$$u^{(\mathrm{I})} = \frac{2|a_3a_5 - a_1a_7|}{\sqrt{3\left(a_1^2+a_5^2\right)\left(a_1a_3-2a_3^2+a_5a_7-2a_7^2\right)}},$$

which locates at $(x_\pm^*(t),\ y^*(t))$.

Correspondingly, for the lump $u^{(\mathrm{II})}$ with parameter constraining conditions (3.3)-(3.4), we have the maximum point $(x_+^*(t),\ y^*(t))$ with

$$u_{\max} = \frac{2|a_3a_6 - a_2a_7|}{\sqrt{3(a_2^2+a_6^2)(a_2a_3-2a_3^2+a_6a_7-2a_7^2)}},$$

the minimum point $(x_-^*(t),\ y^*(t))$ with

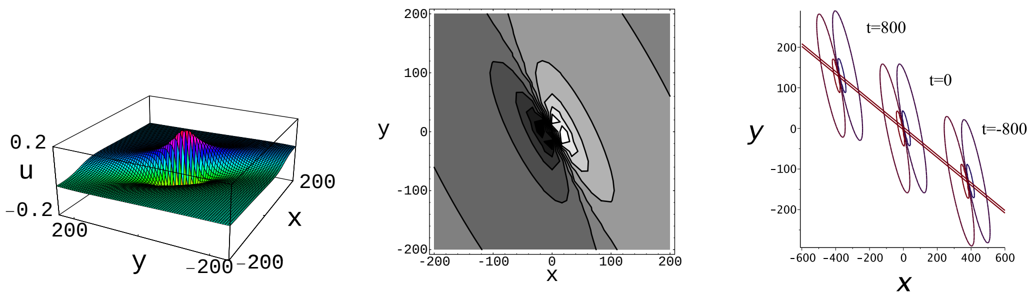
$$u_{\min} = \frac{-2|a_3a_6 - a_2a_7|}{\sqrt{3(a_2^2+a_6^2)(a_2a_3-2a_3^2+a_6a_7-2a_7^2)}},$$

and the amplitude of the lump

$$u^{(\mathrm{II})} = \frac{2|a_3a_6 - a_2a_7|}{\sqrt{3(a_2^2+a_6^2)(a_2a_3-2a_3^2+a_6a_7-2a_7^2)}},$$

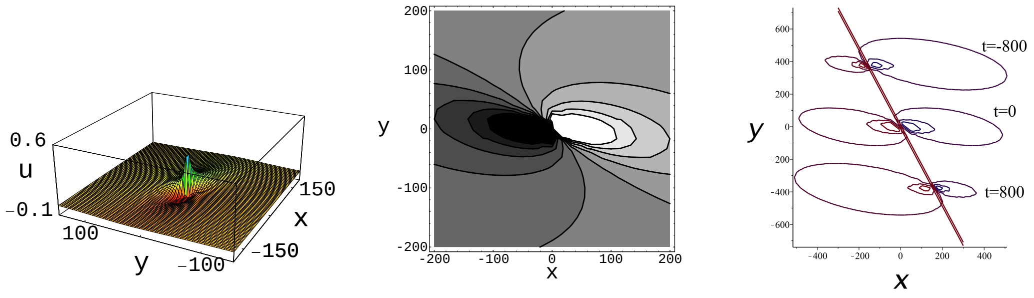
which locates at $(x_\pm^*(t),\ y^*(t))$.

With particular choices of the involved parameters in the lump solution u, the localised characteristics and lump motion can be seen clearly in Figs. 1 and 2 including (a) three-dimensional structure, (b) density plot and (c) the contour plot with routing display.



(a) (b) (c)

Figure 1: Lump dynamic characteristics of $u^{(\mathrm{I})}$ via the Eq. (2.8) with $a_1=-2$, $a_3=-1$, $a_4=0$, $a_5=4$, $a_7=1.5$ and $a_8=0$: (a) 3-dimensional plot with $t=0$, (b) density plot of (a), (c) the contour plot with routing display.



(a) (b) (c)

Figure 2: Lump dynamic characteristics of $u^{(\mathrm{II})}$ via the Eq. (3.6) with $a_2 = 3$, $a_3 = 1$, $a_4 = 0$, $a_6 = -4$, $a_7 = -2$ and $a_8 = 0$: (a) 3-dimensional plot with $t = 0$, (b) density plot of (a), (c) the contour plot with routing display.

5. Concluding Remarks

For two reduced cases of the new $(3+1)$-dimensional generalised KP equation — cf. the Eq. (1.1), we have directly constructed two classes of lump solutions — cf. Eqs. (2.8) and (3.6) via searching for positive quadratic function solutions to the associated bilinear equations (2.1) and (3.1).

It is interesting to consider the following two questions. Firstly, for the reduction with $z = t$, the Eq. (1.1) is reduced into

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} = 0. \tag{5.1}$$

How to derive lump solutions or how to prove the non-existence of lump solutions to the Eq. (5.1) is a further question. Secondly, how to derive lump solutions to the $(3+1)$-dimensional nonlinear evolution equations, e.g., the Eq. (1.1). Within the frame work of this paper, one may suppose

$$f = g^2 + h^2 + k^2 + a_{16}$$

with

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\ h &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\ k &= a_{11} x + a_{12} y + a_{13} z + a_{14} t + a_{15}, \end{aligned}$$

which should be substituted directly into the Eq. (1.3) for the purpose of positive quadratic function solutions so as to lump solutions to the Eq. (1.1).

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities of China (2018RC031), the National Natural Science Foundation of China (71971015), and the Open Fund of CEMEE (CEMEE2020K0201A).

References

[1] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press (1991).

[2] S.J. Chen, X. Lü, X.F. Tang, *Novel evolutionary behaviors of the mixed solutions to a generalized Burgers equation with variable coefficients*, Commun. Nonlinear Sci. Numer. Simul. **95**, 105628 (2021).

[3] S.J. Chen, W.X. Ma, X. Lü, *Bäcklund transformation, exact solutions and interaction behaviour of the* $(3+1)$*-dimensional Hirota-Satsuma-Ito-like equation*, Commun. Nonlinear Sci. Numer. Simul. **83**, 105135 (2020).

[4] S.J. Chen, Y.H. Yin, W.X. Ma, X. Lü, *Abundant exact solutions and interaction phenomena of the* $(2+1)$*-dimensional YTSF equation*, Anal. Math. Phys. **9**, 2329–2344 (2019).

[5] L.N. Gao, Y.Y. Zi, Y.H. Yin, W.X. Ma, X. Lü, *Bäcklund transformation, multiple wave solutions and lump solutions to a* $(3+1)$*-dimensional nonlinear evolution equation*, Nonlinear Dyn. **89**, 2233–2240 (2017).

[6] L.N. Gao, X.Y. Zhao, Y.Y. Zi, J. Yu, X. Lü, *Resonant behavior of multiple wave solutions to a Hirota bilinear equation*, Comput. Math. Appl. **72**, 2334–2342 (2016).

[7] X.J. He, X. Lü, M.G. Li, *Bäcklund transformation, Pfaffian, Wronskian and Grammian solutions to the* $(3+1)$*-dimensional generalized Kadomtsev-Petviashvili equation*, Anal. Math. Phys. **11**, 4 (2021).

[8] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press (2004).

[9] Y.F. Hua, B.L. Guo, W.X. Ma, X. Lü, *Interaction behavior associated with a generalized* $(2+1)$*-dimensional Hirota bilinear equation for nonlinear waves*, Appl. Math. Model. **74**, 184–198 (2019).

[10] B.B. Kadomtsev, V.I. Petviashvili, *On the stability of solitary waves in weakly dispersing media*, Sov. Phys. Dokl. **15**, 539–541 (1970).

[11] F.H. Lin, S.T. Chen, Q.X. Qu, J.P. Wang, X.W. Zhou, X. Lü, *Resonant multiple wave solutions to a new* $(3+1)$*-dimensional generalized Kadomtsev-Petviashvili equation: Linear superposition principle*, Appl. Math. Lett. **78**, 112–117 (2018).

[12] Y.Q. Liu, X.Y. Wen, *Fission and fusion interaction phenomena of mixed lump kink solutions for a generalized* $(3+1)$*-dimensional B-type Kadomtsev-Petviashvili equation*, Mod. Phys. Lett. B, **32**, 1850161 (2018).

[13] X. Lü, S.J. Chen, *Interaction solutions to nonlinear partial differential equations via Hirota bilinear forms: One-lump-multi-stripe and one-lump-multi-soliton types*, Nonlinear Dyn. **103**, 947–977 (2021).

[14] X. Lü, S.T. Chen, W.X. Ma, *Constructing lump solutions to a generalized Kadomtsev-Petviashvili-Boussinesq equation*, Nonlinear Dyn. **86**, 523–534 (2016).

[15] X. Lü, Y.F. Hua, S.J. Chen, X.F. Tang, *Integrability characteristics of a novel* $(2+1)$*-dimensional nonlinear model: Painlevé analysis, soliton solutions, Bäcklund transformation, Lax pair and infinitely many conservation laws*, Commun. Nonlinear Sci. Numer. Simul. **95**, 105612 (2021).

[16] X. Lü, W.X. Ma, *Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation*, Nonlinear Dyn. **85**, 1217–1222 (2016).

[17] X. Lü, W.X. Ma, Y. Zhou, C.M. Khalique, *Rational solutions to an extended Kadomtsev-Petviashvili-like equation with symbolic computation*, Comp. Math. Appl. **71**, 1560–1567 (2016).

[18] W.X. Ma, *Lump solutions to the Kadomtsev-Petviashvili equation*, Phys. Lett. A **379**, 1975–1978 (2015).

[19] W.X. Ma, E.G. Fan, *Linear superposition principle applying to Hirota bilinear equations*, Comp. Math. Appl. **61**, 950–959 (2011).

[20] W.X. Ma, Z.Y. Qin, X. Lü, *Lump solutions to dimensionally reduced p-gKP and p-gBKP equations*, Nonlinear Dyn. **84**, 923–931 (2016).
[21] W.X. Ma, Y. Zhang, Y.N. Tang, J.Y. Tu, *Hirota bilinear equations with linear subspaces of solutions*, Appl. Math. Comput. **218**, 7174–7183 (2012).
[22] S.F. Tian, *Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation*, Appl. Math. Lett. **100**, 106056 (2020).
[23] D.S. Wang, Y.B. Yin, *Symmetry analysis and reductions of the two-dimensional generalized Benney system via geometric approach*, Comput. Math. Appl. **71**, 748–757 (2016).
[24] A.M. Wazwaz, S.A. El-Tantawy, *A new* $(3+1)$*-dimensional generalized Kadomtsev-Petviashvili equation*, Nonlinear Dyn. **84**, 1107–1112 (2016).
[25] X.Y. Wen, Z.Y. Yan, *Higher-order rational solitons and rogue-like wave solutions of the* $(2+1)$*-dimensional nonlinear fluid mechanics equations*, Commun. Nonlinear Sci. Numer. Simul. **43**, 311–329 (2017).
[26] J.W. Xia, Y.W. Zhao, X. Lü, *Predictability, fast calculation and simulation for the interaction solutions to the cylindrical Kadomtsev-Petviashvili equation*, Commun. Nonlinear Sci. Numer. Simul. **90**, 105260 (2020).
[27] H.N. Xu, W.Y. Ruan, Y. Zhang, X. Lü, *Multi-exponential wave solutions to two extended Jimbo-Miwa equations and the resonance behavior*, Appl. Math. Lett. **99** , 105976 (2020).
[28] Y.H. Yin, S.J. Chen, X. Lü, *Study on localized characteristics of lump and interaction solutions to two extended Jimbo-Miwa equations*, Chin. Phys. B. **29**, 120502 (2020).
[29] Y.H. Yin, W.X. Ma, J.G. Liu, X. Lü, *Diversity of exact solutions to a* $(3+1)$*-dimensional nonlinear evolution equation and its reduction*, Comput. Math. Appl. **76**, 1275–1283 (2018).
[30] L.D. Zhang, S.F. Tian, W.Q. Peng, T.T. Zhang, X.J. Yan, *The dynamics of lump, lumpoff and rogue wave solutions of* $(2+1)$*-dimensional Hirota-Satsuma-Ito equations*, East Asian J. Appl. Math., **10**, 243–255 (2020).
[31] X.M. Zhou, S.F. Tian, J.J. Yang, J.J. Mao, *The Riemann-Hilbert approach and N-soliton solutions of a four-component nonlinear Schrödinger equation*, East Asian J. Appl. Math. **11**, 143–163 (2021).