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Research paper

Riemann–Hilbert problems and soliton solutions for a generalized coupled Sasa–Satsuma equation

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ABSTRACT

This paper studies the multi-component Sasa-Satsuma integrable hierarchies via an arbitrary-order matrix spectral problem, based on the zero curvature formulation. A generalized coupled Sasa-Satsuma equation is derived from the multi-component Sasa-Satsuma integrable hierarchies with a bi-Hamiltonian structure. The inverse scattering transform of the generalized coupled Sasa-Satsuma equation is presented by the spatial matrix spectral problem and the Riemann–Hilbert method, which enables us to obtain the N-soliton solutions. And then the dynamics of one- and two-soliton solutions are discussed and presented graphically. Asymptotic analyses of the presented two-soliton solution are finally analyzed.

1. Introduction

The inverse scattering transform is one of the most significant methods to obtain solutions of numerous integrable nonlinear equations. With the in-depth study of integrable systems, the Riemann–Hilbert method is proposed which is more straightforward and simplified compared with the inverse scattering transform method [1]. A large number of integrable nonlinear equations have been studied by using Riemann–Hilbert method, such as the nonlinear Schrödinger (NLS) equation [2–4], the modified Korteweg–de Vries (mKdV) equation [5], the focusing and defocusing Hirota equations [6], the Fokas–Lenells (FL) equation [7,8], and the Dullin–Gottwald–Holm (DGH) equation [9]. This method is also widely investigated and applied to generate soliton solutions of the coupled higher-order nonlinear Schrödinger (HNLS) equations [10] and the multi-component cubic–quintic NLS system [11].

The Sasa-Satsuma equation is derived from the HNLS equation [12–15] by using the suitable transformations, and such equation with higher-order terms still preserves integrability. The Sasa-Satsuma equation can describe the wave development in the complex scalar field and the pulse propagation in birefringent fibers [16,17], so as to increase the bit rate in optical fibers, or achieve wavelength-division multiplexing [18]. And it is worth mentioning that the Sasa-Satsuma equation could contribute different dynamic characteristics of optical solitons such as bright solitons, dark solitons, breather, single-hump, double-hump, rogue waves and W-shaped soliton, and the related studies can be found

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in [19–25]. The Lax pair of the Sasa-Satsuma equation has been constructed in [26], and there are many studies to show its integrability [27,28]. Xu et al. derive N-soliton solutions and construct two different types of femto-second soliton solutions of the coupled Sasa-Satsuma system by using Darboux transformation [29,30]. In both vanishing and non-vanishing backgrounds, the explicit solutions of the coupled Sasa-Satsuma equations are generated through the binary Darboux transformation method [31]. Using Riccati equation mapping technique, periodic solitons and solitary waves for the Sasa-satsuma equation are derived in [32]. Nth-order semi-rational solutions of the Sasa-Satsuma equation are derived via a modified dressing transformation [33]. The nonlinear steepest descent method is applied to analyze the long-time asymptotics for the Sasa-Satsuma equation with decaying initial data [34]. Liu and Guo follow the modifications of nonlinear steepest descent approach and find the long-time asymptotic behaviors for the Sasa-Satsuma equation using Cauchy formula [35]. The inverse scattering transform is constructed for the Sasa-Satsuma equation with nonzero boundary condition in [36]. Geng et al. [17,37,38] study the one-, two- and three-component Sasa-Satsuma equations through the Riemann–Hilbert method, and describe the different dynamic behaviors of soliton solutions. Through a standard dressing procedure, soliton matrices for simple and elementary high-order zeros for Sasa-Satsuma equation are constructed [39]. Wu [40] analyzes spectral and soliton structures of the Sasa-Satsuma equation by solving the Riemann–Hilbert problem. Wang and Han [41] derive N-soliton solutions of a three-component coupled Sasa-Satsuma equation, and discuss the dynamics of the soliton solutions. To the authors' best knowledge, the soliton solutions and asymptotic analysis about the generalized coupled Sasa-Satsuma equation with arbitrary constant coefficients have not been reported until now.

In this paper, we consider the generalized coupled Sasa-Satsuma equation as follows

$$\begin{aligned} &u_t + u_{xxx} + 6(\sigma_1|u|^2 + \sigma_2|v|^2)u_x + 3u(\sigma_1|u|^2 + \sigma_2|v|^2)_x = 0,\\ &v_t + v_{xxx} + 6(\sigma_1|u|^2 + \sigma_2|v|^2)v_x + 3v(\sigma_1|u|^2 + \sigma_2|v|^2)_x = 0, \end{aligned} \tag{1}$$

where u and v represent complex-valued functions for the independent spatial variable x and temporal variable t, the parameters σ_1 and σ_2 are arbitrary real numbers, and the subscript x (or t) denotes the partial derivative with respect to x (or t) of functions u and v. In the generalized coupled Sasa-Satsuma equation, the last three terms account for third order dispersion, Kerr dispersion and self-frequency shift under Raman effect, respectively. In order to explore the origin of the generalized coupled Sasa-Satsuma Eq. (1), the multi-component Sasa-Satsuma integrable hierarchies are constructed from the zero curvature formulation. In the meantime, the corresponding bi-Hamiltonian structure is also constructed. Several types of solutions are obtained via the Riemann–Hilbert method, such as the bright solitons, breather, single-hump solitons and double-hump solitons. And dynamic behaviors are investigated, which enrich the physical features of nonlinear systems. It is noted that the elastic interaction between one (or two) single-hump solution(s) and one (or two) breather-type solution(s) of the generalized coupled Sasa-Satsuma equation are investigated, and polarization-changing collisions between two single-hump solitons are also analyzed. For the purpose of observing phase shifts of the two-soliton solutions before interaction and after interaction, we give the asymptotic expressions of two-soliton solutions through long-time behavior analysis.

The outline of this paper is presented as follows. In Section 2, according to the zero curvature formulation, the multi-component Sasa-Satsuma integrable hierarchies are constructed, which contribute to derive the generalized coupled Sasa-Satsuma equation. And a bi-Hamiltonian structure is also constructed, which displays the integrability of the Sasa-Satsuma system. In Section 3, the inverse scattering transform of the generalized coupled Sasa-Satsuma equation with the spectral analysis is studied through the Riemann–Hilbert method. In Section 4, the N-soliton solutions are derived explicitly via the reflectionless transforms in two cases. The expressions of one- and two-soliton solutions are presented, and some figures are given to describe the dynamic characteristics of them. In Section 5, the long-time asymptotic analysis on two-soliton solutions of the generalized coupled Sasa-Satsuma equation is given. Finally, the conclusions of this paper are stated in Section 6.

2. The multi-component Sasa-Satsuma integrable hierarchies

2.1. Zero curvature formulation

Integrable hierarchies can be constructed by using zero curvature formulation. Firstly, a square spectral matrix $U = U(\boldsymbol{u}, \boldsymbol{u}^*, \lambda)$ can be introduced with vector potentials $\boldsymbol{u}$, $\boldsymbol{u}^*$ and a spectral parameter λ. In order to solve the stationary zero curvature equation

$$V_x = [U,\ V], \tag{2}$$

we assume a solution

$$V = V(\boldsymbol{u}, \boldsymbol{u}^*, \lambda) = \sum_{m=0}^{\infty} V_m \lambda^{-m} = \sum_{m=0}^{\infty} V_m(\boldsymbol{u}, \boldsymbol{u}^*)\lambda^{-m}. \tag{3}$$

Based on the solution V of the stationary zero curvature Eq. (2), we would like to derive an integrable hierarchy from the zero curvature equations

$$U_{t_n} - V_x^{[n]} + \left[U,\ V^{[n]}\right] = 0, \quad n \geq 1. \tag{4}$$

In order to make the left side of the above equations appear to the zero power of λ, a series of Lax matrix expressions $V^{[n]}$ in the above zero curvature Eqs. (4) are constructed as [42]

$$V^{[n]}=V^{[n]}(\boldsymbol{u},\boldsymbol{u}^*,\lambda)=\left(\lambda^n V\right)_+ +\Delta_n,\quad n\geq 1, \tag{5}$$

where the term of $(\lambda^n V)_+$ represents a polynomial part in λ, and Δ_n is the modification term. With different values of n, we select appropriate Δ_n, and then the integrable hierarchy could be given as follows

$$\begin{pmatrix} \boldsymbol{u}_{t_n} \\ \boldsymbol{u}^{*\top}_{t_n}\end{pmatrix}=\mathbf{K}_n(\boldsymbol{u},\boldsymbol{u}^*)=K_n\left(x,t,\boldsymbol{u},\boldsymbol{u}^*,\boldsymbol{u}_x,\boldsymbol{u}^*_x,\ldots\right),\quad n\geq 1, \tag{6}$$

where $\mathbf{K}_n$ is the formal differential operator representing $K_n\left(x,t,\boldsymbol{u},\boldsymbol{u}^*,\boldsymbol{u}_x,\boldsymbol{u}^*_x,\ldots\right)$ [43].

The associated spatial and temporal matrix spectral problems of the nth evolution equation in the hierarchy (6) are given as follows

$$Y_x=UY=U(\boldsymbol{u},\boldsymbol{u}^*,\lambda)Y,\quad Y_{t_n}=V^{[n]}Y=V^{[n]}(\boldsymbol{u},\boldsymbol{u}^*,\lambda)Y,\quad n\geq 1, \tag{7}$$

where Y is the matrix eigenfunction, and the compatibility condition of these matrix spectral problems is the zero curvature equation (4).

Then, we would like to analyze the bi-Hamiltonian structures of the hierarchy (6), which can be usually provided by trace identification [44,45]:

$$\frac{\delta}{\delta\boldsymbol{u}}\int \operatorname{tr}\left(V\frac{\partial U}{\partial\lambda}\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left[\lambda^{\gamma}\operatorname{tr}\left(V\frac{\partial U}{\partial\boldsymbol{u}}\right)\right],\quad \gamma=-\frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\operatorname{tr}\left(V^2\right)\right|, \tag{8}$$

or more generally, the variational identity:

$$\frac{\delta}{\delta\boldsymbol{u}}\int \left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left[\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial\boldsymbol{u}}\right\rangle\right],\quad \gamma=-\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle V,V\rangle|, \tag{9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form [46], and $\frac{\delta}{\delta\boldsymbol{u}}$ stands for the variational derivative which is defined as

$$\frac{\delta}{\delta\boldsymbol{u}}=\sum_{n\geq 0}(-\partial)^n\frac{\partial}{\partial\boldsymbol{u}^{(n)}},\quad \left(\partial=\frac{\partial}{\partial x},\ \boldsymbol{u}^{(n)}=\partial^n\boldsymbol{u}\right). \tag{10}$$

The bi-Hamiltonian structure is provided

$$\begin{pmatrix} \boldsymbol{u}_{t_n} \\ \boldsymbol{u}^{*\top}_{t_n}\end{pmatrix}=\mathbf{K}_n=J_1\frac{\delta\widetilde{H}_{n+1}}{\delta\boldsymbol{u}}=J_2\frac{\delta\widetilde{H}_n}{\delta\boldsymbol{u}},\quad n\geq 1, \tag{11}$$

which could help to analyze the Liouville integrability [43] of the hierarchy (6). The J_1 and J_2 are the Hamiltonian pairs [47].

2.2. The multi-component Sasa-Satsuma integrable hierarchies

In order to generate the Sasa-Satsuma integrable hierarchies with multiple components, the $2n+1$ order matrix spectral problem is constructed as follows

$$Y_x=UY=U(\boldsymbol{u},\boldsymbol{u}^*,\lambda)Y,\quad U=\begin{pmatrix} i\lambda I_{2n} & \boldsymbol{u} \\ -\boldsymbol{u}^* & -i\lambda\end{pmatrix}, \tag{12}$$

where λ is a spectral parameter, $\boldsymbol{u}$ is a $2n$-dimensional column vector and $\boldsymbol{u}^*$ is a $2n$-dimensional row vector

$$\boldsymbol{u}=(u_1,\sigma_1u_1^*,u_2,\sigma_2u_2^*,\ldots,u_n,\sigma_nu_n^*)^\top\quad \boldsymbol{u}^*=(\sigma_1u_1^*,u_1,\sigma_2u_2^*,u_2,\ldots,\sigma_nu_n^*,u_n), \tag{13}$$

where σ_1, σ_2, $\ldots\sigma_n$ are arbitrary real numbers, and the superscript $*$ and $\top$ denote the conjugate and the transpose, respectively.

Then, we introduce a solution V of the corresponding stationary zero curvature Eq. (2)

$$V=\begin{pmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & d\end{pmatrix}, \tag{14}$$

where a is a $2n\times 2n$ matrix, $\boldsymbol{b}$ and $\boldsymbol{c}^\top$ are $2n$-dimensional columns, and d is a scalar. So as to derive the multi-component Sasa-Satsuma integrable hierarchies, we substitute the above matrix (14) into the corresponding stationary zero curvature Eq. (2). Based on the matrix operation, the equations are given as follows

$$\begin{cases} a_x=\boldsymbol{uc}+\boldsymbol{bu}^*, \\ \boldsymbol{b}_x=2i\lambda\boldsymbol{b}+\boldsymbol{u}d-a\boldsymbol{u}, \\ \boldsymbol{c}_x=-\boldsymbol{u}^*a-2i\lambda\boldsymbol{c}+d\boldsymbol{u}^*, \\ d_x=-\boldsymbol{u}^*\boldsymbol{b}-\boldsymbol{cu}.\end{cases} \tag{15}$$

Then, we give the expression of V

$$V=\begin{pmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & d \end{pmatrix}=\sum_{m=0}^{\infty} V_m\lambda^{-m}, \quad V_m=V_m(\boldsymbol{u},\boldsymbol{u}^*)=\begin{pmatrix} a^{[m]} & \boldsymbol{b}^{[m]} \\ \boldsymbol{c}^{[m]} & d^{[m]} \end{pmatrix}, \quad m\geq 0, \tag{16}$$

where $a^{[m]}$, $\boldsymbol{b}^{[m]}$ and $\boldsymbol{c}^{[m]}$ are expressed as

$$\begin{aligned} &a^{[m]}=(a_{ij}^{[m]})_{2n\times 2n}, \quad m\geq 0,\\ &\boldsymbol{b}^{[m]}=(b_1^{[m]},b_2^{[m]},\ldots,b_{2n}^{[m]})^{\top}, \quad m\geq 0,\\ &\boldsymbol{c}^{[m]}=(c_1^{[m]},c_2^{[m]},\ldots,c_{2n}^{[m]}), \quad m\geq 0. \end{aligned} \tag{17}$$

Substituting the Eq. (16) into the equation set (15), the recursion relations are derived as follows

$$\begin{aligned} &\boldsymbol{b}^{[0]}=0, \quad \boldsymbol{c}^{[0]}=0, \quad a_x^{[0]}=0, \quad d_x^{[0]}=0,\\ &\boldsymbol{b}^{[m+1]}=-\frac{i}{2}\boldsymbol{b}_x^{[m]}+\frac{i}{2}\boldsymbol{u}d^{[m]}-\frac{i}{2}a^{[m]}\boldsymbol{u},\\ &\boldsymbol{c}^{[m+1]}=\frac{i}{2}\boldsymbol{c}_x^{[m]}+\frac{i}{2}\boldsymbol{u}^*a^{[m]}-\frac{i}{2}d^{[m]}\boldsymbol{u}^*,\\ &a_x^{[m]}=\boldsymbol{u}\boldsymbol{c}^{[m]}+\boldsymbol{b}^{[m]}\boldsymbol{u}^*,\\ &d_x^{[m]}=-\boldsymbol{u}^*\boldsymbol{b}^{[m]}-\boldsymbol{c}^{[m]}\boldsymbol{u}. \end{aligned} \tag{18}$$

We take the initial values

$$a^{[0]}=\alpha_1 I_{2n}, \quad d^{[0]}=\alpha_2, \tag{19}$$

where α_1, α_2 are arbitrary real constants. The recursion relation (18) can determine all matrices V_m, $m\geq 1$, and the coefficients are derived as follows

$$\begin{aligned} &\boldsymbol{b}^{[1]}=-\frac{i}{2}\alpha\boldsymbol{u}, \quad \boldsymbol{c}^{[1]}=\frac{i}{2}\alpha\boldsymbol{u}^*, \quad a_x^{[1]}=0, \quad d_x^{[1]}=0, \quad a^{[1]}=\alpha_3 I_{2n}, \quad d^{[1]}=\alpha_4,\\ &\boldsymbol{b}^{[2]}=-\frac{1}{4}\alpha\boldsymbol{u}_x-\frac{i}{2}\beta\boldsymbol{u}, \quad \boldsymbol{c}^{[2]}=-\frac{1}{4}\alpha\boldsymbol{u}_x^*+\frac{i}{2}\beta\boldsymbol{u}^*,\\ &a_x^{[2]}=-\frac{1}{4}\alpha(\boldsymbol{u}\boldsymbol{u}_x^*+\boldsymbol{u}_x\boldsymbol{u}^*), \quad d_x^{[2]}=\frac{1}{4}\alpha(\boldsymbol{u}^*\boldsymbol{u}_x+\boldsymbol{u}_x^*\boldsymbol{u}),\\ &a^{[2]}=-\frac{1}{4}\alpha\boldsymbol{u}\boldsymbol{u}^*+\alpha_5 I_{2n}, \quad d^{[2]}=\frac{1}{4}\alpha\boldsymbol{u}^*\boldsymbol{u}+\alpha_6,\\ &\boldsymbol{b}^{[3]}=\frac{i}{8}\alpha\boldsymbol{u}_{xx}-\frac{1}{4}\beta\boldsymbol{u}_x+\frac{i}{4}\alpha\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{u}-\frac{i}{2}\gamma\boldsymbol{u},\\ &\boldsymbol{c}^{[3]}=-\frac{i}{8}\alpha\boldsymbol{u}_{xx}^*-\frac{1}{4}\beta\boldsymbol{u}_x^*-\frac{i}{4}\alpha\boldsymbol{u}^*\boldsymbol{u}\boldsymbol{u}^*+\frac{i}{2}\gamma\boldsymbol{u}^*,\\ &a_x^{[3]}=\frac{i}{8}\alpha(\boldsymbol{u}_x\boldsymbol{u}^*-\boldsymbol{u}\boldsymbol{u}_x^*)_x-\frac{1}{4}\beta(\boldsymbol{u}\boldsymbol{u}^*)_x,\\ &d_x^{[3]}=\frac{i}{8}\alpha(\boldsymbol{u}_x^*\boldsymbol{u}-\boldsymbol{u}^*\boldsymbol{u}_x)_x+\frac{1}{4}\beta(\boldsymbol{u}^*\boldsymbol{u})_x,\\ &a^{[3]}=\frac{i}{8}\alpha(\boldsymbol{u}_x\boldsymbol{u}^*-\boldsymbol{u}\boldsymbol{u}_x^*)-\frac{1}{4}\beta\boldsymbol{u}\boldsymbol{u}^*+\alpha_7 I_{2n},\\ &d^{[3]}=\frac{i}{8}\alpha(\boldsymbol{u}_x^*\boldsymbol{u}-\boldsymbol{u}^*\boldsymbol{u}_x)+\frac{1}{4}\beta\boldsymbol{u}^*\boldsymbol{u}+\alpha_8, \end{aligned} \tag{20}$$

where $\alpha=\alpha_1-\alpha_2$, $\beta=\alpha_3-\alpha_4$, $\gamma=\alpha_5-\alpha_6$ and α_i ($i=3,4,5,6,7,8$) are arbitrary real constants. On the basis of Eq. (18), the recursion relation of $\boldsymbol{b}^{[m]}$ and $\boldsymbol{c}^{[m]}$ is given as follows

$$\begin{pmatrix} \boldsymbol{b}^{[m+1]} \\ \boldsymbol{c}^{[m+1]\top} \end{pmatrix}=\Gamma\begin{pmatrix} \boldsymbol{b}^{[m]} \\ \boldsymbol{c}^{[m]\top} \end{pmatrix}, \quad m\geq 1, \tag{21}$$

where Γ is a $4n\times 4n$ matrix integro-differential operator

$$\Gamma=-\frac{i}{2}\begin{pmatrix} \partial_x+\boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^*+\sum\limits_{j=1}^{n}h_j & \boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^{\top}+(\boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^{\top})^{\top} \\ -(\boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^*)^{\top}-\boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^* & -\partial_x-\sum\limits_{j=1}^{n}h_j-\boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^{\top} \end{pmatrix}. \tag{22}$$

In the above matrix Γ, h_j is presented as

$$h_j = \sigma_j u_j \partial_x^{-1} u_j^* + \sigma_j u_j^* \partial_x^{-1} u_j. \tag{23}$$

Then, we introduce the Lax matrix of temporal matrix spectral problems of the nth evolution equation in the hierarchy (6)

$$V^{[n]} = V^{[n]}(\boldsymbol{u}, \boldsymbol{u}^*, \lambda) = (\lambda^n V)_+ = \sum_{m=0}^{n} V_m \lambda^{n-m}, \quad n \geq 1, \tag{24}$$

where the V_m is defined in (16). According to the zero curvature Eq. (4), the multi-component Sasa-Satsuma integrable hierarchies are given as follows

$$\begin{pmatrix} \boldsymbol{u}_{t_n} \\ \boldsymbol{u}_{t_n}^{*\top} \end{pmatrix} = 2i \begin{pmatrix} \boldsymbol{b}^{[n+1]} \\ \boldsymbol{c}^{[n+1]\top} \end{pmatrix}, \quad n \geq 1. \tag{25}$$

When $n = 3$, we take the functions $u_2 = u_3 = 0$ and let the coefficients $\alpha = 8i$, $\beta = 0$, $\gamma = 0$ and $\alpha_7 = \alpha_8$. The generalized one-component Sasa-Satsuma equation is derived from the above hierarchies (25) in the following form

$$u_{1,t_3} + u_{1xxx} + 6u_{1x}(\sigma_1|u_1|^2) + 3u_1(\sigma_1|u_1|^2)_x = 0. \tag{26}$$

Additionally, we take the function $u_3 = 0$, and let the coefficients $\alpha = 8i$, $\beta = 0$, $\gamma = 0$ and $\alpha_7 = \alpha_8$. The generalized coupled Sasa-Satsuma equation is derived from the above hierarchies (25) as follows

$$\begin{aligned} &u_{1,t_3} + u_{1xxx} + 6u_{1x}(\sigma_1|u_1|^2 + \sigma_2|u_2|^2) + 3u_1(\sigma_1|u_1|^2 + \sigma_2|u_2|^2)_x = 0, \\ &u_{2,t_3} + u_{2xxx} + 6u_{2x}(\sigma_1|u_1|^2 + \sigma_2|u_2|^2) + 3u_2(\sigma_1|u_1|^2 + \sigma_2|u_2|^2)_x = 0. \end{aligned} \tag{27}$$

So as to analyze the Liouville integrability of the multi-component Sasa-Satsuma integrable hierarchies (25), bi-Hamiltonian structures can be presented using the trace identity or the variational identity. According to the matrix U, the traces satisfy

$$\mathrm{tr}(V\frac{\partial U}{\partial \lambda}) = i[\mathrm{tr}(a) - d] = i[\sum_{m=0}^{\infty}(\sum_{j=1}^{2n} a_{jj}^{[m]} - d^{[m]})\lambda^{-m}], \tag{28}$$

$$\mathrm{tr}(V\frac{\partial U}{\partial \boldsymbol{u}}) = \begin{pmatrix} \boldsymbol{c}^\top \\ -\boldsymbol{b} \end{pmatrix} = \sum_{m\geq 0} G_{m-1}\lambda^{-m}, \tag{29}$$

where

$$G_{m-1} = \begin{pmatrix} \boldsymbol{c}^{[m]\top} \\ -\boldsymbol{b}^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{30}$$

Substituting the above (28)–(29) into the trace identity, we get

$$\frac{\delta \tilde{H}_m}{\delta \boldsymbol{u}} = iG_{m-1}, \quad \tilde{H}_m = \frac{i}{m}\int (\sum_{j=1}^{2n} a_{jj}^{[m+1]} - d^{[m+1]})dx, \quad m \geq 1. \tag{31}$$

Then, the bi-Hamiltonian structure for the multi-component Sasa-Satsuma Eqs. (1) is derived as

$$\begin{pmatrix} \boldsymbol{u}_{t_m} \\ \boldsymbol{u}_{t_m}^{*\top} \end{pmatrix} = K_m = J_1 \frac{\delta \tilde{H}_{m+1}}{\delta \boldsymbol{u}} = J_2 \frac{\delta \tilde{H}_m}{\delta \boldsymbol{u}}, \quad m \geq 1, \tag{32}$$

where

$$J_1 = \begin{pmatrix} 0 & -2I_{2n} \\ 2I_{2n} & 0 \end{pmatrix}, \tag{33}$$

and

$$J_2 = i \begin{pmatrix} -\boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^\top - (\boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^\top)^\top & \partial_x + \boldsymbol{u}\partial_x^{-1}\boldsymbol{u}^* + \sum_{j=1}^{n} h_j \\ \partial_x + \sum_{j=1}^{n} h_j + \boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^\top & -(\boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^*)^\top - \boldsymbol{u}^{*\top}\partial_x^{-1}\boldsymbol{u}^* \end{pmatrix}. \tag{34}$$

The bi-Hamiltonian structure is constructed which displays the integrability of multi-component Sasa-Satsuma integrable system. Thus, adjoint symmetry constraints decompose each multi-component Sasa-Satsuma system (25) into two commuting finite-dimensional Hamiltonian systems being Liouville integrable .

3. Inverse scattering for the generalized coupled Sasa-Satsuma equation

In order to obtain the N-soliton solution of the generalized coupled Sasa-Satsuma Eq. (1), the symmetry relations of discrete scattering data should be derived first. In this section, we use the inverse scattering transform method to analyze the problem. We begin our analysis with the following Lax pair which can be derived from a five-order matrix spectral problem when u_3 and u_3^* are zero in (12),

$$Y_x = UY, \quad Y_t = VY, \tag{35}$$

where $Y = Y(x, t, \lambda)$ is the matrix eigenfunction with the complex spectral parameter λ. And

$$U = -i\lambda \Lambda + Q, \tag{36}$$

$$V = -4i\lambda^3 \Lambda + 4\lambda^2 Q - 2i\lambda(Q^2 + Q_x)\Lambda + Q_x Q - QQ_x - Q_{xx} + 2Q^3, \tag{37}$$

where

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & \sigma_1 u^* \\ 0 & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & \sigma_2 v^* \\ -\sigma_1 u^* & -u & -\sigma_2 v^* & -v & 0 \end{pmatrix}, \tag{38}$$

and

$$\Lambda = \mathrm{diag}(-1, -1, -1, -1, 1). \tag{39}$$

Based on the Lax pair (35), we study inverse scattering transform for the generalized coupled Sasa-Satsuma Eq. (1) via Riemann–Hilbert framework. Assume the solution $(u, v) \to 0$ as $x \to \pm\infty$, and the expression of Y can be given as follows

$$Y = Je^{-i\lambda\Lambda x - 4i\lambda^3 \Lambda t}, \tag{40}$$

where the matrix function J is (x, t)-independent at infinity. Substituting (40) into (35), we get

$$J_x = -i\lambda[\Lambda,\ J] + QJ, \tag{41}$$

$$\begin{aligned} J_t &= -4i\lambda^3[\Lambda,\ J] + [4\lambda^2 Q - 2i\lambda(Q^2 + Q_x)\Lambda + Q_x Q - QQ_x - Q_{xx} + 2Q^3]J \\ &= -4i\lambda^3[\Lambda,\ J] + \tilde{Q}J, \end{aligned} \tag{42}$$

where $[\Lambda,\ J] = \Lambda J - J\Lambda$.

Notice that the matrix Q satisfies the following property

$$Q^\dagger = -FQF^{-1}, \tag{43}$$

where the superscript $\dagger$ represents the Hermitian transpose (i.e., conjugate transpose) of a matrix, and

$$F = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma_1} & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{44}$$

In the scattering problem, the matrix Jost solutions $J_\pm(x, \lambda)$ of Eq. (41) are given with the asymptotic condition as x approaches $\pm\infty$

$$J_\pm(x, \lambda) \to I, \quad x \to \pm\infty, \tag{45}$$

where I is the 5×5 unit matrix. The notations are also given as follows

$$J_- E = \Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5), \tag{46}$$

$$J_+ E = \Psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5), \tag{47}$$

where $E(x, \lambda) = e^{-i\lambda\Lambda x}$. Since Φ and Ψ are both fundamental matrices of (35), they could be related as

$$\Phi(x, t, \lambda) = \Psi(x, t, \lambda)S(t, \lambda), \quad \lambda \in \mathbb{R}, \tag{48}$$

where $S(t, \lambda) = [s_{ij}]$ for real λ. In addition, it is easy to verify that the matrix Jost solutions $J_\pm(x, \lambda)$ are uniquely determined by the Volterra integral equations

$$J_- = I + \int_{-\infty}^{x} e^{i\lambda\Lambda(y-x)} QJ_- e^{-i\lambda\Lambda(y-x)} dy, \tag{49}$$

$$J_+ = I - \int_x^{+\infty} e^{i\lambda\Lambda(y-x)} QJ_+ e^{-i\lambda\Lambda(y-x)} dy. \tag{50}$$

Then we analyze the analytic properties of column vectors of $J_\pm(x, \lambda)$. According to the expression (38) of Q, it is obvious that the first to fourth columns of matrix J_- and the fifth column of matrix J_+ can be analytically continued to the lower half plane $\lambda \in \mathbb{C}_-$, while the fifth column of matrix J_- and the first to fourth columns of matrix J_+ can be analytically continued to the upper half plane $\lambda \in \mathbb{C}_+$. The Jost solutions are derived as

$$P^+ = (\psi_1, \psi_2, \psi_3, \psi_4, \phi_5)e^{i\lambda\Lambda x} = J_-H_1 + J_+H_2, \tag{51}$$

which are analytic in $\lambda \in \mathbb{C}_+$. In the above expression (51), the H_1 and H_2 are given as follows

$$H_1 = \mathrm{diag}(0, 0, 0, 0, 1), \quad H_2 = \mathrm{diag}(1, 1, 1, 1, 0). \tag{52}$$

Meanwhile, some notations of the inverse Jost solutions are introduced as

$$J_-^{-1} = E\Phi^{-1}, \quad J_+^{-1} = E\Psi^{-1}, \tag{53}$$

where

$$\Phi^{-1} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3, \hat{\phi}_4, \hat{\phi}_5)^\top, \quad \Psi^{-1} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4, \hat{\psi}_5)^\top. \tag{54}$$

In a similar way, the analytic properties of row vectors of $J_\pm^{-1}(x, \lambda)$ can be analyzed, and the inverse Jost solutions are derived as

$$P^- = e^{-i\lambda\Lambda x}(\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4, \hat{\phi}_5)^\top = H_1J_-^{-1} + H_2J_+^{-1}, \tag{55}$$

which are analytic in $\lambda \in \mathbb{C}_-$. When the λ approaches ∞, these analytical solutions are

$$P^\pm(x, t, \lambda) \to I_5, \quad \lambda \in \mathbb{C}_\pm \to \infty. \tag{56}$$

The relation between P^+ and P^- is given on the real line

$$P^-(x, t, \lambda)P^+(x, t, \lambda) = G(x, t, \lambda), \quad \lambda \in \mathbb{R}, \tag{57}$$

where

$$\begin{aligned} G &= (H_1J_-^{-1} + H_2J_+^{-1})(J_-H_1 + J_+H_2) \\ &= E \begin{pmatrix} 1 & 0 & 0 & 0 & s_{15} \\ 0 & 1 & 0 & 0 & s_{25} \\ 0 & 0 & 1 & 0 & s_{35} \\ 0 & 0 & 0 & 1 & s_{45} \\ \hat{s}_{51} & \hat{s}_{52} & \hat{s}_{53} & \hat{s}_{54} & 1 \end{pmatrix} E^{-1} \end{aligned} \tag{58}$$

and $S^{-1}(t, \lambda) = [\hat{s}_{ij}]$. Eq. (57) determines a matrix Riemann–Hilbert problem under the normalization condition (56).

Then, we consider the expanded form of P^+ as λ approaches ∞

$$P^+(x, t, \lambda) = I + \lambda^{-1}P_1^+(x, t) + O(\lambda^{-2}), \quad \lambda \to \infty. \tag{59}$$

Based on the reason that P^+ is the solution of Eq. (41), we substitute the above expansion (59) into Eq. (41) and compare coefficients of terms $O(1)$, and then the potential Q is derived as

$$Q = i[\Lambda, \ P_1^+]. \tag{60}$$

Hence u, v, $\sigma_1 u^*$, $\sigma_2 v^*$ can be constructed as

$$u = -2i(P_1^+)_{15} = -2i(P_1^+)_{52}, \quad \sigma_1 u^* = -2i(P_1^+)_{25} = -2i(P_1^+)_{51}, \tag{61}$$

$$v = -2i(P_1^+)_{35} = -2i(P_1^+)_{54}, \quad \sigma_2 v^* = -2i(P_1^+)_{45} = -2i(P_1^+)_{53}. \tag{62}$$

4. The N-soliton solutions and their dynamics

The determinants of $P^\pm$ are derived as

$$\det P^+(x, t, \lambda) = s_{55}(t, \lambda), \quad \det P^-(x, t, \lambda) = \hat{s}_{55}(t, \lambda), \tag{63}$$

which get through the definitions of $P^\pm$ and the scattering relation (48). Based on the different assumptions of zeros of $s_{55}(t, \lambda)$ and $\hat{s}_{55}(t, \lambda)$, we analyze the N-soliton solution of generalized coupled Sasa-Satsuma Eq. (1) in two cases. In the first case, we suppose $s_{55}(t, \lambda)$ and $\hat{s}_{55}(t, \lambda)$ have N simple zeros at $\lambda_{1,k} \in \mathbb{C}_+$ and $\bar{\lambda}_{1,k} \in \mathbb{C}_-(1 \le k \le N)$, respectively. In this case, $\lambda_{1,k}$ is pure imaginary, and we call this case as 'purely imaginary case'. In the second case, we assume that $s_{55}(t, \lambda)$ and $\hat{s}_{55}(t, \lambda)$ have 2N simple zeros at $\lambda_{2,k} \in \mathbb{C}_+$ and $\bar{\lambda}_{2,k} \in \mathbb{C}_-(1 \le k \le 2N)$, respectively. In this case, $\lambda_{2,k}$ has no restriction of pure imaginary, and we call the second case as 'imaginary case'.

4.1. The N-soliton solutions in purely imaginary case

From the above discussion, the zeros of $\det P^{+}$ and $\det P^{-}$ have been assumed. In this case, the kernels of $P^{+}(x,t,\lambda_{1,k})$ and $P^{-}(x,t,\bar{\lambda}_{1,k})$ contain only a single column vector $\boldsymbol{\omega}_{1,k}$ and row vector $\bar{\boldsymbol{\omega}}_{1,k}$, respectively,

$$P^{+}(x,t,\lambda_{1,k})\boldsymbol{\omega}_{1,k}=0,\quad \bar{\boldsymbol{\omega}}_{1,k}P^{-}(x,t,\bar{\lambda}_{1,k})=0,\quad 1\le k\le N, \tag{64}$$

where the vectors $\boldsymbol{\omega}_{1,k}$ and $\bar{\boldsymbol{\omega}}_{1,k}$ are (x,t)-dependent.

Combining the symmetry relation of the matrix Q in (43), these symmetry constraints of discrete scattering data are found to be

$$\bar{\lambda}_{1,k}=\lambda_{1,k}^{*},\quad \boldsymbol{\omega}_{1,k}^{\dagger}=\bar{\boldsymbol{\omega}}_{1,k}F^{-1},\quad S^{\dagger}(t,\lambda_{1,k})=FS^{-1}(t,\lambda_{1,k}^{*})F^{-1}. \tag{65}$$

Since P^{+} satisfies the Eqs. (41)–(42), the differential equations of $\boldsymbol{\omega}_{1,k}$ are given as

$$\frac{d\boldsymbol{\omega}_{1,k}}{dx}=-i\lambda_{1,k}\Lambda\boldsymbol{\omega}_{1,k},\quad \frac{d\boldsymbol{\omega}_{1,k}}{dt}=-4i\lambda_{1,k}^{3}\Lambda\boldsymbol{\omega}_{1,k}, \tag{66}$$

Solving the above Eq. (66) and noticing the relationship between $\boldsymbol{\omega}_{1,k}$ and $\bar{\boldsymbol{\omega}}_{1,k}$ in the second formula of (65), we obtain

$$\boldsymbol{\omega}_{1,k}=e^{-i\lambda_{1,k}\Lambda x-4i\lambda_{1,k}^{3}\Lambda t}\boldsymbol{\omega}_{1,k0},\quad \bar{\boldsymbol{\omega}}_{1,k}=\boldsymbol{\omega}_{1,k0}^{\dagger}Fe^{i\lambda_{1,k}^{*}\Lambda x+4i\lambda_{1,k}^{*3}\Lambda t}, \tag{67}$$

where $\boldsymbol{\omega}_{1,k0}$ is a constant column vector.

When the scattering coefficients $(s_{15},s_{25},s_{35},s_{45},\hat{s}_{51},\hat{s}_{52},\hat{s}_{53},\hat{s}_{54})=\mathbf{0}$ and $P^{-}P^{+}=I$, the Riemann–Hilbert problem can be solved explicitly. At this moment, the expression of the solution P^{+} is

$$P^{+}(x,t,\lambda)=I+\sum_{j,k=1}^{N}\frac{\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k}}{\lambda-\bar{\lambda}_{1,k}}, \tag{68}$$

where

$$M_{jk}=\frac{\bar{\boldsymbol{\omega}}_{1,j}\boldsymbol{\omega}_{1,k}}{\bar{\lambda}_{1,j}-\lambda_{1,k}},\quad 1\le j,k\le N. \tag{69}$$

From (59), (61) and (62), the N-soliton solution of the generalized coupled Sasa-Satsuma Eq. (1) is derived as follows

$$u=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{15}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{52}, \tag{70}$$

$$\sigma_{1}u^{*}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{25}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{51}, \tag{71}$$

$$v=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{35}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{54}, \tag{72}$$

$$\sigma_{2}v^{*}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{45}=-2i(\sum_{j,k=1}^{N}\boldsymbol{\omega}_{1,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{1,k})_{53}, \tag{73}$$

and it is obvious that the expressions (70) and (72) are equivalent to (71) and (73), respectively. The normative column eigenvector $\boldsymbol{\omega}_{1,k0}$ and row eigenvector $\boldsymbol{\omega}_{1,k0}^{\dagger}$ are given as

$$\boldsymbol{\omega}_{1,k0}=(a_{k},b_{k},c_{k},d_{k},1)^{\top},\quad \boldsymbol{\omega}_{1,k0}^{\dagger}=(a_{k}^{*},b_{k}^{*},c_{k}^{*},d_{k}^{*},1). \tag{74}$$

For simplification, the notations are given as follows

$$\theta_{1,k}=i\lambda_{1,k}x+4i\lambda_{1,k}^{3}t,\quad \theta_{1,k}^{*}=-i\lambda_{1,k}^{*}x-4i\lambda_{1,k}^{*3}t. \tag{75}$$

Then in this case, the N-soliton solutions of the generalized coupled Sasa-Satsuma Eq. (1) can be given as follows

$$\begin{aligned}\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix}&=-2i\sum_{j,k=1}^{N}\begin{pmatrix}a_{j}\\ c_{j}\end{pmatrix}e^{\theta_{1,j}-\theta_{1,k}^{*}}(M^{-1})_{jk}\\ &=-2i\sum_{j,k=1}^{N}\begin{pmatrix}\frac{1}{\sigma_{1}}b_{k}^{*}\\ \frac{1}{\sigma_{2}}d_{k}^{*}\end{pmatrix}e^{-\theta_{1,j}+\theta_{1,k}^{*}}(M^{-1})_{jk},\end{aligned} \tag{76}$$

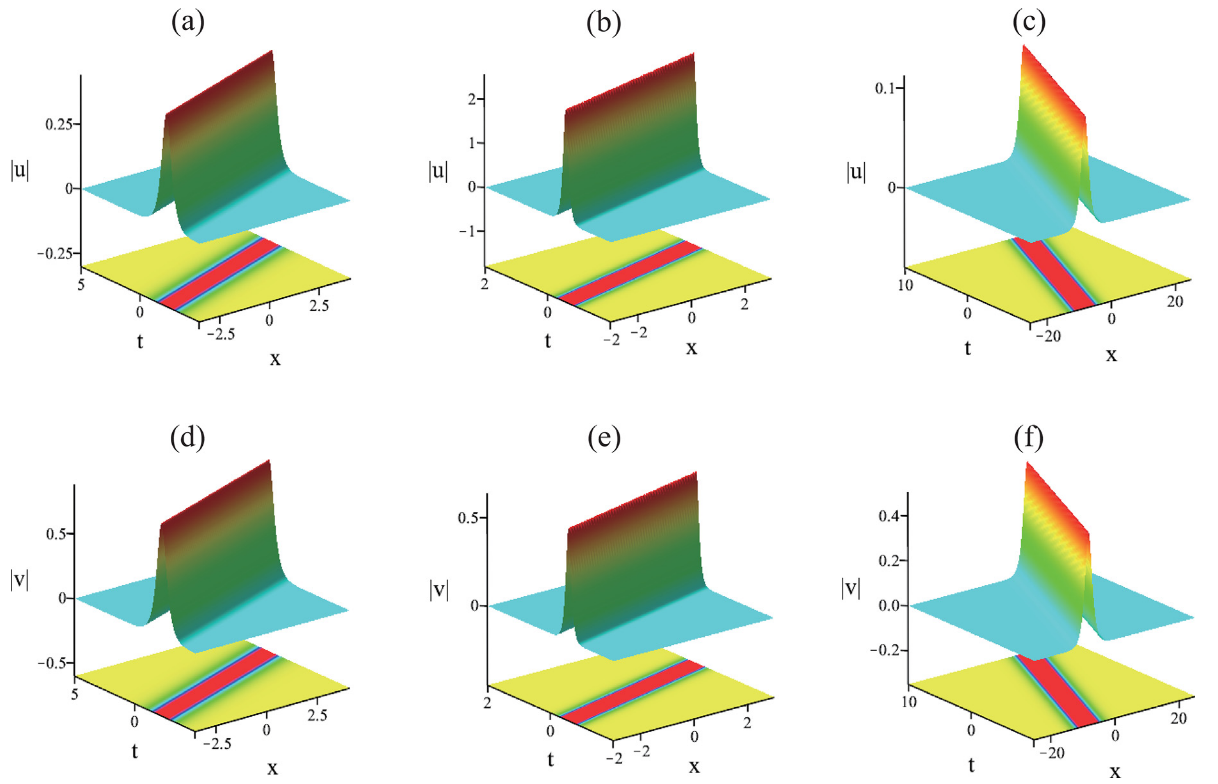

Fig. 1. Single-hump solution of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 0.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -1$, $\sigma_2 = 2$.

where

$$M_{jk} = \frac{(\sigma_1 a_j^* a_k + \frac{1}{\sigma_1} b_j^* b_k + \sigma_2 c_j^* c_k + \frac{1}{\sigma_2} d_j^* d_k) e^{\theta_{1,j}^* + \theta_{1,k}} + e^{-\theta_{1,j}^* - \theta_{1,k}}}{\lambda_{1,j}^* - \lambda_{1,k}}, \quad 1 \le j, k \le N, \tag{77}$$

and the expressions of b_k and d_k are derived from (76)

$$b_k = \sigma_1 a_j^* e^{2\theta_{1,j}^* - 2\theta_{1,k}}, \quad d_k = \sigma_2 c_j^* e^{2\theta_{1,j}^* - 2\theta_{1,k}}. \tag{78}$$

When $N = 1$, from (76) and (77), the single-soliton solution can be obtained with one purely imaginary eigenvalue $\lambda_{1,1}$ and its eigenvector $\boldsymbol{\omega}_{1,1}$. The single-soliton solution is

$$u = -2ie^{\theta_{1,1} - \theta_{1,1}^*} a_1 (M^{-1})_{11} = -ia_1(\lambda_{1,1}^* - \lambda_{1,1}) e^{\theta_{1,1} - \theta_{1,1}^* - \kappa} \operatorname{sech}(\theta_{1,1} + \theta_{1,1}^* + \kappa), \tag{79}$$

$$v = -2ie^{\theta_{1,1} - \theta_{1,1}^*} c_1 (M^{-1})_{11} = -ic_1(\lambda_{1,1}^* - \lambda_{1,1}) e^{\theta_{1,1} - \theta_{1,1}^* - \kappa} \operatorname{sech}(\theta_{1,1} + \theta_{1,1}^* + \kappa), \tag{80}$$

where

$$M_{11} = \frac{(\sigma_1 |a_1|^2 + \frac{1}{\sigma_1} |b_1|^2 + \sigma_2 |c_1|^2 + \frac{1}{\sigma_2} |d_1|^2) e^{\theta_{1,1}^* + \theta_{1,1}} + e^{-\theta_{1,1}^* - \theta_{1,1}}}{\lambda_{1,1}^* - \lambda_{1,1}}, \tag{81}$$

and

$$e^{2\kappa} = \sigma_1 |a_1|^2 + \frac{1}{\sigma_1} |b_1|^2 + \sigma_2 |c_1|^2 + \frac{1}{\sigma_2} |d_1|^2. \tag{82}$$

The figures of single-soliton solution are shown in Figs. 1–2. The single-hump solution and the breather-type solution are obtained by setting three different parameters of $(\lambda_{1,1}, a_1, c_1)$ in Fig. 1 $(0.7i, -i, 2i)$, $(1.2i, 2, 0.5)$, $(0.5i, 0.5+0.5i, 1+3i)$, and $(2.5i, -i, 1.5i)$, $(3i, -1.5i, 1.5i)$, $(2.5i, -i, 1.5i)$ in Fig. 2. The peak amplitudes remain unchanged of both single-hump solution and breather-type solution in Figs. 1–2. From the images, it is not difficult to find that the amplitude and direction of the wave propagation vary in different parameter values of σ_1 and σ_2.

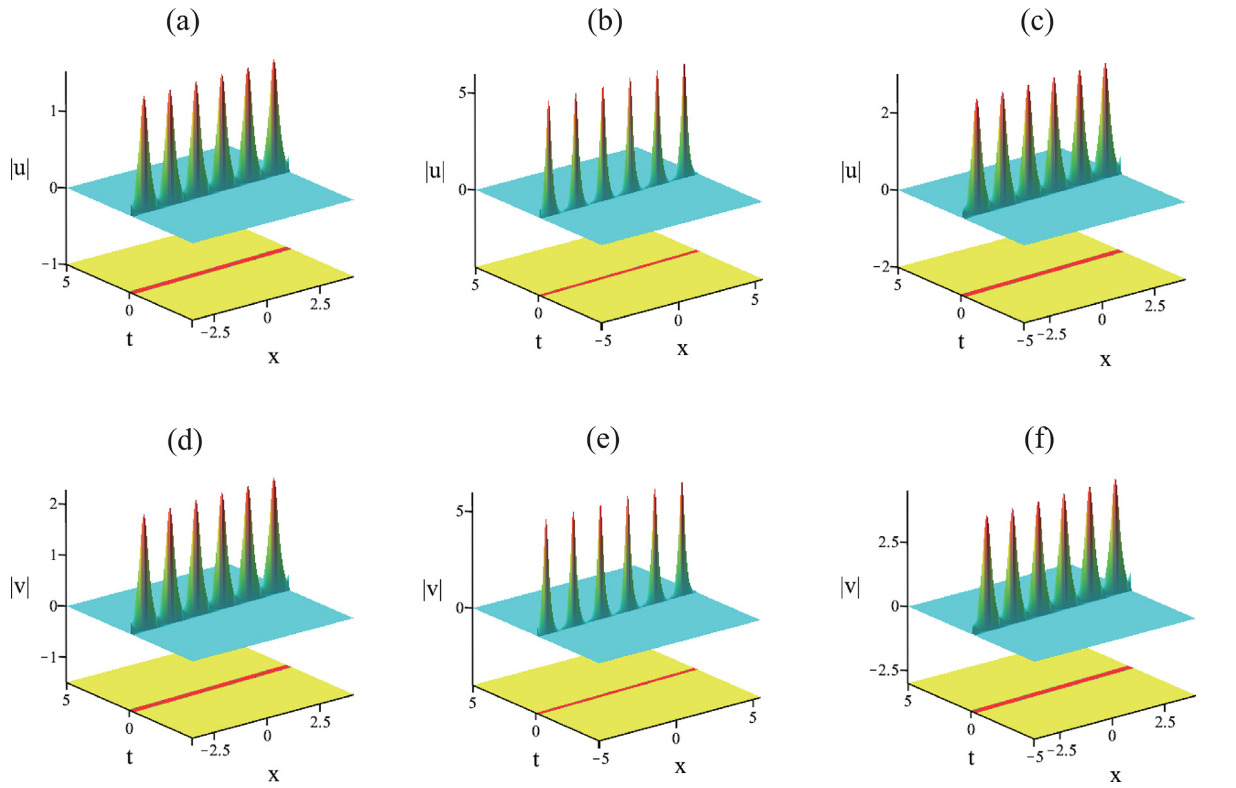

Fig. 2. The breather-type solution of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 2$, $\sigma_2 = 1.5$; (b) and (e) with parameters: $\sigma_1 = 1.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -2$, $\sigma_2 = 1.5$.

When $N = 2$, the two-soliton solution can be obtained with two purely imaginary eigenvalues $\lambda_{1,1}$, $\lambda_{1,2}$ and with their eigenvectors $\boldsymbol{\omega}_{1,1}$, $\boldsymbol{\omega}_{1,2}$, respectively. The two-soliton solution is derived as follows

$$u = -2i[e^{\theta_{1,1}-\theta_{1,1}^*}a_1(M^{-1})_{11} + e^{\theta_{1,1}-\theta_{1,2}^*}a_1(M^{-1})_{12} + e^{\theta_{1,2}-\theta_{1,1}^*}a_2(M^{-1})_{21} + e^{\theta_{1,2}-\theta_{1,2}^*}a_2(M^{-1})_{22}], \tag{83}$$

$$v = -2i[e^{\theta_{1,1}-\theta_{1,1}^*}c_1(M^{-1})_{11} + e^{\theta_{1,1}-\theta_{1,2}^*}c_1(M^{-1})_{12} + e^{\theta_{1,2}-\theta_{1,1}^*}c_2(M^{-1})_{21} + e^{\theta_{1,2}-\theta_{1,2}^*}c_2(M^{-1})_{22}], \tag{84}$$

where

$$\begin{aligned}
M_{11} &= \frac{(\sigma_1|a_1|^2 + \frac{1}{\sigma_1}|b_1|^2 + \sigma_2|c_1|^2 + \frac{1}{\sigma_2}|d_1|^2)e^{\theta_{1,1}^*+\theta_{1,1}} + e^{-\theta_{1,1}^*-\theta_{1,1}}}{\lambda_{1,1}^* - \lambda_{1,1}},\\
M_{12} &= \frac{(\sigma_1 a_1^* a_2 + \frac{1}{\sigma_1}b_1^* b_2 + \sigma_2 c_1^* c_2 + \frac{1}{\sigma_2}d_1^* d_2)e^{\theta_{1,1}^*+\theta_{1,2}} + e^{-\theta_{1,1}^*-\theta_{1,2}}}{\lambda_{1,1}^* - \lambda_{1,2}},\\
M_{21} &= \frac{(\sigma_1 a_2^* a_1 + \frac{1}{\sigma_1}b_2^* b_1 + \sigma_2 c_2^* c_1 + \frac{1}{\sigma_2}d_2^* d_1)e^{\theta_{1,2}^*+\theta_{1,1}} + e^{-\theta_{1,2}^*-\theta_{1,1}}}{\lambda_{1,2}^* - \lambda_{1,1}},\\
M_{22} &= \frac{(\sigma_1|a_2|^2 + \frac{1}{\sigma_1}|b_2|^2 + \sigma_2|c_2|^2 + \frac{1}{\sigma_2}|d_2|^2)e^{\theta_{1,2}^*+\theta_{1,2}} + e^{-\theta_{1,2}^*-\theta_{1,2}}}{\lambda_{1,2}^* - \lambda_{1,2}}.
\end{aligned} \tag{85}$$

The figures of two-soliton solution are shown in Figs. 3–4. The oblique elastic collision behaviors for two single-hump solutions are described in Fig. 3, and the parameters $(\lambda_{1,1}, \lambda_{1,2}, a_1, a_2, c_1, c_2)$ of the figures are $(0.65i, 0.49i, 0.25 + 0.5i, 1.2, 0.5 + 0.25i, 0.5 + 0.25i)$, $(0.65i, 0.49i, 1 + 0.5i, 0.25 + 0.5i, 0.7 + 0.7i, 0.5 + 0.25i)$ and $(0.65i, 0.49i, 0.6, 1.2 + 0.25i, 1 + i, 1.7 + 2i)$, respectively. Based on the two single-hump solutions remaining their individual intensities and velocities after the interaction, the images in Fig. 3 display the elastic interaction, and there is no energy exchange between two different solitons. Elastic collisions between a single-hump soliton and a breather of generalized coupled

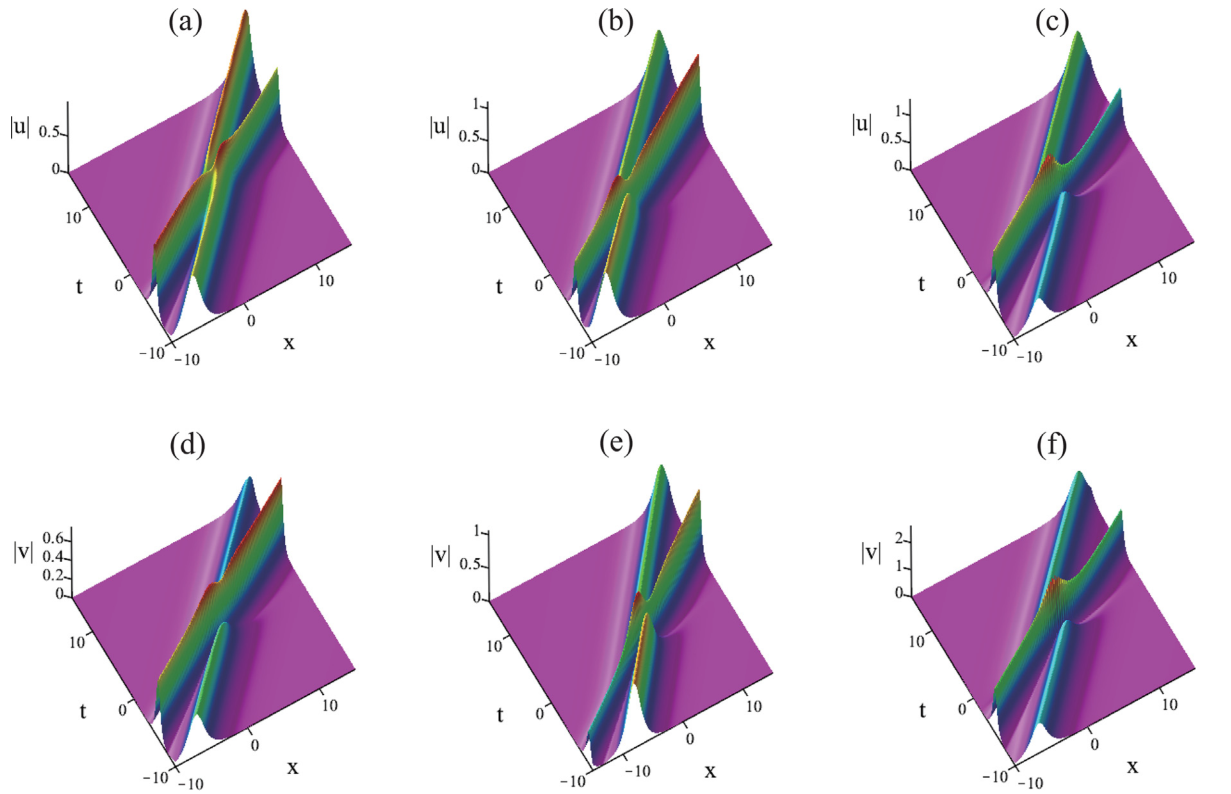

Fig. 3. Oblique elastic collision behaviors for two single-hump solutions of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 0.5$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -0.5$, $\sigma_2 = 0.5$.

Sasa-Satsuma equation are shown in Fig. 4, and the three different parameters of $(\lambda_{1,1}, \lambda_{1,2}, a_1, a_2, c_1, c_2)$ for the figures are $(0.6i, 1.9i, 0.25, 1.2+0.25i, 0.7+0.7i, 0.7+2i)$, $(1.5i, 0.5i, 0.5i, 0.25, 1+i, 0.7+0.7i)$ and $(0.6i, 1.9i, 0.25, 1.2+0.25i, 0.7+0.7i, 1.7+2i)$. It could be observed that the amplitudes of two solitons change after interaction, so energy exchange may take place between some interacting soliton and cause the shape change of such soliton after interaction.

4.2. The N-soliton solutions in imaginary case

Since the potential matrix Q satisfies not only the symmetry relation (43) but also

$$Q^* = F^{-1}F_1QF_1F, \tag{86}$$

where

$$F_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{87}$$

there are some symmetry constraints of $J_\pm$ and S

$$J_\pm^*(-\lambda^*) = F^{-1}F_1J(\lambda)F_1F, \tag{88}$$

$$S^*(-\lambda^*) = F^{-1}F_1S(\lambda)F_1F. \tag{89}$$

From the Eq. (89), we know if λ_k is the zero of $s_{55}(t,\lambda)$, $-\lambda_k^*$ is also the zero of $s_{55}(t,\lambda)$. So, we suppose $s_{55}(t,\lambda)$ has 2N simple zeros at $\lambda_{2,k} \in \mathbb{C}_+$, and $\hat{s}_{55}(t,\lambda)$ has 2N simple zeros at $\bar{\lambda}_{2,k} \in \mathbb{C}_-$. In this case, $\lambda_{2,k}$ satisfies the following conditions

$$\begin{aligned} &(a)\ \bar{\lambda}_{2,k} = \lambda_{2,k}^*, \quad 1 \le k \le 2N, \\ &(b)\ \lambda_{2,k+N} = -\lambda_{2,k}^*, \quad 1 \le k \le N. \end{aligned} \tag{90}$$

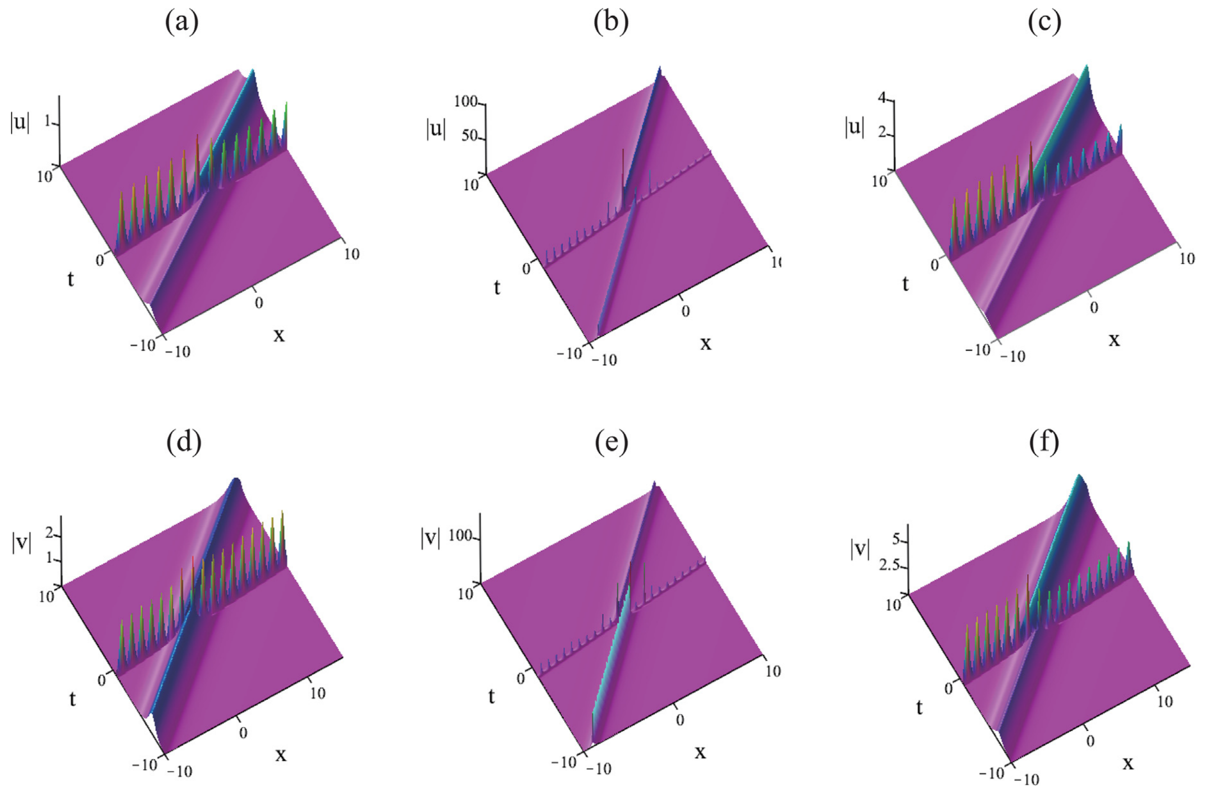

Fig. 4. Elastic collisions between a single-hump soliton and a breather of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 0.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -2$, $\sigma_2 = 1$.

The kernels of $P^+(x, t, \lambda_{2,k})$ and $P^-(x, t, \bar{\lambda}_{2,k})$ contain only a single column vector $\boldsymbol{\omega}_{2,k}$ and row vector $\bar{\boldsymbol{\omega}}_{2,k}$, respectively,

$$P^+(x, t, \lambda_{2,k})\boldsymbol{\omega}_{2,k} = 0, \quad \bar{\boldsymbol{\omega}}_{2,k}P^-(x, t, \bar{\lambda}_{2,k}) = 0, \quad 1 \le k \le 2N. \tag{91}$$

And $\boldsymbol{\omega}_{2,k}$ satisfies the following conditions

$$\begin{aligned} &(a)\ \bar{\boldsymbol{\omega}}_{2,k} = \boldsymbol{\omega}_{2,k}^{\dagger}F, \quad 1 \le k \le 2N, \\ &(b)\ \boldsymbol{\omega}_{2,k+N} = F_1F\boldsymbol{\omega}_{2,k}^{*}, \quad 1 \le k \le N. \end{aligned} \tag{92}$$

In general, the matrix Riemann–Hilbert problem (57) could yield explicit analytical solutions under the normalization condition (56). If $P^-P^+ = I$, the explicit solution P^+ is given as

$$P^+(x, t, \lambda) = I + \sum_{j,k=1}^{2N} \frac{\boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k}}{\lambda - \bar{\lambda}_{2,k}}, \tag{93}$$

where

$$M_{jk} = \frac{\bar{\boldsymbol{\omega}}_{2,j}\boldsymbol{\omega}_{2,k}}{\bar{\lambda}_{2,j} - \lambda_{2,k}}, \quad 1 \le j, k \le 2N. \tag{94}$$

From (59), (61) and (62), the N-soliton solution of the two-component Sasa-Satsuma Eq. (1) is derived as follows

$$u = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{15} = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{52}, \tag{95}$$

$$\sigma_1 u^* = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{25} = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{51}, \tag{96}$$

$$v = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{35} = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{54}, \tag{97}$$

$$\sigma_2 v^* = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{45} = -2i(\sum_{j,k=1}^{2N} \boldsymbol{\omega}_{2,j}(M^{-1})_{jk}\bar{\boldsymbol{\omega}}_{2,k})_{53}, \tag{98}$$

and it is obvious that the expressions (95) and (97) are equivalent to (96) and (98), respectively.

In order to give the precise expressions for the N-soliton solutions of the two-component Sasa-Satsuma equation, the normative column eigenvector $\boldsymbol{\omega}_{2,k0}$ and row eigenvector $\boldsymbol{\omega}_{2,k0}^{\dagger}$ are given as

$$\boldsymbol{\omega}_{2,k0} = (a_k, b_k, c_k, d_k, 1)^{\top}, \quad \boldsymbol{\omega}_{2,k0}^{\dagger} = (a_k^*, b_k^*, c_k^*, d_k^*, 1). \tag{99}$$

For simplification, the notations are introduced as follows

$$\theta_{2,k} = i\lambda_{2,k}x + 4i\lambda_{2,k}^3 t, \quad \theta_{2,k}^* = -i\lambda_{2,k}^* x - 4i\lambda_{2,k}^{*3} t. \tag{100}$$

Then, the expressions are derived as

$$\begin{aligned} u =& -2i[\sum_{j,k=1}^{N} a_j e^{\theta_{2,j}-\theta_{2,k}^*}(M^{-1})_{jk} + \sum_{j=1}^{N}\sum_{k=N+1}^{2N} a_j e^{\theta_{2,j}-\theta_{2,k-N}}(M^{-1})_{jk} + \\ & \sum_{j=N+1}^{2N}\sum_{k=1}^{N} \frac{1}{\sigma_1} b_{j-N}^* e^{\theta_{2,j-N}^*-\theta_{2,k}^*}(M^{-1})_{jk} + \sum_{j,k=N+1}^{2N} \frac{1}{\sigma_1} b_{j-N}^* e^{\theta_{2,j-N}^*-\theta_{2,k-N}}(M^{-1})_{jk}], \end{aligned} \tag{101}$$

$$\begin{aligned} v =& -2i[\sum_{j,k=1}^{N} c_j e^{\theta_{2,j}-\theta_{2,k}^*}(M^{-1})_{jk} + \sum_{j=1}^{N}\sum_{k=N+1}^{2N} c_j e^{\theta_{2,j}-\theta_{2,k-N}}(M^{-1})_{jk} + \\ & \sum_{j=N+1}^{2N}\sum_{k=1}^{N} \frac{1}{\sigma_2} d_{j-N}^* e^{\theta_{2,j-N}^*-\theta_{2,k}^*}(M^{-1})_{jk} + \sum_{j,k=N+1}^{2N} \frac{1}{\sigma_2} d_{j-N}^* e^{\theta_{2,j-N}^*-\theta_{2,k-N}}(M^{-1})_{jk}], \end{aligned} \tag{102}$$

where M_{jk} satisfies the following conditions

$$\begin{aligned} &(a)\, 1 \le j \le N,\ \ 1 \le k \le N \\ &M_{jk} = \frac{\boldsymbol{\omega}_{2,j0}^{\dagger} F e^{i\lambda_{2,j}^* \Lambda x + 4i\lambda_{2,j}^{*3}\Lambda t} e^{-i\lambda_{2,k}\Lambda x - 4i\lambda_{2,k}^3 \Lambda t} \boldsymbol{\omega}_{2,k0}}{\lambda_{2,j}^* - \lambda_{2,k}}, \end{aligned} \tag{103}$$

$$\begin{aligned} &(b)\, 1 \le j \le N,\ \ N+1 \le k \le 2N \\ &M_{jk} = \frac{\boldsymbol{\omega}_{2,j0}^{\dagger} F e^{i\lambda_{2,j}^* \Lambda x + 4i\lambda_{2,j}^{*3}\Lambda t} F_1 F e^{i\lambda_{2,k-N}^* \Lambda x + 4i\lambda_{2,k-N}^{*3} \Lambda t} \boldsymbol{\omega}_{2,k-N0}}{\lambda_{2,j}^* + \lambda_{2,k-N}^*}, \end{aligned} \tag{104}$$

$$\begin{aligned} &(c)\, N+1 \le j \le 2N,\ \ 1 \le k \le N \\ &M_{jk} = \frac{\boldsymbol{\omega}_{2,j-N0}^{\top} e^{-i\lambda_{2,j-N}\Lambda x - 4i\lambda_{2,j-N}^3 \Lambda t} F F_1 F e^{-i\lambda_{2,k}\Lambda x - 4i\lambda_{2,k}^3 \Lambda t} \boldsymbol{\omega}_{2,k0}}{-\lambda_{2,j-N} - \lambda_{2,k}}, \end{aligned} \tag{105}$$

$$\begin{aligned} &(d)\, N+1 \le j \le 2N,\ \ N+1 \le k \le 2N \\ &M_{jk} = \frac{\boldsymbol{\omega}_{2,j-N0}^{\top} e^{-i\lambda_{2,j-N}\Lambda x - 4i\lambda_{2,j-N}^3 \Lambda t} F F_1 F F_1 F e^{i\lambda_{2,k-N}^* \Lambda x + 4i\lambda_{2,k-N}^{*3} \Lambda t} \boldsymbol{\omega}_{2,k-N0}}{-\lambda_{2,j-N} + \lambda_{2,k-N}^*}. \end{aligned} \tag{106}$$

When $N = 1$, from (101)–(106), the single-soliton solution can be obtained with an imaginary eigenvalue $\lambda_{2,1}$ and an eigenvector $\boldsymbol{\omega}_{2,1}$. The single-soliton solution is

$$u = -2i[a_1 e^{\theta_{2,1}-\theta_{2,1}^*}(M^{-1})_{11} + a_1 (M^{-1})_{12} + \frac{1}{\sigma_1} b_1^* (M^{-1})_{21} + \frac{1}{\sigma_1} b_1^* e^{\theta_{2,1}^*-\theta_{2,1}}(M^{-1})_{22}], \tag{107}$$

$$v = -2i[c_1 e^{\theta_{2,1}-\theta_{2,1}^*}(M^{-1})_{11} + c_1 (M^{-1})_{12} + \frac{1}{\sigma_2} d_1^* (M^{-1})_{21} + \frac{1}{\sigma_2} d_1^* e^{\theta_{2,1}^*-\theta_{2,1}}(M^{-1})_{22}], \tag{108}$$

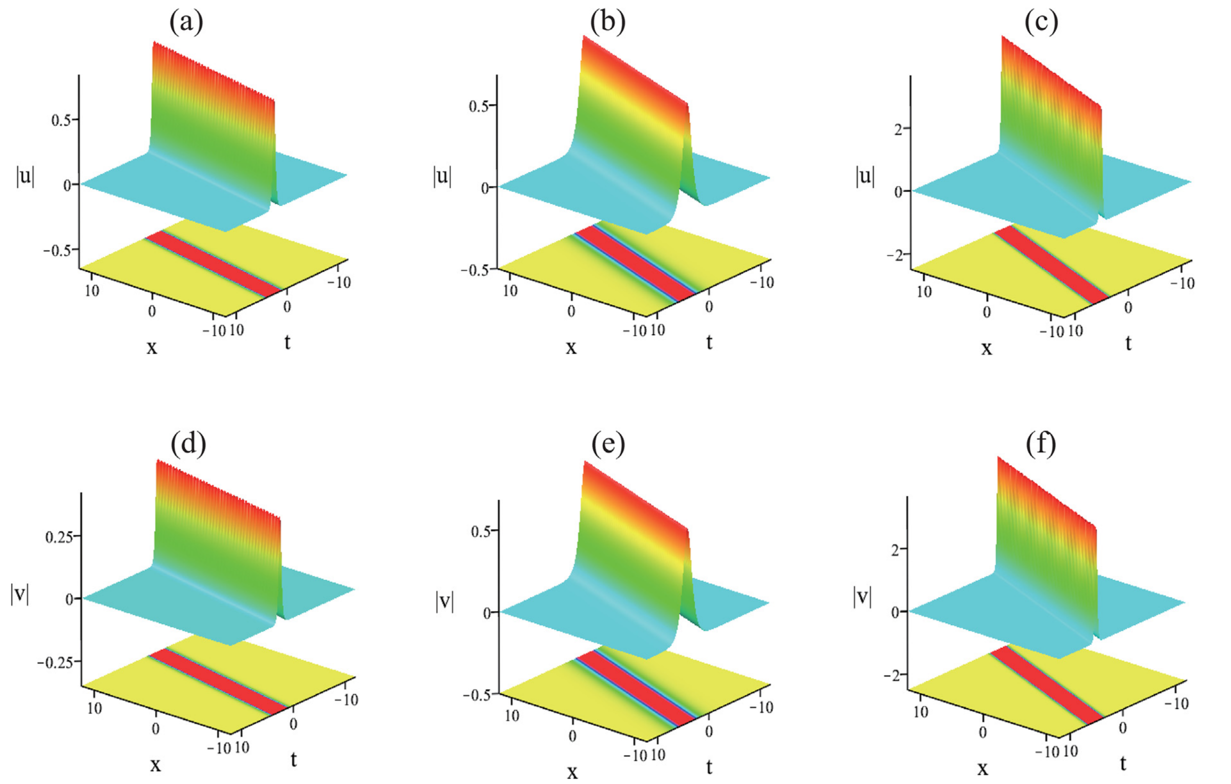

Fig. 5. Single-hump solution of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1$, $\sigma_2 = -0.5$; (c) and (f) with parameters: $\sigma_1 = -0.5$, $\sigma_2 = 1$.

where

$$
\begin{aligned}
M_{11} &= \frac{(\sigma_1|a_1|^2 + \frac{1}{\sigma_1}|b_1|^2 + \sigma_2|c_1|^2 + \frac{1}{\sigma_2}|d_1|^2)e^{\theta_{2,1}+\theta_{2,1}^*} + e^{-\theta_{2,1}-\theta_{2,1}^*}}{\lambda_{2,1}^* - \lambda_{2,1}}, \\
M_{12} &= \frac{(2a_1^*b_1^* + 2c_1^*d_1^*)e^{2\theta_{2,1}^*} + e^{-2\theta_{2,1}^*}}{2\lambda_{2,1}^*}, \\
M_{21} &= \frac{(2a_1b_1 + 2c_1d_1)e^{2\theta_{2,1}} + e^{-2\theta_{2,1}}}{-2\lambda_{2,1}}, \\
M_{22} &= \frac{(\sigma_1|a_1|^2 + \frac{1}{\sigma_1}|b_1|^2 + \sigma_2|c_1|^2 + \frac{1}{\sigma_2}|d_1|^2)e^{\theta_{2,1}+\theta_{2,1}^*} + e^{-\theta_{2,1}-\theta_{2,1}^*}}{\lambda_{2,1}^* - \lambda_{2,1}}.
\end{aligned}
\tag{109}
$$

The figures of single-soliton solution are shown in Figs. 5–7. The single-hump solution, the breather-type solution and the double-hump solution are given in the following figures, which are characterized by seven involved parameters of $\lambda_{2,1}$, a_1, b_1, c_1, d_1, σ_1, σ_2. In Fig. 5, the three different sets of parameters of $(\lambda_{2,1}, a_1, b_1, c_1, d_1)$ are $(0.7 + 0.5i, 0, 1, 0.5, 0)$, $(0.5 + 0.25i, 1, 0, 1, 0)$ and $(1 + 1.55i, 1, 0, 1, 0)$, respectively. In Fig. 6, the three different sets of parameters of $(\lambda_{2,1}, a_1, b_1, c_1, d_1)$ are $(0.7+0.5i, 0.5+i, 1+i, -1+0.15i, 0.5+0.7i)$, $(0.7+0.5i, 0.5+i, 1+i, -1+0.15i, 0.5+0.2i)$ and $(0.22 + 0.42i, 0.5 + 0.5i, 0, 0.3 + 0.5i, 1 + i)$, respectively. In Fig. 7, the parameters of $(\lambda_{2,1}, a_1, b_1, c_1, d_1)$ are all $(0.1 + 0.5i, 0.5 + 1.5i, 0, 1 + i, 0)$. The peak amplitudes of three types of single-soliton solution remain unchanged. For the double-hump soliton, they remain separated between two humps during the propagation, and they are not affected by time position shifts arising from intra-channel interaction in high bit-rate systems [48].

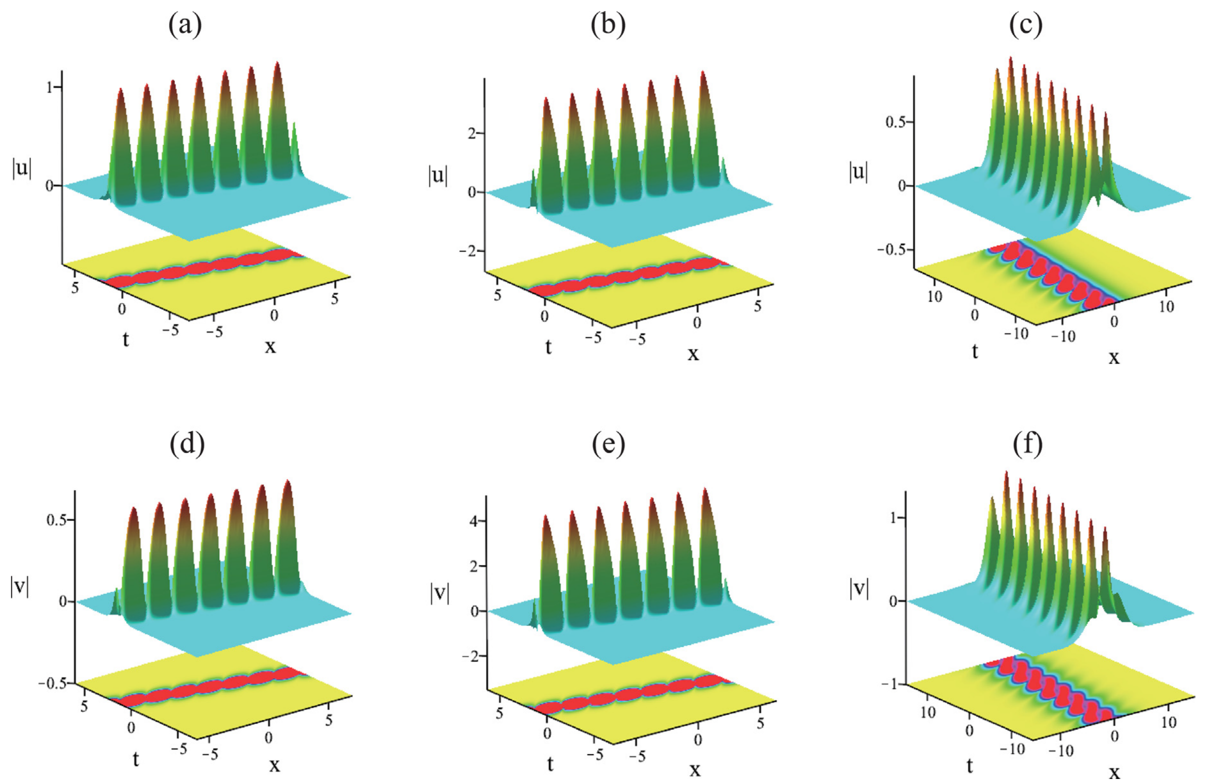

Fig. 6. The breather-type solution of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1$, $\sigma_2 = -0.5$; (c) and (f) with parameters: $\sigma_1 = -1$, $\sigma_2 = 1$.

When $N = 2$, the two-soliton solution can be obtained with two imaginary eigenvalues $\lambda_{2,1}$, $\lambda_{2,2}$ and with their eigenvectors $\boldsymbol{\omega}_{2,1}$ and $\boldsymbol{\omega}_{2,2}$, respectively. The two-soliton solution is derived as

$$\begin{aligned} u = & -2i[a_1 e^{\theta_{2,1}-\theta_{2,1}^*}(M^{-1})_{11} + a_1 e^{\theta_{2,1}-\theta_{2,2}^*}(M^{-1})_{12} + a_1(M^{-1})_{13} + a_1 e^{\theta_{2,1}-\theta_{2,2}}(M^{-1})_{14} \\ & + a_2 e^{\theta_{2,2}-\theta_{2,1}^*}(M^{-1})_{21} + a_2 e^{\theta_{2,2}-\theta_{2,2}^*}(M^{-1})_{22} + a_2 e^{\theta_{2,2}-\theta_{2,1}}(M^{-1})_{23} + a_2(M^{-1})_{24} \\ & + \frac{1}{\sigma_1} b_1^*(M^{-1})_{31} + \frac{1}{\sigma_1} b_1^* e^{\theta_{2,1}^*-\theta_{2,2}^*}(M^{-1})_{32} + \frac{1}{\sigma_1} b_1^* e^{\theta_{2,1}^*-\theta_{2,1}}(M^{-1})_{33} + \frac{1}{\sigma_1} b_1^* e^{\theta_{2,1}^*-\theta_{2,2}} \\ & (M^{-1})_{34} + \frac{1}{\sigma_1} b_2^* e^{\theta_{2,2}^*-\theta_{2,1}^*}(M^{-1})_{41} + \frac{1}{\sigma_1} b_2^*(M^{-1})_{42} + \frac{1}{\sigma_1} b_2^* e^{\theta_{2,2}^*-\theta_{2,1}}(M^{-1})_{43} \\ & + \frac{1}{\sigma_1} b_2^* e^{\theta_{2,2}^*-\theta_{2,2}}(M^{-1})_{44}], \end{aligned} \tag{110}$$

$$\begin{aligned} v = & -2i[c_1 e^{\theta_{2,1}-\theta_{2,1}^*}(M^{-1})_{11} + c_1 e^{\theta_{2,1}-\theta_{2,2}^*}(M^{-1})_{12} + c_1(M^{-1})_{13} + c_1 e^{\theta_{2,1}-\theta_{2,2}}(M^{-1})_{14} \\ & + c_2 e^{\theta_{2,2}-\theta_{2,1}^*}(M^{-1})_{21} + c_2 e^{\theta_{2,2}-\theta_{2,2}^*}(M^{-1})_{22} + c_2 e^{\theta_{2,2}-\theta_{2,1}}(M^{-1})_{23} + c_2(M^{-1})_{24} \\ & + \frac{1}{\sigma_2} d_1^*(M^{-1})_{31} + \frac{1}{\sigma_2} d_1^* e^{\theta_{2,1}^*-\theta_{2,2}^*}(M^{-1})_{32} + \frac{1}{\sigma_2} d_1^* e^{\theta_{2,1}^*-\theta_{2,1}}(M^{-1})_{33} + \frac{1}{\sigma_2} d_1^* e^{\theta_{2,1}^*-\theta_{2,2}} \\ & (M^{-1})_{34} + \frac{1}{\sigma_2} d_2^* e^{\theta_{2,2}^*-\theta_{2,1}^*}(M^{-1})_{41} + \frac{1}{\sigma_2} d_2^*(M^{-1})_{42} + \frac{1}{\sigma_2} d_2^* e^{\theta_{2,2}^*-\theta_{2,1}}(M^{-1})_{43} \\ & + \frac{1}{\sigma_2} d_2^* e^{\theta_{2,2}^*-\theta_{2,2}}(M^{-1})_{44}], \end{aligned} \tag{111}$$

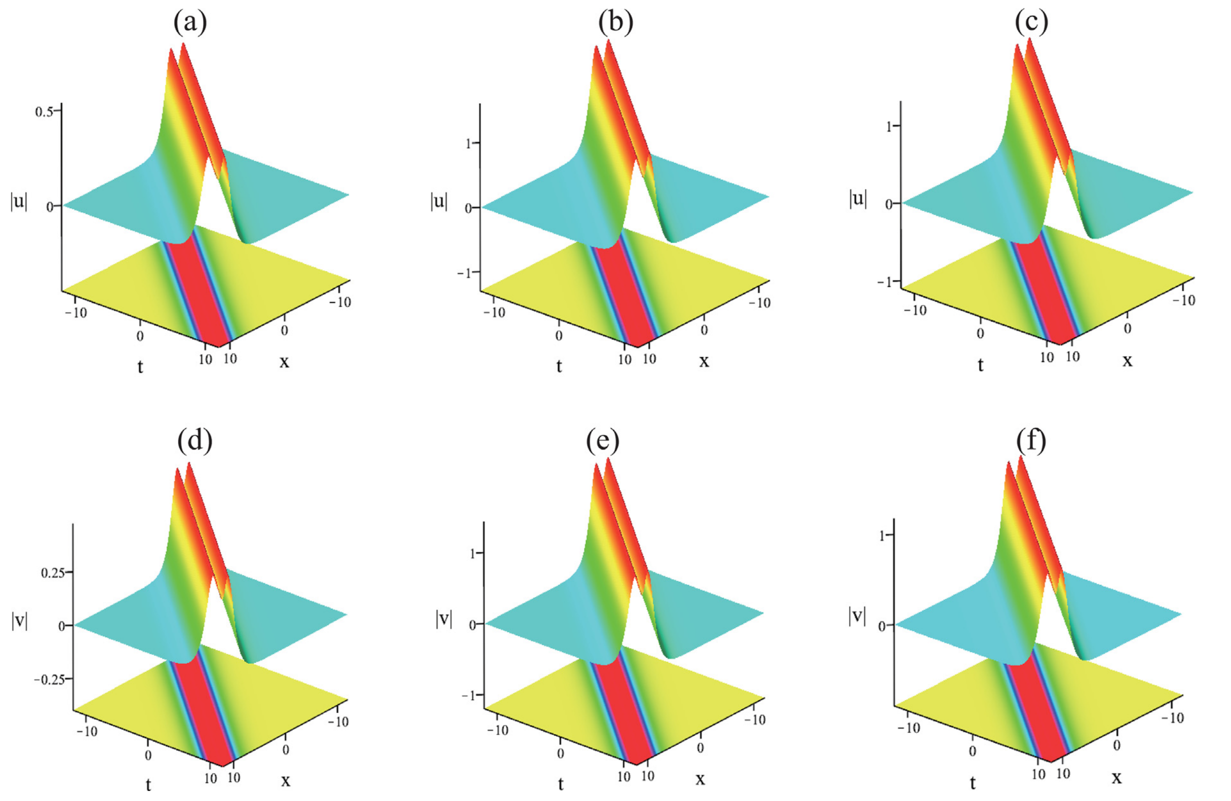

Fig. 7. Double-hump solution of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -0.5$, $\sigma_2 = 1$.

where

$$\begin{aligned}
M_{11} = M_{33} &= \frac{(\sigma_1|a_1|^2 + \frac{1}{\sigma_1}|b_1|^2 + \sigma_2|c_1|^2 + \frac{1}{\sigma_2}|d_1|^2)e^{\theta_{2,1}+\theta_{2,1}^*} + e^{-\theta_{2,1}-\theta_{2,1}^*}}{\lambda_{2,1}^* - \lambda_{2,1}},\\
M_{12} = M_{43} &= \frac{(\sigma_1 a_1^* a_2 + \frac{1}{\sigma_1} b_1^* b_2 + \sigma_2 c_1^* c_2 + \frac{1}{\sigma_2} d_1^* d_2)e^{\theta_{2,1}^*+\theta_{2,2}} + e^{-\theta_{2,1}^*-\theta_{2,2}}}{\lambda_{2,1}^* - \lambda_{2,2}},\\
M_{13} = -M_{31}^* &= \frac{(2a_1^* b_1^* + 2c_1^* d_1^*)e^{2\theta_{2,1}^*} + e^{-2\theta_{2,1}^*}}{2\lambda_{2,1}^*},\\
M_{14} = M_{23} &= \frac{(a_2^* b_1^* + a_1^* b_2^* + d_1^* c_2^* + c_1^* d_2^*)e^{\theta_{2,1}^*+\theta_{2,2}^*} + e^{-\theta_{2,1}^*-\theta_{2,2}^*}}{\lambda_{2,1}^* + \lambda_{2,2}^*},\\
M_{21} = M_{34} &= \frac{(\sigma_1 a_1 a_2^* + \frac{1}{\sigma_1} b_1 b_2^* + \sigma_2 c_1 c_2^* + \frac{1}{\sigma_2} d_1 d_2^*)e^{\theta_{2,1}+\theta_{2,2}^*} + e^{-\theta_{2,1}-\theta_{2,2}^*}}{\lambda_{2,2}^* - \lambda_{2,1}}\\
M_{22} = M_{44} &= \frac{(\sigma_1|a_2|^2 + \frac{1}{\sigma_1}|b_2|^2 + \sigma_2|c_2|^2 + \frac{1}{\sigma_2}|d_2|^2)e^{\theta_{2,2}+\theta_{2,2}^*} + e^{-\theta_{2,2}-\theta_{2,2}^*}}{\lambda_{2,2}^* - \lambda_{2,2}},\\
M_{24} = -M_{42}^* &= \frac{(2a_2^* b_2^* + 2c_2^* d_2^*)e^{2\theta_{2,2}^*} + e^{-2\theta_{2,2}^*}}{2\lambda_{2,2}^*},\\
M_{32} = M_{41} &= -\frac{(a_2 b_1 + a_1 b_2 + d_1 c_2 + c_1 d_2)e^{\theta_{2,1}+\theta_{2,2}} + e^{-\theta_{2,1}-\theta_{2,2}}}{\lambda_{2,1} + \lambda_{2,2}}.
\end{aligned} \tag{112}$$

The figures of two-soliton solution are shown in Figs. 8–12. Choosing parameters $\lambda_{2,1} = 1 + 2i$, $\lambda_{2,2} = 1 + 1.5i$, $a_1 = a_2 = c_1 = c_2 = 1$ and $b_1 = b_2 = d_1 = d_2 = 0$, the elastic collision behaviors for two single-hump solutions of the generalized coupled Sasa-Satsuma Eq. (1) are given in Fig. 8. Polarization-changing collisions between two single-hump

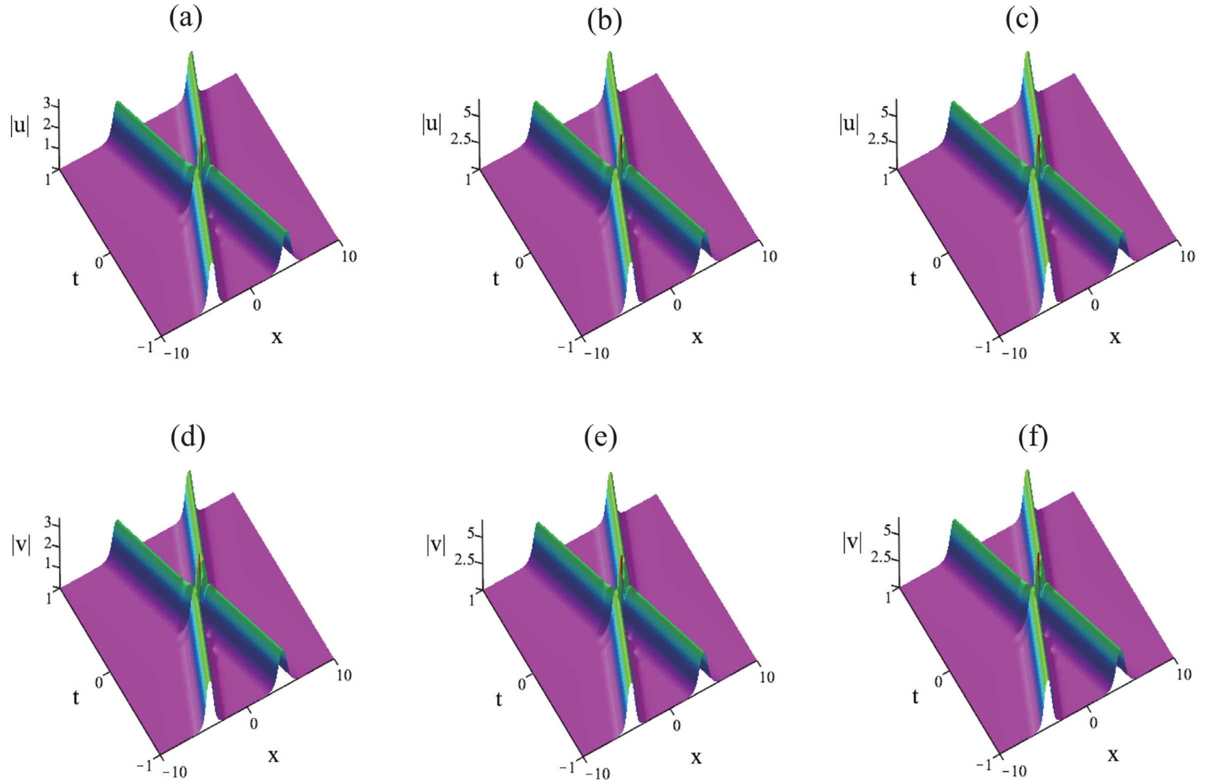

Fig. 8. Elastic collision behaviors for two single-hump solutions of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -0.5$, $\sigma_2 = 1$.

solutions are obtained through choosing the parameters $(1+2i, 1+i, 1, 1, 0, 0, 1, 0, 0, 0)$, $(1+2i, 1+i, 0, 1, 1, 0, 0, 1, 0, 0)$, $(1 + 2i, 1 + i, 1, 1, 0, 0, 0, 1, 0, 0)$ of $(\lambda_{2,1}, \lambda_{2,2}, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2)$ in Fig. 9. Choosing parameters $\lambda_{2,1} = 1 + i$, $\lambda_{2,2} = i$, $a_1 = a_2 = c_1 = c_2 = 1$ and $b_1 = b_2 = d_1 = d_2 = 0$, the collisions between two single-hump solutions are given in Fig. 10. Elastic collisions between a single-hump soliton and a breather are obtained through choosing the parameters $(1 + i, 0.5i, 1, 1, 1, 1, 1, 1, 0, 0)$, $(1 + i, 0.5i, 1, 1, 1, 1, 1, 1, 0, 0)$, $(1 + i, 0.5i, 1, 1, 0, 0, 1, 1, 1, 1)$ of $(\lambda_{2,1}, \lambda_{2,2}, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2)$ in Fig. 11. Elastic collisions between two breather-type solutions are obtained through choosing the parameters $(0.8 + 0.5i, 1.2 + 0.5i, 1, 1, 1, 0, 1, 1, 0, 0)$, $(0.8 + 0.5i, 1.2 + 0.5i, 1, 1, 1, 0, 1, 1, 0, 0)$, $(0.8 + 0.5i, 1.2 + 0.5i, 1, 1, 0, 0, 1, 1, 1, 0)$ of $(\lambda_{2,1}, \lambda_{2,2}, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2)$ in Fig. 12. For the elastic interactions between one (or two) single-hump solution(s) and one (two) breather-type solution(s) of generalized coupled Sasa-Satsuma equation, the individual solitons remain their individual intensities and velocities after the interaction in Figs. 8, 11 and 12. The polarization-changing collisions could contribute to the enhancement of intensity in one soliton, and the intensity of the other soliton is suppressed in Fig. 9. In Fig. 10, one of single-hump solutions transforms into a breather after the interaction, and their density evolution admits a periodic oscillation behavior.

5. Asymptotic analysis

After the solitons collide, there exists possibility of soliton's shape restoration. Asymptotic analyses are made to investigate the elastic and inelastic interactions between two-soliton solutions. Now, we consider the asymptotic analysis of two-soliton solutions of the generalized coupled Sasa-Satsuma Eq. (1) in the pure imaginary case. First, we suppose that

$$\lambda_{1,k} = \xi_{1,k} + i\eta_{1,k}, \quad k = 1, 2. \tag{113}$$

Then the $\mathrm{Re}(\theta_{1,1})$ and $\mathrm{Re}(\theta_{1,2})$ can be derived as follows

$$\mathrm{Re}(\theta_{1,1}) = -\eta_{1,1}[x + (12\xi_{1,1}^2 - 4\eta_{1,1}^2)t], \tag{114}$$

$$\begin{aligned}\mathrm{Re}(\theta_{1,2}) &= -\eta_{1,2}[x + (12\xi_{1,2}^2 - 4\eta_{1,2}^2)t] \\ &= \frac{\eta_{1,2}}{\eta_{1,1}}\,\mathrm{Re}(\theta_{1,1}) + \eta_2[(12\xi_{1,1}^2 - 4\eta_{1,1}^2) - (12\xi_{1,2}^2 - 4\eta_{1,2}^2)]t.\end{aligned} \tag{115}$$

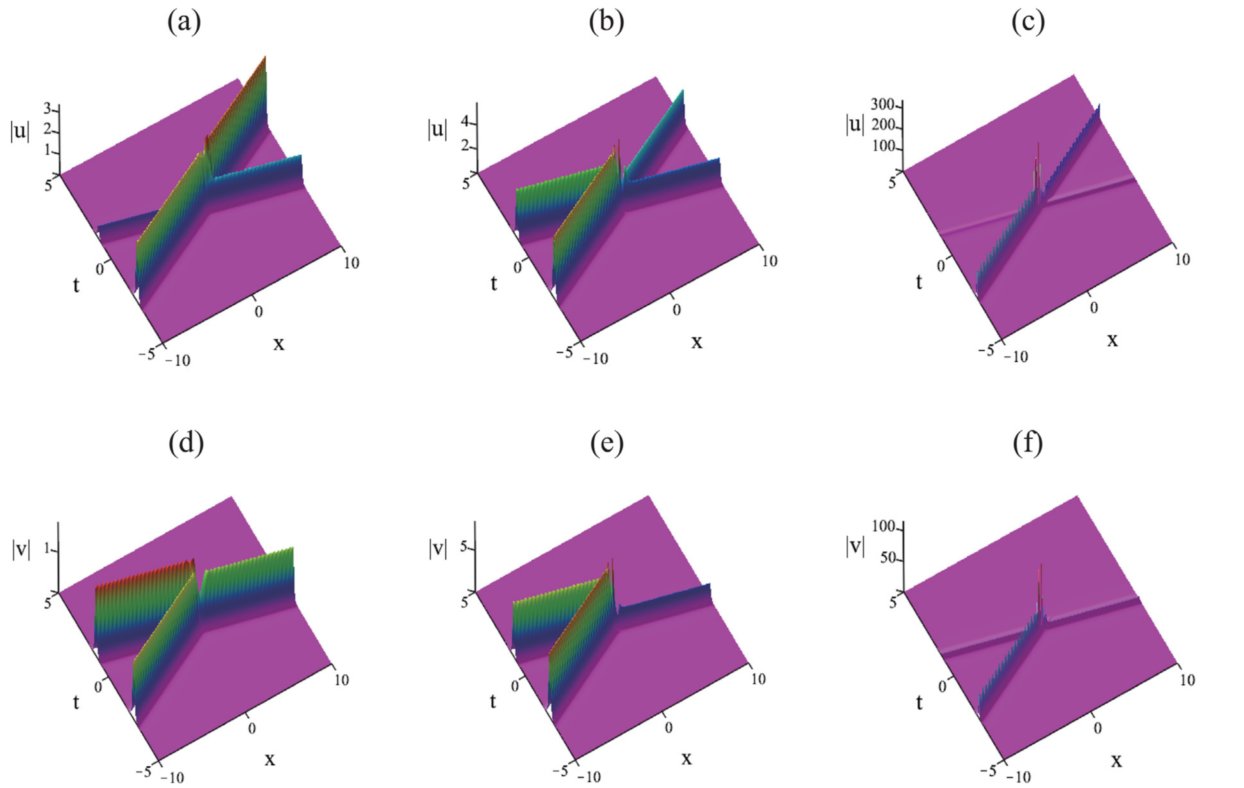

Fig. 9. Polarization-changing collisions between two single-hump solutions of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1.5$, $\sigma_2 = -0.5$; (c) and (f) with parameters: $\sigma_1 = -0.2$, $\sigma_2 = 0.5$.

In order to simplify the above equations, assume that

$$v_{1,k} = 12\xi_{1,k}^2 - 4\eta_{1,k}^2, \quad k = 1, 2. \tag{116}$$

Considering the two-soliton solutions (83) and (84), without loss of generality, we assume that $\eta_{1,1}, \eta_{1,2} > 0$. For fixed $\mathrm{Re}(\theta_{1,1})$, suppose that $v_{1,1} > v_{1,2}$.

(i) Taking limit as $t \to -\infty$: $\mathrm{Re}(\theta_{1,1}) \sim 0$, $\mathrm{Re}(\theta_{1,2}) \sim -\infty$, the dominant terms are those which contain the factor $e^{-\theta_{1,2}-\theta_{1,2}^*}$ in this case. The asymptotic expressions of the two solitons before interaction can be given by

$$\begin{aligned} u^{1-} \sim & -i(\lambda_{1,2}^* - \lambda_{1,1})(\lambda_{1,1}^* - \lambda_{1,1})(\lambda_{1,1}^* - \lambda_{1,2}^*)a_1 e^{\theta_{1,1}-\theta_{1,1}^* - \frac{\alpha_1+\alpha_2}{2}} \\ & \mathrm{sech}(\theta_{1,1} + \theta_{1,1}^* - \frac{\alpha_1}{2} + \frac{\alpha_2}{2}), \end{aligned} \tag{117}$$

$$\begin{aligned} v^{1-} \sim & -i(\lambda_{1,2}^* - \lambda_{1,1})(\lambda_{1,1}^* - \lambda_{1,1})(\lambda_{1,1}^* - \lambda_{1,2}^*)c_1 e^{\theta_{1,1}-\theta_{1,1}^* - \frac{\alpha_1+\alpha_2}{2}} \\ & \mathrm{sech}(\theta_{1,1} + \theta_{1,1}^* - \frac{\alpha_1}{2} + \frac{\alpha_2}{2}), \end{aligned} \tag{118}$$

where

$$\begin{aligned} e^{\alpha_1} &= (\lambda_{1,1}^* - \lambda_{1,2}^*)(\lambda_{1,2} - \lambda_{1,1}), \\ e^{\alpha_2} &= -2(\sigma_1|a_1|^2 + \sigma_2|c_1|^2)(\lambda_{1,2}^* - \lambda_{1,1})(\lambda_{1,2} - \lambda_{1,1}^*). \end{aligned} \tag{119}$$

(ii) Taking limit as $t \to +\infty$: $\mathrm{Re}(\theta_{1,1}) \sim 0$, $\mathrm{Re}(\theta_{1,2}) \sim +\infty$, the dominant terms are those which contain the factor $e^{\theta_{1,2}+\theta_{1,2}^*}$ in this case. The asymptotic expressions of the two solitons after interaction can be given by

$$u^{1+} \sim -i\beta_1 e^{\theta_{1,1}-\theta_{1,1}^* - \frac{\alpha_3+\alpha_4}{2}} \,\mathrm{sech}(\theta_{1,1} + \theta_{1,1}^* - \frac{\alpha_3}{2} + \frac{\alpha_4}{2}), \tag{120}$$

$$v^{1+} \sim -i\beta_2 e^{\theta_{1,1}-\theta_{1,1}^* - \frac{\alpha_3+\alpha_4}{2}} \,\mathrm{sech}(\theta_{1,1} + \theta_{1,1}^* - \frac{\alpha_3}{2} + \frac{\alpha_4}{2}), \tag{121}$$

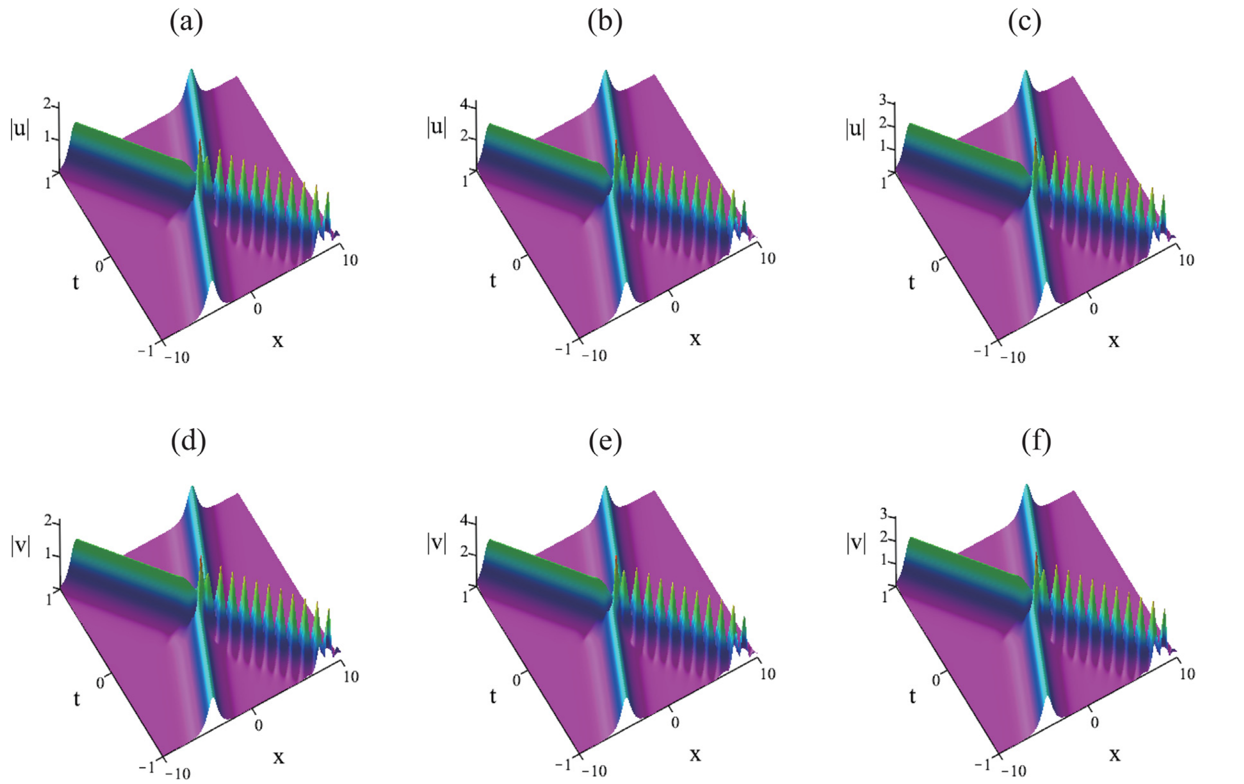

Fig. 10. Collisions between two single-hump solutions of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1.5$, $\sigma_2 = -1$; (c) and (f) with parameters: $\sigma_1 = -1$, $\sigma_2 = 2$.

where

$$\begin{aligned}
e^{\alpha_3} &= -2(\sigma_1|a_2|^2 + \sigma_2|c_2|^2)(\lambda_{1,2}^* - \lambda_{1,1})(\lambda_{1,2} - \lambda_{1,1}^*),\\
e^{\alpha_4} &= \gamma_1(\lambda_{1,1}\lambda_{1,1}^* + \lambda_{1,2}\lambda_{1,2}^*) + \gamma_2(\lambda_{1,1}\lambda_{1,2} + \lambda_{1,1}^*\lambda_{1,2}^*) + \gamma_3(\lambda_{1,1}\lambda_{1,2}^* + \lambda_{1,1}^*\lambda_{1,2}),\\
\beta_1 &=(\lambda_{1,1} - \lambda_{1,1}^*)(\lambda_{1,2} - \lambda_{1,1}^*)[\sigma_1 a_1|a_2|^2(\lambda_{1,2} + \lambda_{1,2}^* - 2\lambda_{1,1}) + 2\sigma_2 a_1|c_2|^2(\lambda_{1,2}^* - \lambda_{1,1})\\
&\quad - \sigma_2 a_1 c_1 c_2^*(\lambda_{1,2}^* - \lambda_{1,2})],\\
\beta_2 &=(\lambda_{1,1} - \lambda_{1,1}^*)(\lambda_{1,2} - \lambda_{1,1}^*)[\sigma_2 c_1|c_2|^2(\lambda_{1,2} + \lambda_{1,2}^* - 2\lambda_{1,1}) + 2\sigma_1 c_1|a_2|^2(\lambda_{1,2}^* - \lambda_{1,1})\\
&\quad - \sigma_1 a_1 a_2^* c_2(\lambda_{1,2}^* - \lambda_{1,2})],
\end{aligned} \tag{122}$$

and

$$\begin{aligned}
\gamma_1 &= -4\sigma_1^2|a_1|^2|a_2|^2 - 4\sigma_2^2|c_1|^2|c_2|^2 - 4\sigma_1\sigma_2(|a_1|^2|c_2|^2 + |a_2|^2|c_1|^2),\\
\gamma_2 &=2\sigma_1^2|a_1|^2|a_2|^2 + 2\sigma_2^2|c_1|^2|c_2|^2 + 2\sigma_1\sigma_2(2|a_1|^2|c_2|^2 + 2|a_2|^2|c_1|^2 - a_1^*a_2c_1c_2^*\\
&\quad - a_1a_2^*c_1^*c_2),\\
\gamma_3 &= 2\sigma_1^2|a_1|^2|a_2|^2 + 2\sigma_2^2|c_1|^2|c_2|^2 + 2\sigma_1\sigma_2(a_1^*a_2c_1c_2^* + a_1a_2^*c_1^*c_2).
\end{aligned} \tag{123}$$

For fixed $\mathrm{Re}(\theta_{1,2})$, similarly suppose that $v_{1,1} > v_{1,2}$.

(iii) Taking limit as $t \to -\infty$: $\mathrm{Re}(\theta_{1,2}) \sim 0$, $\mathrm{Re}(\theta_{1,1}) \sim +\infty$, the dominant terms are those which contain the factor $e^{\theta_{1,1}+\theta_{1,1}^*}$ in this case. The asymptotic expressions of the two solitons before interaction can be given by

$$u^{2-} \sim -i\beta_3 e^{\theta_{1,2}-\theta_{1,2}^*-\frac{\alpha_2+\alpha_4}{2}}\,\mathrm{sech}(\theta_{1,2} + \theta_{1,2}^* - \frac{\alpha_2}{2} + \frac{\alpha_4}{2}), \tag{124}$$

$$v^{2-} \sim -i\beta_4 e^{\theta_{1,2}-\theta_{1,2}^*-\frac{\alpha_2+\alpha_4}{2}}\,\mathrm{sech}(\theta_{1,2} + \theta_{1,2}^* - \frac{\alpha_2}{2} + \frac{\alpha_4}{2}), \tag{125}$$

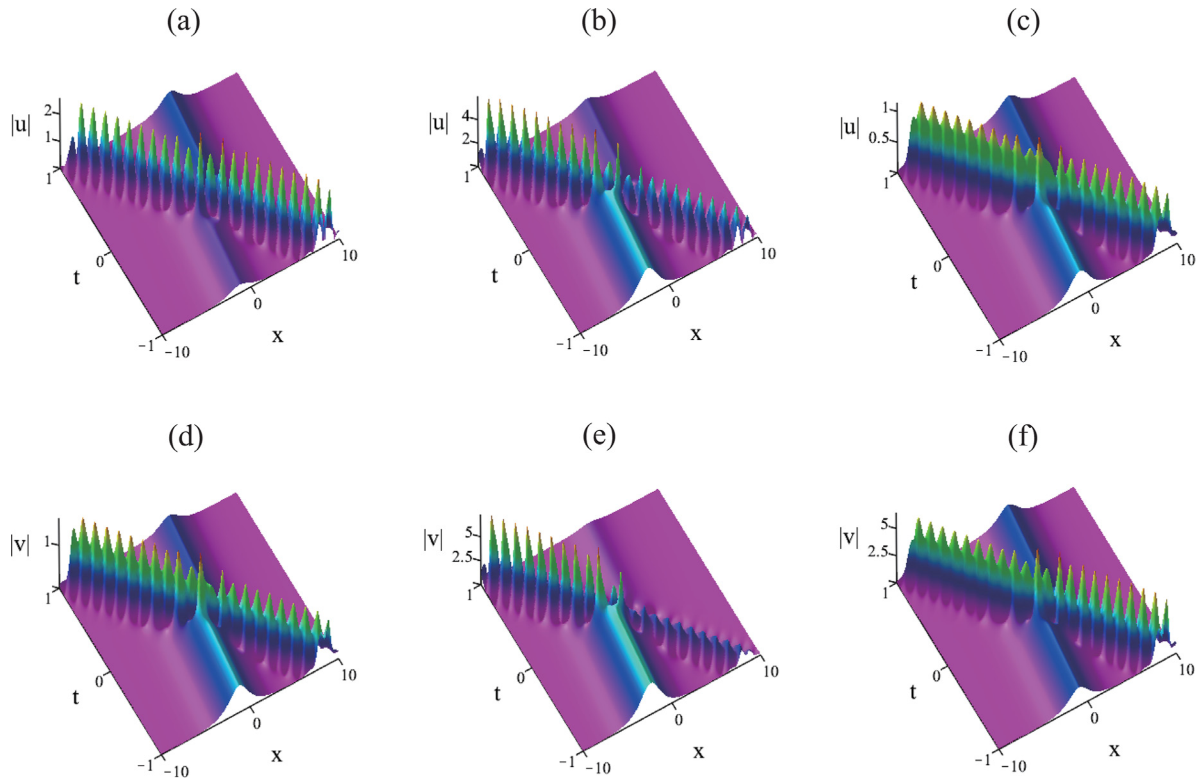

Fig. 11. Elastic collisions between a single-hump soliton and a breather of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 1$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1$, $\sigma_2 = -0.5$; (c) and (f) with parameters: $\sigma_1 = -0.5$, $\sigma_2 = 0.2$.

where

$$\begin{aligned}\beta_3 =&(\lambda_{1,2}^*-\lambda_{1,1})(\lambda_{1,2}^*-\lambda_{1,2})[\sigma_1 a_2|a_1|^2(\lambda_{1,1}+\lambda_{1,1}^*-2\lambda_{1,2})-2\sigma_2 a_2|c_1|^2(\lambda_{1,2}-\lambda_{1,1}^*)\\ &+\sigma_2 a_1 c_1^* c_2(\lambda_{1,1}-\lambda_{1,1}^*)],\\ \beta_4 =&(\lambda_{1,2}^*-\lambda_{1,1})(\lambda_{1,2}^*-\lambda_{1,2})[\sigma_2 c_2|c_1|^2(\lambda_{1,1}+\lambda_{1,1}^*-2\lambda_{1,2})-2\sigma_1 c_2|a_1|^2(\lambda_{1,2}-\lambda_{1,1}^*)\\ &+\sigma_1 a_1^* a_2 c_1(\lambda_{1,1}-\lambda_{1,1}^*)].\end{aligned} \tag{126}$$

(iv) Taking limit as $t \to +\infty$: $\mathrm{Re}(\theta_{1,2}) \sim 0$, $Re(\theta_{1,1}) \sim -\infty$, the dominant terms are those which contain the factor $e^{-\theta_{1,1}-\theta_{1,1}^*}$ in this case. The asymptotic expressions of the two solitons after interaction can be given by

$$\begin{aligned}u^{2+} \sim &-i(\lambda_{1,2}-\lambda_{1,2}^*)(\lambda_{1,1}^*-\lambda_{1,2})(\lambda_{1,1}^*-\lambda_{1,2}^*)a_2 e^{\theta_{1,2}-\theta_{1,2}^*-\frac{\alpha_1+\alpha_3}{2}}\\ &\mathrm{sech}(\theta_{1,2}+\theta_{1,2}^*-\frac{\alpha_1}{2}+\frac{\alpha_3}{2}),\end{aligned} \tag{127}$$

$$\begin{aligned}v^{2+} \sim &-i(\lambda_{1,2}-\lambda_{1,2}^*)(\lambda_{1,1}^*-\lambda_{1,2})(\lambda_{1,1}^*-\lambda_{1,2}^*)c_2 e^{\theta_{1,2}-\theta_{1,2}^*-\frac{\alpha_1+\alpha_3}{2}}\\ &\mathrm{sech}(\theta_{1,2}+\theta_{1,2}^*-\frac{\alpha_1}{2}+\frac{\alpha_3}{2}).\end{aligned} \tag{128}$$

Comparing the single-soliton solution (79) and (80), we can see the asymptotic expressions u^{1-}, v^{1-}, u^{1+}, v^{1+}, u^{2-}, v^{2-}, u^{2+}, v^{2+} are single-soliton. After the collision of two single solitons, each soliton has a position shift or a phase shift. The asymptotic analysis of imaginary case shows similar results, for which we only give the asymptotic expression of the case $\mathrm{Re}(\theta_{2,1}) \sim 0$, $\mathrm{Re}(\theta_{2,2}) \sim -\infty$ (as $t \to -\infty$) in Appendix, and the other cases can be obtained in a similar way.

6. Conclusions

In this paper, the multi-component Sasa-Satsuma integrable hierarchies were obtained firstly by giving a $(2N+1)$-order spectral matrix through the corresponding stationary zero curvature equation. Furthermore, the generalized one-component and coupled Sasa-Satsuma equation were derived from the multi-component Sasa-Satsuma integrable hierarchies when $N = 3$, and a bi-Hamiltonian structure was constructed for the hierarchies, which displayed their

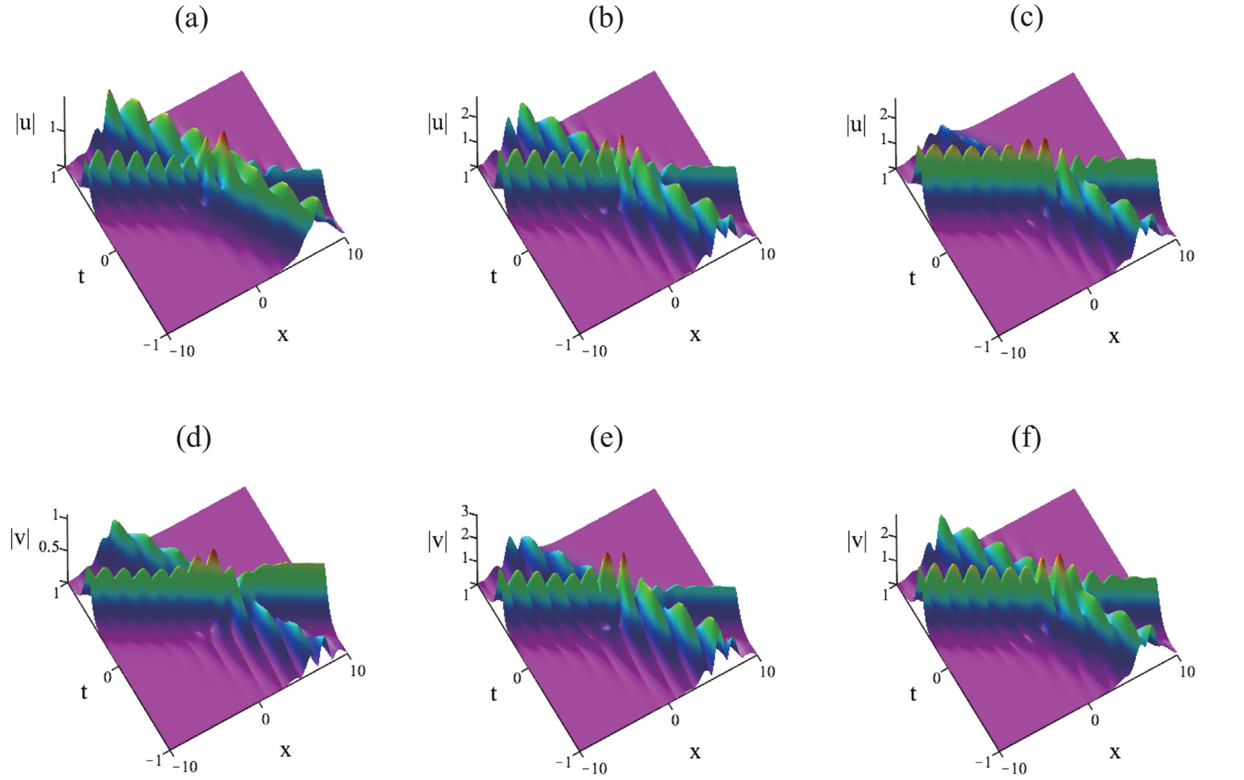

Fig. 12. Elastic collisions between two breather-type solutions of generalized coupled Sasa-Satsuma equation. (a) and (d) with parameters: $\sigma_1 = 0.5$, $\sigma_2 = 1$; (b) and (e) with parameters: $\sigma_1 = 1$, $\sigma_2 = -0.5$; (c) and (f) with parameters: $\sigma_1 = -0.2$, $\sigma_2 = 0.5$.

Liouville integrability. Then the generalized coupled Sasa-Satsuma Eq. (1) was considered by the inverse scattering transform, and the N-soliton solutions were derived via the Riemann–Hilbert method. The one- and two-soliton solutions were presented graphically, and their dynamics was investigated. In these figures, the bright solitons, breather, single-hump solitons, double-hump solitons were shown through selecting different values of the involved parameters. Moreover, the elastic interactions between one (two) single-hump solution(s) and one (two) breather-type solution(s) of the generalized coupled Sasa-Satsuma Eq. (1) were analyzed, and the polarization-changing collisions between two single-hump solitons were explored. Additionally, asymptotic analysis for two-soliton solutions of the generalized coupled Sasa-Satsuma equation was made in the last part. It is expected that all these results would be helpful to understand physical phenomena and develop novel applications of the Riemann–Hilbert technique to other nonlinear systems.

CRediT authorship contribution statement

Yaqing Liu: Writing – original draft, Writing – review & editing, Supervision. **Wen-Xin Zhang:** Writing – original draft, Data curation, Investigation. **Wen-Xiu Ma:** Supervision, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Asymptotic analysis of two solitons solution in imaginary case

Considering the expressions of two solitons solution (110) and (111), the equations can also be written in the following form

$$
\begin{aligned}
u = & -2i\frac{1}{\det M} \\
& (a_1 e^{\theta_{2,1}} \begin{vmatrix} e^{-\theta_{2,1}*} & M_{12} & M_{13} & M_{14} \\ e^{-\theta_{2,2}*} & M_{22} & M_{23} & M_{24} \\ e^{-\theta_{2,1}} & M_{32} & M_{33} & M_{34} \\ e^{-\theta_{2,2}} & M_{42} & M_{43} & M_{44} \end{vmatrix} + a_2 e^{\theta_{2,2}} \begin{vmatrix} M_{11} & e^{-\theta_{2,1}*} & M_{13} & M_{14} \\ M_{21} & e^{-\theta_{2,2}*} & M_{23} & M_{24} \\ M_{31} & e^{-\theta_{2,1}} & M_{33} & M_{34} \\ M_{41} & e^{-\theta_{2,2}} & M_{43} & M_{44} \end{vmatrix} + \frac{1}{\sigma_1} b_1^* \\
& e^{\theta_{2,1}*} \begin{vmatrix} M_{11} & M_{12} & e^{-\theta_{2,1}*} & M_{14} \\ M_{21} & M_{22} & e^{-\theta_{2,2}*} & M_{24} \\ M_{31} & M_{32} & e^{-\theta_{2,1}} & M_{34} \\ M_{41} & M_{42} & e^{-\theta_{2,2}} & M_{44} \end{vmatrix} + \frac{1}{\sigma_1} b_2{}^* e^{\theta_{2,2}*} \begin{vmatrix} M_{11} & M_{12} & M_{13} & e^{-\theta_{2,1}*} \\ M_{21} & M_{22} & M_{23} & e^{-\theta_{2,2}*} \\ M_{31} & M_{32} & M_{33} & e^{-\theta_{2,1}} \\ M_{41} & M_{42} & M_{43} & e^{-\theta_{2,2}} \end{vmatrix}),
\end{aligned}
$$

and the matrix M is given as

$$
M = \begin{vmatrix}
\frac{K_1 e^{\theta_{2,1}+\theta_{2,1}^*} + e^{-\theta_{2,1}-\theta_{2,1}^*}}{\lambda_{2,1}^* - \lambda_{2,1}} & \frac{K_2 e^{\theta_{2,1}^*+\theta_{2,2}} + e^{-\theta_{2,1}^*-\theta_{2,2}}}{\lambda_{2,1}^* - \lambda_{2,2}} & \frac{K_3 e^{2\theta_{2,1}^*} + e^{-2\theta_{2,1}^*}}{2\lambda_{2,1}^*} & \frac{K_4 e^{\theta_{2,1}^*+\theta_{2,2}^*} + e^{-\theta_{2,1}^*-\theta_{2,2}^*}}{\lambda_{2,1}^* - \lambda_{2,2}^*} \\
\frac{K_2^* e^{\theta_{2,1}+\theta_{2,2}^*} + e^{-\theta_{2,1}-\theta_{2,2}^*}}{\lambda_{2,2}^* - \lambda_{2,1}} & \frac{K_5 e^{\theta_{2,2}^*+\theta_{2,2}} + e^{-\theta_{2,2}^*-\theta_{2,2}}}{\lambda_{2,2}^* - \lambda_{2,2}} & \frac{K_4 e^{\theta_{2,1}^*+\theta_{2,2}^*} + e^{-\theta_{2,1}^*-\theta_{2,2}^*}}{\lambda_{2,1}^* + \lambda_{2,2}^*} & \frac{K_6 e^{2\theta_{2,2}^*} + e^{-2\theta_{2,2}^*}}{2\lambda_{2,2}^*} \\
-\frac{K_3^* e^{2\theta_{2,1}} + e^{-2\theta_{2,1}}}{2\lambda_{2,1}} & \frac{K_4^* e^{\theta_{2,1}+\theta_{2,2}} + e^{-\theta_{2,1}-\theta_{2,2}}}{-\lambda_{2,1} - \lambda_{2,2}} & \frac{K_1 e^{\theta_{2,1}+\theta_{2,1}^*} + e^{-\theta_{2,1}-\theta_{2,1}^*}}{\lambda_{2,1}^* - \lambda_{2,1}} & \frac{K_2^* e^{\theta_{2,1}+\theta_{2,2}^*} + e^{-\theta_{2,1}-\theta_{2,2}^*}}{\lambda_{2,2}^* - \lambda_{2,1}} \\
\frac{K_4^* e^{\theta_{2,1}+\theta_{2,2}} + e^{-\theta_{2,1}-\theta_{2,2}}}{-\lambda_{2,1} - \lambda_{2,2}} & -\frac{K_6^* e^{2\theta_{2,2}} + e^{-2\theta_{2,2}}}{2\lambda_{2,2}} & \frac{K_2 e^{\theta_{2,1}^*+\theta_{2,2}} + e^{-\theta_{2,1}^*-\theta_{2,2}}}{\lambda_{2,1}^* - \lambda_{2,2}} & \frac{K_5 e^{\theta_{2,2}^*+\theta_{2,2}} + e^{-\theta_{2,2}^*-\theta_{2,2}}}{\lambda_{2,2}^* - \lambda_{2,2}}
\end{vmatrix},
$$

where

$$
\begin{aligned}
& K_1 = \sigma_1 |a_1|^2 + \frac{1}{\sigma_1}|b_1|^2 + \sigma_2 |c_1|^2 + \frac{1}{\sigma_2}|d_1|^2, \\
& K_2 = \sigma_1 a_1^* a_2 + \frac{1}{\sigma_1} b_1^* b_2 + \frac{1}{\sigma_2} c_1^* c_2 + \frac{1}{\sigma_2} d_1^* d_2, \\
& K_3 = 2a_1^* b_1^* + 2c_1^* d_1^*, \quad K_4 = a_1^* b_2^* + a_2^* b_1^* + c_1^* d_2^* + c_2^* d_1^*, \\
& K_5 = \sigma_1 |a_2|^2 + \frac{1}{\sigma_1}|b_2|^2 + \sigma_2 |c_2|^2 + \frac{1}{\sigma_2}|d_2|^2, \quad K_6 = 2a_2^* b_2^* + 2c_2^* d_2^*.
\end{aligned}
$$

Make a similar assumption to the one in above subsection,

$$
\lambda_{2,k} = \xi_{2,k} + i\eta_{2,k}, \quad k = 1, 2.
$$

Then the $\mathrm{Re}(\theta_{2,1})$ and $\mathrm{Re}(\theta_{2,2})$ can be derived as follows

$$
\begin{aligned}
\mathrm{Re}(\theta_{2,1}) &= -\eta_{2,1}[x + (12\xi_{2,1}^2 - 4\eta_{2,1}^2)t], \\
\mathrm{Re}(\theta_{2,2}) &= -\eta_{2,2}[x + (12\xi_{2,2}^2 - 4\eta_{2,2}^2)t] \\
&= \frac{\eta_{2,2}}{\eta_{2,1}} \mathrm{Re}(\theta_{2,1}) + \eta_2[(12\xi_{2,1}^2 - 4\eta_{2,1}^2) - (12\xi_{2,2}^2 - 4\eta_{2,2}^2)]t.
\end{aligned}
$$

In order to simplify the above equations, suppose that

$$
v_{2,k} = 12\xi_{2,k}^2 - 4\eta_{2,k}^2, \quad k = 1, 2.
$$

Considering the above two solitons solution Eqs. (110)–(111), without loss of generality, we assume that $\eta_{2,1}, \eta_{2,2} > 0$. For fixed $\mathrm{Re}(\theta_{2,1})$, suppose that $v_{2,1} > v_{2,2}$. Taking limit as $t \to -\infty$: $\mathrm{Re}(\theta_{2,1}) \sim 0$, $\mathrm{Re}(\theta_{2,2}) \sim -\infty$, the dominant terms are those which contain the factor $e^{-2\theta_{2,2}-2\theta_{2,2}^*}$ in this case. The numerator of asymptotic expression u^{1-} for the two solitons u before interaction can be given by

$$
\begin{aligned}
& -2i[(\frac{a_1 K_1}{\lambda_{2,1}^* - \lambda_{2,1}} \hat{\mathfrak{R}}_{33} - \frac{b_1^* K_3^*}{2\sigma_1 \lambda_{2,1}} \tilde{\mathfrak{R}}_{31}) e^{2\theta_{2,1}} + (a_1 \hat{R} + \frac{a_1 K_3}{2\lambda_1^*} \hat{\mathfrak{R}}_{13}) e^{-2\theta_{2,1}^*} + \frac{b_1^*}{\sigma_1} \tilde{R} e^{-2\theta_{2,1}} \\
& + \frac{b_1^* K_1}{\sigma_1(\lambda_{2,1}^* - \lambda_{2,1})} \tilde{\mathfrak{R}}_{11} e^{2\theta_{2,1}^*}],
\end{aligned}
$$

where

$$\hat{R}=\begin{vmatrix} 1 & \frac{1}{\lambda_{2,1}^*-\lambda_{2,2}} & \frac{1}{2\lambda_{2,1}^*} & \frac{1}{\lambda_{2,1}^*+\lambda_{2,2}^*} \\ 1 & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*+\lambda_{2,2}^*} & \frac{1}{2\lambda_{2,2}^*} \\ 1 & \frac{1}{-\lambda_{2,1}-\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,1}} \\ 1 & \frac{1}{-2\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,2}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} \end{vmatrix}, \quad \tilde{R}=\begin{vmatrix} \frac{1}{\lambda_{2,1}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,2}} & 1 & \frac{1}{\lambda_{2,1}^*+\lambda_{2,2}^*} \\ \frac{1}{\lambda_{2,2}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} & 1 & \frac{1}{2\lambda_{2,2}^*} \\ \frac{1}{-2\lambda_{2,1}} & \frac{1}{-\lambda_{2,1}-\lambda_{2,2}} & 1 & \frac{1}{\lambda_{2,2}^*-\lambda_{2,1}} \\ \frac{1}{-\lambda_{2,1}-\lambda_{2,2}} & \frac{1}{-2\lambda_{2,2}} & 1 & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} \end{vmatrix},$$

and $\hat{\mathfrak{R}}_{ij}$ and $\tilde{\mathfrak{R}}_{ij}$ are the cofactor of $\hat{R}_{ij}$ and $\tilde{R}_{ij}$, respectively. The denominator of asymptotic expression u^{1-} for the two solitons u before interaction can be given by

$$\bar{R}e^{-2\theta_{2,1}-2\theta_{2,1}^*}+\left(\frac{K_1^2}{(\lambda_{2,1}^*-\lambda_{2,1})^2}+\frac{K_2K_3}{4|\lambda_{2,1}|^2}\right)\left(\frac{1}{(\lambda_{2,2}^*-\lambda_{2,2})(\lambda_{2,2}^*-\lambda_{2,2})}+\frac{1}{4\lambda_{2,2}^*\lambda_{2,2}}\right)e^{2\theta_{2,1}+2\theta_{2,1}^*}$$
$$-\frac{K_3}{2\lambda_{2,1}}\bar{\mathfrak{R}}_{31}e^{2\theta_{2,1}-2\theta_{2,1}^*}+\frac{K_3}{2\lambda_{2,1}^*}\bar{\mathfrak{R}}_{13}e^{2\theta_{2,1}^*-2\theta_{2,1}}+\frac{2K_1}{\lambda_{2,1}^*-\lambda_{2,1}}\bar{\mathfrak{R}}_{33},$$

where

$$\bar{R}=\begin{vmatrix} \frac{1}{\lambda_{2,1}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,2}} & \frac{1}{2\lambda_{2,1}} & \frac{1}{\lambda_{2,1}^*+\lambda_{2,2}^*} \\ \frac{1}{\lambda_{2,2}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*+\lambda_{2,2}^*} & \frac{1}{2\lambda_{2,2}^*} \\ \frac{1}{-2\lambda_{2,1}} & \frac{1}{-\lambda_{2,1}-\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,1}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,1}} \\ \frac{1}{-\lambda_{2,1}-\lambda_{2,2}} & \frac{1}{-2\lambda_{2,2}} & \frac{1}{\lambda_{2,1}^*-\lambda_{2,2}} & \frac{1}{\lambda_{2,2}^*-\lambda_{2,2}} \end{vmatrix},$$

and $\bar{\mathfrak{R}}_{ij}$ is the cofactor of $\bar{R}_{ij}$.

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