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The string equation and the τ-function Witt constraints for the discrete Kadomtsev-Petviashvili hierarchy

Shaowei Liu[1,a)] and Wen-Xiu Ma[2,b)]
[1]College of Mathematics and Statistics, Southwest University, Beibei 400715, Chongqing, People's Republic of China
[2]Department of Mathematics and Statistics, University of South Florida, Tampa, Florida 33260-5700, USA

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Using the tool of pseudo-difference operators, the string equation for the discrete Kadomtsev-Petviashvili (KP) hierarchy is introduced and a general expression for the string equation is formulated. Furthermore, the operators of the constraints that the string equation imposes on the τ-function of the p-reduced dKP hierarchy are presented. The algebra, which part of the constraints spans, and eigenvalues of part of the constraints corresponding to the τ-function are calculated. It is also shown that this algebra is exactly a Witt algebra not only for $p = 2$, but also for a general $p \in \mathbb{N}$, and that the τ-function of the p-reduced dKP hierarchy constrained by the string equation is a vacuum vector of a Witt algebra. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4826357]

I. INTRODUCTION

Formulating the constraint operators, which the string equation, $[\tilde{Q}, \tilde{P}] = 1$, imposes on the τ-function of the p-reduced discrete KP hierarchy,

$$\begin{cases} L = \triangle + \sum_{j=0}^{\infty} f_j(n)\triangle^{-j}, \\ \frac{\partial L}{\partial t_i} = [(L^i)_+, L], \\ (L^p)_- = 0, \end{cases} \tag{1}$$

and calculating the special algebraic structure of the algebra that the constraint operators span are the main problems we considered here, especially, in which the p is an arbitrary natural number. String equation connects the solvable string theory and the intersection theory with the integrable hierarchies. In the 2D quantum gravity, Kontsevich[1] proved that the partition function for the intersection theory of moduli spaces is the logarithm of the τ-function[2] which satisfies both the string equation and the 2-reduced KP hierarchy (the Korteweg-de Vries hierarchy, i. e., the KdV hierarchy). Based on this fact, he proved the Witten's conjecture[3] that two 2D quantum gravity models are equal. Meanwhile, the authors in Refs. 4 and 5 showed that the above τ-function is equivalent to a vacuum vector for a Virasoro algebra. Using this fact, the partition function was calculated and proved to be equal to the above τ-function. Later, some works were devoted to generalizing the fact in Refs. 4 and 5. Goeree[6] extended it to the case of the 3-reduced KP hierarchies and the cases for an arbitrary p had been investigated in several different integrable systems.[5,7,8] So, solving our problems will contribute to finding solutions of the partition function of the 2D discrete quantum gravity and it is helpful in exploring the discretization of the 2D quantum gravity models, especially for the case of big p. In addition, the string equation comes from the invariance under an additional symmetry transformation with respect to the $t^*_{1,-p+1}$ flow. The infinitesimal generators of the additional symmetries on τ-function span a W_∞ algebra which tightly connects with these

a)Author to whom correspondence should be addressed. Electronic mail: swliu001@swu.edu.cn
b)mawx@cas.usf.edu

constraints operators. Therefore, formulating these constraints and calculating the related algebraic structure will show the influence of the string equation imposing on the Lie group of the additional symmetries. It is helpful in understanding both the string equation especially from the view of systemic symmetries and the additional symmetry group itself. Moreover, we think the methods used here have generality and can be applied to some other integrable systems, such as the KP hierarchy, the KP hierarchy of B-type (the BKP hierarchy), etc.

We start our study from the discrete KP (dKP)[9,10] hierarchy since discrete integrable systems and the discretization of integrable systems are extensively studied in mathematical physics recently. An obvious advantage of this kind of discretization is that the dKP hierarchy keeps integrability. The dKP hierarchy can be described by either the difference operator Δ or the shift operator Γ.[11–13] We would like to use the difference operator Δ because this will allow us to apply a powerful tool of the pseudo-difference operators. Though some of the structures of both the dKP hierarchy and the KP hierarchy[14] are similar, such as Hamiltonian forms, the τ-function, and the gauge transformation operators,[9,10,15–18] the difference operator Δ introduces much more complicated systems of equations and certain complexities, for example, the existence of zero-order functions and extra shift on commuting between functions and operators, which the KP hierarchy does not have. In what follows, we shall introduce the string equation and calculate the corresponding constraints and their algebraic structure for the dKP hierarchy. Here, these constraints are in the form of operators, an eigenfunction of which the corresponding τ-function becomes. We shall introduce the string equation through the theory of pseudo-difference operators and the additional symmetries.[19] Moreover, we shall formulate a general expression for this string equation. Based on the Adler-Shiota-van Moerbeke (ASvM) formula for the dKP hierarchy[19] and the resulting general expression for the string equation, we shall calculate the constraint operators that the string equation imposes on the τ-function of the p-reduced dKP hierarchy. These constraint operators span a lie algebra. But their eigenvalues are still unknown, which makes the structure of the algebra of constraints not to be determined totally. We will use the commutator relations to calculate directly the eigenvalues of part of the constraint operators, the second order portion. With those determined eigenvalues, we will show that the second order constraints, as a subalgebra of the algebra of all the constraints, can span exactly a Witt algebra. It means that the τ-function of the p-reduced dKP hierarchy constrained by the string equation is a vacuum vector for a Witt algebra. Moreover, this fact holds not only for $p = 2$, but also for any $p \in \mathbb{N}$. Note that although there is a shift connection between the τ-function of the dKP hierarchy in the form of $\triangle$ and the τ-function of the KP hierarchy, we do not use it here because this connection holds only in some kinds of cases. Hence, to draw more general conclusions, we start the deduction from the pseudo-difference operators.

The organization of the paper is as follows. In Sec. II, we shall give a brief description of the dKP hierarchy to make the paper self-contained. In Sec. III, we will introduce the string equation for the dKP hierarchy through the additional symmetries, and we will formulate a general expression for this string equation. In Sec. IV, based on the ASvM formula for the dKP hierarchy, we will calculate the constraints that the string equation imposes on the τ-function of the p-reduced dKP hierarchy and part of the corresponding eigenvalues. Further, we show that the second order constraints span a Witt algebra. Section V is devoted to conclusions and discussion.

II. THE DISCRETE KP HIERARCHY

To make the paper self-contained, we give a brief introduction to the dKP hierarchy based on a detailed research in Refs. 10 and 19.

Let F be an associative ring of functions which includes a discrete variable $n \in \mathbb{Z}$ and many time variables $t_i \in \mathbb{R}$:

$$F = \{f(n) = f(n;t) = f(n, t_1, t_2, \cdots, t_i, \cdots);\ n \in \mathbb{Z},\ t_i \in \mathbb{R}\}. \tag{2}$$

Here $f(n)$ is at least first-order differentiable at some t_i.

The shift operator Γ and the difference operator $\triangle$ acting on F are defined as follows:

$$(\Gamma f)(n) = f(n+1), \tag{3}$$

and

$$(\triangle f)(n) = f(n+1) - f(n) = ((\Gamma - I)f)(n), \tag{4}$$

where I is the identity operator. Moreover, we define the powers of Γ and positive powers of Δ:

$$(\Gamma^i f)(n) = f(n+i), \quad i \in \mathbb{Z} \quad \text{and} \quad \Delta^i f = \Delta^{i-1}(\Delta f), \quad i \in \mathbb{N}, \tag{5}$$

and the multiplication "$\circ$" between a function and the difference operator:

$$(\Delta \circ f)(n) = f(n+1)\Delta + (\Delta f)(n)I, \tag{6}$$

namely,

$$((\triangle \circ f)(g))(n) = f(n+1)(\triangle g)(n) + (\triangle f)(n)g(n). \tag{7}$$

The definition of (6) is equivalent to another definition which is $(\Delta \circ f)g = \Delta(fg)$. When functions are located on the left-hand side of difference operators, we omit the "$\circ$." Here, note that the difference operator is unlike the differential operator in the KP hierarchy. There is an extra shift after changing the order of functions and operators, which causes complexities in the dKP hierarchy. This formula can be extend to any order of $\triangle$, including negative orders. That is, for any $j \in \mathbb{Z}$, we have

$$(\Delta^j \circ f)(n) = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i f)(n+j-i)\Delta^{j-i}, \quad \binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{8}$$

Here for positive j, the sum has finite terms, but for negative j, it has infinitely many terms. With (8), we can introduce an associative ring $f(\triangle)$ of formal pseudo-difference operators, which includes the two operations, the addition "$+$" and the multiplication "$\circ$":

$$F(\triangle) = \left\{ R = R(n) = \sum_{j=-\infty}^{d} f_j(n)\Delta^j, \ f_j \in F, \ d \geq 0 \right\}. \tag{9}$$

This ring $f(\triangle)$ includes two subrings:

$$F_+(\triangle) = \{R_+ = R_+(n) = \sum_{j=0}^{d} f_j(n)\Delta^j, \ f_j \in F, \ d \geq 0\}, \tag{10}$$

the ring of deference operators, and

$$F_-(\triangle) = \{R_- = R_-(n) = \sum_{j=-\infty}^{-1} f_j(n)\Delta^j, \ f_j \in F\}, \tag{11}$$

the ring of Volterra operators.

Let L be a general first-order pseudo-difference operator in the ring $f(\triangle)$,

$$L = \triangle + \sum_{j=0}^{\infty} f_j(n)\Delta^{-j}, \tag{12}$$

then, the dKP hierarchy[10] can be expressed as

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L], \ i \geq 1, \tag{13}$$

where P_+ denotes the positive part of $P \in f(\Delta)$, belonging to $F_+(\Delta)$. Comparing the orders of $\triangle$ on both sides of the above Lax equations in (13), we obtain the dKP evolution equations for the functions f_i as dependent variables. For example, through the t_1 and t_2 flows on the functions f_1 and f_2, after canceling f_1, we can obtain the following equation for f_0:

$$\begin{aligned} &\partial_{t_1}(\Delta^2 f_0)(n) + \partial_{t_1}(\triangle f_0^2)(n) - \partial_{t_2}(\triangle f_0)(n) - \partial_{t_1}(\triangle f_0)(n) - \partial_{t_1}(\triangle f_0)(n+1) \\ &= -\partial_{t_1}^2 f_0(n+1) - \partial_{t_1}^2 f_0(n). \end{aligned} \tag{14}$$

Note that we can see a difference between the KP hierarchy and the dKP hierarchy here: in the dKP hierarchy, the function f_0 can not be canceled out by transformation.

Now, we introduce some equivalent forms of the dKP equations in (13).

Let an operator $W(n) = W(n;t)$,

$$W(n) = W(n;t) = 1 + \sum_{j=1}^{\infty} w_j(n;t)\Delta^{-j}, \quad w_j \in F, \tag{15}$$

be an addressing operator of L:

$$L = W \circ \Delta \circ W^{-1}. \tag{16}$$

Then, we can obtain the following equations about W:

$$\frac{\partial W}{\partial t_i} = -(L^i)_- \circ W, \ i \geq 1, \tag{17}$$

which are called the Sato equations and sufficient to the dKP equations in (13).

The dKP hierarchy is also equivalent to the compatibility conditions of the following equations:

$$L\tilde{w}(n;t,z) = z\tilde{w}(n;t,z), \ \partial_{t_m}\tilde{w}(n;t,z) = (L^m)_+\tilde{w}(n;t,z), \ z \in \mathbb{C}, \tag{18}$$

where $\tilde{w}$ is the eigenfunction of L. Using the dressing operator W, we obtain the formal solutions for (18):

$$\begin{aligned}\tilde{w}(n;t,z) &= W(n;t)(1+z)^n exp(\sum_{i=1}^{\infty} t_i z^i)\\ &= (1 + \frac{w_1(n;t)}{z} + \frac{w_2(n;t)}{z^2} + \cdots)(1+z)^n exp(\sum_{i=1}^{\infty} t_i z^i),\end{aligned} \tag{19}$$

which are called the Baker or wave function. Here the n in $(1 + z)^n$ is the same discrete variable as n in $\tilde{w}(n;t,z)$.

The wave function $\tilde{w}(n;t,z)$ can be expressed by a single function, the τ-function $\tau_\triangle = \tau(n;t)$,[10] that is,

$$\tilde{w}(n;t,z) = \frac{\tau(n;t-[z^{-1}])}{\tau(n;t)}(1+z)^n exp(\sum_{i=1}^{\infty} t_i z^i), \tag{20}$$

where $[z] = (z, z^2/2, z^3/3, \ldots)$. Equation (20) means that all functions $w_i(n;t)$ in the dressing operator W can be generated by a single function $\tau(n;t)$. Further, all functions $f_i(n;t)$ in the operator L can be generated by the same τ-function.[15]

For simplicity, we introduce $G(z)$ and $G^*(z)$ operators, whose actions on functions are

$$(G(z)f)(t) = f(t - [z^{-1}]), \ (G^*(z)f)(t') = f(t' + [z^{-1}]), \tag{21}$$

respectively.

Now, let us introduce a kind of additional symmetries for the dKP hierarchy. We denote the infinitesimal operators for these symmetries by $\partial^*_{m,l} = \frac{\partial}{\partial t^*_{m,l}}$. Their actions on $W(n)$ are as follows:

$$\partial^*_{m,l}W(n) = -(M_\triangle^m L^l)_- \circ W(n), \ m \geq 0, \ l \in \mathbb{Z}, \tag{22}$$

where

$$M_\triangle = W(n) \circ \Gamma_\triangle(n) \circ W^{-1}(n), \ \Gamma_\triangle(n) = \sum_{i=1}^{\infty}[it_i + (-1)^{i-1}n]\Delta^{i-1}. \tag{23}$$

Their actions on the operator L are

$$\partial^*_{m,l}L = -[(M_\triangle^m L^l)_-, L], \ m \geq 0, \ l \in \mathbb{Z}. \tag{24}$$

The vertex operators defined by

$$X(n;\lambda,\mu)=\left(\frac{1+\mu}{1+\lambda}\right)^n :\exp\sum_{i=-\infty}^{\infty}(\frac{P_i}{i\lambda^i}-\frac{P_i}{i\mu^i}):, \tag{25}$$

where the symbol of "::" means the normal ordering and

$$P_i=\begin{cases}\partial_i=\frac{\partial}{\partial t_i}, & i>0,\\ 0, & i=0,\\ -it_{-i}, & i<0\end{cases} \tag{26}$$

can be used to describe the additional symmetries. Here the normal ordering means that P_i with negative i should be placed to the left of P_j with positive j.

Taking the Taylor expansion of the $X(n; \lambda, \mu)$ in μ at the point of λ, we have

$$X(n;\lambda,\mu)=\sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{l=-\infty}^{\infty}\lambda^{-l-m}\tilde{W}_l^{(m)}(n), \tag{27}$$

where

$$\sum_{l=-\infty}^{\infty}\lambda^{-l-m}\tilde{W}_l^{(m)}(n)=\partial_\mu^m|_{\mu=\lambda}X(n;\lambda,\mu). \tag{28}$$

Based on the ASvM formula of the dKP hierarchy,[19] we obtain the actions of the additional symmetries on the wave function of the dKP hierarchy:

$$\partial^*_{m,m+l}\tilde{w}(n;t,z)=\left((G(z)-1)\frac{\frac{\tilde{W}_l^{(m+1)}(n)}{m+1}\tau(n;t)}{\tau(n;t)}\right)\tilde{w}(n;t,z), \tag{29}$$

which holds for $m \geq 0$ and $l \in \mathbb{Z}$.

III. THE STRING EQUATION FOR THE DKP HIERARCHY

In the string theory of modern physics, it is an important problem to find two differential operators satisfying the equation $[Q, P] = 1$. This resembles a quantum problem of finding two commuting differential operators, $[Q, P] = 0$. Now we extend this from continuousness to discreteness. That is to find two difference operators satisfying the equation $[\tilde{Q}, \tilde{P}] = 1$. In integrable systems, this equation is called the string equation. Finding such pairs of difference operators is closely related to the determination of the additional symmetries for the dKP hierarchy.

In this section, we will construct some such pairs of difference operators to satisfy the string equation for the dKP hierarchy. Further, we will extend the string equation to more general forms, which are useful for calculating the algebra of the constraints on the τ-function for the p-reduced dKP hierarchy.

Let us first introduce the p-reduced dKP hierarchy.

Definition 3.1. If the generating operator of the dKP hierarchy, L, satisfies $(L^p)_- = 0$, where $p \geq 2$ and $p \in \mathbb{N}$, then the corresponding dKP hierarchy is called the p-reduced dKP hierarchy.

Here, the 2-reduced dKP hierarchy is the discrete KdV hierarchy and the p-reduced dKP hierarchy is the discrete pth Gelfand–Dickey hierarchy, following the meaning of discreteness in our dKP hierarchy.

In addition, we have the following proposition.

Proposition 3.2. The τ-function of the p-reduced dKP hierarchy does not depend on the variables of $\{t_{mp}|m \in \mathbb{N}\}$.

Proof: Since $(L^p)_- = 0$, we have

$$(L^{mp})_+ = ((L^p)^m)_+ = (((L^p)_+)^m)_+ = ((L^p)_+)^m = (L^p)^m = L^{mp}, \quad m \geq 1. \tag{30}$$

Substituting it into (13), we have

$$\partial_{mp} L = [(L^{mp})_+, L] = [L^{mp}, L] = 0, \tag{31}$$

which means that L is independent of $\{t_{mp}|m \in \mathbb{N}\}$. Then the τ-function $\tau(n; t)$ of L does not depend on $\{t_{mp}|m \in \mathbb{N}\}$ too. □

Since f_i can be generated by the τ-function, all functions, f_i, in the p-reduced dKP hierarchy do not depend on the variables $t_{mp}, m \geq 1$.

From now on, let us consider two kinds of difference operators, $\frac{1}{p}(M_\triangle L^{1-p})_+$ and L^p. Here L is the generating operator for the p-reduced dKP hierarchy. We will try to determine the commutator relation between them. To this end, we make the following proposition.

Proposition 3.3. Let L be the generating operator for the p-reduced KP. If $\partial^*_{1,-p+1} L^p = 0$, then $[L^p, \frac{1}{p}(M_\triangle L^{1-p})_+] = 1$, where $M_\triangle$ is defined as in (23).

Proof: By $[M_\triangle, L] = -1$, we obtain

$$[M_\triangle L^{-p+1}, L^p] = [M_\triangle, L^p]L^{-p+1} = \left(\sum_{i=0}^{p-1} L^i [M_\triangle, L] L^{p-1-i}\right) \circ L^{-p+1} = -p. \tag{32}$$

Meanwhile, by the action of ∂^* on L, we have

$$\partial^*_{1,-p+1} L = -[(M_\triangle L^{-p+1})_-, L]. \tag{33}$$

Then,

$$\partial^*_{1,-p+1} L^p = -[(M_\triangle L^{-p+1})_-, L^p] = [(M_\triangle L^{-p+1})_+, L^p] - [M_\triangle L^{-p+1}, L^p]. \tag{34}$$

Substituting (32) into the above identity, we obtain

$$\partial^*_{1,-p+1} L^p = [(M_\triangle L^{-p+1})_+, L^p] + p. \tag{35}$$

It now follows from (35) that $\partial^*_{1,-p+1} L^p = 0$ is equivalent to $[L^p, \frac{1}{p}(M_\triangle L^{1-p})_+] = 1$. □

With the requirement of independence of the additional variable $t^*_{1,-p+1}$, those two kinds of difference operators satisfy $[Q, P] = 1$. Now, we define the string equation for the dKP hierarchy.

Definition 3.4. $[L^p, \frac{1}{p}(M_\triangle L^{1-p})_+] = 1$, in which $(L^p)_- = 0$, is called the string equation of the dKP hierarchy.

In addition, by (33), we obtain that if $(L^p)_- = 0$ and the identity

$$(M_\triangle L^{-p+1})_- = \frac{p-1}{2} L^{-p} \tag{36}$$

holds, so does the string equation. Thus, (36) together with $(L^p)_- = 0$ is equivalent to the string equation for the dKP hierarchy.

Substituting (36) into (22), we can obtain the equivalent identity of (36) in terms of $W(n)$, that is,

$$\partial^*_{1,-p+1} W(n) = -(M_\triangle L^{-p+1})_- \circ W(n) = -\frac{p-1}{2} L^{-p} \circ W(n). \tag{37}$$

Now, we extend Eq. (36) to more general forms, which are useful for calculating the algebra of the constraints on the τ-function. This is the following theorem.

Theorem 3.5. If the p-reduced dKP hierarchy generator, L, satisfies Eq. (36), then we have the following general string equations,

$$(M_\triangle^j L^{kp+j})_- = \begin{cases} \prod_{r=0}^{j-1}(\frac{p-1}{2} - r)L^{-p}, & \text{when } j \in \mathbb{N},\ k=-1, \quad (38a) \\ 0, & \text{when } j \in \mathbb{N},\ k = 0, 1, 2, \cdots. \quad (38b) \end{cases}$$

Proof: We first show that if (38a) holds, then (38b) holds too.

If (38a) holds, then, for $k = 0, 1, 2, \ldots$ and any $j \in \mathbb{N}$, we have

$$\begin{aligned} (M_\triangle^j L^{kp+j})_- &= (M_\triangle^j L^{-p+j+(k+1)p})_- \\ &= ((M_\triangle^j L^{-p+j})L^{(k+1)p})_- \\ &= ((M_\triangle^j L^{-p+j})_- L^{(k+1)p})_- \ \ (\text{due to } (L^{(k+1)p})_- = 0) \\ &= (constant \cdot L^{-p} L^{(k+1)p})_- \\ &= constant \cdot (L^{kp})_- \\ &= 0. \end{aligned} \tag{39}$$

It means that (38b) holds.

Second, we calculate the following commutator relation:

$$\begin{aligned} &[L^{mp+j}, M_\triangle(n)] \\ =&W(n) \circ \left[\triangle^{mp+j}, \quad \sum_{i=1}^{\infty}(it_i \triangle^{i-1} + (-1)^{i-1} n \triangle^{i-1})\right] \circ W^{-1}(n) \\ =&W(n) \circ \left[\triangle^{mp+j}, \quad n\sum_{i=1}^{\infty}(-1)^{i-1}\triangle^{i-1}\right] \circ W^{-1}(n) \\ =&W(n) \circ \left(\left((n+mp+j)\triangle^{mp+j} + (mp+j)\triangle^{mp+j-1} - n\triangle^{mp+j}\right)\sum_{i=1}^{\infty}(-1)^{i-1}\triangle^{i-1}\right) \circ W^{-1}(n) \\ =&W(n) \circ \left((mp+j)\Big(\sum_{i=1}^{\infty}(-1)^{i-1}\triangle^{mp+j+i-1} + \sum_{i=1}^{\infty}(-1)^{i-1}\triangle^{mp+j+i-2}\Big)\right) \circ W^{-1}(n) \\ =&(mp+j)W(n) \circ \triangle^{mp+j-1} \circ W^{-1}(n) \\ =&(mp+j)L^{mp+j-1}. \end{aligned} \tag{40}$$

By the above commutator relation, we can obtain

$$\begin{aligned} (M_\triangle^j L^{mp+j})(M_\triangle L^{lp+1}) =& M_\triangle^j (M_\triangle L^{mp+j} + [L^{mp+j}, M_\triangle])L^{lp+1} \\ =& M_\triangle^{j+1} L^{mp+lp+j+1} + (mp+j) M_\triangle^j L^{mp+lp+j}. \end{aligned} \tag{41}$$

Setting $l = -1$ and $m = 0$ in the above relation, using the string equation (36), we obtain that if $M_\triangle^j L^j$ is a difference operator, we can obtain the relation

$$\begin{aligned} M_\triangle^{j+1} L^{-p+j+1} =&((M_\triangle^j L^j)(M_\triangle L^{-p+1}))_- - j(M_\triangle^j L^{-p+j})_- \\ =&((M_\triangle^j L^j)(M_\triangle L^{-p+1})_-)_- - j(M_\triangle^j L^{-p+j})_- \\ =&\frac{p-1}{2}(M_\triangle^j L^j L^{-p})_- - j(M_\triangle^j L^{-p+j})_- \\ =&(\frac{p-1}{2} - j)(M_\triangle^j L^{-p+j})_-. \end{aligned} \tag{42}$$

Now, with these preparations, we can use the mathematical induction on j to prove Theorem 3.5.

When $k = -1$ and $j = 1$, (38a) is just Eq. (36). So, (38a) holds for $j = 1$, $k = -1$. Substituting it into (39), we know that (38b) holds for $j = 1$, $k = 0, 1, 2, \ldots$, too. Then, Theorem 3.5 holds for $j = 1$, $k = -1, 0, 1, 2, \ldots$.

Assume that (38a) holds for $k = -1$ and $j = j_0$.

Substituting it into (39), we know that (38b) holds for $k = 0, 1, 2, \ldots$ and $j = j_0$. Then, Theorem 3.5 holds for $j = j_0$, $k = -1, 0, 1, 2, \ldots$.

Especially, setting $k = 0$, (38b) becomes

$$(M_\triangle^{j_0} L^{j_0})_- = 0. \tag{43}$$

It means that $M_\triangle^{j_0} L^{j_0}$ is a difference operator. So, the requirements of (42) hold. With (42), we have

$$M_\triangle^{j+1} L^{-p+j+1} = (\frac{p-1}{2} - j)(M_\triangle^j L^{-p+j})_- = (\frac{p-1}{2} - j)\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)L^{-p}. \tag{44}$$

It means that (38a) holds for $k = -1$ and $j = j_0 + 1$. By this conclusion and (39), we know that (38b) holds for $j = j_0 + 1$, $k = 0, 1, 2, \ldots$. Then, Theorem 3.5 holds for $j = j_0 + 1$, $k = -1, 0, 1, 2, \ldots$.

Now, Theorem 3.5 holds from the mathematical induction. □

IV. THE CONSTRAINTS AND RELATED WITT ALGEBRA

In this section, we will calculate the constraints that the string equation imposes on the τ-function of the p-reduced KP hierarchy. These constraints are in the form of linear operators and the τ-function is one eigenfunction of all these operators. These operators, as generators, can span a Lie algebra. But the eigenvalues of these operators corresponding to the τ-function are unknown. We use the commutator relation to calculate directly part of them, i.e., the eigenvalues of the second order constraints (the concept of the order will be defined later). With these determined eigenvalues, we show after a proper transformation, the second order constraints can span a Witt algebra. It implies that the τ-function of the p-reduced dKP hierarchy constrained by the string equation is a vacuum vector for a Witt algebra. This fact holds not only for $p = 2$, but also for any $p \in \mathbb{N}$.

First, using additional symmetries and from (38a) and (38b), we calculate the constraint equations that the general string equations impose on the τ-function.

Substituting the general string equations (i.e., (38a) and (38b)) into the definition of the additional symmetries (22), we obtain

$$\partial^*_{j,kp+j} W(n) = \begin{cases} -\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)L^{-p} \circ W(n), & \text{when } j \in \mathbb{N},\ k = -1, \quad (45a) \\ 0, & \text{when } j \in \mathbb{N},\ k = 0, 1, 2, \cdots. \quad (45b) \end{cases}$$

Substituting the above equations into the definition of the wave function, (19), we obtain

$$\partial^*_{j,kp+j} w(n;t,z) = \begin{cases} -\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)z^{-p} w(n;t,z), & \text{when } j \in \mathbb{N},\ k = -1, \quad (46a) \\ 0, & \text{when } j \in \mathbb{N},\ k = 0, 1, 2, \cdots. \quad (46b) \end{cases}$$

Then, based on the ASvM formula (29), we obtain the constraint equations for the τ-function which satisfies both the string equation and the p-reduced dKP hierarchy. That is, the following lemma.

Lemma 4.1. If the τ-function $\tau(n; t)$ satisfies both the string equation and the p-reduced dKP hierarchy, then it satisfies the equations of

$$(G(z)-1)\frac{\frac{\tilde{W}^{(j+1)}_{kp}}{j+1}\tau(n;t)}{\tau(n;t)} = \begin{cases} -\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)z^{-p}, & \text{when } j \in \mathbb{N},\ k = -1, \quad (47a) \\ 0, & \text{when } j \in \mathbb{N},\ k = 0, 1, 2, \cdots. \quad (47b) \end{cases}$$

Now, we start to extract the concrete operators of constraints from the above equation. First, we are going to calculate the $\tilde{W}^m_l$'s in (47a). We transform the definition of (25) as follows:

$$\begin{aligned} X(n;\lambda,\mu) &= :\exp\left(n\ln(1+\mu) - n\ln(1+\lambda) + \sum_{i=-\infty}^{\infty}(\frac{P_i}{i\lambda^i} - \frac{P_i}{i\mu^i})\right): \\ &= :\exp\left(\sum_{i=-\infty}^{\infty}\frac{P_i}{i\lambda^i} - n\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\lambda^i}{i} - \sum_{i=-\infty}^{\infty}\frac{P_i}{i\mu^i} + n\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\mu^i}{i}\right): \\ &= :\exp\left(\sum_{i=-\infty}^{\infty}\frac{\tilde{P}_i(n)}{i\lambda^i} - \sum_{i=-\infty}^{\infty}\frac{\tilde{P}_i(n)}{i\mu^i}\right):, \end{aligned} \quad (48)$$

in which

$$\tilde{P}_i(n) = \begin{cases} \partial_i, & i > 0, \\ 0, & i = 0, \\ -it_{-i} + (-1)^{-i-1}n, & i < 0. \end{cases} \quad (49)$$

Here n is the discrete variable which is the same as the discrete variable defined in (2). For convenience, we do not write down the variable n explicitly in the following deduction.

Now, substituting (48) into (28), after an onerous straightforward calculation, we can obtain the expressions of $\tilde{W}^{(m)}_l$'s. For convenience, we omit the multiply symbol of "$\circ$" between operators in the following deduction. We list the first few items as follows:

$$\tilde{W}^{(0)}_l = \delta_{l,0}, \quad (50a)$$

$$\tilde{W}^{(1)}_l = \tilde{P}_l, \quad (50b)$$

$$\tilde{W}^{(2)}_l = \sum_{i+j=l} :\tilde{P}_i\tilde{P}_j: -(l+1)\tilde{P}_l, \quad (50c)$$

$$\tilde{W}^{(3)}_l = \sum_{i+j+k=l} :\tilde{P}_i\tilde{P}_j\tilde{P}_k: -\frac{3}{2}(l+2)\sum_{i+j=l} :\tilde{P}_i\tilde{P}_j: +(l+1)(l+2)\tilde{P}_l, \quad (50d)$$

$$\tilde{W}^{(4)}_l = \tilde{Q}^{(4)}_l - 2(l+3)\tilde{Q}^{(3)}_l + (2l^2+9l+11)\tilde{Q}^{(2)}_l - (l+1)(l+2)(l+3)\tilde{Q}^{(1)}_l, \quad (50e)$$

$$\cdots,$$

$$\tilde{W}^{(n)}_l = \sum_{i=1}^{n} C^{(n)}_i(l)\tilde{Q}^{(i)}_l, \quad C^{(n)}_i(l)\text{'s are constants depending on } l, \quad (50f)$$

$$\cdots,$$

where

$$\tilde{Q}_l^{(0)} = \delta_{l,0}, \tag{51a}$$

$$\tilde{Q}_l^{(1)} = \tilde{P}_l, \tag{51b}$$

$$\tilde{Q}_l^{(2)} = \sum_{i+j=l} :\tilde{P}_i\tilde{P}_j:, \tag{51c}$$

$$\tilde{Q}_l^{(3)} = \sum_{i+j+k=l} :\tilde{P}_i\tilde{P}_j\tilde{P}_k:, \tag{51d}$$

$$\tilde{Q}_l^{(4)} = \sum_{i+j+k+r=l} :\tilde{P}_i\tilde{P}_j\tilde{P}_k P_r: - \sum_{i+j=l} :ij\tilde{P}_i\tilde{P}_j:, \tag{51e}$$

$$\cdots.$$

Here, we introduce the concept of order for operators. We define the number in the bracket in superscript as the order of the operator. For example, the operators $\tilde{Q}_l^{(i)}$ and $\tilde{W}_l^{(i)}$ are both of ith order operators.

Now, notice that the $\tilde{W}_l^{(n)}$ is a linear combination of $\tilde{Q}_l^{(n)}, \cdots, \tilde{Q}_l^{(1)}$. And the transformation matrix between them is a triangle matrix with all diagonal elements being equal to 1. So, the transformation matrix is invertible. That means $\tilde{Q}_l^{(n)}$ can also be expressed as a linear combination of $\tilde{W}_l^{(n)}, \cdots, \tilde{W}_l^{(1)}$. That is, the following theorem.

Proposition 4.2. We have

$$\tilde{Q}_l^{(i)} = \tilde{W}_l^{(i)} + constant\cdot\tilde{W}_l^{(i-1)} + \cdots + constant\cdot\tilde{W}_l^{(1)}. \tag{52}$$

Before continuing to calculate the constraints, recall that the τ-function of the dKP hierarchy does not depend on the variables t_{mp} (Proposition 3.2). So, to obtain the compact forms of the operators of these constraints, we need to get rid of the redundant variables of t_{mp} or $\tilde{P}_{-mp}$ in the constraints. Here, before doing this, we introduce some notations. Every $\tilde{Q}_l^{(i)}$ is a sum of items of the normal ordering. For each $\tilde{Q}_l^{(i)}$, these items can be divided into two categories: one includes all items that contain none of variables in $\{\tilde{P}_{-mp}|m\in\mathbb{N}\}$ and the other includes all items that contain at least one variable in $\{\tilde{P}_{-mp}|m\in\mathbb{N}\}$. We denote the sum of all items in the former category by $\bar{Q}_l^{(i)}$ and the sum of all items in the latter category by $\hat{Q}_l^{(i)}$. Therefore, for every $\tilde{Q}_l^{(i)}$, we have

$$\tilde{Q}_l^{(i)} = \bar{Q}_l^{(i)} + \hat{Q}_l^{(i)},\ \bar{Q}_l^{(i)} = \tilde{Q}_l^{(i)}|_{\tilde{P}_{-mp}=0}. \tag{53}$$

Furthermore, without the variables of t_{mp}'s, $\partial_{mp}(\tilde{P}_{mp,m\in\mathbb{N}})$'s are also redundant. So, to get rid of both t_{mp}'s and ∂_{mp}'s, we similarly introduce another decomposition for $\tilde{Q}_l^{(i)}$. One category includes all items that contain none of variables in $\{\tilde{P}_{-mp}, \tilde{P}_{mp}|m\in\mathbb{N}\}$, denoted by $\mathring{Q}$; and the other category includes all items that contain at least one variable in $\{\tilde{P}_{-mp}, \tilde{P}_{mp}|m\in\mathbb{N}\}$, denoted by $\check{Q}_l^{(i)}$. Then, we have

$$\tilde{Q}_l^{(i)} = \mathring{Q}_l^{(i)} + \check{Q}_l^{(i)},\ \mathring{Q}_l^{(i)} = \tilde{Q}_l^{(i)}|_{\substack{\tilde{P}_{-mp}=0\\ \partial_{mp}=0}} = \tilde{Q}_l^{(i)}|_{\tilde{P}_{lp}=0,l\in\mathbb{Z}}. \tag{54}$$

We will use the same decompositions for $\{\tilde{W}_{kp}^{(i)}\}$:

$$\tilde{W}_l^{(i)} = \mathring{W}_l^{(i)} + \check{W}_l^{(i)},\ \mathring{W}_l^{(i)} = \tilde{W}_l^{(i)}|_{\substack{\tilde{P}_{-mp}=0\\ \partial_{mp}=0}} = \tilde{W}_l^{(i)}|_{\tilde{P}_{lp}=0,l\in\mathbb{Z}}, \tag{55}$$

$$\tilde{W}_l^{(i)} = \bar{W}_l^{(i)} + \hat{W}_l^{(i)},\ \bar{W}_l^{(i)} = \tilde{W}_l^{(i)}|_{\tilde{P}_{-mp}=0}. \tag{56}$$

Now, with these preparations, we can calculate the constraints that the string equation imposes on $\tau(n;t)$ from Lemma 4.1. These constraints are operators, of which the τ is a eigenfunction. It is the following theorem.

Theorem 4.3. If the τ-function $\tau(n;t)$ satisfies both the p-reduced KP hierarchy and the string equation (36), then

$$\bar{Q}^{(i)}_{kp}\tau(n;t) = \mathring{Q}^{(i)}_{kp}\tau(n;t) = c^{(i)}_k\tau(n;t), \quad i \in \mathbb{N},\ k = -1, 0, 1, 2\cdots. \tag{57}$$

Here each $c^{(i)}_k$ is a constant, which is the eigenvalue for the τ-function.

Proof: From Eq. (47b), by the properties of the operator $G(z)$, we can obtain

$$\tilde{W}^{(j+1)}_{kp}\tau(n;t) = constant \cdot \tau(n;t) \quad \text{when } k = 0, 1, 2, \cdots,\ j \in \mathbb{N}. \tag{58}$$

As for Eq. (47a), consider a fact that

$$-\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)(G(z) - 1)(\frac{\tilde{P}_{-p}\tau(n;t)}{\tau(n;t)}) = \prod_{r=0}^{j-1}(\frac{p-1}{2} - r)z^{-p}. \tag{59}$$

Add (59) to (47a) and we can obtain

$$(G(z) - 1)\frac{(\frac{\tilde{W}^{(j+1)}_{kp}}{j+1} - \prod_{r=0}^{j-1}(\frac{p-1}{2} - r)\tilde{P}_{-p})\tau(n;t)}{\tau(n;t)} = 0, \quad \text{when } k = -1,\ j \in \mathbb{N}. \tag{60}$$

By the properties of the operator $G(z)$, from the above equations, we can obtain

$$\big(\tilde{W}^{(j+1)}_{kp} - (j+1)\prod_{r=0}^{j-1}(\frac{p-1}{2} - r)\tilde{P}_{-p}\big)\tau(n;t) = constant \cdot \tau(n;t), \quad \text{when } k = -1,\ j \in \mathbb{N}. \tag{61}$$

Consider (58) and (61) together. We are going to get rid of the redundant variables in these two equations. Let $t_{mp} = \frac{(-1)^{mp}\cdot n}{mp}$ on both sides of the two identities. After doing this, the two identities still hold. And on the left-hand sides of these two equations, the items which include at least one variable in $\{P_{-mp}|m \in \mathbb{N}\}$ are canceled. As for the right-hand sides, these have no effect, because the τ-function $\tau(n;t)$ of the p-reduced dKP hierarchy is independent of the variables of $\{t_{mp}|m \in \mathbb{N}\}$. So, we obtain

$$\bar{W}^{(j+1)}_{kp}\tau(n;t) = constant \cdot \tau(n;t), \quad \text{when } k = -1, 0, 1, 2, \cdots,\ j \in \mathbb{N}. \tag{62}$$

Now, we transfer the constraints from $\bar{W}$ to $\bar{Q}$, a more compact expression. By Proposition 4.2, we have

$$\tilde{Q}^{(i)}_{kp} = \tilde{W}^{(i)}_{kp} + constant \cdot \tilde{W}^{(i-1)}_{kp} + \cdots + constant \cdot \tilde{W}^{(1)}_{kp}. \tag{63}$$

Taking $t_{mp} = \frac{(-1)^{mp}\cdot n}{mp}$ on both sides, we obtain

$$\bar{Q}^{(i)}_{kp} = \bar{W}^{(i)}_{kp} + constant \cdot \bar{W}^{(i-1)}_{kp} + \cdots + constant \cdot \bar{W}^{(1)}_{kp}. \tag{64}$$

Substituting the above into (62), we obtain

$$\bar{Q}^{(i)}_{kp}\tau(n;t) = c^{(i)}_k\tau(n;t), \quad k = -1, 0, 1\ldots,\ i \in \mathbb{N}.$$

Finally, because $\partial_{t_{mp}}\tau(n;t) = 0$, we can also get rid of the items that include at least one item in $\{\partial_{mp}\}$. So, we obtain

$$\mathring{Q}^{(i)}_{kp}\tau(n;t) = \bar{Q}^{(i)}_{kp}\tau(n;t) = c^{(i)}_k\tau(n;t), \quad k = -1, 0, 1\ldots,\ i \in \mathbb{N}.$$

The proof is finished. □

Now, we obtain the operators of the constraints, $\{\mathring{Q}^{(i)}_{kp}\}$ or $\{\bar{Q}^{(i)}_{kp}\}$. But the eigenvalues for them are still unknown. We will use the commutator relation to determine part of them, the eigenvalues for the second order constraints, $c^{(2)}_k$.

First, we calculate the commutator relations of the algebra $\{\tilde{Q}^{(2)}_{kp}\}$ and the algebra $\{\mathring{Q}^{(2)}_{kp}\}$.

Lemma 4.4. Algebra $\{\tilde{Q}^{(2)}_{kp}|k=-1,0,1,\cdots\}$ and algebra $\{\mathring{Q}^{(2)}_{kp}|k=-1,0,1,\cdots\}$ have the same commutator relations, (71).

Proof: By a straightforward and onerous computation, we can obtain the following commutator relation for the algebra $\{\tilde{Q}^{(2)}_{kp}\}$:

$$[\frac{1}{2}\tilde{Q}^{(2)}_{mp},\frac{1}{2}\tilde{Q}^{(2)}_{np}]=\frac{1}{2}(mp-np)\tilde{Q}^{(2)}_{(m+n)p}+\frac{1}{12}((mp)^3-mp)\delta_{mp+np,0}\quad m,n\in\{-1,0,,\cdots\}. \tag{65}$$

Now, in order to obtain the commutator relation for the algebra $\{\mathring{Q}^{(2)}_{kp}\}$, we need to get rid of the redundant variables in the above commutator relation.

We consider $\check{Q}^{(2)}_{kp}$. Because $k\geq -1$, we obtain

$$\check{Q}^{(2)}_{kp}=2\sum_{m\in\mathbb{N}}\tilde{P}_{-mp}\tilde{P}_{(k+m)p}+Q^{+(2)}_{kp}, \tag{66}$$

where

$$Q^{+(2)}_{kp}=\begin{cases}\sum_{l=1}^{k}\tilde{P}_{lp}\tilde{P}_{(k-l)p}, & \text{when}\quad k\geq 1, \quad (67a)\\ 0, & \text{when}\quad k=-1,0. \quad (67b)\end{cases}$$

Without loss of generality, we choose the more complicated case of (67a) to calculate. The other cases including (67b) are similar and we omit them. Substituting (67a) into (53), we obtain

$$\tilde{Q}^{(2)}_{kp}=\mathring{Q}^{(2)}_{kp}+2\sum_{m\in\mathbb{N}}\tilde{P}_{-mp}\tilde{P}_{(k+m)p}+Q^{+(2)}_{kp}. \tag{68}$$

We substitute the above into (65). The right-hand side is

$$\begin{aligned}R.H.S.=&\frac{1}{2}(np-mp)\mathring{Q}^{(2)}_{np+mp}+(n-m)p\sum_{l_3\in\mathbb{N}}\tilde{P}_{-l_3p}\tilde{P}_{(l_3+n+m)p}\\&+\frac{1}{2}(n-m)pQ^{+(2)}_{(n+m)p}+\frac{1}{12}((np)^3-np)\delta_{np+mp,0}.\end{aligned} \tag{69}$$

As for the left-hand side, because $\{\mathring{Q}^{(2)}_{kp}\}$ do not include the variables of $\{t_{mp}\}$, they can commutate with $\{\tilde{P}_{kp},k\in\mathbb{Z}\}$. So, we obtain

$$\begin{aligned}L.H.S=&[\frac{1}{2}\mathring{Q}^{(2)}_{np}+\sum_{l_1\in\mathbb{N}}\tilde{P}_{-l_1p}\tilde{P}_{(l_1+n)p}+\frac{1}{2}Q^{+(2)}_{np},\frac{1}{2}\mathring{Q}^{(2)}_{mp}+\sum_{l_2\in\mathbb{N}}\tilde{P}_{-l_2p}\tilde{P}_{(l_2+m)p}+\frac{1}{2}Q^{+(2)}_{mp}]\\
=&[\frac{1}{2}\mathring{Q}^{(2)}_{np},\frac{1}{2}\mathring{Q}^{(2)}_{mp}]+[\sum_{l_1\in\mathbb{N}}\tilde{P}_{-l_1p}\tilde{P}_{(l_1+n)p},\sum_{l_2\in\mathbb{N}}\tilde{P}_{-l_2p}\tilde{P}_{(l_2+m)p}]\\
&+[\sum_{l_1\in\mathbb{N}}\tilde{P}_{-l_1p}\tilde{P}_{(l_1+n)p},\frac{1}{2}Q^{+(2)}_{mp}]+[\frac{1}{2}Q^{+(2)}_{np},\sum_{l_2\in\mathbb{N}}\tilde{P}_{-l_2p}\tilde{P}_{(l_2+m)p}]\\
=&[\frac{1}{2}\mathring{Q}^{(2)}_{np},\frac{1}{2}\mathring{Q}^{(2)}_{mp}]+\sum_{n+l_1=l_2}(n+l_1)p\tilde{P}_{-l_1p}\tilde{P}_{(m+l_2)p}-\sum_{m+l_2=l_1}(m+l_2)p\tilde{P}_{-l_2p}\tilde{P}_{(n+l_1)p}\\
&+\sum_{l_4=1}^{m}(-l_4p)\tilde{P}_{(n+l_4)p}\tilde{P}_{(m-l_4)p}+\sum_{l_5=1}^{n}l_5p\tilde{P}_{(m+l_5)p}\tilde{P}_{(n-l_5)p}\\
=&[\frac{1}{2}\mathring{Q}^{(2)}_{np},\frac{1}{2}\mathring{Q}^{(2)}_{mp}]+(n-m)p\sum_{l_1\in\mathbb{N}}\tilde{P}_{-l_1p}\tilde{P}_{(m+n+l_1)p}+\frac{1}{2}(n-m)pQ^{+(2)}_{(n+m)p}.\end{aligned} \tag{70}$$

We cancel the same items on both sides. Then, we obtain the commutator relation for the algebra $\{\mathring{Q}^{(2)}_{kp}\}$,

$$[\frac{1}{2}\mathring{Q}^{(2)}_{np}, \frac{1}{2}\mathring{Q}^{(2)}_{mp}] = [\frac{1}{2}\tilde{Q}^{(2)}_{np}, \frac{1}{2}\tilde{Q}^{(2)}_{mp}]|_{\tilde{P}_{lp}=0, l\in\mathbb{Z}}$$
$$= \frac{1}{2}(np - mp)\mathring{Q}^{(2)}_{np+mp} + \frac{1}{12}((np)^3 - np)\delta_{np+mp,0}, \quad m, n \in \{-1, 0, , \cdots\}, \quad (71)$$

which is the same as the one for $\{\frac{1}{2}\tilde{Q}^{(2)}_{kp}\}$. □

With the above lemma, we can determine the eigenvalues of $\{c^{(2)}_k\}$.

Theorem 4.5. Let $c^{(2)}_k$ be defined as in (57). Then we have

$$c^{(2)}_k = 0, \quad k = -1, 1, 2, \cdots, \quad (72)$$

$$c^{(2)}_0 = -\frac{1}{12}(p^2 - 1). \quad (73)$$

Proof: Let both sides of (71) act on $\tau(n; t)$. By Theorem 4.3, we obtain

$$0 = [\frac{1}{2}c^{(2)}_n, \frac{1}{2}c^{(2)}_m] = \frac{1}{2}(np - mp)c^{(2)}_{n+m} + \frac{1}{12}((np)^3 - np)\delta_{np+mp,0}. \quad (74)$$

Then, with different values of m and n, we can obtain $\{c^{(2)}_k = 0|k = -1, 1, 2, \cdots\}$ and $c^{(2)}_0 = -\frac{1}{12}(p^2 - 1)$. □

With these values of $c^{(2)}_k$, we go further to calculate the algebraic structure of $\{\mathring{Q}^{(2)}\}$. Notice that the commutator relation (71) cannot show that $\mathring{Q}^{(2)}$ is a Virasoro algebra. It only shows $\mathring{Q}^{(2)}$ being a subalgebra of a Virasoro algebra. So, to obtain an exact Virasoro algebra, we need to transform the $\mathring{Q}^{(2)}$'s. Define

$$\tilde{L}_k = \frac{1}{2p}(\mathring{Q}^{(2)}_{kp} - c^{(2)}_k), \quad k = -1, 0, 1, 2, \cdots. \quad (75)$$

Then, substituting $\tilde{L}_k$ into (65), we obtain

$$[\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n}, \quad (76)$$

which is just the centerless Virasoro commutator relation. So we obtain the following theorem.

Theorem 4.6. $\{\tilde{L}_k|k = -1, 0, 1, 2, \cdots\}$ spans a Witt (centerless Virasoro) algebra (76).
Meanwhile, substituting (75) into Theorem 4.3, we can obtain

$$\tilde{L}_k\tau(n; t) = 0, \quad k = -1, 0, 1, 2, \cdots. \quad (77)$$

It means that the τ-function of the p-reduced dKP hierarchy constrained by the string equation is a vacuum vector for a Witt algebra $\{\tilde{L}_k\}$.

V. CONCLUSIONS AND DISCUSSION

In this paper, using the tool of pseudo-difference operators, we defined the string equation for the dKP hierarchy through the additional symmetries. Furthermore, we formulated a general expression for this string equation. Then, based on the ASvM formula for the dKP hierarchy and the general string equations, we calculated the operators of the constraints that the string equation imposes on the τ-function of the p-reduced dKP hierarchy. And using the commutator relation, we determine the eigenvalues of part of these constraints. With those determined eigenvalues, we showed that the algebra which the second order constraints span is exactly a Witt algebra.

In our discussion, we did not use the shift connection between the τ-function of the dKP hierarchy and the τ-function of the KP hierarchy to obtain our results, because this connection is applicable only for some kinds of cases. So to obtain more general results, we totally used the method of pseudo-difference operators to draw our conclusions.

There are still a lot of eigenvalues being unknown. We will try to identify more of them in near future, and we will further study the mathematical structure of the algebra that the constraints span.

We remark that it is open to us how to define additional symmetries and the string equation for the extended soliton equations and the compatibility equations generated from the extended Lax operators (see Refs. 20 and 21).

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