Journal of Ambient Intelligence and Humanized Computing (2023) 14:10431–10440
https://doi.org/10.1007/s12652-022-03700-8

ORIGINAL RESEARCH

A nonlinear evolutionary equation modelling a dockless bicycle-sharing system

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Received: 14 February 2021 / Accepted: 10 January 2022 / Published online: 30 January 2022
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Abstract
We model the operation of a dockless bicycle-sharing system by two groups of interacting components, bicycles on a sidewalk and users of the system. The model illustrates the bicycle-sharing system by a nonlinear evolutionary equation about the density of bikes on the sidewalk. Users' behaviours, which are some straightforward and necessary actions, determine coefficients of the nonlinear evolutionary equation. Some nontrivial solutions to the equation show that even if every user has no malice, and the environment is stable, it is still possible that heaps of shared-bicycles appear somewhere along the road. Based on the data of heaps, parameters of users' psychological models can be obtained. A numerical simulation shows how to calculate features of users and change the supply of bicycles into the system.

Keywords Dockless bicycle-sharing system · Nonlinear evolutionary equation · Soliton solution

1 Introduction

As a crucial last-mile service, a bicycle-sharing system is healthy and reduces pollution in a city (Benedini et al. 2020; Caspi and Noland 2019; Vieira et al. 2020). Fishman (2016) is an early review of bicycle-sharing systems, and (Galatoulas et al. 2016; Sun et al. 2021) are two recent reviews of bicycle-sharing systems as well as other related issues. Among more than 2900 bicycle-sharing systems worldwide, dockless bicycle-sharing systems attract more and more interest of researchers because of their dramatic growth (Bakogiannis et al. 2019; Xu 2020). In a typical Chinese city, such a massive dockless bicycle-sharing system often has millions of bicycles. A user of the dockless bicycle-sharing system can walk little time to get a shared-bicycle, and the user can park the shared-bicycle anywhere along the road near his or her destination. Although many systems encourage a user to park a shared-bicycle in rectangle bays painted on the sidewalk, lots of bicycles are parked here and there for users' convenience. Notice that convenience is the primary motivator for a user of a bicycle-sharing system, as pointed out by the study Chen et al. (2020a) of an optimal pricing strategy for bicycle-sharing systems.

Those cities with massive dockless bicycle-sharing systems provide us with much empirical knowledge about the advantages and disadvantages of practical massive dockless bicycle-sharing systems. We refer to Chen et al. (2020b)

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for a good review of features, components and regulations of a real dockless bicycle-sharing system, and the corresponding challenges. A common unfavourable phenomenon among challenges, a clogging sidewalk, is that lots of parked shared-bicycles sometimes appear on a narrow sidewalk. Gao et al. (2020) pointed out that the phenomenon brings tremendous pressure on the city's management, and Yin et al. (2019) presents a pessimistic idea that a clogging sidewalk made by dockless bicycle-sharing systems, as well as their users, is value co-destruction. Although there are millions of bicycles in a city, and hence Wang et al. (2019) discusses their overuse issues, users often have to face too many or too few shared-bicycles along sidewalks and thus fail to use shared-bicycles (Zhai et al. 2019; Yi et al. 2019). The disorderly heaps of parked shared-bicycles can be partly controlled by collaborative efforts of users, systems, and the government (Zhao and Wang 2019).

We now consider who causes the issue of clogging or empty sidewalks with a dockless bicycle-sharing system. Neither the operating system nor the user group is satisfied with a clogging sidewalk or unsensible aggregation of bicycles. Hence in Liu et al. (2020), we proposed a model of a dockless bicycle-sharing system where no user has any malice, and the environment is stable, and we find that there is still a clogging sidewalk in the model. In this paper, we will present a more general model for a dockless bicycle-sharing system, which is the model in Liu et al. (2020) when a parameter is zero. We will investigate how to obtain model's parameters from the data of parked shared-bicycles, and will discuss possible remedies for some issues.

In this paper, the proposed model for a dockless bicycle-sharing system focuses on two kinds of interacting components, lots of bicycles on a long road and many users of the system. The living period of a user starts from the user's appearing independently and randomly on the road. The user wanders and picks up a shared-bicycle, then the user becomes a rider who rides the bicycle to his or her destination located at random position on the road, and finally, the rider parks the bicycle and departs from the system. Other actions of a user consist of walking and choosing a shared-bicycle, and the following details are assumed. The chance of a user choosing a shared-bicycle depends on the density of parked bicycles at the place, as there are more choices for the user near more parked bicycles. Because a user checks and searches a bicycle according to the situation of parked shared-bicycles showed on a mobile phone application, we assume that a walking user follows a diffusion process whose drift coefficient depends on the density of parked shared-bicycles.

The density of parked shared-bicycles along the road plays a vital role in the model. The action of a user relies on density. On the other hand, the action influences the density: the density decreases when a user chooses a shared-bicycle, and it increases when a rider parks a shared-bicycle. Hence, a nonlinear evolutionary equation about the density can be established based on the above assumptions. The equation's solution is used to show heaps along sidewalks, indicates methods to obtain the model's parameters from observation data, and derives remedies for some issues.

Notations—For a function $f : \mathbb{R} \to \mathbb{R}$, f' denotes the derivative of f. Taking $f(x) = x^2$ as an instance, we have $f'(x) = 2x$ and $f'(x^2) = 2x^2$. We use the term $\frac{d}{dx}(f)$ for the derivative when the expression of f is complex. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ with two independent variable, such as $f(t, y)$, f_t denotes the partial derivative $\frac{\partial}{\partial t}(f(t, y))$, f_{ty} denotes the two-order partial derivative $\frac{\partial^2}{\partial t \partial y}(f(t, y))$, and so on. For a real number $B \in \mathbb{R}$, $f(B - 0)$ denotes the left-hand limit of the function $f : \mathbb{R} \to \mathbb{R}$ as the variable x approaches from values to the left of B.

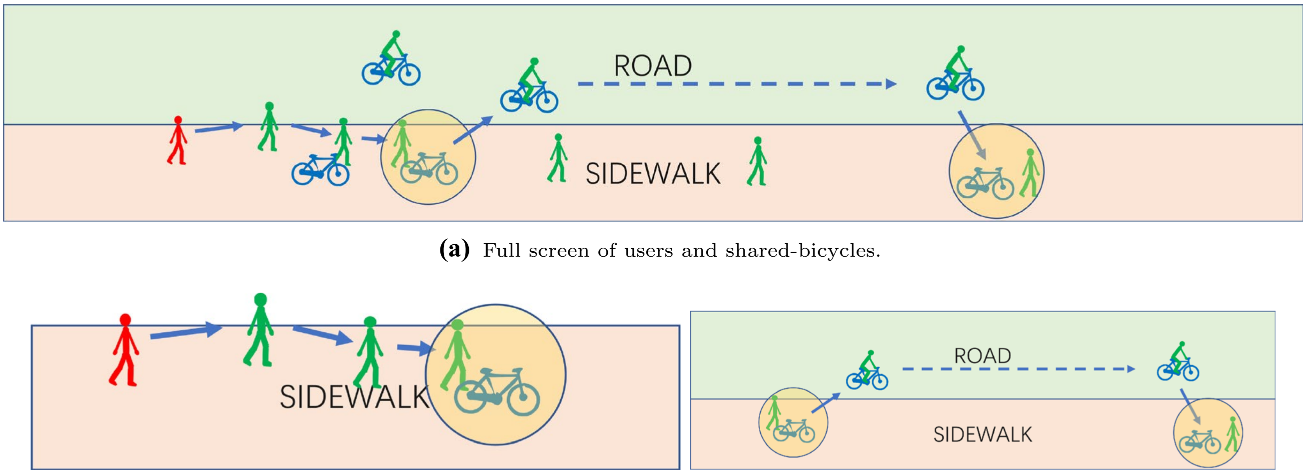
The rest of the paper is organized as follows. In Sect. 2, we present the model of a dockless bicycle-sharing system as well as environment and construct an evolutionary equation for the density of parked shared-bicycles. Section 3 investigates solutions of the evolutionary equation in an ideal case that parked shared-bicycles in the environment do not influence every user's behaviors, and presents the asymptotical density of parked shared-bicycles. The soliton solutions of the evolutionary equation for more cases and features of the density are discussed in Section 4. Section 5 concludes the paper.

2 An evolutionary equation about the density of parked bicycles

2.1 Dockless bicycle-sharing system

Different from traditional bike-sharing systems that offer rented bicycles going between docking stations, dockless bicycle-sharing systems are based on mobile payments and tracking techniques. With systems' mobile applications, a user can locate parked bicycles, unlock and pick up a bicycle, and pay for his or her rental By regulations of most systems, a user will be barred if he or she improperly parked a bicycle. However, most sidewalks are proper places, and hence a user can leave a bicycle anywhere along a sidewalk.

Figure 1 illustrates a long road with parked and ridden shared-bicycles of a dockless bicycle-sharing system and people who are users of the system. From the system's perspective, there are three kinds of objects: a user, a parked bicycle, and a compound rider that is a user riding a bicycle. The input of the systems is new users coming from their homes, substations, and so on. The output is riders who leave their bicycles on the road. The life of a user includes three stages, as illustrated in Fig. 1b. At the first stage, a person, as an input of the



(a) Full screen of users and shared-bicycles.

(b) Life of a user: appearing, wandering, and picking up a shared-bicycles.

(c) Departure and reappearance of a parked shared-bicycles

Fig. 1 Users' behaviours and evolution of shared-bicycles along a long road

system, appears on the road. During the second stage, the user wanders along the road and looks for a suitable shared-bicycle. At the third stage, the user picks up a shared-bicycle and becomes a rider. A parked shared-bicycle departs and becomes part of a compound rider when a wandering user picks it up. As an output of the system, a rider may leave his or her shared-bicycle on the road; then, the bicycle reappears in the system as a parked bicycle, as illustrated by Fig. 1c.

We assume that numbers of users, parked shared-bicycles, input, and output are vast, and we can discuss their densities at y, where $y \in \mathbb{R}$ denotes a position on the road. At a given time t, $w(t, y)$ and $v(t, y)$ represents 'densities' of input and output. Concretely, $\int_{s_1}^{s_2}\int_{x_1}^{x_2} w(t, y)dtdy$ is the number of new users appear in $[x_1, x_2]$ and from s_1 to s_2, and $\int_{s_1}^{s_2}\int_{x_1}^{x_2} v(t, y)dtdy$ is the number of riders who park a shared-bicycle in the same region and during the same period. In general, the the output function $v(t, y)$ relies on the input function $w(s, y)$, where $s \in (-\infty, t]$. However, if the input function $w(t, y)$ is constant, and the system comes to a steady-state after a burning time, Liu et al. (2020) showed that the output function $v(t, y)$ becomes a constant, too. The difference between output and input is equal to the number of people, including wandering users and riders on a bicycle, in the system. Hence any difference between output and input implies a blow-up or a disappearance of people in the system. This paper assumes that the system is steady and the input and output are constant. That is,

$$w(t, y) = v(t, y) \equiv v \in \mathbb{R}^+. \tag{1}$$

Let $u(t, y)$ denote the density function of parked bicycles. According to Fig. 1c, when a rider leaves his or her shared-bicycle on the road, the parked bicycle increases. That is, the output v contributes to u_t. On the other hand, when a user picks up a shared-bicycle, the parked bicycle decreases. We assume that $q(t, y)$ is the density caused by a user picking up a shared-bicycle. Then we have that

$$u_t = v - q(t, y). \tag{2}$$

We will model a users' behaviour and obtain $q(t, y)$.

2.2 User's behaviour

We follow a classical kinematic theory of traffic flow introduced by Lighthill and Whitham (1955) to model user's behaviour. We refer to Silvia et al. (2021) for a recent review of the traffic flow theory. We assume that the crowd of users has a density $p(t, y)$ per unit length and a flux $\tilde{p}(t, y)$ per unit time. It follows from the theory of traffic flow that the flow velocity $\mu(t, y)$ satisfies

$$\mu(t, y) = \frac{\tilde{p}(t, y)}{p(t, y)}, \tag{3}$$

and when $p(t, x)$ has continuous derivatives, there is a conservation equation

$$p_t(t, y) + \tilde{p}_y(t, y) = v - q(t, y). \tag{4}$$

Both sides of (4) have the same meaning, representing the rate of change of wandering users near y and t. The left-hand side is written by terms of density and flux, and the right-hand side is by terms of input and $q(t, y)$, which denotes the status transformation of a wandering user to a rider.

Notice that the flow velocity $\mu(t, y)$ is the speed of a wandering user at position $y \in \mathbb{R}$ and time t. The psychological model of a wandering user's behaviour relies on local information about the density of parked bicycles showing on his or her mobile phone application (Xu et al. (2020)). For a wandering user at position $y \in \mathbb{R}$ and time t, his or her local information about the density can be described by $u(t, y), u_y(t, y), u_{yy}(t, y), \ldots$. After neglecting high-order local information $u_{yy}(t, y), u_{yyy}(t, y), \ldots$, we assume that the flow velocity $\mu = b(u(t, y), u_y(t, y))$, where the psychological model b is a function $b : \mathbb{R}^2 \to \mathbb{R}$.

Then we have the following simple model for a user.

$$\frac{\partial}{\partial t}p(t, y) = v - q(t, y) - \frac{\partial}{\partial y}(b(u(t, y), u_y(t, y))p(t, y)). \tag{5}$$

However, model (5) is strange because it assumes that every wandering user at the same place walks at the same speed. To overcome it, the traffic flow theory includes dependence of flux $\tilde{p}(t, y)$ on $p_y(t, y)$ and produces a diffusion of the flow. Because it means slight randomness of users' behaviour, we can investigate it from a user's perspective.

Let $\{B(t)\}$ be a standard Brownian motion, and we denote the movement of a wandering user who is finding a shared-bicycle by a stochastic differential equation. A stochastic diffusion process $X(t)$ denotes the concrete position of a wandering users at time t. The stochastic differential equation of $X(t)$ is

$$dX(t) = \sigma dB(t) + bdt. \tag{6}$$

Here $\sigma dB(t)$ represents the user's random walk, and the drift item bdt represents, within the same situation, wandering users' average behaviour, including direction and speed for finding a bicycle. From the discussion about model (5), σ is far less than the absolute value of $b = b(u(t, y), u_y(t, y)$.

The number of wandering users on the road is vast, then the density $p(t, y)$ of wandering users can be derived by the probability function of the stochastic diffusion process $X(t)$. According to (6), the density satisfies the following heat equation (Liu (2016); Ross (2014)).

$$p_t = \frac{\sigma^2}{2}p_{yy} - (b\,p)_y + v - q(t, y). \tag{7}$$

The last item $q(t, y)$ in the right-hand side of (7) represents the decrement of density as a wandering user gets a shared-bicycle and becomes a rider. Notice that the item $q(t, y)$ appears in (2) too. We assume that the rate of a user picking up a shared-bicycle at y and t is $\lambda(t, y)$, where $\lambda(t, y) \in (0, 1)$. Then we have that

$$q(t, y) = \lambda(t, y)\,p(t, y). \tag{8}$$

2.3 Evolutionary equation

Users and parked shared-bicycles in the dockless bicycle-sharing system can be described by three Eqs. (2), (7), and (8). We can establish unique evolutionary equation for the system for some particular function $\lambda(t, y)$.

As the discussion about b, we know that $\lambda(t, y)$ relies on many complex factors, such as densities $u(t, y)$ and $p(t, y)$, the time between t and a user's deadline, and the distance from y to a user's destination. When destinations locate randomly on the road, the issue turns to a simple case that $\lambda = \lambda(t)$ is a function of t. Such a function $\lambda(t)$ corresponds to a case that the rate of a user picking up a shared-bicycle varies according to time, and an example is a period near the beginning of office hours. When there is not a clear deadline, $\lambda(t)$ can be a constant λ_0.

From (2) and (8), we have that $\lambda p = q = v - u_t$, and hence $\lambda bp = bv - bu_t$, $\lambda p_t = -u_{tt}$, $\lambda p_{yy} = -u_{tyy}$. Then multiplying (7) by λ and substituting the above three equalities, we have that

$$\frac{\sigma^2}{2}u_{tyy}(t, y) - u_{tt}(t, y) - (b(u, u_y)u_t(t, y))_y - \left(\frac{\lambda'(t)}{\lambda(t)} - \lambda(t)\right)u_t(t, y) + v\,b_y = \frac{\lambda'(t)}{\lambda(t)}v. \tag{9}$$

Notice that $b(u, u_y) = b(u(t, y), u_y(t, y))$. When we obtain $u(t, y)$ from (9), we have that $p = \frac{v-u_t}{\lambda}$.

If the rate $\lambda(t)$ is a constant λ_0, model (9) becomes

$$\frac{\sigma^2}{2}u_{tyy}(t, y) - u_{tt}(t, y) - (b(u, u_y)u_t(t, y))_y - \lambda_0 u_t(t, y) + v\,b_y = 0. \tag{10}$$

If the parameter v tends to 0, the term $v\,b_y$ disappears and (10) becomes the model in Liu et al. (2020).

Compared with $\lambda = \lambda_0$, the assumption that λ relies on time is more realistic. But the corresponding model (9) is complex. Because conditions of model (10) are valid in particular cases, we will focus on model (10), study its properties, and manage to obtain its parameters for observed real-world data. Most of parameters are still valid for model (9), and hence model (9) can be investigated by numerical methods.

3 Evolution of density with constant drift parameter

A constant drift parameter $b \in \mathbb{R}$ corresponds to an ideal case that wandering users' behaviour does not rely on the environment, that is, a user independently walks and chooses a shared-bicycle no matter more or less the density of parked-bicycles nearby is. If the mobile phone application of the bicycle-sharing system does not provide a user with any information about nearby parked-bicycles, an unplanning user searching a bicycle follows the case. Moreover, a drift parameter $b \neq 0$ corresponds to a common favour direction of movement of all wandering users. We can imagine some scenarios of the case, such as the crowd near a transportation hub, and commuters along a particular route.

Suppose that a function $\psi(y)$ has a second derivative. It is obvious $u(t, y) = \psi(y)$ is a trivial solution of (10) with constant drift parameter b, and then from (7) and (8) we have that $p(t, y) = \frac{v}{\lambda_0}$. For any continuous initial condition $u(0, y) = \psi_1(y)$ and a bounded initial fluctuation $u_t(0, y) = \psi_2(y)$, the following proposition shows general solutions of (10).

Theorem 1 *Assume that the drift parameter b is constant. The parameter λ_0 of the exponential distribution satisfies $\lambda_0 > 0$. Let $\psi_i(y), i = 1, 2$, be continuous functions and $\psi_2(y)$ is bounded. We write the bound of initial fluctuation in time as $M_2 = \max\{|\psi_2(y)|, y \in \mathbb{R}\} < \infty$.*

1. *There is a unique solution $u(t, y)$ to (10) with initial conditions $u(0, y) = \psi_1(y)$ and $u_t(0, y) = \psi_2(y)$.*
2. *When $b < 2\lambda_0$, we have that $\lim_{t\to+\infty} u(t, y) = 0$.*
3. *When $b = 0$, we have that*

$$\max\{|\psi_1(y) - u(t, y)|, y \in \mathbb{R}, t > 0\} \leq \frac{M_2}{\lambda_0}. \tag{11}$$

Proof When b is a constant, model (10) becomes $\frac{\sigma^2}{2}u_{tyy}(t, y) - u_{tt}(t, y) - bu_{ty}(t, y)) - \lambda_0 u_t(t, y) = 0$. The equation suggests an auxiliary function $\tilde{u} = u_t$, and hence $\tilde{u}_t = \frac{\sigma^2}{2}\tilde{u}_{yy} - b\,\tilde{u}_y - \lambda_0\tilde{u}$. According to the Feymann-Kac formulae, we can write

$$\tilde{u}(t, y) = w(t, y)\exp\left(\frac{b}{\sigma^2}y + \left(-\frac{b^2}{2\sigma^2} - \lambda_0\right)t\right), \tag{12}$$

and then we have a classical heat equation.

$$w_t = \frac{\sigma^2}{2}w_{yy}. \tag{13}$$

(12) implies that

$$w(0, y) = \exp\left(-\frac{by}{\sigma^2}\right)\psi_2(y). \tag{14}$$

Then for the above initial condition about $w(0, y)$, the solution to (13) is unique. Concretely, for $t > 0$ and $y \in \mathbb{R}$, we have that

$$w(t, y) = \int_{\mathbb{R}} \frac{w(0, x)}{\sqrt{2\pi t}\sigma}\exp\left(-\frac{(x - y)^2}{2\sigma^2 t}\right)dx. \tag{15}$$

Therefore, with initial conditions $u(0, y) = \psi_1(y)$ and $u_t(0, y) = \psi_2(y)$, there is the following unique solution $u(t, y)$ to (10).

$$u(t, y) = \psi_1(y) + \int_0^t \tilde{u}(s, y)ds. \tag{16}$$

Now we turn to cases 2) and 3), and assume that $b < 2\lambda_0$. It follows from (14) that $|w(0, x)| \leq M_2\exp\left(-\frac{bx}{\sigma^2}\right)$ and hence from (15),

$$|w(t, y)| \leq M_2\exp\left(\frac{b^2 t - 2by}{\sigma^2}\right). \tag{17}$$

Moreover, from definitions of $\tilde{u}$ and w, we have that the above inequality implies that the fluctuation is restricted by its initial bound M_2.

$$|u_t(t, y)| = |\tilde{u}(t, y)| \leq M_2\exp\left(-\lambda_0 t\right). \tag{18}$$

Therefore, for any $0 \leq T \leq s \leq t$, $|u(s, y) - u(t, y)| \leq \int_0^t |\tilde{u}(s, y)|ds$ and hence

$$|u(s, y) - u(t, y)| \leq \frac{M_2}{\lambda_0}\exp\left(-\lambda_0 T\right). \tag{19}$$

As $\frac{b}{2} - \lambda_0 < 0$, the right hand-side of (19) tends to 0 as $T \to +\infty$, and we have that $\lim_{t\to+\infty} u(t, y)$ exists and equals 0. Moreover, let $T = s = 0$, (19) implies that

$$|u(t, y) - \psi_1(y)| \leq \frac{M_2}{\lambda_0}. \tag{20}$$

Let $b = 0$, we obtain the result 3).

Theorem 1 illustrates the limiting state for the bicycle-sharing system with the drift parameter being constant. Moreover, with a constant drift parameter, from the heat equation (7) and (8), we can obtain a limiting solution of the density $p(+\infty, y)$ of wandering users.

If there is no common direction of movement of wandering users, that is, $b = 0$, Theorem 1 illustrates an ideal scenario of a bicycle-sharing system. For any initial density $\psi_1(y)$ of parked-bicycles, there are little differences between the initial density and the limiting density $u(+\infty, y)$ as well as the density at any time. So we can plan and design the initial density of parked-bicycles carefully, and then we are sure that the situation of parked-bicycles is under control at any time.

When the drift parameter $b \neq 0$ and there is a recognizable mean movement of users, although Theorem 1 ensures a limiting density curve of parked-bicycles, but (19) shows that it is away from the initial density.

Besides other reasons, features of dockless bicycle-sharing systems may rely on users' behaviour based on information about parked shared-bicycles. Details of nearby parked shared-bicycles support a user's carefully searching and picking up a shared-bicycle. Users' specific and necessary behaviour finally cause fluctuation of the density of parked shared-bicycles. Theorem 1 studies a particular mobile phone application, which does not provide a user with any information about nearby parked-bicycles, and shows that the solution to the model is under control. Of course, we think no users would appreciate this application.

4 Solutions of density in the case of unconstant drift

Competitive bicycle-sharing systems always improve their mobile phone applications, and we can find the applications more and more appealing because they provide with more and more details of parked shared-bicycles. For those cases, a constant drift parameter is not a sensible assumption. Here we will study two cases which have bounded densities of parked-bicycles. That is, at any time, the parked-bicycles of those two cases are under control in some sense. However, the bad news is that the densities are not constant, and hence there may be too many or too few parked-bicycles somewhere.

4.1 Sinusoidal solutions of the density

The following proposition investigates solutions to (10) with an unconstant drift parameter.

Theorem 2 *For given two constants $b_0 > 0$ and $A > \lambda_0 + 2b_0^2\sigma^{-2}$, denote the drift parameter by*

$$b(u, u_y) = b_0 - \frac{Au}{u_y + \frac{v}{b_0}}. \tag{21}$$

Let $\omega = \sqrt{2\sigma^{-2}(A - \lambda_0)}$ and $C_1 = v(A - \lambda_0)^{-1}$. Then for any parameter $\phi \in [0, 2\pi)$ and $C_2 \in \mathbb{R}$ such that $|C_2| < C_1$,

$$u(t, y) = C_2 \cos(\omega(y - b_0 t) + \phi) + C_1 \tag{22}$$

is a bounded positive sinusoidal solution to (10).

Proof It follows from $A > \lambda_0$ that $C_1 > 0$ and hence $0 < u(t, y) < 2C_1$, that is, $u(t, y)$ is a bounded positive function. Moreover, it follows from $A > \lambda_0 + 2b_0^2\sigma^{-2}$ that $C_1\omega < \frac{v}{b_0}$. Then it follows from the fact and $|u_y| \le C_2\omega < C_1\omega$ that $u_y + \frac{v}{b_0} > 0$.

Now we turn to confirm that $u(t, y)$ is a solution to (10). (22) suggests that (10) has a traveling wave solution such that

$$u_t = -b_0 u_y. \tag{23}$$

(23), when substituted into (10), yields

$$\frac{\sigma^2}{2}u_{yy} - (b - b_0)u_y - \lambda_0 u - \frac{vb}{b_0} = C, \tag{24}$$

where C is a constant.

We have that from (22),

$$\frac{\sigma^2}{2}u_{yy} - \lambda_0 u = -\left(\frac{\omega^2\sigma^2}{2} + \lambda_0\right)u + \frac{\omega^2\sigma^2}{2}C_1. \tag{25}$$

It follows from the definition of ω that $\frac{\omega^2\sigma^2}{2} = A - \lambda_0$, and hence from $C_1 = \frac{v}{A-\lambda_0}$, we have $\frac{\omega^2\sigma^2}{2}C_1 = v$. Therefore,

$$\frac{\sigma^2}{2}u_{yy} - \lambda_0 u = -Au + v. \tag{26}$$

On the other hand, from the definition of $b(u, u_y)$, we have that

$$(b - b_0)u_y + \frac{vb}{b_0} = b\left(\frac{v}{b_0} + u_y\right) - b_0 u_y = v - Au. \tag{27}$$

It follows from (26) and (27) that $u(t, y)$ satisfies (24) with $C = 0$.

Theorem 2 presents an unconstant drift parameter which corresponds to a bounded sinusoidal density of parked-bicycles. In general, conditions of model (10) are ideal because there are many restrictions about conditions such as the input and output of the system, destinations and deadlines of users, and we must know the expression of $b(u, u_y)$. However, in the following, we will show a method provided by Theorem 2 to deduce some parameters of $b(u, u_y)$ from observations of the parked shared-bicycles.

It follows from (22) that the density $u(t, y)$ varies on space and time. At a given time t, the maximal density is $C_1 + C_2$ and minimal density is $C_1 - C_2$ along the road. When we observe a maximum point of the density, that is, one of heaps of parked shared-bicycles, we can find that it is moving because wandering users pick up bicycles behind the heap and park bicycles ahead. The velocity of a heap, or a maximum point, is the constant part b_0 of the mean speed $b(u, u_y)$ of a wandering users. In the following, we will obtain more details of $b(u, u_y)$ from features of parked shared-bicycles.

Due to the property of sinusoidal density in (22), the mean density of parked shared-bicycles is C_1 on the road. Hence C_1 can be obtained from data of bicycles. We can also obtain v because it is a parameter of the input (1) for the system. From expressions of the mean density C_1 and the angular frequency ω, we have that a relation among them and the parameter σ.

$$\sigma\omega = \sqrt{\frac{2v}{C_1}}. \tag{28}$$

When we measure the distance from a heap of parked shared-bicycles to the next heap, which is the wavelength $L = \frac{2\pi}{\omega}$, we have that the parameter σ is

$$\sigma = \sqrt{\frac{v}{2C_1}}\frac{L}{\pi}. \tag{29}$$

It is interesting to study the wavelength L from (22) and compare it with our experience. It follows from (29) that

$$L = \sqrt{2}\pi\sqrt{\frac{C_1}{v}}\sigma. \tag{30}$$

Here σ is the standard deviation of our walking speed when finding a bicycles, and its order of magnitude is 1 km per hr. $\sqrt{2}\pi$ is about 4.4. The supply capacity of the system, C_1/v, is the proportion of parked bicycles to new users per unit time, its order of magnitude is about 1 because our experience is that there are enough bicycles on the road although few is around us. Then we have L is 4.4 km, which is slightly beyond a person's sight range. Hence we often see a heap of parked bicycles located at a 'mean' place about 1 km from us, and it is moving quicker than us because $b(u, u_y) < b_0$.

With initial conditions other than (22), model (10) becomes difficult to solve analytically, and we can use the finite difference method (FDM) to solve it numerically (Blazek (2015)). Concretely, the numerical simulation utilizing the FDM replaces u_t, u_y, b_y, u_{tt}, and u_{tyy} respectively by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t}, \frac{u_{j+1}^n - u_j^n}{\Delta y}, \frac{b_{j+1}^n - b_j^n}{\Delta y}, \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}, \tag{31}$$

and

$$\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^n) - (u_{j+1}^{n+1} - 2u_j^n + u_{j-1}^n)}{\Delta t \Delta y^2}, \tag{32}$$

where n and j are indices of time t and space y, and Δt and Δy are constant increments of time and space, respectively. Then we can solve the corresponding linear algebraic equation of (10) numerically.

The numerical simulation shows other features of the model (10) with $b(u, u_y)$ given by (21). The parameters are $\sigma = 0.8$ km/h, $b_0 = 1.1$ km/h, and a large input $v = 3.2$ per km. Moreover, we introduce a fluctuation to the initial values of the density of parked shared-bicycles. The numerical solution to (10) presented in Fig. 2 tends to sinusoidal wave in about 0.4 h, and the asymptotic sinusoidal wave has a wavelength about 2.9 km and a mean density C_1 about 0.9 per km. And the supply capacity C_1/v of the system is about 0.28.

The numerical simulation corresponds to a scenrio with a large pressure on the system, when there is about two parked

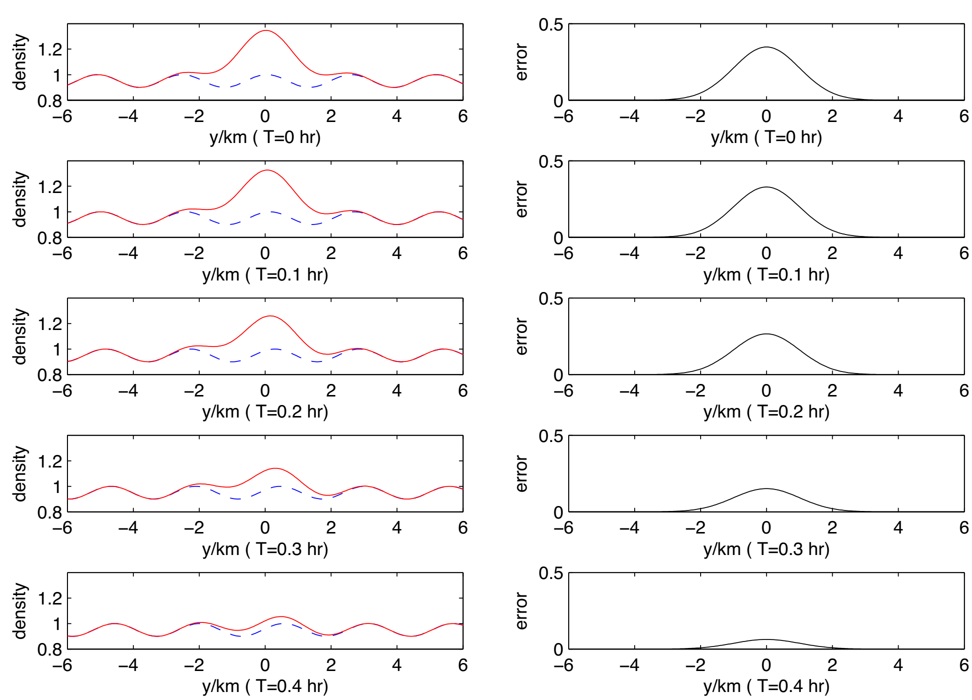


Fig. 2 The influence of a fluctuation to initial values vanishes quickly. Dashed curves are of (22), and solid curves show the evolution of density with an initial fluctuation. The difference between the two curves is in the right column

shared-bicycles for every seven new users. Although a supply of bicycles, corresponding to the fluctuation of the initial values, disappears in 0.4 h due to new users, there is still unbalance in the system and the distance between two heaps is about 2.9 km. In the simulation, the ratio of maximum to minimum of the density is 1.3. The ratio means the system's unbalancing and can be studied by numerical simulations.

4.2 Other travelling waves of the density

The expression of $b(u, u_y)$ relies on the mean behaviour of users and it follows a psychological model (Xu et al. (2020)). We must carefully check the validity of the expression based on observed data because the solution to the system may differ for different $b(u, u_y)$. The following proposition shows that a weak solution to (10) with a linear drift parameter is completely different from sinusoidal solutions in Theorem 2.

Theorem 3 *For a constant $\beta \in \mathbb{R}$, denote the drift parameter by*

$$b(u, u_y) = \beta u + \nu \lambda_0^{-2} \beta^2 u_y. \tag{33}$$

There exists a bounded nonnegative function $h : \mathbb{R} \to \mathbb{R}$ such that

$$u(t, y) = h\left(y + \frac{\nu\beta}{\lambda_0} t \right) \tag{34}$$

is a weak solution to (10).

Here a weak solution u to a partial differential equation $\mathcal{L}u = f$ means that $\mathcal{L}u$ equals to f in the sense of distributions. Concretely, we allow the solution u contains singularities where u_t or u_y do not exist. However, for any test function $\phi \in C_c^\infty(R \times (0, \infty))$ such that ϕ has compact support, we have that,

$$\int_{\mathbb{R}^2} \left(\frac{\sigma^2}{2} u_{tyy} - u_{tt} - (bu_t)_y - \lambda_0 u_t + \nu b_y \right) \phi dtdy = 0. \tag{35}$$

The proof of Theorem 3 is in Appendix 2, and it is based on the following Lemma 1.

Lemma 1 *For constants $\alpha, \gamma, \eta \in (0, +\infty)$, such that $\eta < \gamma\alpha^{-2}$ and denote a function $f(u)$ as*

$$f(u) = \frac{\exp(-\alpha u)}{\eta - \gamma\alpha^{-2}(1 + \alpha u)\exp(-\alpha u)}. \tag{36}$$

1. *There is $B > 0$ such that for $u \in [0, B)$, $f(u) < 0$ and $f(B - 0) = -\infty$.*
2. *For $u \in [0, B)$,*

$$f'(u) + \gamma u f^2(u) + \alpha f(u) = 0. \tag{37}$$

3. *For $u \in [0, B)$, write*

$$F(u) = \int_0^u f(x)dx. \tag{38}$$

Then $F(0) = 0$ and $F(B - 0) = -\infty$. For any $u_1, u_2 \in [0, B)$ such that $u_1 < u_2$, we have that $0 \geq F(u_1) > F(u_2)$. That is, $F : [0, B) \to (-\infty, 0]$ is decreasing.

The proof of Lemma 1 is in Appendix 1.

5 Conclusion and future works

Dockless bicycle-sharing systems improve users' feeling of convenience, and they bring tremendous pressure on the city's management because of disorderly heaps of parked shared-bicycles. To investigate the phenomenon of clogging or empty sidewalks, we proposed a model of a dockless bicycle-sharing system where no user has any malice, and the environment is stable. In some situations with ideal conditions, we can obtain the exact solutions to the nonlinear evolutionary equation derived by the model.

The ideal conditions include the following assumptions and limitations: The environment is a long road without any fork or cross. There are lots of bicycles and users in the system so that a continuous function can describe their densities. Both the flow of users entering the system and the flow of leaving users are steady at any time. The departure points and destinations are distributed uniformly on the road so that flows of entering and leaving are balanced and two flows are steady in space.

Even in ideal conditions, the results show that an ambitious and competitive bicycle-sharing system, who wants to provide a good user experience by its mobile applications, must face the problem of unbalance. Based on the data of parked shared-bicycles, we proposed a method to obtain parameters of users' psychological models. Numerical simulation provides the ratio of the maximum density to the minimum, which can use to change the supply of bicycles into the system. Another theorem points out that we must carefully check users model's validity based on observed data because the solution to the system may differ for different users models.

There are many problems worth investigating in future research. The ideal conditions of investigated model are valid in some particular cases, the swarming behaviour of users and distributions of bicycles may be different under environments and flows such as a complex road network, unbalanced flows of entering and leaving by commuters, specific flows illustrating transfer between transit system, and so on. It is an interesting problem whether and which of the obtained parameters under ideal conditions are still valid for realistic models. The problem is important for exploring the realistic model by numerical methods and for providing more suggestions to dockless bicycle-sharing systems in the future. Finally, a theoretical problem is the physical meaning of weak solutions of the system.

Appendix 1 Proof of Lemma 1

Proof For any u, the numerator $f(u)$ is positive and it does not influence the sign of $f(u)$. We consider the denominator $\phi(u)$ of $f(u)$:

$$\phi(u) = \eta - \gamma\alpha^{-2}(1+\alpha u)\exp(-\alpha u). \tag{39}$$

As $0 < \eta < \gamma\alpha^{-2}$, it is obvious that $\phi(0) < 0$. Moreover, $\phi(\infty) = \eta > 0$. And for $u > 0$, $\phi'(u) = \gamma u \exp(-\alpha u) > 0$. So there is a unique $B > 0$ such that $\phi(B) = 0$ and for $u \in [0, B)$, $\phi(u) < 0$. Then the result of 1) follows.

We write $g(u) = f(u)\exp(\alpha u)$. It follows from

$$f'(u) = -\alpha f(u) + g'(u)\exp(-\alpha u) \tag{40}$$

and (37) that

$$\frac{1}{g^2(u)}g'(u) = -\gamma u \exp(-\alpha u). \tag{41}$$

Therefore

$$\frac{1}{g(u)} = \eta - \gamma\alpha^{-2}(1+\alpha u)\exp(-\alpha u). \tag{42}$$

Notice that it follows from 1) that for $u \in [0, B)$, the above function is not zeros at u and hence (36) follows. That is, f is a solution to (37) on $[0, B)$.

As $f(u)$ is continuous at any $u \in [0, B)$, (38) is a proper definition to $F(u)$. It follows from $f(u) < 0$ that $F(u)$ is decreasing. From its definition, we have that $F(0) = 0$. Finally, to prove that $F(B-0) = -\infty$, we only need to prove that for some $\delta > 0$,

$$F(B-0) - F(B-\delta) = \int_{B-\delta}^{B} f(u)du = -\infty. \tag{43}$$

Now we study

$$\frac{1}{f(u)} = \exp(\alpha u)\phi(u), \tag{44}$$

where $\phi(u)$ is given in (39). As for $u > 0$, $\phi'(u) = \gamma u \exp(-\alpha u) > 0$,

$$\left.\frac{d((f(u))^{-1})}{du}\right|_{u=B} = \exp(\alpha B)\phi'(B) \neq 0. \tag{45}$$

Moreover, it follows from $f(B-0) = -\infty$ that $1/f(B-0) = 0$. Therefore, there is a constant C and $\delta > 0$ such that for $u \in [B-\delta, B]$

$$\frac{1}{f(u)} \geq C(u-B), \tag{46}$$

and hence $f(u) \leq C^{-1}(u-B)^{-1}$. So we can obtain

$$\int_{B-\delta}^{B} f(u)du \leq \int_{B-\delta}^{B} \frac{1}{C(u-B)}du = -\infty. \tag{47}$$

Appendix 2 Proof of Theorem 3

.

We write $\alpha = \frac{2\nu\beta^2}{\sigma^2\lambda_0^2} > 0$, $\gamma = \frac{2\beta}{\sigma^2} > 0$ and let η be a positive number such that $\eta < \gamma\alpha^{-2}$. It follows from 3) of Lemma 1 that F has an inverse $F^{-1} : (-\infty, 0] \to [0, B)$ such that $F^{-1}(-\infty) = B$ and $F^{-1}(0) = 0$. We write

$$h(y) = \begin{cases} F^{-1}(y), & y \leq 0, \\ 0, & y \geq 0. \end{cases} \tag{48}$$

We will show that $u(t, y)$ given by (34) satisfies (10) at any (t, y) such that $\lambda_0 y + \nu\beta t \neq 0$.

Let $k = \nu\beta/\lambda_0$. It is clear that $u_t(t, y) = ku_y(t, y)$ for $\lambda_0 y + \nu\beta t \neq 0$. Substituting $u_t(t, y) = ku_y(t, y)$ into (10), we have

$$\begin{aligned}&\frac{\sigma^2 k}{2}u_{yyy} - k^2 u_{yy} \\ &- k(bu_y)_y - \lambda_0 k u_y + \nu b_y = 0.\end{aligned} \tag{49}$$

And hence

$$\frac{\sigma^2 k}{2}u_{yy} - k^2 u_y - k(bu_y) - \lambda_0 k u + \nu b = C, \tag{50}$$

where C is a constant.

We will show that for $C = 0$, (50) has two solutions: $u(t, y) = 0$ and $u(t, y) = F^{-1}\left(y + \frac{\nu\beta}{\lambda_0}t\right)$ and hence $u(t, y)$ given by (34) is a weak solution to (50). It is clear that $u(t, y) = 0$ is a trivial solution. Substituting $b(u, u_y)$ given in (33) into (50) and notice $C = 0$, we have that $\bar{u}(y) = u(0, y)$ satisfies

$$\begin{aligned}&\frac{\sigma^2 k}{2}\bar{u}_{yy} - k^2\bar{u}_y - k\beta\bar{u}\bar{u}_y - k\frac{\nu\beta^2}{\lambda_0^2}(\bar{u}_y)^2 \\ &- \lambda_0 k\bar{u} + \nu\beta\bar{u} + \frac{\nu^2\beta^2}{\lambda_0^2}\bar{u}_y = 0.\end{aligned} \tag{51}$$

The coefficient of $\bar{u}$ in the above (51) is $-\lambda_0 k + \nu\beta = 0$, that follows from $k = \nu\beta/\lambda_0$, and the coefficient of $\bar{u}_y$ is $-k^2 + \nu^2\beta^2\lambda_0^{-2} = 0$. Hence (51) yields

$$\frac{\sigma^2}{2}\bar{u}_{yy} - \beta\bar{u}\bar{u}_y - \frac{\nu\beta^2}{\lambda_0^2}(\bar{u}_y)^2 = 0. \quad (52)$$

Consider the inverse $y = y(\bar{u})$ of $\bar{u} = \bar{u}(y)$. Substituting

$$\bar{u}_y = \frac{1}{y_{\bar{u}}}, \quad \bar{u}_{yy} = -\frac{y_{\bar{u}\bar{u}}}{(y_{\bar{u}})^3} \quad (53)$$

into (52), we have that

$$\frac{\sigma^2}{2}y_{\bar{u}\bar{u}} + \beta\bar{u}y_{\bar{u}}^2 + \frac{\nu\beta^2}{\lambda_0^2}y_{\bar{u}} = 0. \quad (54)$$

Let $f(\bar{u}) = y_{\bar{u}}$, then we have

$$f' + \frac{2\beta}{\sigma^2}\bar{u}f^2 + \frac{2\nu\beta^2}{\sigma^2\lambda_0^2}f = 0. \quad (55)$$

As $\alpha = \frac{2\nu\beta^2}{\sigma^2\lambda_0^2}$ and $\gamma = \frac{2\beta}{\sigma^2}$, the above (55) and (37) is the same one. It follows from Lemma 1 that $y(\bar{u}) = F(\bar{u})$. Therefore $\bar{u} = F^{-1}(y)$ is a solution to (51), and hence $u(t, y) = F^{-1}\left(y + \frac{\nu\beta}{\lambda_0}t\right)$ is a solution to (50).

Acknowledgements We would like to extend sincere gratitude to two anonymous reviewers for their careful comments and inspired suggestions that have helped improve this paper substantially. This work is supported by national natural science foundation of China (Grants no. 11991023 and no. 11771352), Research achievement Cultivation Fund of Northwest University, and fundamental research funds for the central universities.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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