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Evolutionary equations of a sharing-bicycle system and their solutions

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ABSTRACT

The development of sharing-bicycle systems is accompanied by serious phenomena such as indiscriminate parking of shared bikes. This study aims to explore causes of parking behaviour problems in a sharing-bicycle system. Based on user demand and the characteristics of the shared bicycle system, we establish the evolutionary equations for the two variables, the density of user groups on the road, and the density of parked shared bicycles on the road, and study conditions satisfied by a shock wave solution. The results of the model's shock wave solution, time-invariant solution, and travelling wave solution allow us to reveal the phenomenon of aggregation of sharing-bicycles on the sidewalk due to the architecture of a sharing-bicycle system, as well as the fluctuation of the variables that still exists in a stable system. The results have a wide range of potential impacts on sharing-bicycle system operators. In terms of operation strategy, operators can predict the gathering situation of sharing-bicycle system at different times and places according to our model, so as to optimize vehicle scheduling and parking management and reduce the phenomenon of disorderly parking. In addition, in terms of cost control, operators can use resources more effectively and reduce operating costs through more accurate forecasting and scheduling. Our research findings, when combined the basic technologies of modern sharing-bicycle system (such as mobile payment and location technology), promise to significantly enhance system optimization, control, and management, thereby fostering the sustainable growth of the sharing-bicycle industry.

1. Introduction

With the growing concerns over urban traffic congestion and environmental pollution, sharing-bicycles, as a new form of eco-friendly transportation, have been catching the attention and admiration of an increasing number of people [1]. The modern sharing-bicycle system can ensure efficient management and operation of sharing-bicycles by utilizing two key technologies, namely mobile payment and tracking technology. The mobile payment technology enables users to conveniently and quickly unlock, rent, and pay for sharing-bicycles by scanning a QR code via a mobile app. Furthermore, the mobile payment technology can facilitate the evaluation and management of user credit, in addition to allowing for more personalized services. The tracking technology is crucial to the sharing-bicycle system. It enables real-time management and scheduling of sharing-bicycles through bike monitoring

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and positioning. When users park their bikes on the sidewalk, the system automatically records the parking location information. This information streamlines subsequent bikes scheduling and maintenance. Furthermore, the tracking technology is also highly effective in preventing bicycle theft and damage [2].

The development of the sharing-bicycle system has caused a significant problem with unorganized and haphazard parking, causing inconvenience to both urban management and residents' lives. The regulation of the parking behaviour of sharing-bicycles has become a major concern that needs immediate attention [3]. To protect the rights and interests of users and maintain urban order, most sharing-bicycle systems have regulations on users' incorrect parking of bicycles. Such regulations generally stipulate that users must park their bikes in the designated parking areas after use. Failure to follow the regulations may lead to penalties, such as a deduction of credit scores or restrictions in use. Many sharing-bicycle systems prohibit random parking on the sidewalk [4]. In practical terms, however, users usually employ sharing-bicycles for rapid commutes and frequently park them on a sidewalk closer to their final destination. Even though sidewalks are usually spacious enough to hold multiple sharing-bicycles, they are sometimes obstructed by sharing-bicycles. This occurrence not only reduces roadway efficiency but also jeopardizes pedestrian safety [5,6]. Numerous researchers have investigated the reasons behind this trend and mostly concur that users have an inadequate civic responsibility, the parking region for the sharing-bicycle system is unreasonably arranged, and the administration and planning of sharing-bicycle and parking in urban planning are lacking [7,8]. It is deemed necessary to enhance the civic responsibility of users, improve the parking area settings of the sharing-bicycle system, enforce an effective penalty mechanism, and thoroughly review the management of sharing-bicycle and parking arrangements in urban planning.

Researchers have conducted a rigorous analysis of sharing-bicycle parking aggregation, focusing on user demand and market behaviour. For instance, [4] investigated the user demand perspective and identified the reasons for sharing-bicycle parking aggregation. Their findings revealed that users tend to select suitable parking locations based on their convenience and requirements while using sharing-bicycles. Furthermore, market behaviour affects sharing-bicycle parking aggregation. [9] suggested that sharing-bicycle companies encourage users to park their bikes in specific areas by optimizing their bike distribution strategies to increase their market share. Additionally, other researchers investigated the phenomenon of sharing-bicycle parking aggregation from the perspective of system dynamics. It is widely believed that the aggregation of parked sharing-bicycles is a complex process driven by system dynamics [10]. [11] utilized a system dynamics approach to investigate the formation mechanism of the sharing-bicycle parking aggregation phenomenon. Their findings reveal that the performance of the sharing-bicycle system is influenced by an array of factors, such as user demand, market behaviour, and bike distribution strategy. These factors are interdependent and jointly influence the occurrence of the sharing-bicycle parking aggregation phenomenon. To investigate the sharing-bicycle parking aggregation phenomenon comprehensively, [5,6] employed partial equations as mathematical models, which use transportation theories and stochastic analysis to simulate the operating conditions of the sharing-bicycle system. The models indicate that bikes parked alongside the road can aggregate under certain extreme but ideal conditions caused by stochastic factors, in the absence of malicious intent from users.

Historically, significant work has been conducted on traffic flow fluctuation theory, primarily focusing on motor vehicles. [12] laid the groundwork for traffic flow theory by introducing concepts that correlate with wave dynamics. [13], often referred to as the LWR model, further developed this theory by describing how waves of density and flow propagate along a highway. More recently, the review [14]provided an overview of traffic flow theory, incorporating wave dynamics and shock wave phenomena in traffic systems. These foundational works, despite their primary focus on motor vehicles, offer valuable insights and research methodologies that can be adapted and extended to the study of sharing-bicycle systems.

This paper's research is based on transportation theory and presents a concise mathematical model that demonstrates the tendency of sharing-bicycle to aggregate across commonly occurring environments, excluding the impact of random factors. Although previous studies have attempted to link the sharing-bicycle system with the solutions of differential equations, they often struggle to capture a comprehensive understanding of sharing-bicycle dynamics. Either a complete picture of sharing-bicycle changes is elusive, or random influences must be introduced during the solution process, resulting in solutions that are inherently stochastic and whose underlying patterns are less clear compared to deterministic solutions. The model proposed in this paper addresses this limitation. We prove that, even in the absence of random driving factors, there will still be wave solutions representing the aggregation phenomenon within a sharing-bicycle system. Compared to stochastic solutions, the temporal variation of this wave solution is more distinct and predictable, offering us a more precise control over the evolution of sharing-bicycle aggregation phenomena in the future. This not only enhances our comprehension of the causes and features of the sharing-bicycle parking aggregation phenomenon but also supports sharing-bicycle enterprises in devising feasible operational tactics. Therefore, this study not only enriches the theoretical framework of the sharing-bicycle system but also provides valuable practical guidance for operators in the sharing-bicycle industry.

The mathematical model of the system is outlined in Section 2, which provides details on the variables, fundamental assumptions, and parameters used in its representation. Section 3 establishes the evolutionary equations that govern the model variables, along with the mathematical criteria that shock waves must fulfil. Section 4 presents various applications, beginning with straightforward examples that demonstrate the fundamental techniques for solving the evolutionary equations and understanding shock waves. This is followed by examples that describe the processing techniques of a sharing-bicycle system located near a public transport hub. The solutions presented in these examples illustrate the aggregation of shared bicycles parked near a transportation hub and the role of shock waves in constructing a travelling wave solution for the system. Finally, Section 5 concludes the paper.

2. Model

2.1. System components

Fig. 1-a displays how users, parked bikes, and composite riders (users who are riding the bikes) are distributed in a sharing-bicycle system. This system has three principal components that interact with each other: users, parked bicycles, and riders. The identities of system objects may change sometimes, and the users entering and leaving the system reflect the interactions between the system and its external environment. Users who enter the system consist of new users originating from homes, stations, and other locations. User's leaving the system refers to the behaviour of a rider who parks the bike on the road and then leave.

As per the above description and Fig. 1-b, a user's lifecycle is divided into three stages. The first phase is characterized by the appearance of new users of the system onto the road. In the second phase, these users walk on the road while searching for the required bike. In the third phase, users select a sharing-bicycle and ride it, thus becoming riders. At this point, the parked sharing-bicycle and the walking user disappear from the system and combine to form a new rider. Upon reaching the destination, the rider parks the sharing-bicycle on the road. Then, as the rider disappears from the system, a parked bike reappears in the system, while the user, having reached their destination, also leaves and is no longer visible within the system. Fig. 1-c illustrates this stage. Those processes continue cycling through the system, enabling the sharing-bicycle system to operate and meet the needs of more users.

2.2. Variables in the model

Let $x \in Z \subseteq R$ denote a location in a region Z on a long, straight, one-way road, and $t \geq 0$ denote time. Commonly used regions are $Z = R$, half a straight line $Z = [0, \infty)$, and intervals $Z = [0, M_z]$, etc. $u(x,t)$ denotes the density of parked bicycles on the roadside at that moment, and $p(x,t)$ denotes the density of users on the road (excluding riders) at that moment.

$V(x,t)$ denotes the walking speed of users on the road at that time. In our model, a person chooses the speed at that moment based on the density of users around them and the density of parked bicycles. That is, we assume that $V(x,t)$ is a function of $u(x,t)$ and $p(x,t)$.

$$V(x,t) = V(u(x,t), p(x,t)). \tag{1}$$

The exact expression of $V(u,p)$ depends on the measured data, and this problem can be solved by current technology: the tracking technology that supports the sharing-bicycle system can provide us with the exact expression of V. Although we do not give the expression of V here, we can summarize some of its properties. Among crowded people, the walking speed decreases, i.e. for $p_2 \geq p_1 > 0$ we have that

$$V\left(u, p_2\right) \leq V\left(u, p_1\right). \tag{2}$$

This is a consensus assumption in classical traffic flow analysis [15].

It is difficult to clearly see the effect of the density u of parked sharing-bicycles on walking speed. It is possible that pedestrians reduce their speed because there are more sharing-bicycles to choose from, or they increase their speed to find the sharing-bicycles. In a simple walking speed model which is used in some examples of Section 4, we linearize within a certain range.

$$V(u,p) = (au + b)\left(M_p - p\right) \tag{3}$$

Here M_p is the maximum density of pedestrians, and the pedestrians cannot move forward. a reflects the impact of sharing-bicycle density u, when the sharing-bicycle density u is 0, the speed formula degrades to the classical traffic flow analysis of the common model. b is a parameter of the common model.

$T(x,t)\,dx\,dt$ represents the count of individuals who select a sharing-bicycle and become a rider in the vicinity of (x,t) within space horizon dx and time horizon dt. It is assumed that $T(x,t)$ is a function of $u(x,t)$ and $p(x,t)$, as follows.

$$T(x,t) = T(u(x,t), p(x,t)) \tag{4}$$

It was assumed in earlier papers that $T(x,t)$ and $p(x,t)$ were proportional, though it is evident that this assumption was oversimplified. In general, $T(u,p)$ has the following four essential characteristics.

1. When curbside bike parking remains constant, there is an increase in the number of sharing-bicycles selected and ridden away within the same range, along with human density. In other words, for $p_2 \geq p_1 > 0$, we have

$$T\left(u, p_2\right) \geq T\left(u, p_1\right). \tag{5}$$

2. When the density of parked bikes remains fixed, the number of sharing-bicycles chosen and ridden away never exceeds the number of parked bikes. That is, for a fixed u, there is an upper limit on $T(u,p)$.

3. For a fixed p, as the availability of bikes increases with the rise in u, more bikes are selected. In other words, for $u_2 > u_1 > 0$, we have that

$$T\left(u_2, p\right) \geq T\left(u_1, p\right). \tag{6}$$

The inequality sign above is strict when $p \neq 0$.

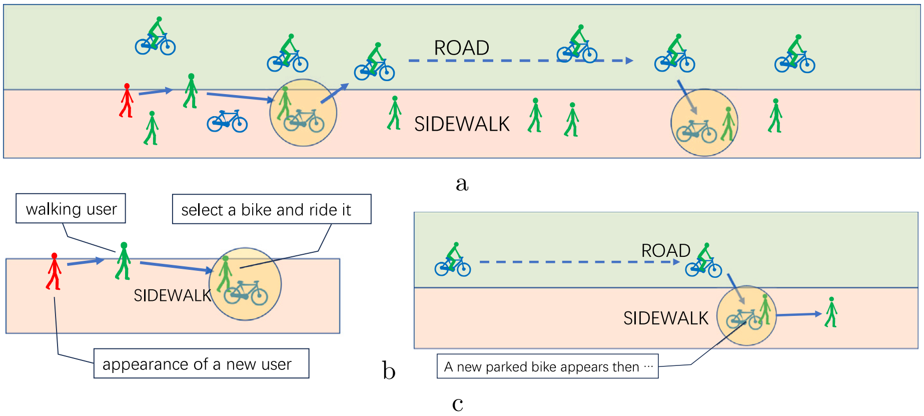

Fig. 1. Evolution of shared-bicycle system along a road.

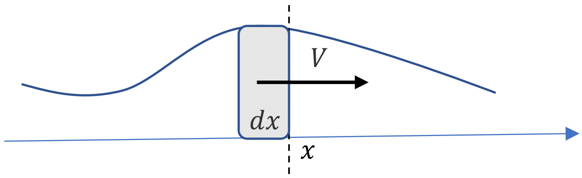

Fig. 2. Flow at the boundary of a region.

4. $T(u,0) = T(0,p) = 0$ represents the scenario where no one selects a bike or when there is no bike available.

To illustrate, in Examples 4 and 6 we use a choice model:

$$T(u,p) = \frac{up}{M_T + p} \tag{7}$$

Here, M_T is a parameter and it is related to $\frac{\partial^2}{\partial u \partial p} T(0,0)$.

Let $g(x,t)$ be the number of parked bikes that return to the system as a result of a rider leaving their sharing-bicycle on the road in the dx space and dt time horizon around the spatio-temporal point (x,t). As these riders come from someone who acquired a bike before, farther away and cycled to the sharing-bicycle, there exists a complex time-lag relationship between $g(x,t)$ and $u(x,t), p(x,t)$, which can be assumed to be not related to $u(x,t), p(x,t)$ for a short period of time. Also, each individual must reach their destination within a specific time frame. Therefore, it is assumed that $g(x,t)$ represents the number of general users destined for this place, which depends on both time t and location x, but is unrelated to $u(x,t)$ and $p(x,t)$.

Moreover, for a spatio-temporal point (x,t) inside a region Z, $f(x,t)$ denotes the number of newly emerging users in the dx space and dt time horizon near the spatio-temporal point (x,t). Similarly to $g(x,t)$, we assume that $f(x,t)$ is determined by time t and location x and bears no relation to $u(x,t)$ and $p(x,t)$.

The number of users emerging is also related to the flow of users. As illustrated in Fig. 2, the density of users at the spatio-temporal point (x,t) is $p(x,t)$, and the velocity is $V(x,t)$ in the right direction. The number of users within a small range of $dx = V(x,t)dt$ to the left of the point x is $p(x,t)dx$. These users cross the point x from the left to right during the following dt period. Therefore, the flow of users crossing the point x at spatio-temporal point (x,t) is defined as

$$q(x,t) = \frac{p(x,t)V(x,t)dt}{dt} = V(x,t)p \tag{8}$$

If $V(x,t) < 0$, the above argument still holds, so that $q(x,t) < 0$. In other words, the sign of $q(x,t)$ indicates the direction in which the motion is taking place.

We write

$$Q(u,p) = V(u,p)p. \tag{9}$$

In general, the flow $q(x,t)$ at the spatio-temporal point (x,t) satisfies the following equation.

$$q(x,t) = Q(u(x,t), p(x,t)). \tag{10}$$

The user emerging on the edge of region Z is also connected to the flow at the boundary. For illustration, if $Z = [0,\infty)$, the quantity of users arriving (or departing, contingent on the direction of the velocity $V(0,t)$) during time $(t, t+dt)$ is $q(0,t)\,dt$. In such a region, we have boundary conditions:

$$q(0,t) = q_0(t). \tag{11}$$

Here $q_0(t)$ is the number of users entering region Z from the boundary 0 at time t.

3. An evolutionary equation

This section presents the equations that the density $p(x,t)$ of users and the density $u(x,t)$ of bikes parked on the side of the road satisfy.

3.1. Density of users

Consider two points x_1, x_2 within region Z, and assume that $x_2 \geq x_1$, $t_2 \geq t_1$. We have that $\int_{x_1}^{x_2} p(x,t_i)\,dx$ be the number of users in interval $[x_1,x_2]$ at time t_i, and $\int_{x_1}^{x_2} p(x,t_2)\,dx - \int_{x_1}^{x_2} p(x,t_1)\,dx$ be the change in the number of users within the interval $[x_1,x_2]$ over the time period $[t_1,t_2]$. These changes stem from several sources. $\int_{t_1}^{t_2} q(x_j,t)\,dt$ is the number of users crossing from the left side of x_j to its right side during the time period $[t_1,t_2]$. Thus, $j=1$ corresponds to the entry of the interval $[x_1,x_2]$, while $j=2$ to the departure from the interval $[x_1,x_2]$. The amount

$$\int_{t_1}^{t_2} q(x_1,t)\,dt - \int_{t_1}^{t_2} q(x_2,t)\,dt \tag{12}$$

represents the number of users entering the left and right ends of interval $[x_1,x_2]$ within the time slot $[t_1,t_2]$. Also within the time period $[t_1,t_2]$ and the space interval $[x_1,x_2]$, the expression

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} (-T(x,t) + g(x,t))\,dt dx \tag{13}$$

refers to the number of riders who have selected and boarded a sharing-bicycle to leave and the number of users entering the system from within the interval. By combining the above information provided, it can be determined that

$$\begin{aligned} &\int_{x_1}^{x_2} p(x,t_2)\,dx - \int_{x_1}^{x_2} p(x,t_1)\,dx - \int_{t_1}^{t_2} q(x_1,t)\,dt + \int_{t_1}^{t_2} q(x_2,t)\,dt \\ &= \int_{t_1}^{t_2}\int_{x_1}^{x_2} (-T(x,t) + f(x,t))\,dt dx. \end{aligned} \tag{14}$$

If both $p(x,t)$ and $q(x,t)$ have derivative functions within the region under consideration, dividing $(t_2-t_1)(x_2-x_1)$ at both ends of the above equation while simultaneously setting x_1, x_2 to approach x, and t_1, t_2 to approach t, will lead to the conclusion that

$$p_t(x,t) + q_x(x,t) = -T(x,t) + f(x,t). \tag{15}$$

As per the shock wave theory, a shock wave will persist over time if a discontinuous point arises within $p(x,t)$.

As shown in Fig. 3, consider the function $p(x,t)$ or $q(x,t)$ to be discontinuous at the curve $x=s(t)$. Both $p(x,t)$ and $q(x,t)$ are continuous in the yellowish region below (or to the right of) the curve $x=s(t)$. Additionally, as a point (x,t) converges in this region to a point on the curve $x=s(t)$, with two limits $p(s^+(t),t)$, $q(s^+(t),t)$ of both $p(x,t)$ and $q(x,t)$ exist, respectively. Similarly, both $p(x,t)$ and $q(x,t)$ are continuous in the light grey region above (or to the left of) the curve $x=s(t)$. In this region, as a point (x,t) converges to the point on the curve $x=s(t)$, we assume that the limits $p(s^-(t),t)$, $q(s^-(t),t)$ of both $p(x,t)$ and $q(x,t)$ exist. It is not necessarily true that $p(s^+(t),t) = p(s^-(t),t)$ and $q(s^+(t),t) = q(s^-(t),t)$.

Suppose that a point (x,t) lies on the curve $x=s(t)$, and we write $U=s'(t)$. The red line in Fig. 3 represents the tangent to the curve $x=s(t)$ at (x,t). Its slope, U, corresponds to the velocity of the shock wave at this point. By selecting $x_2 \geq x_1$ and $t_2 \geq t_1$ as shown in Fig. 3, we can determine that

$$x_2 - x_1 = U(t_2 - t_1). \tag{16}$$

In (14), we let both x_1, x_2 converge to x, and t_1, t_2 converge to t, and keep the relation $x_2 - x_1 = U(t_2 - t_1)$, then we have

$$\int_{x_1}^{x_2} p(x,t_2)\,dx = p(s^-(t),t)(x_2-x_1) + o(x_2-x_1), \tag{17}$$

$$\int_{x_1}^{x_2} p(x,t_1)\,dx = p(s^+(t),t)(x_2-x_1) + o(x_2-x_1), \tag{18}$$

$$\int_{t_1}^{t_2} q(x_1,t)\,dt = q(s^-(t),t)(t_2-t_1) + o(t_2-t_1), \tag{19}$$

$$\int_{t_1}^{t_2} q(x_2,t)\,dt = q(s^+(t),t)(t_2-t_1) + o(t_2-t_1). \tag{20}$$

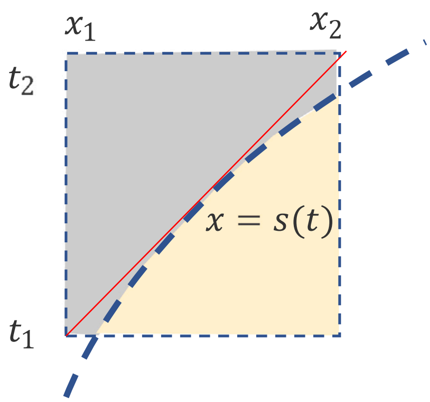

Fig. 3. Analyse properties of functions around a shock wave.

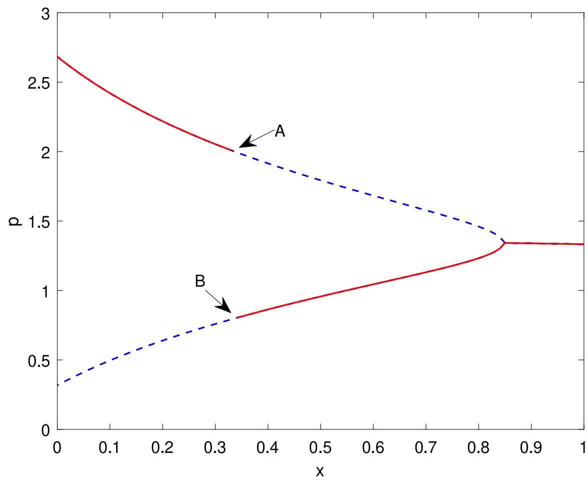

Fig. 4. Jumping of sharing-bicycle density between points A and B in a shock wave.

If both $T(x,t), f(x,t)$ are bounded within the region being studied, then

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}(-T(x,t)+f(x,t))\,dt\,dx = O\left(\left(t_2-t_1\right)\left(x_2-x_1\right)\right), \tag{21}$$

and therefore (14) becomes

$$U\left(p\left(s^-(t),t\right)-p\left(s^+(t),t\right)\right)-\left(q\left(s^-(t),t\right)-q\left(s^+(t),t\right)\right)=0. \tag{22}$$

For any function $h(s,t)$, we write the jump of the function on both sides of the shock wave is $[h]$, which can be expressed as follows.

$$[h](s,t) = h\left(x^+,t\right)-h\left(x^-,t\right). \tag{23}$$

Therefore, we can deduce from (22) that

$$U[p]-[q]=0. \tag{24}$$

3.2. Density of parked bikes

In this subsection, we will establish the equation satisfied by the density $u(x,t)$ of bicycles parked on the roadside. For two points x_1, x_2, in the region Z, and $x_2 \geq x_1$, as well as times $t_2 \geq t_1$, we have

$$\int_{x_1}^{x_2} u\left(x,t_2\right)dx-\int_{x_1}^{x_2} u\left(x,t_1\right)dx=\int_{t_1}^{t_2}\int_{x_1}^{x_2}(-T(x,t)+g(x,t))\,dt\,dx. \tag{25}$$

If the functions involved are continuous in the region under discussion, let both x_1, x_2 converge to x and we have from (25) that

$$u\left(x,t_2\right)-u\left(x,t_1\right)=\int_{t_1}^{t_2}\left(-T\left(x,t\right)+g\left(x,t\right)\right)dt. \tag{26}$$

Let t_1, t_2 both converge to t. From the right-hand side of (26), we have that the partial derivative of u concerning t exists at (x,t), and moreover, we have

$$u_t\left(x,t\right)=-T\left(x,t\right)+g\left(x,t\right). \tag{27}$$

We will analyse the continuity of the function $u\left(x,t\right)$ at the location of a shock wave. It can be seen that if $T\left(x,t\right), g\left(x,t\right)$ is bounded, we have that

$$U\left(u\left(s^-\left(t\right),t\right)-u\left(s^+\left(t\right),t\right)\right)=0, \tag{28}$$

or the equivalent form

$$U[u]=0 \tag{29}$$

holds. That is, if the velocity of a shock wave $U\neq 0$, we have that

$$u\left(s^-\left(t\right),t\right)=u\left(s^+\left(t\right),t\right). \tag{30}$$

That is, $u\left(x,t\right)$ is continuous at the shock wave.

4. Applications

Example 1. There are no bicycles on the road. That is, the density $u\left(x,t\right)=0$.

SOLUTION: $T\left(x,t\right)dxdt$ represents the count of users who select a sharing-bicycle and become a rider in the vicinity of (x,t) within space horizon dx and time horizon dt. Therefore, as there are no bicycles on the roadside, we have $T\left(0,p\right)=0$. Hence, the density $p\left(x,t\right)$ of user satisfies (24) and

$$p_t\left(x,t\right)+q_x\left(x,t\right)=f\left(x,t\right), \tag{31}$$

and from (9) and (10),

$$q=Q(0,p)=pV(0,p). \tag{32}$$

We can utilize the characteristic curve technique from classical shock wave analysis for the case [16]. Let $c(p)$ denotes the derivative function of $Q(0,p)$. Let us consider a curve $x=x(t)$ in the (x,t) plane satisfying

$$\frac{dx}{dt}=c(p(x\left(t\right),t)), \tag{33}$$

and

$$\frac{dp(x\left(t\right),t)}{dt}=f(x\left(t\right),t). \tag{34}$$

The solution $x=x(t)$ represents a characteristic curve, and $p(x\left(t\right),t)$ denotes the value on this characteristic curve. If different characteristic curves intersect, the system undergoes a shock wave, and its motion is recorded by (24).

Example 2 (*Continuing from Example 1*).. Now we have $f\left(x,t\right)=0$.

SOLUTION: From (34) and $f\left(x,t\right)=0$, we have that on a characteristic curve

$$\frac{dp(x\left(t\right),t)}{dt}=0, \tag{35}$$

and hence $p(x\left(t\right),t)$ is constant on this characteristic curve. Thus on this characteristic curve we have

$$\frac{dx}{dt}=c(p(x\left(0\right),0)). \tag{36}$$

Therefore

$$x\left(t\right)=x\left(0\right)+t\ c(p(x\left(0\right),0)). \tag{37}$$

That is, the characteristic curve from the position $x=\xi$ and from the moment $t=0$ which has an initial value $p(\xi,0)$ is

$$x\left(t\right)=\xi+t\ c(p(\xi,0)). \tag{38}$$

and on this characteristic curve $p(x\left(t\right),t)$ is the constant $p(\xi,0)$. This problem is discussed in detail in the classical monographs [17] on shock waves.

The following Example 3 explores the density distribution of user and parked bicycles at a local transport hub (e.g. metro station). Our focus lies on the changes in the local area's users' and parked bikes' densities due to commuters exiting and entering the transport hub.

Example 3. There is a transport hub at the origin of a half-plane $Z = [0, \infty)$. A uniform flow of commuters can be observed as they exit the transport hub and head towards a sharing-bicycle. That is there is a constant q_0 and the flow of users arriving on bikes satisfies the constant boundary condition

$$q(0,t) = q_0. \tag{39}$$

Moreover, the flow of those leaving sharing-bicycles and entering the transport hub is time-invariant

$$g(x,t) = g(x). \tag{40}$$

Thirdly, assumes that no other users are present in the region,

$$f(x,t) = 0. \tag{41}$$

We will investigate the time-invariant solutions

$$p(x,t) = p(x), \ u(x,t) = u(x), \tag{42}$$

of the system and their derived $q(x,t) = q(x)$.

SOLUTION: First, a point not located at a shock wave is considered. We have that time-invariant solutions satisfy

$$u_t(x,t) = 0, \tag{43}$$

and

$$p_t(x,t) = 0. \tag{44}$$

From (27), we have that

$$T(x,t) = g(x). \tag{45}$$

From (45) and $p_t(x,t) = 0$, $f(x,t) = 0$, we have that (15) yields

$$q_x(x) = -g(x). \tag{46}$$

Secondly, let us further investigate the point located at a shock wave. With a time-invariant solution, a shock wave is accompanied by a shock wave velocity $U = 0$. Then we have $U[u] = 0$. Consequently, from the discussion in the paragraph around (28), $u(x)$ may be intermittent at a shock wave. Moreover, (24) becomes

$$[q] = 0. \tag{47}$$

That is, $q(x)$ remains continuous at the shock wave. Together with the non-shock-wave point analysis, $q(x)$ remains a continuous function in the region Z.

By (4) and (45), we can establish the following equation

$$T(u,p) = g(x). \tag{48}$$

In general, the density

$$u = u(x,p) \tag{49}$$

can be found from (48). According to the third property (6) of the function $T(u,p)$, there is a unique solution of u exists for a given pair of x, p.

As $q(x)$ remains continuous, and (46) holds at a non-shock-wave point, we have that

$$q(x) = q(+\infty) + \int_x^{+\infty} g(x)\,dx. \tag{50}$$

Here $q(\infty)$ represents the flow of users who depart from the transport hub and do not find a suitable sharing-bicycle. Additionally, $q(0) = q_0$ represents the flow of users who leave the transport hub and enter the road. $\int_x^{\infty} g(x)\,dx$ is the flow of users who enter the transportation hub by bike per unit time. If density u is time-invariant, the number of bikes left behind equals the number of bikes being ridden away. Thus, it is sensible that (50) is valid, from the physic viewpoint.

From (8), that is, $q = Q(u,p)$, and $q(x)$ from (50), as well as u from (49), we can established the following equation for q,

$$q(x) = Q(u(x,p), p). \tag{51}$$

As a result, the density $p(x)$ of user can be determined at a point x. It is worth noting (51) may have multiple solutions.

For the problem of a sharing-bicycle system near a transport hub, Example 3 above gives the principles to follow in the processing. In order to show the full details of the processing, the following Example 4 will analyse some special cases in detail.

Example 4. Continuation of the above example. Users arriving at this transport hub, leaving a sharing-bicycle and entering the hub use this transport hub as their destination and the place where they leave their bike is right next to the hub. This example simply assumes that they only park within $[0, 1)$, meaning that for $x \geq 1$ we have $g(x) = 0$. For $0 \leq x < 1$ we assume that $g(x) = 1 - x$. Take $T(u,p)$ to satisfy (7) and $V(u,p)$ to satisfy (3). We are still looking for time-invariant solutions.

SOLUTION: For $x \geq 1$, since $f(x) = g(x) = 0$, the problem degenerates to Example 2. So if $x \geq 1$,

$$q(x) = q(\infty). \tag{52}$$

We explain (52) from the physical viewpoint. In the region $x \geq 1$, since $g(x) = 0$, there will be no new parked bikes in this region. At the moment $t = 0$, even if there are bikes in this region, they will all gradually be taken away and used by users over time. This means that the time-invariant solution of the system must correspond to the absence of parked bikes in this region, i.e. $u(x) = 0$. Since there are no parked bikes in this region, users do not become riders, and $f(x) = 0$, the density of users remains constant, and the flow $q(x)$ of users is therefore constant.

Next, we assume $x \leq 1$. from $g(x) = 1 - x$ and $g(1) = 0$ we have $\int_x^\infty g(x)\,dx = \frac{1}{2}(1-x)^2$. Hence we get

$$q(x) = q(\infty) + \frac{1}{2}(1-x)^2. \tag{53}$$

In the above equation letting $x = 0$ and due to (39), we have

$$q_0 = q(\infty) + \frac{1}{2}. \tag{54}$$

Hence it can be obtained that in order to ensure that the system has a time-invariant solution, the system parameter q_0 should satisfy the condition

$$q_0 \geq \frac{1}{2}. \tag{55}$$

Furthermore, no matter how many users are leaving the transport hub, after passing the area where the bikes are parked, the flow of users decreases by a constant amount of 0.5 by (54).

The definition of $g(x)$ and (45) imply that $T(x,t) = 1 - x$. Further, by (7),

$$u(x,p) = (1-x)\frac{M_T + p}{p}. \tag{56}$$

And thus according to the expression of V given by (7), (51) implies

$$Q(u,p) = (aup + bp)\left(M_p - p\right). \tag{57}$$

And hence

$$q(x,p) = \left(a(1-x)M_T + (b + a(1-x))p\right)\left(M_p - p\right). \tag{58}$$

is obtained by substituting $u(x,p)$ given by (56) into (57). Then according to the expression of $q(x)$, we can solve for $p(x)$ by

$$-(b + a(1-x))\,p^2 + \left(bM_p + a(1-x)(M_p - M_T)\right)p + a(1-x)\,M_T M_p = q(x). \tag{59}$$

We consider the left-hand side of (59), and its maximum value $h(p^*)$ is reached at

$$p^* = \frac{bM_p + a(1-x)(M_p - M_T)}{2\,(b + a(1-x))}. \tag{60}$$

If $p^* > 0$ and

$$a(1-x)\,M_T M_p \leq q(x) \leq h(p^*), \tag{61}$$

two sets of solutions to (59) can be obtained, and we denote these two sets of solutions as $p^+(x)$, $p^-(x)$. If $q(x) = h(p^*)$, we have that $p^+(x) = p^-(x)$, and in other cases we assume that $p^+(x) > p^-(x)$.

It follows that the function $p(x)$ may have interruptions. For $0 \leq x \leq 1$, we have

$$p(x) \in \left\{p^+(x), p^-(x)\right\}. \tag{62}$$

For $x > 1$, we have $p(x) = p(1)$. Furthermore, for $x \geq 0$, $u(x)$ is given by (56). When the density $p(x)$ of users has two solutions $p^+(x), p^-(x)$, the density $u(x)$ of sharing-bicycles parked on the roadside also has two solutions $u^+(x), u^-(x)$. $p(x), u(x)$ constitute the time-invariant solution of this sharing-bicycle system, and shock waves may lie at any of the interruption points of $p(x)$.

For example, for parameter settings $M_T = 0.5, M_p = 2, b = 1, a = 2$, and $q_0 = 0.7$, we can get two solutions for the density of sharing-bicycles in Fig. 4 with blue dashed lines, and the red solid line shows a solution with the presence of a shock wave between points A and B. Where there is a shock wave, there is a jump in the density of sharing-bicycles parked on the sidewalk. From an observer's point of view, this example shows that there is an aggregation of sharing-bicycles parked on the sidewalk near a transport hub: certain places have a higher density of sharing-bicycles and neighbouring areas have a lower density of sharing-bicycles.

Building upon the aforementioned discussion, we have also conducted a series of simulation experiments. It was observed that stabilizing the jump points in these experiments posed quite a challenge. However, when there were fluctuations in the passenger flow at the subway station exits, a shock wave in simulations, as shown in Fig. 5, could be readily realized. In this context, the positions of points A and B relative to the shock wave were observed to shift over time. To provide a comprehensive understanding, we have included a complete animation of the shock wave as an electronic supplement.

Examples 3 and 4 above examine a system of sharing-bicycles near a transport hub, using assumption $f(x,t) = 0$. Example 5 below relaxes this assumption and examines the travelling wave solution of the system.

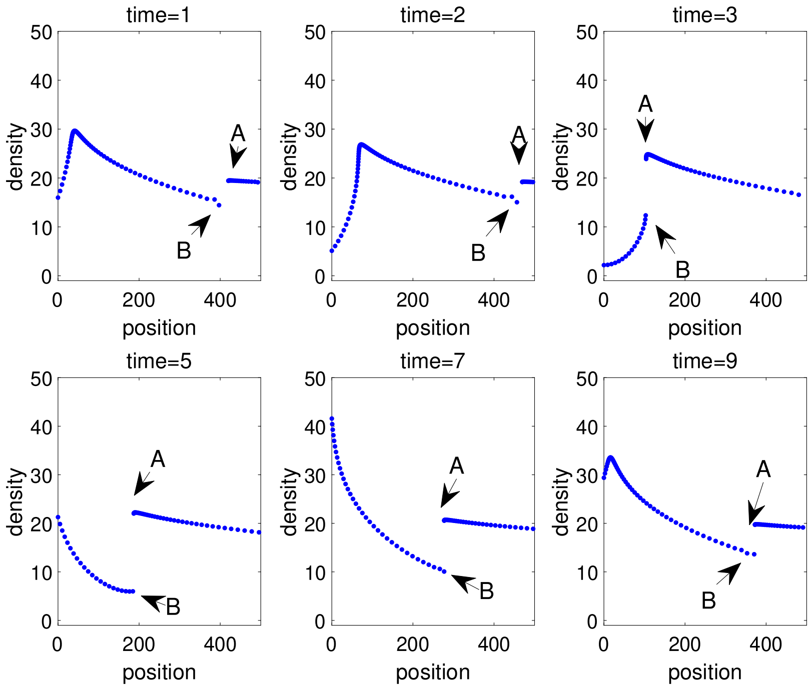

Fig. 5. Simulation with jumping points A and B relative to the shock wave shifting over time.

Example 5. The region is a half-plane $Z = [0, \infty)$ and the transport hub is at the origin. There is a uniform flow of users walking from the transport hub to choose a sharing-bicycle, i.e. the flow satisfies the boundary condition (39) determined by the constant q_0. We also assume that

$$g(x,t) = c_g, f(x,t) = c_f, \tag{63}$$

where $c_g, c_f > 0$ are two constants. We study a travelling wave solution. That is, for two constants $k_u, k_p \in R - \{0\}$, we look for solutions of the form

$$u(x,t) = u\left(x + k_u t\right), \tag{64}$$

and

$$p(x,t) = p\left(x + k_p t\right). \tag{65}$$

SOLUTION: First consider the case of a point (x,t) which is not near a point on a shock wave. Substituting (65) in (15) we get

$$k_p p_x + q_x = -T(u,p) + c_f. \tag{66}$$

Substituting (64) in (27) gives

$$k_u u_x = -T(u,p) + c_g. \tag{67}$$

From (10) we have that

$$q_x = V_u u_x p + \left(V_p p + V\right) p_x = \frac{pV_u}{k_u}\left(-T(u,p) + c_g\right) + \left(V_p p + V\right) p_x. \tag{68}$$

The derivatives of the two densities can then be obtained as follows.

$$p_x = \frac{\left(V_u p - k_u\right) T(u,p) - V_u p c_g + k_u c_f}{k_u\left(k_p + V_p p + V\right)}, \tag{69}$$

$$u_x = \frac{-T(u,p) + c_g}{k_u}. \tag{70}$$

Suppose we have found a function $F(u,p)$ that satisfies

$$F_u u_x + F_p p_x = 0. \tag{71}$$

Then, for each parameter C, a curve determined by

$$F(u,p) = C \tag{72}$$

is a trajectory of (69) and (70).

Next, we analyse the case of a shock wave. Since for the travelling wave solution $p(x,t)$, a shock wave is with speed $U = k_p \neq 0$, and for $u(x,t)$, a shock wave is with speed $k_u \neq 0$, we have that

$$k_p[p] - [q] = 0, \tag{73}$$

as well as

$$k_u[u] = 0. \tag{74}$$

Since the speed $k_u \neq 0$, we get

$$[u] = 0, \tag{75}$$

which means that $u(x,t)$ is a continuous function. We write p, q on each side of a shock wave as p^+, p^-, q^+, q^-. There is

$$q^{\pm} = V\left(u, p^{\pm}\right) p^{\pm}. \tag{76}$$

Thus, we have the condition of a shock wave:

$$k_p\left(p^+ - p^-\right) = \left(V\left(u, p^+\right) p^+ - V\left(u, p^-\right) p^-\right). \tag{77}$$

For fixed u, we can obtain from (77) the function

$$p^+ = P^+(u, p^-). \tag{78}$$

If u, p^- as well as p^+ obtained from (78) satisfy a trajectory inscribed in (72). That is,

$$F\left(u, p^-\right) = F\left(u, P^+(u, p^-)\right) \tag{79}$$

holds. Then the corresponding travelling wave solution can have a shock at p^+, p^-.

Example 6. Continuing from Example 5 above. Specifically, take the functions $T(u,p)$ which satisfies (7) and $V(u,p)$ which satisfies (7).

SOLUTION: From (7) and (67) we get

$$k_u u_x = -\frac{up}{M_T + p} + c_g. \tag{80}$$

From (9) we get

$$q_x = V_u u_x + \left(V_p + V\right) p_x. \tag{81}$$

Substituting (81) in (66) we get

$$k_p p_x + V_u u_x + (V_p + V) p_x = -\frac{up}{M_T + p} + c_f. \tag{82}$$

(80) and (82) are equivalent to

$$u_x = -\frac{up}{k_u\left(M_T + p\right)} + \frac{c_g}{k_u} = \frac{c_g k_u\left(M_T + p\right) - up}{k_u\left(M_T + p\right)} \tag{83}$$

and

$$p_x = \frac{1}{k_u\left(V_p + V + k_p\right)}\left[\left(V_u - k_u\right)\frac{up}{M_T + p} + \left(k_u c_f + c_g V_u\right)\right]. \tag{84}$$

Since $V(u,p)$ satisfies (7), we get

$$V_u = a\left(M_p - p\right), V_p = -(au + b). \tag{85}$$

Substituting these two expressions in (84) we get

$$p_x = \frac{\left(a\left(M_p - p\right) - k_u\right) up + \left(k_u c_f + c_g a\left(M_p - p\right)\right)\left(M_T + p\right)}{k_u\left((au + b)\left(M_p - p - 1\right) + k_p\right)\left(M_T + p\right)}. \tag{86}$$

Next, we analyse the shock waves. Note that (73) is equivalent to

$$k_p\left(p^+ - p^-\right) = \left(q^+ - q^-\right). \tag{87}$$

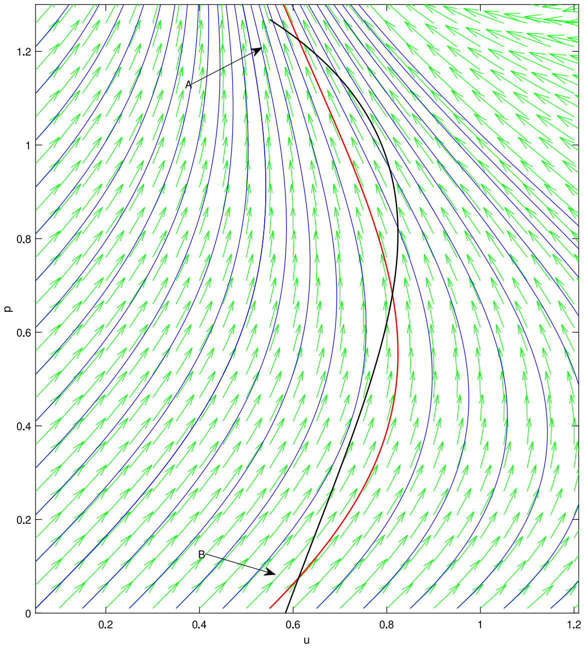

Fig. 6. Vector field and streamlines on $u - p$ plane for travelling wave solutions.

(7) and (76) imply

$$\left(q^{+}-q^{-}\right)=(au+b)\left(p^{+}-p^{-}\right)\left(M_{p}-p^{+}-p^{-}\right). \tag{88}$$

So we get from (87)

$$p^{-}=M_{p}-\frac{k_{p}}{au+b}-p^{+}. \tag{89}$$

For the set parameter values, we can obtain a travelling wave solution using numerical calculation methods. To illustrate the process, an example is given in Fig. 6. We set the parameter values $a = 0.7, b = 1, M_p = 2, M_T = 4, k_u = k_p = 1, C_g = C_f = 0.1$. A vector field in the $u - p$ plane is established according to (83) and (86). That is, for each point (u, p) in the plane, assign a vector $\left(u_x, p_x\right)$. These vectors are represented by green arrows in Fig. 6. From the vector field, we plot streamlines in Fig. 6, where the streamlines are depicted by blue curves, each of which corresponds to one of the curves of (72), with different parameters C for different streamlines. We have arbitrarily taken one of the streamlines and marked it in red. Denote this curve as p^+ and calculate p^- from (89), Fig. 6 plots the curve p^- in black. The black curve and the red curve have intersections, and we mark two intersections A and B, which have the same p value, and that the jump corresponding to a shock wave occurs between the intersections A and B. A segment of the (u, p) curve between the intersection points A and B is intercepted, and from this curve and the formula (83), we can calculate $\frac{dx}{dp}$ for each point on the curve, and from the formula (86) we can also calculate $\frac{dx}{du}$. We thus obtain expressions for the variable u and the variable p concerning the variable x. Then, according to the periodicity of the travelling wave solution, we can obtain the travelling wave solution $u(x, t)$ and $p(x, t)$. Fig. 7 plots the travelling wave solutions for variable u and variable p when $t = 0$. It can be seen that the travelling wave solution for variable p has jumps, and the jump points correspond to the locations where the shock wave occurs. As (75) indicates, the travelling wave solution of variable u is continuous at the position of shock waves, but the derivative function of u is not continuous.

5. Conclusion

The problem of understanding and controlling the overall behaviour of a sharing-bicycle system has become increasingly important as such systems continue to grow and become more popular. The model developed in this paper considers the effects of user riding behaviour and user demand by two variables: the density of potential user groups of sharing-bicycles on the road, and the density of sharing-bicycles parked along the sidewalk. Unlike traditional models, our model does not use stochastic assumptions, so the results of the model are deterministic and can be easily validated.

The results of the model's shock waves, time-invariant, and travelling wave solutions allow us to reveal the phenomenon of aggregation of sharing-bicycles on the sidewalk due to the architecture of the sharing-bicycles system, as well as the phenomenon of fluctuation of variables that still exists in a stable system. These results provide an important reference for us to gain a deeper understanding of the operation mechanism of the sharing-bicycles system.

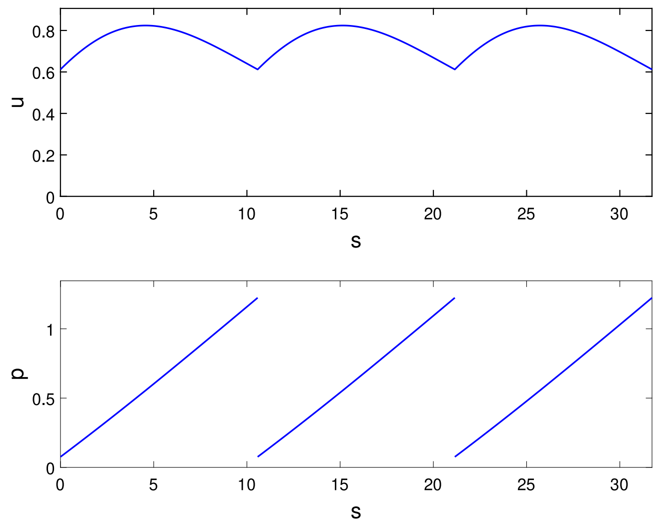

Fig. 7. Travelling wave solutions for u and p, exhibiting jump locations corresponding to waves.

However, from a theoretical point of view, our model has shortcomings in terms of descriptive capability and needs further improvement and extension. Currently, our study is limited to the problem on a long road and only considers the case where there is only one traffic hub on the road. If the destinations or origins of users in the studied road are more complex and diverse, the impact on the density of users and parked sharing-bicycles would be an important research direction. In addition, the impact of complex urban road networks on the density of users and sharing-bicycles needs to be studied in depth. These are the focuses of our next work.

Regarding potential future extensions, we plan to explore several directions: Firstly, we aim to apply the model to sharing-bicycle systems in various market environments, assessing its applicability and effectiveness across different cultural and economic contexts. Secondly, by leveraging the latest positioning technology and big data analysis, we can track the real-time dynamics of users and bicycles, providing more precise data support for the model and enabling the development of more intelligent dispatch and management strategies. Furthermore, with the ongoing advancements in autonomous driving and the Internet of Things (IoT), we will investigate how these emerging technologies can integrate with the sharing-bicycle system to further enhance its efficiency and safety.

These future extensions will not only propel the research on sharing-bicycle systems forward but also offer novel approaches and methodologies to address emerging practical challenges. We are confident that through continuous exploration and innovation, we can provide more comprehensive and effective support for the control and management of sharing-bicycle systems, thereby fostering their sustainable development, enhancing user experience, and maximizing social benefits.

In conclusion, the model presented in this paper offers a significant perspective on understanding the behaviour of sharing-bicycle systems and points to key directions for future research. By incorporating modern technologies, we are confident in achieving further breakthroughs in optimizing and managing sharing-bicycle systems.

CRediT authorship contribution statement

Junrong Liu: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation. **Wen-Xiu Ma:** Writing – review & editing, Supervision, Project administration, Methodology, Investigation, Conceptualization.

Ethical considerations

We affirm that the research conducted in this study complies with all relevant ethical standards and regulations, including those related to human subjects (if applicable), animal welfare, and research integrity.

We have obtained any necessary ethical approvals and informed consent (if applicable) for the research described in this manuscript.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.wavemoti.2025.103498.

Data availability

Data will be made available on request.

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