Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 177 (2020) 516–526
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

The evolution of a clogging sidewalk caused by a dockless bicycle-sharing system: A stochastic particles model

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Received 29 March 2019; received in revised form 26 December 2019; accepted 28 April 2020
Available online 12 May 2020

Abstract

The operation of a dockless bicycle-sharing system is modeled by a stochastic system characterized by two kinds of interacting components, the distribution of bicycles on a sidewalk and behaviors of riders, whose simple and basic actions lead to complex results. From the model, the collective behavior emerging in the bicycle-sharing system is described by a nonlinear evolutionary equation for the density of bicycles on the sidewalk, which has a non-trivial lump solution. The solution describes a heap of bicycles at somewhere on the sidewalk, and the width and the movement of the heap is determined by the mean behaviors of riders. Such a lump solution implies that the phenomenon of clogging sidewalks may be an endogenous processes within some dockless bicycle-sharing system.

Keywords: Dockless bicycle-sharing system; Diffusion; Nonlinear evolutionary equation; Lump solution

1. Introduction

As an important last-mile service, a bicycle-sharing system is healthy and can reduce pollution in a city [2,4,20]. Recently, huge dockless bicycle-sharing systems with millions of bicycles appear in China. In a city with such a system, a rider does not walk several minutes to come across a share bicycle, and she or he can park it anywhere along the road near the destination. Although many systems encourage a rider to park a share bicycle in rectangle bays painted on the sidewalk, lots of bicycles are parked here and there for riders' convenience. Notice that

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https://doi.org/10.1016/j.matcom.2020.04.030
0378-4754/

convenience is a major motivator for a rider, as pointed out by the review [6] of bicycle-sharing systems. Those cities provide us with much empirical knowledge about advantages and disadvantages of practical huge dockless bicycle-sharing systems. Here we consider the phenomenon of a clogging sidewalk that, sometimes, many share bicycles are parked at a narrow sidewalk.

There is an extensive literature about bicycle-sharing systems. [6] is a good review of bicycle-sharing systems before 2015. [5] concludes that a share bicycle can compete with a car in some conditions. [3] and [24] provide a statistical and spatial quantitative analysis for bicycle-sharing systems. [22] and [8] establish different models for bicycle-sharing systems to study the operation of a system and how to keep a balance between expenditure and revenue for a system. A recent model in [9] investigates on safety problems on road segments shared by automobiles and bikes from bicycle-sharing systems. However, those macro level models do not involve the phenomenon of a clogging sidewalk. We assume that neither operators of a bicycle-sharing system nor any riders manage to block a sidewalk intentionally. With the preceding assumption, the phenomenon of a clogging sidewalk may be a result of stigmergic or collective behaviors [17] of riders within a bicycle-sharing system. Stigmergy is a system characterized by interacting components, whose simple and basic actions lead to an unexpected collective behavior that in general is not readily predictable from actions of components. We refer to [19] for an interesting discussion and an example of stigmergic behaviors.

In this paper, we establish a stochastic system modeling the operation of a dockless bicycle-sharing system along a long road. The system is characterized by the density of parked share bicycles along the road, and many independent riders who appear randomly on the road, walk following a diffusion process, pick up a share bicycle, ride to a random destination along the road, park the bicycle at the destination, and leave the system permanently. We assume a rider's origin and destination are randomly at the road, hence from the system we cannot infer any exogenous clogging sidewalks which sometimes appears near population centers, major commercial malls and subway stations. Till a rider picks up a share bicycle, the movement of the walking rider is a diffusion process with a constant diffusion coefficient and a drift coefficient determined by the density of parked share bicycles nearby. That is, the expectation of movement in a small time interval of a walking rider is in relation with the density of bicycles. Moreover, we assume that the chance of picking up a share bicycle at a place is in relation with the density of bicycles at the place too. From those assumptions, to describe a rider's behaviors, we can use a stochastic differential equation with coefficients determined by the density of parked share bicycles along the road. On the other hand, a rider's actions of picking up and parking a share bicycle can affect randomly the density. We assume that there are a lot of independent riders, and then from the law of large number, the effect of all riders on the density is not randomly. Therefore, we establish a nonlinear evolutionary equation to the density. The nonlinear evolutionary equation has a nontrivial lump solution, which is a rationally decaying solution in all space directions. We refer to [21,23] for lump-kink interaction solutions and [1,15,25] for lump-soliton interaction solutions and the methodology to other similar models. The lump solution of the density of parked share bicycles has a peak, where parked bicycles are assembled and we think that it illustrates a clogging phenomenon. The peak moves with human's walking speed, and its width varies when all people search bicycles anxiously.

A list of notations is provided in the following:

· $u(t, y)$: the density of share bicycles parked along a road. $u(t, y)\,dy$ denotes the amount of bicycles parked from position y on the road to $y + dy$ at time t.

· (ξ_t, η_t): the diffusion process describes the walking of a rider. Given a rider starts from position x on the road at time s, ξ_t denotes the rider's position at time t. η_t is an auxiliary process, which is only used in the full model, describes the chance of the rider's picking up a share bicycle.

· σ: the diffusion coefficient of ξ_t, which is set to 1 in the basic model and is a positive constant in the full model.

· b_0: the constant part of the drift coefficient b of ξ_t and denotes a rider's movement intention when the density of parked bicycles is constant.

· $\lambda(u)$: the rate of a walking rider's picking up a share bicycle at a position where the density is u. It is constant in the basic model and $\lambda(u) = \lambda_0 + \beta u$ in the full model.

· $m(s, x)$ and $n(t, y)$: denote respectively the rate of riders' appearing at a position x and time s, and the rate of riders' reaching her or his destination y at time t. We assume that those two functions are equal and are constants, that is, the model does not try to describe phenomena of clogging sidewalk due to any exogenous reasons.

The rest of the paper is organized as follows. In Section 2, we present a basic model which only includes key concepts of the full model, show the idea to construct an evolutionary equation for the basic model, and illustrate how to obtain a lump solution to such equations. Section 3 contains a full model which includes details of riders and the dockless bicycle-sharing system, establishes the evolutionary equation for the full model, and presents equations to describe expectations of walking time and displacement of a rider. Features of those equations are discussed in Section 4. Section 5 concludes the paper.

2. Basic model of dockless bicycle-sharing system

2.1. Features of share bicycles and riders

We study the operation of dockless share bicycles along a long line R, and $u(t, y)$ denotes the density of share bicycles parked at $y \in R$ and time t. Given that a rider starting from $x \in R$ at time s, till picking up a share bicycle, she or he walks in the following form of diffusion process.

$$\xi_t = x + \int_s^t \sigma \, dB_r + \int_s^t b \, dr. \tag{1}$$

Here ξ_t denotes the position of the rider at time t, B_r is a Wiener process. σ is the rider's diffusion coefficient which is set to 1 in this basic model. b is the rider's drift coefficient, which denotes the rider's movement intention, and we assume it is determined by the density of parked bicycles nearby. Concretely, suppose that $\xi_t = y$, that is, the position of a walking rider is y at time t, where the density of share bicycles is $u = u(t, y)$ and let $u_y = \frac{\partial u(t,y)}{\partial y}$ which represents the rate of change of the density nearby. We assume that

$$b(u, u_y) = b_0 + b_1(u)u_y + b_3(u)u_y^3 + b_5(u)u_y^5 + \cdots \tag{2}$$

Here b_0 denotes the rider's movement intention when the density is constant, and we assume $b_0 = 1$ in this basic model. The form of (2) ensures that $b(u, -u_y) - b_0 = -(b(u, u_y) - b_0)$, that is, the remaining part $b - b_0$ of a rider's movement intention is different according to whether there are increasing or decreasing bicycles at front of the rider. Then the transition density $p(s, t, x, y)$ of ξ_t, the position of a walking rider, satisfies Fokker–Planck equation [18].

$$p_t = \frac{1}{2}(\sigma^2 p)_{yy} - (b\,p)_y. \tag{3}$$

There are many independent riders and start points of riders are independently and randomly at the road. We assume that there appear $m(s, x)\,dx\,ds$ riders in $(x, x + dx)$ on the road during a time interval $(s, s + ds)$. If none of riders pick up a share bicycle forever, the amount of riders finding bicycle around $(y, y + dy)$ on the road at time t is

$$p(s, t, x, y)\,m(s, x)\,dx\,ds\,dy.$$

However, a rider picks up a share bicycle sooner or later. In this basic model, it costs a walking rider an exponential random time with mean $1/\lambda_0$ to pick up a share bicycle, $\lambda_0 > 0$. That is, given that a rider starting from $x \in R$ at time s, the probability of that she or he still walks on the road till t is $e^{-\lambda_0(t-s)}$, where $t > s$. The memoryless property of an exponential distribution implies that the hazard rate function is the constant λ_0. Hence the probability of her or his picking up a share bicycle during $(t, t + dt)$ is $\lambda_0 e^{-\lambda_0(t-s)}dt$. Therefore, due to riders' picking, the density of share bicycles parked at position y and time t decreases with a rate $\lambda_0 q(t, y)$ and

$$q(t, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} e^{-\lambda_0(t-s)}\, p(s, t, x, y)\,m(s, x)\,dx\,ds. \tag{4}$$

On the other hand, every rider parks a bicycle at her or his destination and the density of share bicycles increases because of the actions. Hence, the density of share bicycles satisfies

$$u_t = n(t, y) - \lambda_0 q(t, y). \tag{5}$$

Here $n(t, y)dt\,dy$ is the amount of rider who parks a bicycle around $(y, y + dy)$ on the road during a time interval $(t, t + dt)$. Notice that a rider's origin and destination are randomly at the road for eliminating some predictable exogenous phenomena, that is, we assume that both $n(t, y)$ and $m(s, x)$ are constants. Furthermore, every rider finds a share bicycle sooner or later and the rider parks the bicycle when she or he reaches the destination. So we have $n(t, y) = m(s, x)$.

It follows from (3), (5) and $p(t,t,x,y)=\delta(x-y)$ that

$$q_t = m(t,y) - \lambda_0 q + \frac{1}{2}q_{yy} - (b(u,u_y)q)_y. \tag{6}$$

Then it follows from (5) and $n(t,y)=m(s,x)$ that the density $u(t,y)$ satisfies the following nonlinear evolutionary equation.

$$\frac{1}{2}u_{tyy} - u_{tt} - (b(u,u_y)u_t)_y - \lambda_0 u_t = 0. \tag{7}$$

2.2. *Exact solution of density*

The solution of (7) is not unique. $u \equiv constant$, a trivial solution to (7), represents an ideal case. For a specific drift coefficient, a nontrivial lump solution to (7) is presented in the following theorem.

Theorem 1. *For the following drift coefficient*

$$b(u,u_y) = 1 + \left(\frac{3}{4u^2} + \frac{1}{u}\right)u_y + \frac{u_y^3}{8\lambda_0 u^6},$$

the density function $u(t,y) = \frac{1}{1+\lambda_0(y-t)^2}$ *is a solution to nonlinear evolutionary equation* (7).

The proof of Theorem 1 is straightforward by substituting the expression of the density function $u(t,y)$ into (7). Here we present the proceed of solving (7), to exhibit a method to solve similar equations.

We reduce the drift coefficient through a rational transformation $u=\frac{1}{v}$ as

$$\begin{aligned} b &= b_0 - b_1\left(\frac{1}{v}\right)\frac{v_y}{v^2} - b_3\left(\frac{1}{v}\right)\frac{v_y^3}{v^6} + \cdots \\ &= b_0 - \tilde{b}_1(v)\,v_y - \tilde{b}_3(v)\,v_y^3 + \cdots \end{aligned} \tag{8}$$

We manage to solve (7) for a drift coefficient b with simple forms of $\tilde{b}_1, \tilde{b}_3, \ldots$. In fact, the drift coefficient b in Theorem 1 corresponds to $\tilde{b}_1 = \frac{3}{4}+\frac{1}{v}$ and $\tilde{b}_1 = \frac{1}{\lambda_0}$.

Substitution of the dependent variable transformation $u=\frac{1}{v}$ with $v=v(t,y)$ into (7) yields a partial differential equation about v as

$$\sum_{i=0}^{6}\sum_{j=0}^{6} c_{i,j}\,(\partial_i v)\left(\partial_j v\right) + \sum_{i=0}^{6}\sum_{j=0}^{6}\sum_{k=0}^{6} c_{i,j,k}\,(\partial_i v)\left(\partial_j v\right)(\partial_k v) + \cdots = 0. \tag{9}$$

Here $c_{i,j}$ and $c_{i,j,k}$ are constants, and $\partial_0 v = v$, $\partial_1 v = v_t$, $\partial_2 v = v_y$, $\partial_3 v = v_{tt}$, $\partial_4 v = v_{ty}$, $\partial_5 v = v_{yy}$, $\partial_6 v = v_{tyy}$. We guess that the summation of the first part and few other terms in (9) equals 0. Concretely, we guess that

$$-2v_t v_y + 2vv_{ty} - \frac{1}{2}vv_{tyy} + \lambda_0 vv_t + 2vv_{tt} + \lambda_0 vv_y - 2vv_t^2 = 0.$$

The preceding equation has a bilinear representation.

$$\left(D_t^2 + D_y D_t - \frac{1}{4}D^2_{\langle 2,3\rangle,y}D_{\langle 1,3\rangle,t} + \frac{\lambda_0}{2}(D_{\langle 1,1\rangle,t} + D_{\langle 1,1\rangle,y})\right) v\cdot v = 0. \tag{10}$$

Here D_t, D_y, $D_{\langle 2,3\rangle,y}$, $D_{\langle 1,3\rangle,t}$, $D_{\langle 1,1\rangle,t}$ and $D_{\langle 1,1\rangle,y}$ are bilinear derivative operators [7]. We refer to [11,12] for an introduction to bilinear representations and bilinear equations. Many nontrivial solutions to (10) are obtained by following the technique established in [10,13,14,16]. Substitution of those solutions into (9), we can eliminate some extraneous solutions of (10), and finally obtain the solution $u(t,y) = \frac{1}{1+\lambda_0(y-t)^2}$ to (9).

3. Full model of dockless bicycle-sharing system

3.1. *Evolutionary equation for system*

In this practical model, two unreasonable assumptions $\sigma^2 =$ and $b_0 = 1$ in the basic model are canceled. Moreover, we assume that walking time of a rider has a nonconstant hazard rate function $\lambda = \lambda(u(t,y))$, that is, the probability for a rider picking up a share bicycle is determined by the density of bicycles parked nearby. It is obvious that $\lambda(u) \geq 0$ and we assume that $\inf_u \lambda(u) > 0$ to avoid the case that a rider never picks up any share bicycle.

In this model, given that a rider starting from $x \in R$ at time s, the probability of that she or he still walks on the road and searches a bicycle till $t > s$

$$E\left[\exp\left(-\int_s^t \lambda(u(r,\xi_r))dr\right)\bigg|\xi_s = x\right]. \tag{11}$$

Here ξ_r, $s \leq r \leq t$ is a diffusion process defined by (1), and ξ_r satisfies the following stochastic differential equation.

$$d\xi_t = \sigma dB_t + b(u(t,\xi_t), u_y(t,\xi_t))dr. \tag{12}$$

To establish formula corresponding to (4) and (3) for this practical model, we introduce a stochastic process $\{\eta_t\}$ such that

$$d\eta_t = \lambda(u(t,\xi_t))dr. \tag{13}$$

For example, we can then rewrite the probability of (11) as

$$E\left[\exp(-\eta_t)|\,\xi_s = x, \eta_s = 0\right]. \tag{14}$$

$\{(\xi_t, \eta_t)\}$ is a 2-dimensional Markov process whose transition density $p(s,t,x,y,z)$, defined by

$$p(s,t,x,y,z)dydz = P(\xi_t \in (y, y+dy), \eta_t \in (z, z+dz)|\xi_s = x, \eta_s = 0), \tag{15}$$

satisfies the following partial differential equation.

$$p_t = \frac{1}{2}\left(\sigma^2 p\right)_{yy} - (bp) - y - (\lambda p)_z, \tag{16}$$

with initial conditions $p(s,s,x,y,z) = \delta_{y-x}\delta_z$. Moreover, it follows from the fact $\inf_u \lambda(u) > 0$ that $p(s,t,x,y,0) = 0$ for any $t > s$.

Now we turn to compute the decreasing rate of the density $u(t,y)$ due to riders' picking up bicycles. Similar in the basic model, we assume that there appear $m(s,x)\,dx\,ds$ riders in $(x, x+dx)$ on the road during a time interval $(s, s+ds)$, and $n(t,y)dt\,dy$ is the amount of rider who parks a bicycle around $(y, y+dy)$ on the road during a time interval $(t, t+dt)$, where $n(t,y) = m(s,x) \equiv constant$. And we assume that the constant is so large that, due to the law of large number, the effect of all riders on the density is not randomly. Furthermore, we use $\tilde{q}(t,y,z)dydz$ to denote the amount of share bicycles parked at $(y, y+dy)$ at time t and the rider parking the bicycle has $\eta_t \in (z, z+dz)$. It follows from (14) and (15) that

$$\tilde{q}(t,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{t} \mathrm{e}^{-z}\, m(s,x)\, p(s,t,x,y,z)\,dx\,ds. \tag{17}$$

Then we can obtain derivatives of $\tilde{q}(t,y,z)dydz$ from (17). Concretely,

$$\tilde{q}_t = \int_{-\infty}^{+\infty}\int_{-\infty}^{t} \mathrm{e}^{-z}\, m(s,x)\, p_t(s,t,x,y,z)\,dx\,ds + \int_{-\infty}^{+\infty} \mathrm{e}^{-z}\, m(t,x)\, p(t,t,x,y,z)\,dx.$$

It follows from $p(t,t,x,y,z) = \delta_{y-x}\delta_z$ that the second term of the right hand side of the preceding equation is $\mathrm{e}^{-z}\, m(t,y)\delta_z$. Hence the preceding equation and (16) imply that

$$\tilde{q}_t = m(t,y)\mathrm{e}^{-z}\delta_z - \lambda\tilde{q} + \frac{1}{2}(\sigma^2\tilde{q})_{yy} - (b\tilde{q})_y - (\lambda\tilde{q})_z. \tag{18}$$

From the definition of $\tilde{q}(t,y,z)$ and $n(t,y)$, we have that

$$u_t = n(t,y) - \lambda(u(t,y))\int_{-\infty}^{+\infty} \tilde{q}(t,y,z)\,dz.$$

Hence, it follows from (18) and the facts $\int_{-\infty}^{+\infty}\int_{-\infty}^{t} p(s,t,x,y,0)\,ds\,dx = 0$ and $\lim_{z\to+\infty} p(s,t,x,y,z) = 0$ that

$$\frac{\sigma^2}{2}u_{tyy} - u_{tt} - (b(u,u_y)u_t)_y - (\lambda(u)u)_t = 0. \tag{19}$$

Notice that the preceding evolutionary equation of the density $u(t, y)$ reduces to (7) of the basic model when $\sigma = 1$ and $\lambda(u) = \lambda_0$.

3.2. Density, walking time, and displacement

The solution of (19) is not unique. Following the same method, we obtain a nontrivial lump solution to (7) is presented in the following theorem.

Theorem 2. *For $\lambda(u) = \lambda_0 + \beta u$, $\beta \geq 0$, and the following drift coefficient*

$$b(u, u_y) = b_0 + \sigma^2 \frac{u_y}{u} + \left(b_0^2 + \frac{\beta b_0^2}{\lambda_0} + 2\sigma^2\right) \frac{u_y}{4u^2} + \frac{b_0^4}{2}\left(1 + \frac{\sigma^2}{b_0^2} + \frac{\beta}{\lambda_0}\right) \frac{u_y^3}{8\lambda_0 u^6}, \quad (20)$$

the nonlinear evolutionary equation (19) has a nontrivial solution

$$u(t, y) = \frac{b_0^2}{b_0^2 + \lambda_0 (y - b_0 t)^2}. \quad (21)$$

Notice that the preceding density $u(t, y)$ reduces to the solution to (7) given in Theorem 1 when $\sigma = 1$, $b_0 = 1$ and $\beta = 0$.

In the basic model, the time τ it takes for a rider to pick up a shared bicycle has a constant expectation $1/\lambda_0$. However, in the full model, it is not straightforward to compute $E[\tau | \xi_s = x, \eta_s = 0]$ and the displacement between x and $\xi_{s+\tau}$, the position where the rider picks up a bicycle. From the following Theorem, we can obtain those expectations.

Theorem 3. *Let $f(s, x, t, y)$ be the conditional probability density function of $(s + \tau, \xi_{s+\tau})$, given $\xi_s = x$ and $\eta_s = 0$. For a function $h(y) \in C_0^2(R)$, we write*

$$w(s, t, x) = \int_{-\infty}^{+\infty} h(y) f(s, x, t, y) dy.$$

Then $w(s, s, x) = \lambda(u(s, x))h(x)$ and for $t \geq s$,

$$w_t = \frac{1}{2}\sigma^2 w_{xx} + b(u(t, x), u_x(t, x)) w_x - \lambda(u(t, x)) w. \quad (22)$$

The preceding Theorem 3 is based on Feymann–Kac formulas and we place its proof in Appendix.

For $h(y) \equiv 1$, $\int_{-\infty}^{+\infty} h(y) f(s, x, t, y) dy$ is the density of $s + \tau$ of a rider's walking time given that the rider starts from position x at time s, and $\int_s^{+\infty} t w(s, t, x) dt - s$ is the expectation of the rider's walking time. For $h(y) = y$, $\int_s^{+\infty} w(s, t, x) dt - x$ the expectation of the rider's displacement from her or his starting point to the position where the rider picks up a share bicycle, given that he rider starts from position x at time s. Although $1, y \notin C_0^2(R)$, we can use a sequence of $h_n \in C_0^2(R)$ to tend to 1 or y, and obtain corresponding expectations with solutions to (22).

4. Clogging phenomena in dockless bicycle-sharing system

4.1. General features of density

Fig. 1 shows a lump density function $u(t, y)$ defined by (21) at time $t = 1$ and time $t = t_2$. $u(t, y)$ is the density of share bicycles parked at position y and time t, it has a peak and gets its maximum at $b_0 t$. It is obvious from Fig. 1 that $u(t, y)$ is a traveling wave with respect to time t. Form $t = 1$ to $t = t_2$, the displacement of the density is $b_0(t_2 - 1)$, and hence the density function $u(t, \cdot)$ travels with speed b_0. b_0 denotes a rider's movement intention in an environment where bicycles are parked at a constant density, that is, b_0 is a human's typical walking speed, about 4.0–5.0 km/h.

The width of the peak, defined by the distance between two positions where the density is half of its maximum, is $2b_0/\sqrt{\lambda_0}$. λ_0 is the minimum chance of a rider's picking up a share bicycle. Hence, if b_0 is constant, the width decreases when many people search bicycles anxiously. However, a rider's walking speed maybe increase in the case and b_0 is increase too. Hence, the width may vary complex when everyone searches bicycles anxiously.

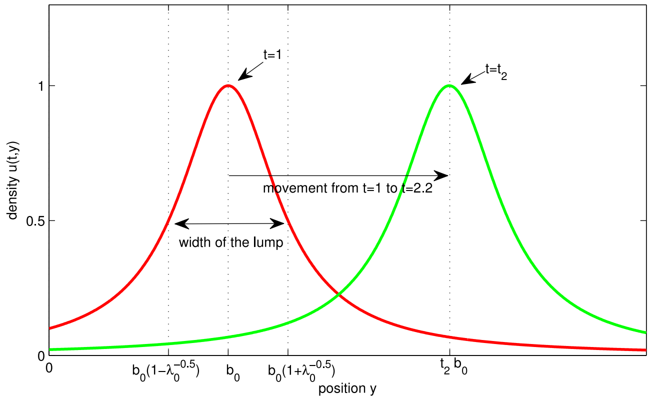

Fig. 1. The density functions $u(1, y)$ and $u(t_2, y)$ of share bicycles parked along the road at time $t = 1$ and $t = t_2$.

4.2. Walking time and displacement of riders

We set $\sigma = b_0 = \lambda_0 = \beta = 1$, and from Theorem 3, we compute numerically the probability density function $f_\tau(t|x)$ of a rider's walking time τ, given that the rider starts from x at time 0. It seems that $f_\tau(t|x) = \exp(-g(t, x) - 1.2t)$ and Fig. 2 shows the contour and surface maps of $g(t, x)$. Notice that the peak of the density $u(t, x)$ appears at $x = t$, and we draw the peak's curve $x = t$ on the contour map. Fig. 2 shows that the maximum and minimum of $g(t, x)$ do not touch the peak's curve, although a rider faces the greatest concentrations of share bicycles at the peak.

In another numerical experiment, with $\sigma = b_0 = \lambda_0 = 1$, and for different $\beta = 0, 1, 2, 3$, we obtain expectations of walking time and displacement of a rider starting from position x and time 0. Here $\beta = 0$ is a benchmark for those expectations. With $\beta = 0$, the walking time is uncorrelated with the density $u(t, y)$ of parked bicycles. Therefore expectation of walking time $E\tau = 1/\lambda_0$ is constant for $\beta = 0$, and the fluctuations in displacement are tiny. When $\beta > 0$, because $\lambda(u) = \lambda_0 + \beta u \geq \lambda_0$, the expectation of walking time decreases. Fig. 3 illustrates the fact and another intricate fact that $E\tau$ reaches its maximum near $x = 0$, where the density u is large. Moreover, $E\tau < 1$ for all $\beta > 0$, and the peak of the density never reaches 1 before $E\tau$. However, the expectation of displacement between a rider's start pointing and destination gets its minimum at a position at right side of 1.

5. Conclusion

We study a stochastic system which describes many riders and dockless shared bicycles along a long road. The model is reasonable when the amount of independent riders is large and their influence on the density of parked bicycles is not random. The nonlinear evolutionary equation of the density has a nontrivial lump solution. The solution has a peak which illustrates a clogging phenomenon, and the peak moves at human's walking speed. The model illustrates that the phenomenon of clogging sidewalks may be an endogenous processes within some dockless bicycle-sharing system, as it does not include any exogenous factors which imply predictable clogging phenomena. An open problem is whether other kinds of bicycle-sharing systems and autonomous-cars systems have a similar issue. Another open problem is how to change and control the phenomena of clogging sidewalks by some new rules for a dockless bicycle-sharing system. Although bicycle-sharing systems and other functions of smart cities will completely overturn old ideas of a town, it seems that we must firstly deal with many difficult issues like clogging sidewalks.

Acknowledgments

Liu would like to acknowledge the funding of Natural Science Foundation of Shaanxi Provincial Department of Education, PR China (18JK0789) and Research Achievement Cultivation Fund in School of Mathmatics and Education and Teaching Reform Research Project of Northwest University, PR China. Duan would like to acknowledge the funding of the National Natural Science Foundation of China (Grant no. 11991023).

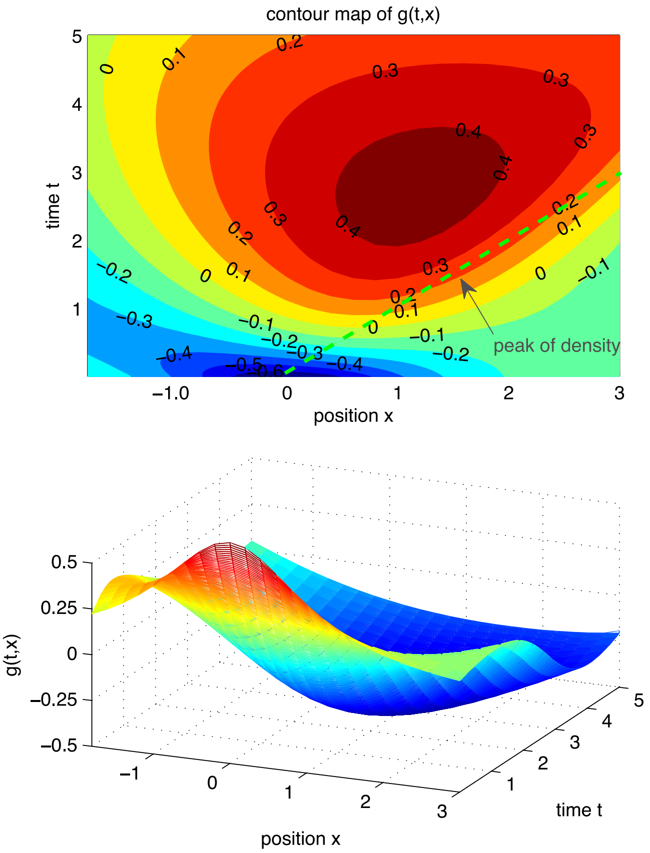

Fig. 2. The contour and surface maps of $g(t, x)$.

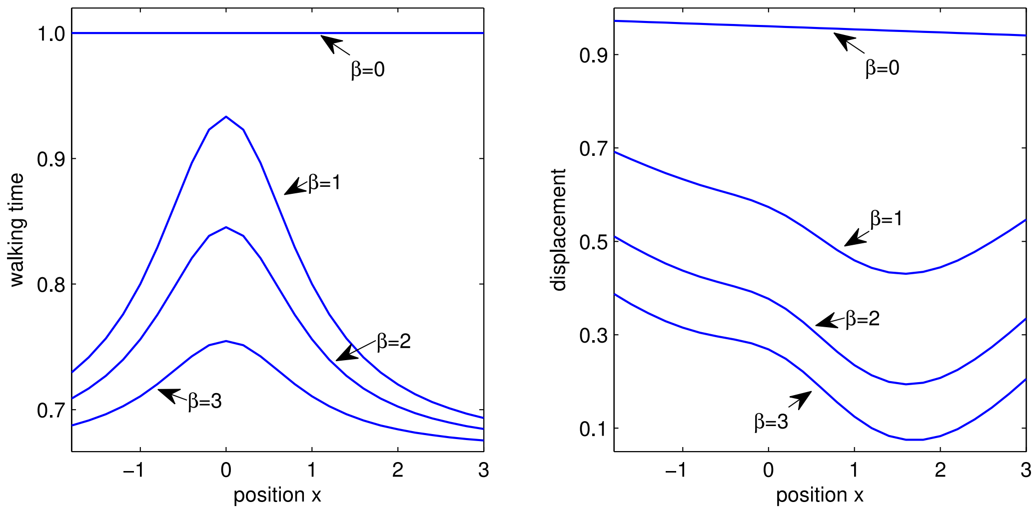

Fig. 3. Expectations of walking time and displacement of a rider starting from position x and time 0.

Appendix. Proof of Theorem 3

Consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the real-world probability. We suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is rich enough to model the randomness of the investigated processes. Consider the right-continuous, complete versions of the following filtrations.

$$\mathcal{F} = \{\mathcal{F}_t\}, \qquad \mathcal{F}_t = \sigma\{\xi_s | s \in [0, t]\}.$$

Notice $\eta_s \in \mathcal{F}_t$ for $s \leq t$. Without loss of generality, we assume that $s = 0$, $\xi_0 = x$, $\eta_0 = 0$.

Lemma 1. *For any $r, v \geq 0$, we have that*

$$E(h(\xi_r)1_{\tau\geq v}) = E\left(h(B_r)\exp\left(-\int_0^v \lambda(u(\rho,\xi_\rho))d\rho\right)\right).$$

Proof. It follows from $h(B_r) \in \mathcal{F}_{v\vee r}$ and

$$E\left(1_{\tau\geq v}\middle| \mathcal{F}_{v\vee r}\right) = \exp\left(-\int_0^v \lambda(u(\rho,\xi_\rho))d\rho\right)$$

that

$$E\left(h(B_r)1_{\tau\geq v}\middle| \mathcal{F}_{v\vee r}\right) = h(B_r)\exp\left(-\int_0^v \lambda(u(\rho,\xi_\rho))d\rho\right).$$

And the result follows by taking expectation to the preceding equity.

Lemma 2. *For $S < T$, we have that*

$$\begin{aligned} E(h(\xi_\tau)1_{S\leq\tau<T}) = {} & E\left(\int_S^T \left(b\,h' + \frac{\sigma^2}{2}h''\right)\exp(-\int_0^v \lambda(u(\rho,\xi_\rho))d\rho)dv\right) \\ & - E\left(h(B_r)\exp\left(-\int_0^r \lambda dv\right)\right)\Big|_{r=S}^{T}. \end{aligned}$$

where λ, b, h', h'' denote the values of corresponding functions at time v and position ξ_v.

Proof. We assume $\xi_0 = x$ hereafter, write $\alpha = (T - S)/n$ and

$$\begin{aligned} I_n &= \sum_{i=0}^{n-1}\left(h(\xi_{S+(i+1)\alpha}) - h(\xi_{S+i\alpha})\right)1_{\tau\geq S+i\alpha} \\ &= \sum_{i=0}^{n-1} h(\xi_{S+(i+1)\alpha})1_{S+(i+1)\alpha>\tau\geq S+i\alpha} - h(\xi_s)1_{\tau\geq S} + h(\xi_T)1_{\tau\geq T}. \end{aligned}$$

As $h(y)$ is continuous and ξ_t has a continuous orbit, we have that almost surely

$$\lim_{n\to\infty} I_n = h(\xi_\tau)1_{S\leq\tau<T} - h(\xi_s)1_{\tau\geq S} + h(\xi_T)1_{\tau\geq T}.$$

Due to the dominated convergence theorem, $\lim_{n\to\infty} EI_n = E\lim_{n\to\infty} I_n$ follows from the fact that $h(\cdot)$ is bounded. Hence,

$$E(h(\xi_\tau)1_{S\leq\tau<T}) = \lim_{n\to\infty} EI_n + Eh(\xi_s)1_{\tau\geq S} - Eh(\xi_T)1_{\tau\geq T}. \tag{23}$$

It follows from Lemma 1 that

$$\begin{aligned} & Eh(\xi_s)1_{\tau\geq S} - Eh(\xi_T)1_{\tau\geq T} \\ = {} & -E\left(h(\xi_v)\exp\left(-\int_0^v \lambda(u(\rho,\xi_\rho))d\rho\right)\right)\Big|_{v=S}^{T}. \end{aligned} \tag{24}$$

On the other hand, Lemma 1 follows that

$$EI_n = \sum_{i=0}^{n-1} E\left((h(\xi_{S+(i+1)\alpha}) - h(\xi_{S+i\alpha}))1_{\tau\geq S+i\alpha}\right)$$
$$= \sum_{i=0}^{n-1} E\left(h(\xi_v)\Big|_{v=S+i\alpha}^{S+(i+1)\alpha} \exp\left(-\int_0^{S+i\alpha}\lambda dv\right)\right) \tag{25}$$

As we have that

$$E\left(h(\xi_v)\Big|_{v=S+i\alpha}^{S+(i+1)\alpha} \exp\left(-\int_0^{S+i\alpha}\lambda dv\right)\right)$$
$$= E\left(\exp\left(-\int_0^{S+i\alpha}\lambda dv\right) E\left(h(\xi_v)\Big|_{v=S+i\alpha}^{S+(i+1)\alpha}\Big|\mathcal{F}_{S+i\alpha}\right)\right),$$

it follows from

$$h(\xi_v)\Big|_{v=S+i\alpha}^{S+(i+1)\alpha} = \int_{S+i\alpha}^{S+(i+1)\alpha}\left(h'(\xi_\rho)d\xi_\rho + \frac{\sigma^2}{2}h''(\xi_\rho)d\rho\right)$$

and $d\xi_v = \sigma dB_v + b\,dv$ that

$$E\left(\exp\left(-\int_0^{S+i\alpha}\lambda dv\right) E\left(h(\xi_v)\Big|_{v=S+i\alpha}^{S+(i+1)\alpha}\Big|\mathcal{F}_{S+i\alpha}\right)\right)$$
$$= E\int_{v=S+i\alpha}^{S+(i+1)\alpha}\left(b\,h'(\xi_v) + \frac{\sigma^2}{2}h''(\xi_v)\right)\exp\left(-\int_0^{S+i\alpha}\lambda(u(\rho,\xi_\rho))d\rho\right)dv.$$

Therefore, it follows from (25) that

$$EI_n = E\left(\int_S^T\left(b\,h' + \frac{\sigma^2}{2}h''\right)\sum_{i=0}^{n-1}1_i\exp\left(-\int_0^{S+i\alpha}\lambda(u(\rho,\xi_\rho))d\rho\right)dv\right),$$

where $1_i = 1_{S+i\alpha\leq v<S+(i+1)\alpha}$. As $0 \leq \sum_{i=0}^{n-1}1_i\exp\left(-\int_0^{S+i\alpha}\lambda(u(\rho,\xi_\rho))d\rho\right) \leq 1$, $\lim_{n\to\infty}\sum_{i=0}^{n-1}1_i\exp\left(-\int_0^{S+i\alpha}\lambda(u(\rho,\xi_\rho))d\rho\right) = \exp\left(-\int_0^v\lambda(u(\rho,\xi_\rho))d\rho\right)$, and $b\,h' + \frac{\sigma^2}{2}h''$ is bounded, we have that

$$\lim_{n\to\infty}EI_n = E\left(\int_S^T\left(b\,h' + \frac{\sigma^2}{2}h''\right)\exp\left(-\int_0^v\lambda(u(\rho,\xi_\rho))d\rho\right)dv\right).$$

Then we can prove Lemma 2 from (23), (24) and the preceding equity.

Proof of Theorem 3. As $f(0,x,t,y)$ denotes the conditional probability density function of $(s+\tau, \xi_{s+\tau})$, we have that

$$E(h(\xi_\tau)1_{\tau<t}) = \int_0^t\int_{-\infty}^{+\infty}h(y)f(0,x,u,y)\,dy\,du.$$

$\int_{-\infty}^{+\infty}h(y)f(0,x,t,y)\,dy$ is continuous with respect to t, and hence

$$\frac{d}{dt}E(h(\xi_\tau)1_{\tau<t}) = \int_{-\infty}^{+\infty}h(y)f(0,x,t,y)\,dy.$$

As the variable t does not appear in $E(h(\xi_{s+\tau}))$, the preceding equation implies

$$\frac{d}{dt}E(h(\xi_\tau)1_{\tau\geq t}) = -\int_{-\infty}^{+\infty}h(y)f(0,x,t,y)\,dy.$$

Therefore Lemma 2 follows that

$$\frac{d}{dt}E(h(\xi_\tau)1_{\tau\geq t}) = -E\left(h(\xi_t)\lambda(u(t,\xi_t))\exp\left(-\int_0^t\lambda(u(\rho,\xi_\rho))d\rho\right)\right).$$

That is,

$$w(0,t,x) = E\left(h(\xi_t)\lambda(u(t,\xi_t))\exp\left(-\int_0^t \lambda(u(\rho,\xi_\rho))d\rho\right)\right).$$

And the result follows from Feymann–Kac formulas.

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