Modern Physics Letters B
Vol. 37, No. 9 (2023) 2250203 (23 pages)

DOI: 10.1142/S0217984922502037

Dynamic analysis of lump solutions based on the dimensionally reduced generalized Hirota bilinear KP-Boussinesq equation

Meng-Meng Liu and Jian-Ping Yu

Department of Applied Mathematics,
University of Science and Technology Beijing,
Beijing 100083, P. R. China

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University,
Jinhua 321004, Zhejiang, P. R. China

Department of Mathematics,
King Abdulaziz University,
Jeddah, Saudi Arabia

Department of Mathematics and Statistics,
University of South Florida,
Tampa, FL 33620, USA

Department of Mathematical Sciences,
International Institute for Symmetry Analysis
and Mathematical Modelling,
North-West University, Mafikeng Campus,
Private Bag X 2046, Mmabatho 2735, South Africa

Chaudry Masood Khalique

Department of Mathematical Sciences,
International Institute for Symmetry Analysis
and Mathematical Modelling,
North-West University, Mafikeng Campus,
Private Bag X 2046, Mmabatho 2735, South Africa

Yong-Li Sun*

Department of Mathematics,
Beijing University of Chemical Technology,
Beijing 100029, P. R. China
sunyl@mail.buct.edu.cn

*Corresponding author.

Received 2 February 2022
Revised 7 April 2022
Accepted 25 April 2022
Published 22 February 2023

In this paper, a $(3+1)$-dimensional generalized KP-Boussinesq equation is introduced and its associate Hirota bilinear form is also given. Based on finding the positive quadratic function solutions of the associate Hirota bilinear equation, the lump solutions of the proposed $(3+1)$-dimensional generalized KP-Boussinesq equation and its corresponding reduced equations in $(2+1)$ dimensions are obtained. Furthermore, the sufficient and necessary conditions for guaranteeing the analyticity and rational localization of lump solutions are derived and expressed in the form of free parameters, which are involved in lump solutions and play a key role in controlling the dynamic properties of lump solutions. The localized properties are also analyzed and shown graphically.

Keywords: Dynamic analysis; lump solutions; Hirota bilinear method; dimensionally reduced Hirota bilinear equations.

1. Introduction

It has been known that soliton solutions exist in all integrable systems, which have been used to model nonlinear phenomena in nature. Particularly, the research on exact and analytic solutions has attracted many researchers, such as breathers and rogue waves, which are rational solutions and exponentially localized in both space and time. Compared with rogue waves and breathers, lump solutions are a particular type of rational solutions, localized in all spatial directions.[1–10] In 1977, the lump solutions were first found since they have important physical meanings.[11] Therefore, it is natural and meaningful to study lump solutions of partial differential equations (PDEs) by using different approaches, for instance, the long-wave-limit method[12] and the Hirota bilinear method,[13] and so on Refs. 14–38. Up until now, it has been found that numerous integrable systems have lump solutions such as the KPI equation,[12] the $(2+1)$-dimensional Sawada–Kotera equation,[39] and the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa equation,[40] and so forth.

In 2015, based on the Hirota bilinear operator theory, Ma proposed a novel direct method for constructing lump solutions of the KP equation.[41] So far, this method has been applied to finding lump solutions of many nonlinear PDEs since it is natural and effective to search for lump solutions. For example, the KPI equation,[41] the $(2+1)$-dimensional KdV equation,[42] the dimensionally reduced p-gKP and p-gBKP equations[43] and the BKP equation.[44]

Inspired by the aforementioned discussions and physical concerns, we, in this paper, focus on a (3+1)-dimensional generalized KP-Boussinesq equation that reads

$$u_{xxxy} + 3(u_x u_y)_x + \alpha(u_x + u_y + u_t)_t + \beta u_{zz} = 0, \tag{1}$$

where α, β are arbitrary real constants. If $\alpha = 1$, $\beta = -1$, then the equation (1) becomes the generalized KP-Boussinesq equation proposed and studied in the reference.[45]

Substituting a dependent variable transformation

$$u = 2(\ln f)_x \tag{2}$$

into Eq. (1) yields the associate Hirota bilinear equation

$$\begin{aligned}
&[D_x^3 D_y + \alpha(D_x D_t + D_y D_t + D_t^2) + \beta D_z^2] f \cdot f \\
&\quad = 2[f_{xxxy} f - f_{xxx} f_y - 3 f_{xxy} f_x + 3 f_{xx} f_{xy} + \alpha(f_{xt} f \\
&\qquad - f_x f_t + f_{yt} f - f_y f_t + f_{tt} f - f_t^2) + \beta(f_{xx} f - f_x^2)] \\
&\quad = 0
\end{aligned} \tag{3}$$

which is identified as the generalized Hirota bilinear KP-Boussinesq equation. As a matter of fact, the transformation equation (2) plays an important role in establishing Bell polynomial theories of soliton equations.[46,47] It is obvious that if f solves Eq. (3), then $u = 2(\ln f)_x$ solves Eq. (1). The following are the Hirota bilinear operators defined by

$$\begin{aligned}
D_x^m D_y^n D_t^l (f \cdot g) = &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \\
&\times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l f(x, y, t) g(x', y', t')|_{x'=x, y'=y, t'=t},
\end{aligned} \tag{4}$$

where m, n, l are all non-negative integers.[13]

In this paper, we study the lump solutions and dynamic properties of the introduced $(3+1)$-dimensional generalized KP-Boussinesq equation (1) and its dimensionally reduced forms by using symbolic computations with Maple. We would like to search for positive quadratic function solutions of the bilinear equation (3) and its dimensionally reduced forms, which in turn yields lump solutions of Eq. (1) and its dimensionally reduced forms with free parameters, by which the sufficient and necessary conditions to guarantee analyticity and rational localization of the obtained solutions are also derived. Finally, the work ends up with some conclusions and remarks.

2. Lump Solutions of Eq. (1)

In this section, we aim to derive the lump solutions of the $(3+1)$-dimensional generalized KP-Boussinesq equation (1) in light of its corresponding bilinear equation. Hence, we first study the quadratic function solutions of Eq. (3) as

$$\begin{cases}
f = g^2 + h^2 + l^2 + a_{16}, \\
g = a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\
h = a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\
l = a_{11} x + a_{12} y + a_{13} z + a_{14} t + a_{15},
\end{cases} \tag{5}$$

where $a_i (1 \leq i \leq 16)$ are real parameters to be determined later. It is evident that if $f(x, y, z, t)$ solves Eq. (3), then $u = 2(\ln f)_x$ is a solution of Eq. (1).

Plugging function f in Eq. (5) directly into Eq. (3) results in three sets of constraint equations for the parameters. All the details can be found in Appendix A.

For Case 1, which needs to satisfy the conditions

$$\alpha\beta a_3^4 a_{14}(a_3^2 + a_8^2) \neq 0, \tag{6}$$

$$\frac{M_1}{\alpha\beta a_{14}} > 0, \tag{7}$$

to guarantee the well-definedness and the positiveness of f and the localization of u in all spatial directions, respectively. A class of positive quadratic function solution of Eq. (3) is derived according to the parameters in Case 1 as

$$f = \left(a_1 x - \frac{\alpha a_1 a_{14} + 2\beta a_3 a_{13}}{\alpha a_{14}} y + a_3 z + a_5\right)^2 + \left(\frac{a_1 a_8}{a_3} x - \frac{a_8(\alpha a_1 a_{14} + 2\beta a_3 a_{13})}{\alpha a_3 a_{14}} y + a_8 z + a_{10}\right)^2 + \left(a_{11} x - \frac{(a_{13}^2 - a_3^2 - a_8^2)\beta + \alpha a_{14}(a_{11} + a_{14})}{\alpha a_{14}} y + a_{13} z + a_{14} t + a_{15}\right)^2 + \frac{M_1}{\alpha\beta a_3^4 a_{14}(a_3^2 + a_8^2)} \tag{8}$$

which in turn further yields a class of lump solution of Eq. (1) via the transformation $u = 2(\ln f)_x$ as follows:

$$u^{(\mathrm{I})} = \frac{4(a_1 g + \frac{a_1 a_8}{a_3} h + a_{11} l)}{f}, \tag{9}$$

where f is defined by Eq. (8) and g, h, l are given in the following forms:

$$\begin{cases} g = a_1 x - \dfrac{\alpha a_1 a_{14} + 2\beta a_3 a_{13}}{\alpha a_{14}} y + a_3 z + a_5, \\ h = \dfrac{a_1 a_8}{a_3} x - \dfrac{a_8(\alpha a_1 a_{14} + 2\beta a_3 a_{13})}{\alpha a_3 a_{14}} y + a_8 z + a_{10}, \\ l = a_{11} x - \dfrac{(a_{13}^2 - a_3^2 - a_8^2)\beta + \alpha a_{14}(a_{11} + a_{14})}{\alpha a_{14}} y \\ \qquad + a_{13} z + a_{14} t + a_{15}. \end{cases} \tag{10}$$

It should be noted that nine parameters $a_1, a_3, a_5, a_8, a_{10}, a_{11}, a_{13}, a_{14}$ and a_{15} are involved in the solution $u^{(I)}$, among which $a_1, a_5, a_{10}, a_{11}, a_{13}$ and a_{15} are free parameters, but a_3, a_8, a_{14} are required to satisfy conditions (6) and (7) to ensure that the lump solution $u^{(I)}$ exist shown by Fig. 1.

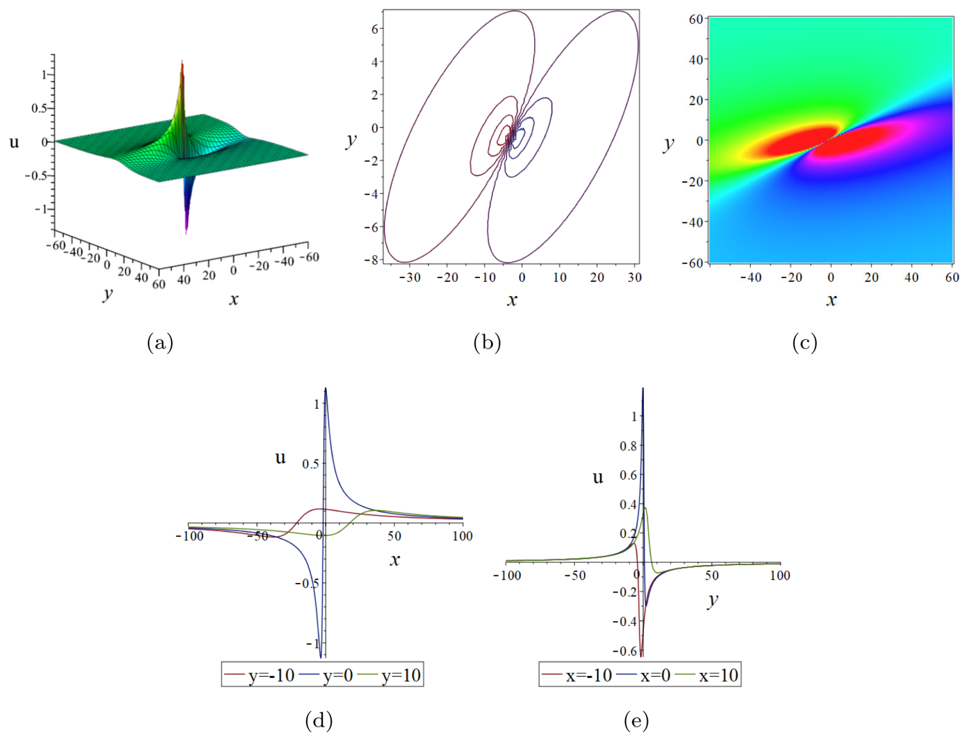


Fig. 1. (Color online) Lump dynamic characteristics of $u^{(\mathrm{I})}$ via (9) with $a_1 = 1, a_3 = 1, a_5 = 2, a_8 = 3, a_{10} = 1, a_{11} = 1, a_{13} = 1, a_{14} = 2, a_{15} = 2, \alpha = 2, \beta = 3, z = 1, t = 1$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) y-curve.

For Case 2, in order to guarantee the well-definedness and the positiveness of f and the localization of u in all directions in the space, the parameters need to satisfy conditions

$$a_{14}(a_{13}^2(a_9^2 + a_{14}^2) + a_3^2 a_9^2) \neq 0, \tag{11}$$

$$a_{16} > 0. \tag{12}$$

A class of positive quadratic function solution of Eq. (3) is generated by the parameters in Case 2 as

$$f = \left(\frac{a_3 a_{11}}{a_{13}} x - \frac{a_3 a_{11}}{a_{13}} y + a_3 z + a_5\right)^2 + \left(-\frac{a_{11} a_{14}}{a_9} x \right.$$
$$\left. + \frac{a_{11} a_{14}^2 + a_9^2 a_{11} + a_9^2 a_{12}}{a_9 a_{14}} y - \frac{a_{13} a_{14}}{a_9} z + a_9 t + a_{10}\right)^2$$
$$+ \left(a_{11} x + a_{12} y + a_{13} z + a_{14} t\right.$$

$$+\frac{a_5a_{13}(a_9^2+a_{14}^2)+a_3a_9a_{10}a_{14}}{a_3a_9^2}\Bigg)^2$$
$$+\frac{3a_1^4a_{14}(a_{13}^2(a_9^2+a_{14}^2)^2+a_3^2a_9^2)^2}{\alpha a_9^4a_{13}^4(a_{14}^2+a_9^2)(a_{11}+a_{12}+a_{14})}, \tag{13}$$

under the transformation $u = 2(\ln f)_x$, which further generates a class of lump solution of Eq. (1) as

$$u^{(\mathrm{II})} = \frac{4(\frac{a_3a_{11}}{a_{13}}g - \frac{a_{11}a_{14}}{a_9}h + a_{11}l)}{f}, \tag{14}$$

where f is presented by Eq. (13) and g, h, l are given in the following forms:

$$\begin{cases} g = \dfrac{a_3a_{11}}{a_{13}}x - \dfrac{a_3a_{11}}{a_{13}}y + a_3z + a_5, \\[2ex] h = -\dfrac{a_{11}a_{14}}{a_9}x + \dfrac{a_{11}a_{14}^2 + a_9^2a_{11} + a_9^2a_{12}}{a_9a_{14}}y - \dfrac{a_{13}a_{14}}{a_9}z + a_9t + a_{10}, \\[2ex] l = a_{11}x + a_{12}y + a_{13}z + a_{14}t + \dfrac{a_5a_{13}(a_9^2+a_{14}^2)+a_3a_9a_{10}a_{14}}{a_3a_9^2}. \end{cases} \tag{15}$$

It is worth noting that eight parameters $a_3, a_5, a_9, a_{10}, a_{11}, a_{12}, a_{13}$ and a_{14} are involved in the solution $u^{(\mathrm{II})}$, among which the rest have to satisfy the conditions Eqs. (11) and (12) for the existence of the lump solution $u^{(\mathrm{II})}$ illustrated by Fig. 2.

For Case 3, which must satisfy the following conditions:

$$\alpha\beta a_9a_{13}^4(a_3^2+a_{13}^2) \neq 0, \tag{16}$$

$$\frac{M_2}{\alpha\beta a_9} > 0, \tag{17}$$

to ensure the well-definedness and the positiveness of f and the localization of u in all directions in the space, respectively. The parameters in Case 3 lead to the following class of positive quadratic function solution of Eq. (3):

$$f = \left(-\frac{a_3a_{12}}{a_{13}}x + \frac{a_3a_{12}}{a_{13}}y + a_3z + a_5\right)^2 + \Bigg(a_6x + \frac{-\alpha a_9(a_6+a_9)+\beta(a_3^2+a_{13}^2)}{\alpha a_9}y$$
$$+ a_9t + a_{10}\Bigg)^2 + (-a_{12}x + a_{12}y + a_{13}z + a_{15})^2 + \frac{M_2}{\alpha\beta a_9a_{13}^4(a_3^2+a_{13}^2)}, \tag{18}$$

through the transformation $u = 2(\ln f)_x$, which generates a class of lump solution of Eq. (1) as follows:

$$u^{(\mathrm{III})} = \frac{4(-\frac{a_3a_{12}}{a_{13}}g + a_6h - a_{12}l)}{f}, \tag{19}$$

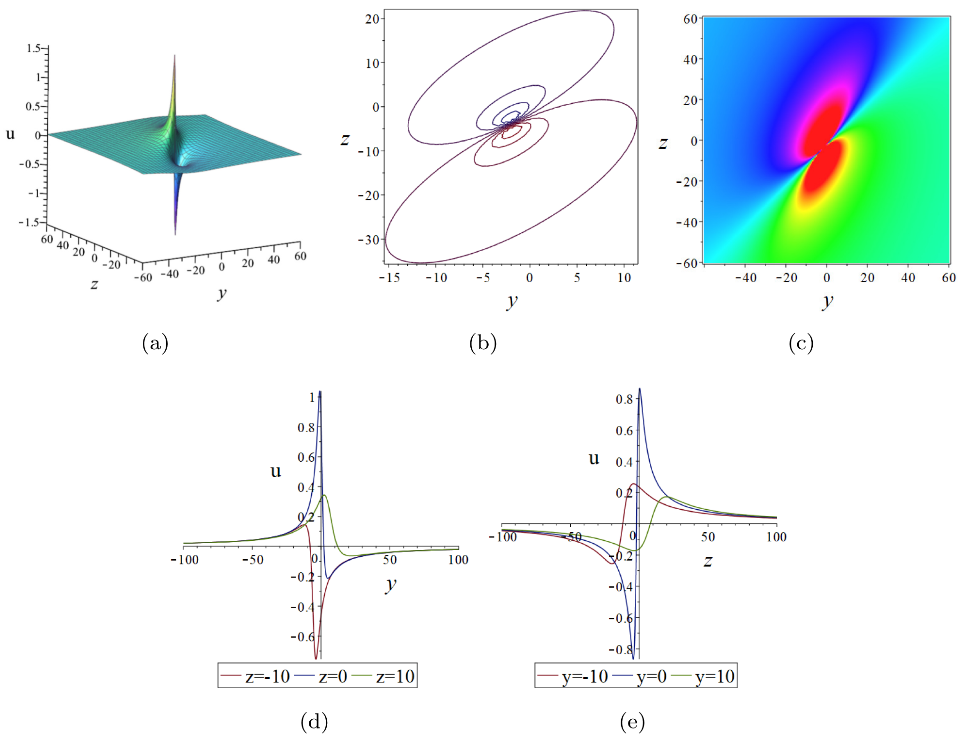


Fig. 2. (Color online) Lump dynamic characteristics of $u^{(\mathrm{II})}$ via (14) with $a_3 = 1, a_5 = 1, a_9 = 2, a_{10} = 1, a_{11} = 2, a_{12} = 1, a_{13} = 2, a_{14} = 3, \alpha = 1, \beta = 2, x = 1, t = 0$. (a) 3D plot; (b) contourplot; (c) density plot; (d) y-curve; (e) z-curve.

with f defined by Eq. (18) and g, h and l are defined as follows:

$$\begin{cases} g = -\dfrac{a_3 a_{12}}{a_{13}} x + \dfrac{a_3 a_{12}}{a_{13}} y + a_3 z + a_5, \\ h = a_6 x + \dfrac{-\alpha a_9 (a_6 + a_9) + \beta(a_3^2 + a_{13}^2)}{\alpha a_9} y + a_9 t + a_{10}, \\ l = -a_{12} x + a_{12} y + a_{13} z + a_{15}. \end{cases} \tag{20}$$

It should be noticed that eight parameters $a_3, a_5, a_6, a_9, a_{10}, a_{12}, a_{13}$ and a_{15} are involved in $u^{(\mathrm{III})}$, among which a_3, a_9, a_{13} have to meet the conditions (16) and (17) to ensure the existence of the lump solution $u^{(\mathrm{III})}$.

3. Lump Solutions of the Reduction with $z = x$

In this section, let us consider the reduction of the $(3+1)$-dimensional generalized KP-Boussinesq equation (1) with $z = x$, then Eq. (1) is reduced to

$$u_{xxxy} + 3(u_x u_y)_x + \alpha(u_x + u_y + u_t)_t + \beta u_{xx} = 0. \tag{21}$$

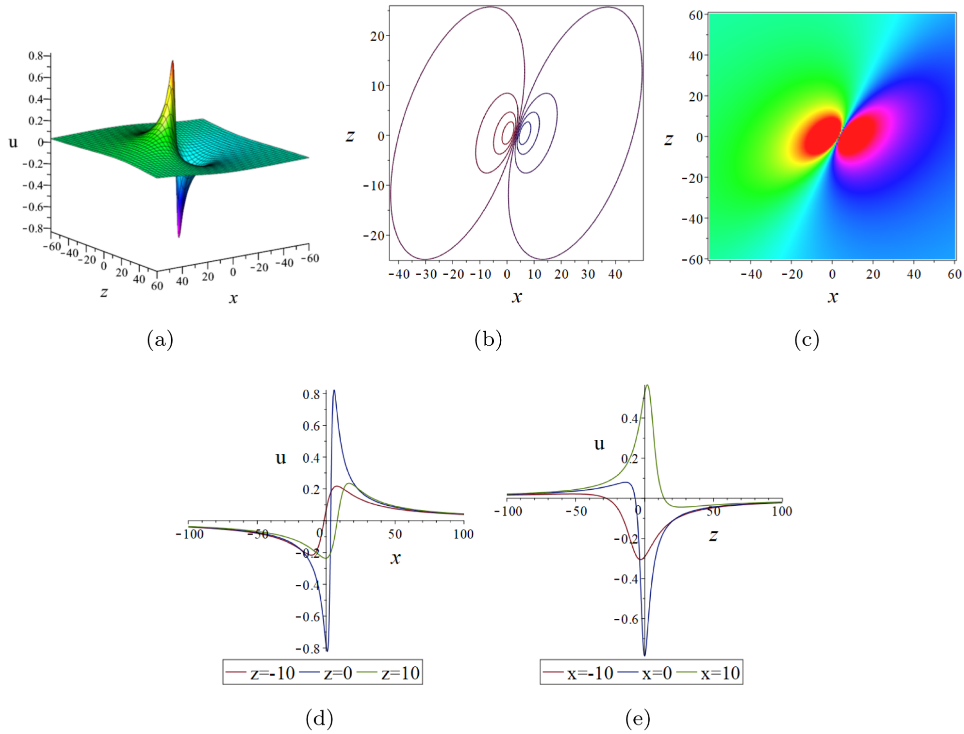


Fig. 3. (Color online) Lump dynamic characteristics of $u^{(\mathrm{III})}$ via (19) with $a_3 = 1, a_5 = 2, a_6 = 2, a_9 = 2, a_{10} = 2, a_{12} = 1, a_{13} = 2, a_{15} = 1, \alpha = -1, \beta = 2, y = 1, t = 0$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) z-curve.

Furthermore, this equation in turn leads to dimensionally reduced form of the Hirota bilinear equation (3) as follows:

$$[D_x^3 D_y + \alpha(D_x D_t + D_y D_t + D_t^2) + \beta D_x^2] f \cdot f = 0. \tag{22}$$

It is obvious that if $f = f(x, y, t)$ solves the Hirota bilinear equation (22), then $u = 2(\ln f)_x$ will solve Eq. (21). Therefore, we will derive positive quadratic function solutions of the dimensionally reduced Hirota bilinear equation (22) by starting with

$$\begin{cases} f = g^2 + h^2 + a_9, \\ g = a_1 x + a_2 y + a_3 t + a_4, \\ h = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{23}$$

where the real parameters $a_i (1 \le i \le 9)$ will be determined later. The direct substitution of f in (23) into Eq. (22) yields the following set of constraining equations for the parameters:

$$\left\{ a_2 = \frac{M_1}{\alpha(a_3^2 + a_7^2)}, a_6 = \frac{M_2}{\alpha(a_3^2 + a_7^2)}, a_9 = \frac{M}{\alpha\beta(a_1 a_7 - a_3 a_5)^2} \right\} \tag{24}$$

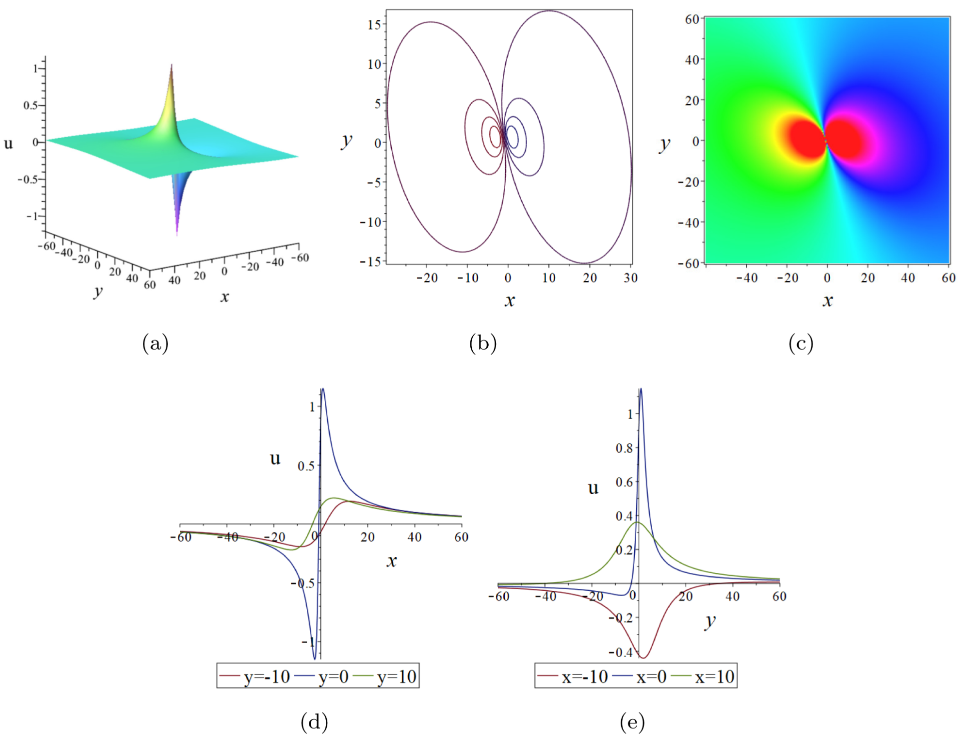


Fig. 4. (Color online) The amplitude of $u^{(\mathrm{V})}$ via (29) is $\frac{4\sqrt{93}}{31}$ and situated in $(-\frac{20t}{37}-\frac{56}{11}\pm\frac{\sqrt{93}}{6}, \frac{8t}{37}+\frac{52}{111})$ with $t=1, a_1=2, a_3=2, a_4=3, a_5=5, a_7=2, a_8=1, \alpha=2, \beta=-2$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) y-curve.

with

$$\begin{cases} M = 3(a_1^2+a_5^2)(a_3^2+a_7^2)(a_1^2+a_1a_3+a_5^2+a_5a_7)\alpha + 3(a_1^2 \\ \qquad + a_5^2)^2(a_1a_3+a_5a_7)\beta, \\ M_1 = -\alpha(a_1+a_3)(a_3^2+a_7^2) + \beta(a_3a_5^2 - a_1^2a_3 - 2a_1a_5a_7), \\ M_2 = -\alpha(a_3^2+a_7^2)(a_5+a_7) + \beta(a_1^2a_7 - a_5^2a_7 - 2a_1a_3a_5) \end{cases} \tag{25}$$

which have to satisfy the following conditions:

$$\Delta = \begin{vmatrix} a_1 & a_3 \\ a_5 & a_7 \end{vmatrix} = a_1a_7 - a_3a_5 \neq 0, \tag{26}$$

$$\frac{M}{\alpha\beta} > 0, \tag{27}$$

to guarantee the well-definedness and the positiveness of f and the localization of u in all directions in the space.

A class of positive quadratic function solution of (22) is obtained by the parameters in the set (24), which is

$$f = (a_1 x + \frac{M_1}{\alpha(a_3^2 + a_7^2)} y + a_3 t + a_4)^2 + (a_5 x + \frac{M_2}{\alpha(a_3^2 + a_7^2)} y + a_7 t + a_8)^2 + \frac{M}{\alpha\beta(a_1 a_7 - a_3 a_5)^2}. \tag{28}$$

Through transformation $u = 2(\ln f)_x$, a class of lump solution is generated for Eq. (21), which is

$$u^{(\mathrm{IV})} = \frac{4(a_1 g + a_5 h)}{f}, \tag{29}$$

where Eq. (28) defines the function f, and the functions g and h are presented in the following form:

$$\begin{cases} g = a_1 x + \dfrac{M_1}{\alpha(a_3^2 + a_7^2)} y + a_3 t + a_4, \\ h = a_5 x + \dfrac{M_2}{\alpha(a_3^2 + a_7^2)} y + a_7 t + a_8. \end{cases} \tag{30}$$

It should be noted that the solution $u^{(\mathrm{IV})}$ involves six parameters a_1, a_3, a_4, a_5, a_7 and a_8, among which a_4 and a_8 are arbitrary parameters, and other parameters must meet conditions (26) and (27) to ensure that $u^{(\mathrm{IV})}$ is a lump solution.

It is also observed that the corresponding lump solution $u^{(\mathrm{IV})}$ tends to zero while the determinant Δ in (26) tends to zero. Particularly, taking

$$\{a_1 = 1, a_3 = 3, a_4 = 0, a_5 = 1, a_7 = 1 + \varepsilon, a_8 = 0, \alpha = 1, \beta = 2\} \tag{31}$$

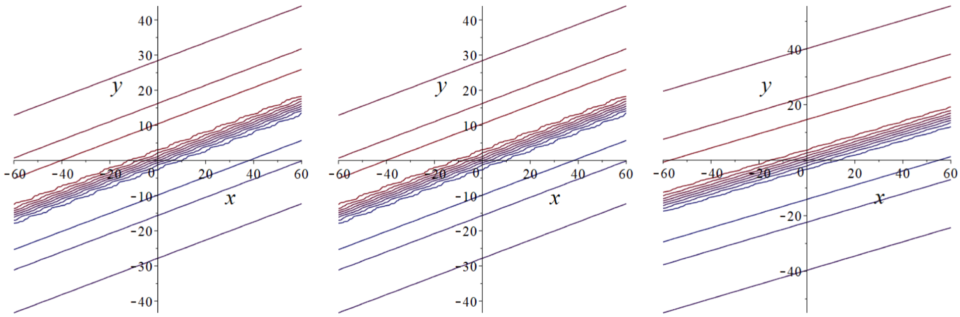
which implies $\Delta = \varepsilon$. As the following is the lump solution obtained from Eq. (29),

$$u = \frac{4\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}, \tag{32}$$

where

$$\begin{cases} p(\varepsilon) = (t - y)\varepsilon^3 + (2x - 6y + 4t)\varepsilon^2 + (4x - 14y + 6t)\varepsilon \\ \qquad + 4x - 16y + 4t, \\ q(\varepsilon) = 96 + (t - y)^2\varepsilon^6 + (4t^2 + (2x - 10y)t - 2xy + 6y^2 \\ \qquad + 3)\varepsilon^5 + (8t^2 + (8x - 24y)t + 2x^2 - 12xy + 18y^2 \\ \qquad + 24)\varepsilon^4 + (8t^2 + (12x - 44y)t + 4x^2 - 28xy + 48y^2 \\ \qquad + 84)\varepsilon^3 + (4t^2 + (8x - 32y)t + 4x^2 - 32xy + 64y^2 \\ \qquad + 168)\varepsilon^2 + 180\varepsilon. \end{cases} \tag{33}$$

It is obvious that the limit of this lump solution u in (32) is zero while ε tends to zero. Three different contour plots are displayed in Fig. 5 with $t = 1$ and three different values of Δ.



(a) (b) (c)

Fig. 5. (Color online) Lump dynamic characteristics of u via (32) with $t = 1$. (a) $\Delta = 0.3$; (b) $\Delta = 0.2$; (c) $\Delta = 0.1$.

4. Lump Solutions of the Reduction with $z = y$

The reduction of the $(3 + 1)$-dimensional generalized KP-Boussinesq equation (1) with $z = y$ will be discussed in this section, and Eq. (1) is reduced as

$$u_{xxxy} + 3(u_x u_y)_x + \alpha(u_x + u_y + u_t)_t + \beta u_{yy} = 0. \tag{34}$$

Moreover, the corresponding reduction form of Hirota bilinear equation (3) is as follows:

$$[D_x^3 D_y + \alpha(D_x D_t + D_y D_t + D_t^2) + \beta D_y^2] f \cdot f = 0. \tag{35}$$

Obviously, if $f = f(x, y, t)$ is the solution of the Hirota bilinear equation (35), then $u = 2(\ln f)_x$ will be able to solve Eq. (34). In order to find the lump solution of Eq. (34), we begin with the quadratic solution of the Hirota bilinear equation (35) as follows:

$$\begin{cases} f = g^2 + h^2 + a_9, \\ g = a_1 x + a_2 y + a_3 t + a_4, \\ h = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{36}$$

where a_i $(1 \leq i \leq 9)$ are real parameters to be determined later. By the direct substitution of f in Eq. (36) into Eq. (35), the set of constraint equations for the parameters are obtained as follows:

$$\left\{ a_1 = \frac{M_1}{\alpha(a_3^2 + a_7^2)}, a_5 = \frac{M_2}{\alpha(a_3^2 + a_7^2)}, a_9 = \frac{M}{\alpha^3 \beta(a_3^2 + a_7^2)(a_2 a_7 - a_3 a_6)^2} \right\} \tag{37}$$

with

$$\begin{cases} M = 3((a_3^2 + a_7^2)(a_2^2 + a_2a_3 + a_6^2 + a_6a_7)\alpha \\ \qquad + \beta(a_2^2 + a_6^2)(a_2a_3 + a_6a_7))((a_3^2 + a_7^2)(a_2^2 + 2a_2a_3 + a_3^2 + a_6^2 \\ \qquad + 2a_6a_7 + a_7^2)\alpha^2 + 2\beta((a_2^2 - a_6^2)a_3^2 + a_2(a_2^2 + a_6^2 \\ \qquad + 4a_6a_7)a_3 + a_7((-a_2^2 + a_6^2)a_7 + a_2^2a_6 + a_6^3))\alpha \\ \qquad + \beta^2(a_2^2 + a_6^2)^2), \\ M_1 = -\alpha(a_3^2 + a_7^2)(a_2 + a_3) + \beta(a_3a_6^2 - a_2^2a_3 - 2a_2a_6a_7), \\ M_2 = -(a_3^2 + a_7^2)(a_6 + a_7)\alpha + \beta(a_2^2a_7 - a_6^2a_7 - 2a_2a_3a_6) \end{cases} \tag{38}$$

which need to satisfy the conditions

$$\Delta = \begin{vmatrix} a_2 & a_3 \\ a_6 & a_7 \end{vmatrix} = a_2a_7 - a_3a_6 \neq 0, \tag{39}$$

$$\frac{M}{\alpha^3\beta} > 0, \tag{40}$$

to ensure the good definition and the positiveness of f and the locality of u in all directions of space, respectively. The parameters in the set (37) generate the following class of positive quadratic function solution of Eq. (35) as

$$f = \left(\frac{M_1}{\alpha(a_3^2 + a_7^2)}x + a_2y + a_3t + a_4\right)^2 + \left(\frac{M_2}{\alpha(a_3^2 + a_7^2)}x + a_6y + a_7t + a_8\right)^2 + \frac{M}{\alpha^3\beta(a_3^2 + a_7^2)(a_2a_7 - a_3a_6)^2}. \tag{41}$$

By means of the transformation $u = 2(\ln f)_x$, a class of lump solution of Eq. (34) can be further obtained, that is

$$u^{(\mathrm{V})} = \frac{4(a_1g + a_5h)}{f}, \tag{42}$$

where f is defined by Eq. (41), then g and h are given as follows:

$$\begin{cases} g = \dfrac{M_1}{\alpha(a_3^2 + a_7^2)}x + a_2y + a_3t + a_4, \\ h = \dfrac{M_2}{\alpha(a_3^2 + a_7^2)}x + a_6y + a_7t + a_8. \end{cases} \tag{43}$$

It is worth noting that six parameters a_2, a_3, a_4, a_6, a_7 and a_8 are involved in the solution $u^{(\mathrm{V})}$, among which the rest are required to satisfy the conditions (39) and (40) to guarantee both analyticity and localization of $u^{(\mathrm{V})}$.

Based on Eq. (42), we can find that the lump solution $u^{(\mathrm{V})}$ tends to zero when the determinant in Eq. (39) tends to zero. We take the following particular parameters:

$$\{a_2 = 1, a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 1 + \varepsilon, a_8 = 0, \alpha = 1, \beta = 2\} \tag{44}$$

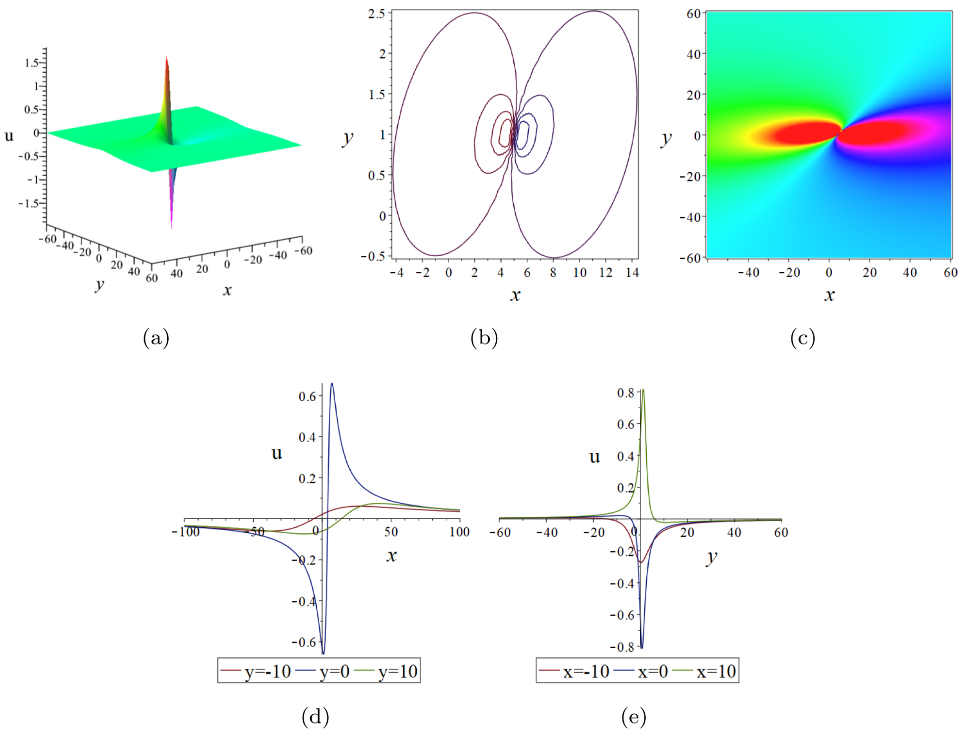


Fig. 6. (Color online) The amplitude of $u^{(\mathrm{V})}$ via (42) is $2\sqrt{6}$ and situated in $(5\mathrm{t} \pm \frac{\sqrt{6}}{6}, \mathrm{t})$ with $a_2 = -1, a_3 = 2, a_4 = 0, a_6 = 1, a_7 = 1, a_8 = 0, \alpha = 1, \beta = 2, t = 1$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) y-curve.

which indicates $\Delta = \varepsilon$, from Eq. (42) we can obtain the lump solution as follows:

$$u = \frac{-8\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}, \tag{45}$$

where

$$\begin{cases} p(\varepsilon) = (t-x)\varepsilon^4 + (5t - 6x + y)\varepsilon^3 + (12t - 18x + 6y)\varepsilon^2 \\ \qquad + (22t - 48x + 14y)\varepsilon - 64x + 16y + 16t, \\ q(\varepsilon) = 3072 + 3\varepsilon^7 + (2t^2 - 4xt + 2x^2 + 36)\varepsilon^6 + (8t^2 \\ \qquad + (-20x + 4y)t + 12x^2 - 4xy + 204)\varepsilon^5 + (16t^2 \\ \qquad + (-48x + 16y)t + 36x^2 - 24xy + 4y^2 + 768)\varepsilon^4 \\ \qquad + (16t^2 + (-88x + 24y)t + 96x^2 - 56xy + 8y^2 \\ \qquad + 2100)\varepsilon^3 + (8t^2 + (-64x + 16y)t + 128x^2 \\ \qquad - 64xy + 8y^2 + 4032)\varepsilon^2 + 4992\varepsilon. \end{cases} \tag{46}$$

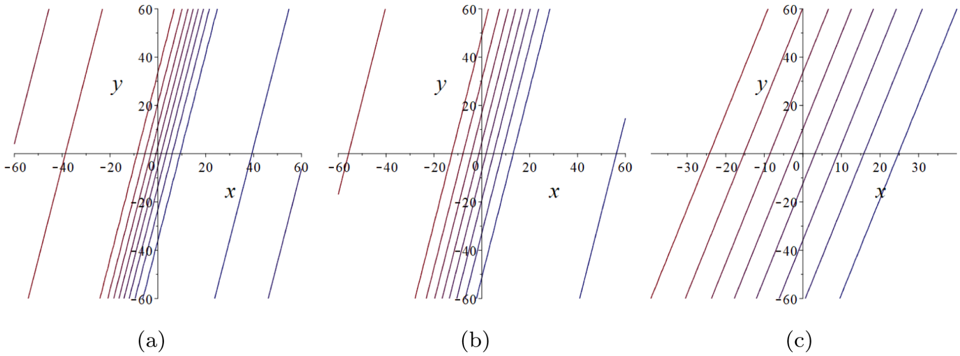


Fig. 7. (Color online) Lump dynamic characteristics of u via (45) with $t = 1$. (a) $\Delta = 0.3$; (b) $\Delta = 0.2$; (c) $\Delta = 0.1$.

Obviously, the limit of this lump solution u in Eq. (45) tends to zero while ε tends to zero. The contour plots are displayed in Fig. 7 when $t = 1$ with different values of Δ.

5. Lump Solutions of the Reduction with $y = x$

In this section, we will continue to discuss the reduction of Eq. (1). For $y = x$, Eq. (1) is reduced to

$$u_{xxxxx} + 3(u_x^2)_x + \alpha(2u_x + u_t)_t + \beta u_{zz} = 0, \tag{47}$$

and the bilinear equation (3) is reduced to the following form:

$$[D_x^4 D_y + \alpha(2D_x D_t + D_t^2) + \beta D_z^2] f \cdot f = 0. \tag{48}$$

It is clear that if $f = f(x, z, t)$ solves the bilinear equation (48), then $u = 2(\ln f)_x$ will solve Eq. (47). To study lump solutions of Eq. (47), we suppose the quadratic function solutions of the bilinear equation (48) as follows:

$$\begin{cases} f = g^2 + h^2 + a_9, \\ g = a_1 x + a_2 y + a_3 t + a_4, \\ h = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{49}$$

where $a_i (1 \leq i \leq 9)$ are real parameters will be determined later. By using symbolic computation with Maple on the direct substitution of f in Eq. (49) into Eq. (48), which yields the following set of constraining equations for the parameters

$$\left\{ a_1 = \frac{M_1}{2\alpha(a_3^2 + a_7^2)}, a_5 = \frac{M_2}{2\alpha(a_3^2 + a_7^2)}, a_9 = -\frac{M}{16\alpha^4 \beta(a_3^2 + a_7^2)(a_2 a_7 - a_3 a_6)^2} \right\}, \tag{50}$$

where

$$\begin{cases} M = 3((a_3^2 + a_7^2)^2\alpha^2 + 2\beta((a_2 + a_6)a_3 - a_7(a_2 - a_6)) \\ \qquad \times ((a_2 - a_6)a_3 + a_7(a_2 + a_6))\alpha + \beta^2(a_2^2 + a_6^2)^2)^2, \\ M_1 = -\alpha a_3(a_3^2 + a_7^2) + \beta((-a_2^2 + a_6^2)a_3 - 2a_2a_6a_7, \\ M_2 = -\alpha a_7(a_3^2 + a_7^2) + \beta((a_2^2 - a_6^2)a_7 - 2a_2a_3a_6). \end{cases} \tag{51}$$

In order to guarantee the good definiteness and the positiveness of f and the localization of u in all directions in the space, the parameters in (50) need to satisfy the following conditions:

$$\Delta = \begin{vmatrix} a_2 & a_3 \\ a_6 & a_7 \end{vmatrix} = a_2a_7 - a_3a_6 \neq 0, \tag{52}$$

$$\beta < 0. \tag{53}$$

The parameters in the set (50) generate the following class of quadratic function solution of the bilinear equation (48):

$$f = \left(\frac{M_1}{2\alpha(a_3^2 + a_7^2)}x + a_2z + a_3t + a_4\right)^2 + \left(\frac{M_2}{2\alpha(a_3^2 + a_7^2)}x + a_6z + a_7t + a_8\right)^2 - \frac{M}{16\alpha^4\beta(a_3^2 + a_7^2)(a_2a_7 - a_3a_6)^2}. \tag{54}$$

By virtue of the transformation $u = 2(\ln f)_x$, a class of lump solutions to equation (47) is derived as

$$u^{(\mathrm{VI})} = \frac{4(a_1g + a_5h)}{f}, \tag{55}$$

where f is defined by Eq. (54), and g and h are presented in the following forms:

$$\begin{cases} g = \dfrac{M_1}{2\alpha(a_3^2 + a_7^2)}x + a_2z + a_3t + a_4, \\ h = \dfrac{M_2}{2\alpha(a_3^2 + a_7^2)}x + a_6z + a_7t + a_8. \end{cases} \tag{56}$$

It is observed that six parameters a_2, a_3, a_4, a_6, a_7 and a_8 are involved in $u^{(\mathrm{VI})}$, among which the rest need to satisfy the conditions (52) and (53) to guarantee the existence of the lump solution $u^{(\mathrm{VI})}$ illustrated by Fig. 8.

On account of (55), we find that the lump solution $u^{(\mathrm{VI})}$ approaches to zero as the determinant Δ in (52) tends to zero. Generally, we take the following special parameters:

$$\{a_2 = 1, a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 1 + \varepsilon, a_8 = 0, \alpha = 1, \beta = -2\} \tag{57}$$

which gives a rise to $\Delta = \varepsilon$, substituting Eq. (57) into Eq. (55), we have the following lump solution:

$$u = \frac{-64\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}, \tag{58}$$

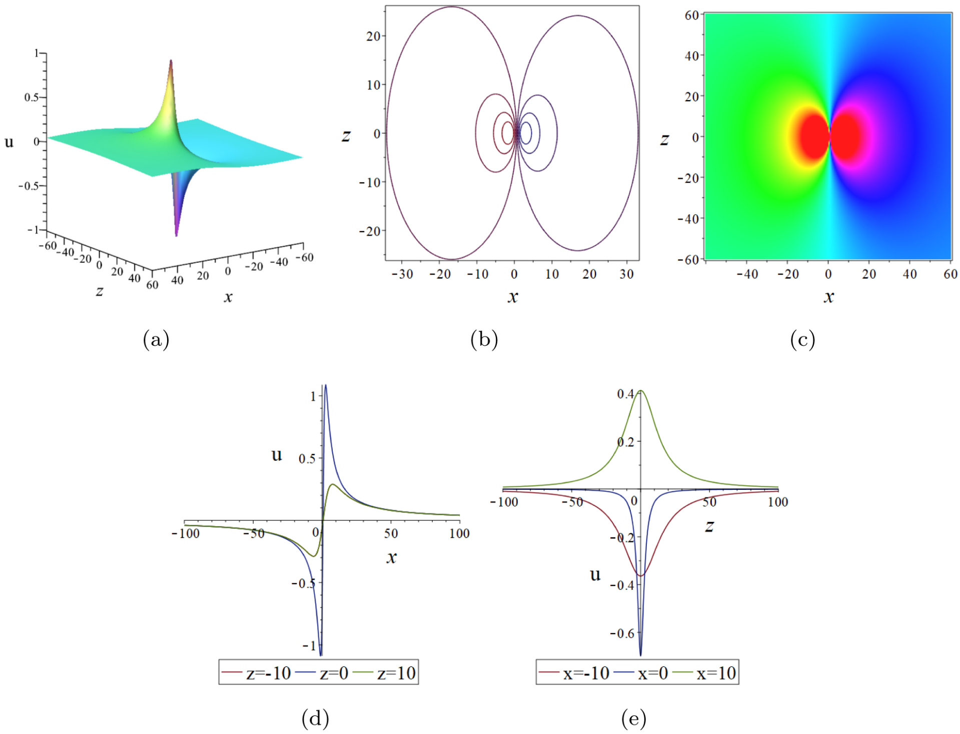
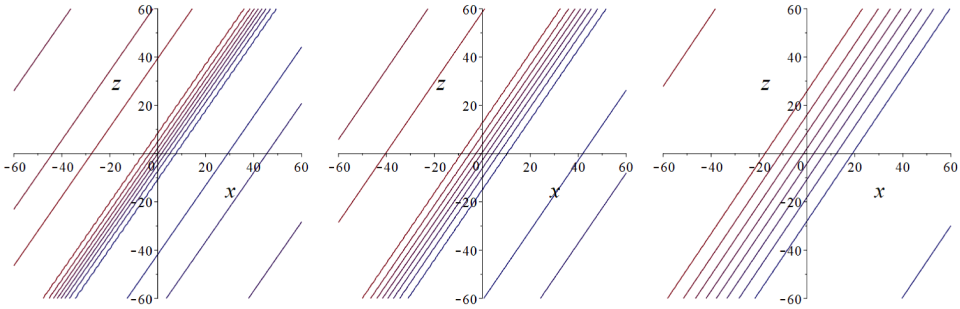


Fig. 8. (Color online) The amplitude of $u^{(\mathrm{VI})}$ via (55) is $\frac{4\sqrt{6}}{9}$ and situated in $(\frac{2}{3} \pm \frac{3\sqrt{6}}{4}, 0)$ with $a_2 = -1, a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 1, a_8 = 0, \alpha = 1, \beta = -2, t = 1$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) z-curve.

where

$$\begin{cases} p(\varepsilon) = \left(t - \dfrac{x}{2}\right)\varepsilon^4 + (4t - 2x + z)\varepsilon^3 + (8t - 4x + 4z)\varepsilon^2 + (16t \\ \qquad - 12x + 10z)\varepsilon + 12t - 18x + 12z, \\ q(\varepsilon) = 3888 + 3\varepsilon^8 + 24\varepsilon^7 + (32t^2 - 32xt + 8x^2 + 96)\varepsilon^6 + (128t^2 \\ \qquad + (-128x + 64z)t + 32x^2 - 32xz + 336)\varepsilon^5 + (256t^2 \\ \qquad + (-256x + 256z)t + 64x^2 - 128xz + 64z^2 + 984)\varepsilon^4 \\ \qquad + (256t^2 + (-512x + 384z)t + 192x^2 - 320xz + 128z^2 \\ \qquad + 2016)\varepsilon^3 + (128t^2 + (-384x + 256z)t + 288x^2 - 384xz \\ \qquad + 128z^2 + 3456)\varepsilon^2 + 5184\varepsilon. \end{cases} \tag{59}$$

Apparently, when ε approaches to zero, the limit of lump solution u in Eq. (58) is zero. This is fully illustrated by Fig. 9 with different values of Δ.



(a) (b) (c)

Fig. 9. (Color online) Lump dynamic characteristics of u via (58) with $t = 1$. (a) $\Delta = 0.3$; (b) $\Delta = 0.2$; (c) $\Delta = 0.1$.

6. Lump Solutions of the Reduction with $y = t$

Equation (1) is reduced to the following equation in $(2+1)$ dimensions by considering $y = t$ as follows:

$$u_{xxxt} + 3(u_x u_t)_x + \alpha(u_x + 2u_t)_t + \beta u_{zz} = 0, \tag{60}$$

of which the corresponding Hirota bilinear equation is

$$[D_x^3 D_t + \alpha(D_x D_t + 2D_t^2) + \beta D_z^2] f \cdot f = 0. \tag{61}$$

It is clear that if $f = f(x, z, t)$ solves Eq. (61), then $u = 2(\ln f)_x$ is the solution of Eq. (60). Hence, we first take the positive quadratic function solution of Eq. (61) by starting with

$$\begin{cases} f = g^2 + h^2 + a_9, \\ g = a_1 x + a_2 y + a_3 t + a_4, \\ h = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{62}$$

where the real parameters $a_i (1 \leq i \leq 9)$ will be determined later. By substituting f in Eq. (62) into Eq. (61), a set of constraint equations for the parameters are obtained as follows:

$$\left\{ a_1 = \frac{M_1}{\alpha(a_3^2 + a_7^2)}, a_5 = \frac{M_2}{\alpha(a_3^2 + a_7^2)}, a_9 = \frac{M}{\alpha^3 \beta (a_3^2 + a_7^2)(a_2 a_7 - a_3 a_6)^2} \right\} \tag{63}$$

with

$$\begin{cases} M = 3(((a_2^2 + a_6^2)^2 \beta^2 + 4((a_2 + a_6)a_3 - a_7(a_2 - a_6)) \\ \qquad \times ((a_2 - a_6)a_3 + a_7(a_2 + a_6))\alpha\beta + 4\alpha^2(a_3^2 + a_7^2)^2) \\ \qquad \times (((a_2^2 - a_6^2)a_3^2 + 4a_2 a_3 a_6 a_7 + (-a_2^2 + a_6^2)a_7^2)\beta \\ \qquad + 2\alpha(a_3^2 + a_7^2)^2)), \\ M_1 = -2\alpha a_3(a_3^2 + a_7^2) + \beta(-a_2^2 a_3 + a_3 a_6^2 - 2a_2 a_6 a_7), \\ M_2 = -2\alpha a_7(a_3^2 + a_7^2) + \beta((a_2^2 - a_6^2)a_7 - 2a_2 a_3 a_6). \end{cases} \tag{64}$$

In order to guarantee the well-definedness and the positiveness of f and the localization of u in all directions in the space, respectively, the parameters in the set (63) are required to satisfy the following conditions:

$$\Delta = \begin{vmatrix} a_2 & a_3 \\ a_6 & a_7 \end{vmatrix} = a_2a_7 - a_3a_6 \neq 0, \tag{65}$$

$$\frac{M}{\alpha^3\beta} > 0. \tag{66}$$

The set (63) yields the following class of positive quadratic function solution of Eq. (61):

$$f = \left(\frac{M_1}{\alpha(a_3^2 + a_7^2)}x + a_2z + a_3t + a_4\right)^2 + \left(\frac{M_2}{\alpha(a_3^2 + a_7^2)}x + a_6z + a_7t + a_8\right)^2 + \frac{M}{\alpha^3\beta(a_3^2 + a_7^2)(a_2a_7 - a_3a_6)^2}. \tag{67}$$

Via the transformation $u = 2(\ln f)_x$, a class of lump solution of Eq. (60) is obtained as follows:

$$u^{(\mathrm{VII})} = \frac{4(a_1g + a_5h)}{f}, \tag{68}$$

where f is defined by Eq. (67) g and h are defined as

$$\begin{cases} g = \dfrac{M_1}{\alpha(a_3^2 + a_7^2)}x + a_2z + a_3t + a_4, \\ h = \dfrac{M_2}{\alpha(a_3^2 + a_7^2)}x + a_6z + a_7t + a_8. \end{cases} \tag{69}$$

It should be noted that the solution $u^{(\mathrm{VII})}$ involves six parameters a_2, a_3, a_4, a_6, a_7 and a_8, among which a_4 and a_8 are arbitrary parameters, and other parameters must satisfy conditions (65) and (66) to ensure that $u^{(\mathrm{VII})}$ is a lump solution illustrated by Fig. 10.

To sum up, according to Eq. (68), we can find that the corresponding lump solution $u^{(\mathrm{VII})}$ approaches zero while the determinant Δ tends to zero. We take special parameters as follows:

$$\{a_2 = 1, a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 1 + \varepsilon, a_8 = 0, \alpha = -2, \beta = 2\} \tag{70}$$

which indicates $\Delta = \varepsilon$, then the lump solution of Eq. (68) is derived as follows:

$$u = \frac{-8\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}, \tag{71}$$

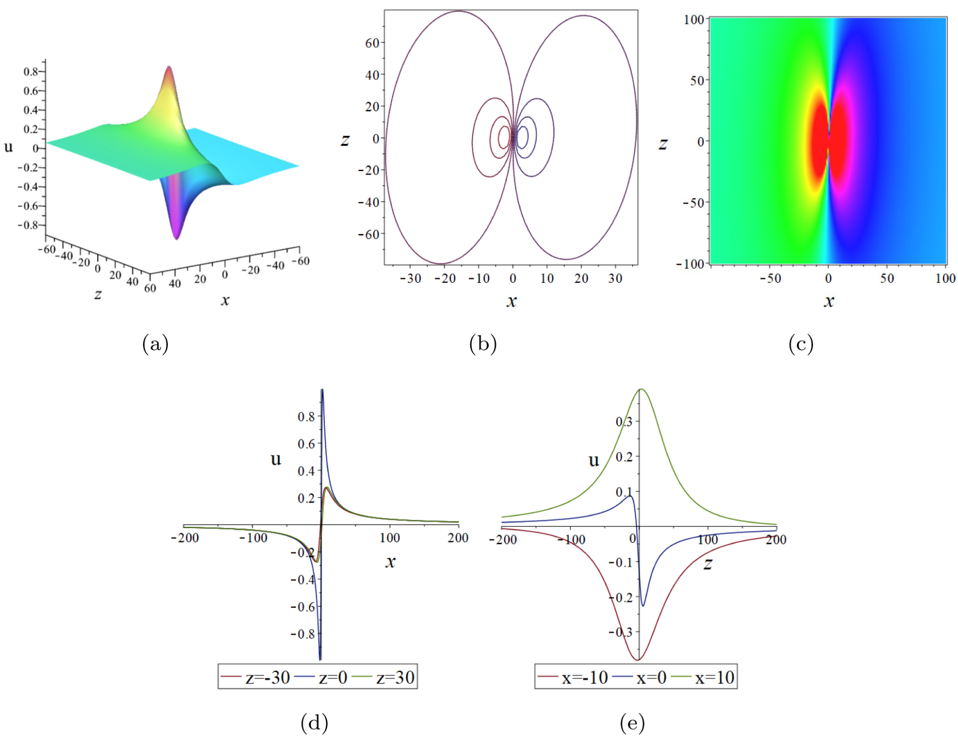
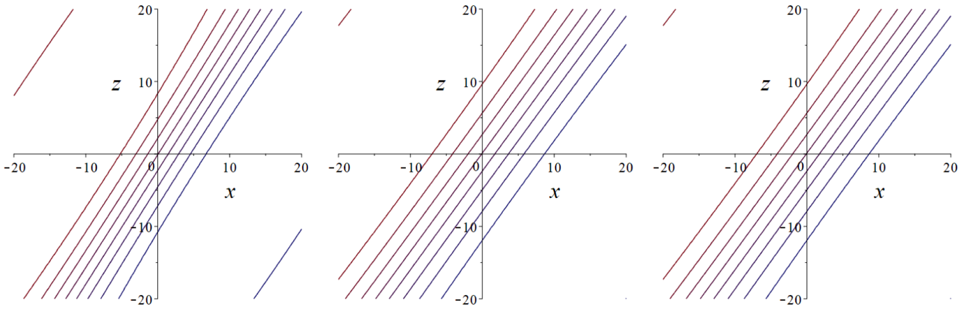


Fig. 10. (Color online) The amplitude of $u^{(\mathrm{VII})}$ via (68) is 1 and situated in $(\frac{t}{7} \pm 2,\ \frac{2t}{7})$ with $a_2 = -1, a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 1, a_8 = 0, \alpha = -1, \beta = 2, t = 1$. (a) 3D plot; (b) contourplot; (c) density plot; (d) x-curve; (e) z-curve.

where

$$\begin{cases} p(\varepsilon) = (t-2x)\varepsilon^4 + (4t-8x+z)\varepsilon^3 + (8t-16x+4z)\varepsilon^2 \\ \qquad\quad + (6t-8x+5z)\varepsilon + 2t - 2x + 2z, \\ q(\varepsilon) = 24 + 12\varepsilon^8 + 96\varepsilon^7 + (t^2 - 4xt + 4x^2 + 384)\varepsilon^6 + (4t^2 \\ \qquad\quad + (-16x+2z)t + 16x^2 - 4xz + 888)\varepsilon^5 + (8t^2 \\ \qquad\quad + (-32x+8z)t + 32x^2 - 16xz + 2z^2 + 1284)\varepsilon^4 \\ \qquad\quad + (8t^2 + (-24x+12z)t + 16x^2 - 20xz + 4z^2 \\ \qquad\quad + 1104)\varepsilon^3 + (4t^2 + (-8x+8z)t + 4x^2 - 8xz \\ \qquad\quad + 4z^2 + 576)\varepsilon^2 + 168\varepsilon. \end{cases} \tag{72}$$

Obviously, the limit of the lump solution u in Eq. (71) equals zero while ε tends to zero, which is illustrated by Fig. 11 with $t = 1$ and three different values of Δ.



(a) (b) (c)

Fig. 11. (Color online) Lump dynamic characteristics of u via (71) with $t = 1$. (a) $\Delta = 0.3$; (b) $\Delta = 0.2$; (c) $\Delta = 0.1$.

7. Conclusions

Using the Hirota bilinear operator theory and symbolic computation with Maple, we have studied the positive quadratic functions solutions of the (3+1)-dimensional generalized KP-Boussinesq equation (1) and its four reduction forms in (2 + 1) dimensions. As a consequence, some lump solutions of these equations have been derived. Furthermore, the restriction conditions for guaranteeing the analyticity and positiveness and localization of these obtained solutions has been also derived. The dynamic properties have been illustrated by the corresponding graphs with specific parameters. It should be noted that the lump solutions of Eq. (1) are constructed by using the sum of three positive quadratic functions. As a direct and simple and robust method, this method can be used to construct rogue wave solutions in terms of positive polynomial solutions of the associated bilinear equations. This will be our future research project.

Appendix A.

Case 1.

$$\Big\{a_2 = -\frac{\alpha a_1 a_{14} + 2\beta a_3 a_{13}}{\alpha a_{14}}, a_4 = 0, a_6 = \frac{a_1 a_8}{a_3}, a_7 = -\frac{a_8(\alpha a_1 a_{14} + 2\beta a_3 a_{13})}{\alpha a_3 a_{14}},$$

$$a_9 = 0, a_{12} = -\frac{(a_{13}^2 - a_3^2 - a_8^2)\beta + \alpha a_{14}(a_{11} + a_{14})}{\alpha a_{14}},$$

$$a_{16} = \frac{M_1}{\alpha\beta a_3^4 a_{14}(a_3^2 + a_8^2)}\Big\},$$

where

$$M_1 = \Big(- a_3^4 a_{14}\beta(a_3 a_{10} - a_5 a_8)^2 + 3a_3^4 a_{14}(a_1^2 + a_{11}^2)(a_1^2 + a_{11}^2 + a_{11}a_{14})$$

$$+ 6a_1^2a_8^2a_{14}\left(a_1^2(a_3^2+\frac{a_8^2}{2})+a_3^2a_{11}\left(a_{11}+\frac{a_{14}}{2}\right)\right)\right)\alpha + 6a_3((a_1^2+a_{11}^2)a_3^2$$

$$+ a_1^2a_8^2)\left(-\frac{a_3^2a_{11}}{2}+a_1a_3^2a_{13}+a_3a_{11}\left(-\frac{a_8^2}{2}+\frac{a_{13}^2}{2}\right)+a_1a_8^2a_{13}\right)\beta.$$

Case 2.

$$\Big\{a_1=\frac{a_3a_{11}}{a_{13}}, a_2=-\frac{a_3a_{11}}{a_{13}}, a_4=0, a_6=-\frac{a_{11}a_{14}}{a_9}, a_7=\frac{a_{11}a_{14}^2+a_9^2a_{11}+a_9^2a_{12}}{a_9a_{14}},$$

$$a_{15}=\frac{a_5a_{13}(a_9^2+a_{14}^2)+a_3a_9a_{10}a_{14}}{a_3a_9^2}, \beta=\frac{\alpha a_9^2(a_9^2+a_{14}^2)(a_{11}+a_{12}+a_{14})}{a_{14}(a_{13}^2(a_9^2+a_{14}^2)+a_3^2a_9^2)}$$

$$a_8=-\frac{a_{13}a_{14}}{a_9}, a_{16}=\frac{3a_1^4a_{14}(a_{13}^2(a_9^2+a_{14}^2)^2+a_3^2a_9^2)^2}{\alpha a_9^4a_{13}^4(a_{14}^2+a_9^2)(a_{11}+a_{12}+a_{14})}\Big\}.$$

Case 3.

$$\Big\{a_1=-\frac{a_3a_{12}}{a_{13}}, a_2=\frac{a_3a_{12}}{a_{13}}, a_4=0, a_7=\frac{-\alpha a_9(a_6+a_9)+\beta(a_3^2+a_{13}^2)}{\alpha a_9},$$

$$a_8=0, a_{11}=-a_{12}, a_{14}=0, a_{16}=\frac{M_2}{\alpha\beta a_9a_{13}^4(a_3^2+a_{13}^2)}\Big\},$$

where

$$M_2=\Big(-\beta a_9a_{13}^4(-a_5a_{13}+a_3a_{15})^2+3a_9a_{13}^4(a_6^2+a_{12}^2)(a_6^2+a_{12}^2+a_6a_9)$$

$$+6a_3^2a_9a_{12}^2a_{13}^2\left(a_{12}^2+a_6\left(a_6+\frac{a_9}{2}\right)\right)+3a_3^4a_9a_{12}^4\Big)\alpha-3\beta a_6a_{13}^2(a_3^2$$

$$+a_{13}^2)(a_{13}^2(a_6^2+a_{12}^2)+a_3^2a_{12}^2).$$

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11971067, 11101029, 12001556), National Science Foundation (DMS-1664561), Beijing Nova program (Nos. Z1311090, 00413029), Beijing Social Science Fund Project (15JGC184), the Program for Innovation Research in Central University of Finance and Economics (CUFE), the Disciplinary Funds and the Emerging Interdisciplinary Project of CUFE.

References

1. X. Lü, S. T. Chen and W. X. Ma, *Nonlinear Dyn.* **86** (2016) 523.
2. B. Ren, *Phys. Scr.* **90** (2015) 065206.
3. B. Ren, W. X. Ma and J. Yu, *Nonlinear Dyn.* **96** (2019) 717.

4. J. P. Yu, F. D. Wang, W. X. Ma, Y. L. Sun and C. M. Khalique, *Nonlinear Dyn.* **25** (2019) 1687.
5. Z. L. Zhao and L. C. He, *Nonlinear Dyn.* **100** (2020) 2753.
6. L.-L. Zhang, J.-P. Yu, W.-X. Ma, C. M. Khalique and Y.-L. Sun, *Nonlinear Dyn.* **104** (2021) 4317.
7. Z. L. Zhao and L. C. He, *Appl. Math. Lett.* **111** (2021) 106612.
8. Y. L. Sun, W. X. Ma, J. P. Yu and C. M. Khalique, *Mod. Phys. Lett. B* **32** (2018) 1850282.
9. J. P. Yu and Y. L. Sun, *Nonlinear Dyn.* **87** (2017) 2755.
10. S. J. Chen, Y. H. Yin, W. X. Ma and X. Lü, *Analy. Math. Phys.* **9** (2019) 2329.
11. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its and V. B. Matveev, *Phys. Lett. A* **63** (1977) 205.
12. J. Satsuma and M. J. Ablowitz, *J. Math. Phys.* **20** (1979) 1496.
13. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
14. L. N. Gao, Y. Y. Zi, Y. H. Yin, W. X. Ma and X. Lü, *Nonlinear Dyn.* **89** (2017) 2233.
15. X. Lü, Y. F. Hua, S. J. Chen and X. F. Tang, *Commun. Nonlinear Sci. Numer. Simul.* **95** (2021) 105612.
16. S. J. Chen, X. Lü and X. F. Tang, *Commun. Nonlinear Sci. Numer. Simul.* **95** (2021) 105628.
17. S. J. Chen, X. Lü, M. G. Li and F. Wang, *Commun. Phys. Scr.* **96** (2021) 095201.
18. R. Z. Gong and D. S. Wang, *Appl. Math. Lett.* **126** (2022) 107795.
19. D. S. Wang, Y. R. Shi, W. X. Feng and L. Wen, *Physica D* **351–352** (2017) 30.
20. D. Bilman, R. Buckingham and D. S. Wang, *J. Differ. Equ.* **297** (2021) 320.
21. D. S. Wang, L. Xu and Z. X. Xuan, *J. Nonlinear Sci.* **32** (2022) 3.
22. X. Lü and S. J. Chen, *Nonlinear Dyn.* **103** (2021) 947.
23. X. Lü and S. J. Chen, *Commun. Nonlinear Sci. Numer. Simul.* **103** (2021) 105939.
24. M. Z. Yin, Q. W. Zhu and X. Lü, *Nonlinear Dyn.* 106 (2021) 1347.
25. X. Lü *et al.*, *Nonlinear Dyn.* **106** (2021) 1491.
26. B. Ren, X. P. Cheng and J. Lin, *Nonlinear Dyn.* **86** (2016) 1855.
27. S. Kumar, M. Niwas, M. S. Osman and M. A. Abdou, *Commun. Theor. Phys.* **73** (2021) 105007.
28. S. Kumar, V. Jaduan and W. X. Ma, *Eur. Phys. J. Plus.* **136** (2021) 843.
29. S. Kumar, V. Jaduan and W. X. Ma, *Chinese J. Phys.* **69** (2021) 1.
30. B. Ren, J. Lin and Z. W. Lou, *Appl. Math. Lett.* **105** (2020) 106326.
31. D. S. Wang and J. Liu, *Appl. Math. Lett.* **79** (2018) 211.
32. D. S. Wang and X. L. Wang, *Nonlinear Anal.*: *Real World Appl.* **41** (2018) 334.
33. Z. L. Zhao and L. C. He, *Eur. Phys. J. Plus.* **135** (2020) 639.
34. D. S. Wang, B. L. Guo and X. L. Wang, *J. Differ. Equ.* **266** (2019) 5209.
35. J. P. Yu and Y. L. Sun, *Nonlinear Dyn.* **90** (2017) 2263.
36. Z. L. Zhao and L. C. He, *Theor. Math. Phys.* **206** (2021) 142.
37. J. P. Yu, W. X. Ma, Y. L. Sun and C. M. Khalique, *Mod. Phys. Lett. B* **32** (2018) 1850409.
38. S. J. Chen, W. X. Ma and X. Lü, *Commun. Nonlinear Sci. Numer. Simul.* **83** (2020) 105135.
39. L. L. Huang and Y. Chen, *Commun. Theor. Phys.* **67** (2017) 473.
40. F. Guo and J. Lin, *Nonlinear Dyn.* **96** (2019) 1233, doi:10.1007/s11071-019-04850-9.
41. W. X. Ma, *Phys. Lett. A* **379** (2015) 1975.
42. C. J. Wang, *Nonlinear Dyn.* **84** (2016) 697.
43. W. X. Ma, Z. Y. Qin and X. Lü, *Nonlinear Dyn.* **84** (2016) 923.

44. J. Y. Yang, W. X. Ma, *Int. J Mod Phys. B* **30** (2016) 1640028.
45. L. Kaur and A. M. Wazwaz, *Phys. Scr.* **93** (2018) 075203.
46. W. X. Ma, J. *Phys. Conf. Ser.* **411** (2013) 012021.
47. C. Gilson, F. Lambert, J. Nimmo and R. Willox, *Proc. R. Soc. Lond. A* **452** (1996) 223.