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# An effective approach for constructing novel KP-like equations

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## ABSTRACT

In this paper, an effective algorithm for constructing nonlinear evolution equations (NLEEs) has been proposed. Particularly, the existence of resonant multi-soliton solutions in the newly generated NLEEs is verified and demonstrated, and the accuracy of the extracted resonant multi-soliton solutions has been proved at the same time. Firstly, via the linear superposition principle along with reverse engineering two new NLEEs arising from the B-type Kadomtsev–Petviashvili (BKP) equation are established and investigated as well. The first new NLEE is constructed with three time derivative terms, and the second one is constructed with three space dissipative terms, respectively. Besides, the infinite resonant multi-soliton solutions are extracted which enjoy a variety of inelastic interactions due to the fact that they are constructed with variable parameters. Then, the reliable judgments to the multi-soliton solutions are carried out. Finally, the Painlevé test is applied to examine the new equations and none of them passed the test. It is important to highlight that the presented method and NLEEs could be extended to diversify the problem of physical nature.

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resonant multi-soliton;  
inelastic; Painlevé test

## 1. Introduction

In the past decades, the field of nonlinear evolution equations (NLEEs) has been one of the most active multidisciplinary areas of research due to the fact that related integrable equations describe the real properties and reveal the mysterious nature of the nonlinearity in various sciences [1–10]. It is well known that the main property of integrable equations is the existence of multi-soliton solutions, which always comes with resonant multi-soliton solutions. In other words, once the resonant multi-soliton solutions are extracted confirming the existence of the multi-soliton solutions and the integrability of investigated equations this way [11]. Moreover, if the exact  $N$ -wave solitary solutions are variable they facilitate the numerical solvers in comparison and assist in the stability analysis. Therefore, a variety of powerful methods used to search multi-soliton solutions have attracted

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intensive interest researchers' attention, such as the Hirota method [7,11–14], the multiple exp-function method [15,16] and so on [17–39]. In the related studies, the famous one is the Kadomtsev–Petviashvili (KP) hierarchy, which is used to describe certain interesting  $(2 + 1)$  and  $(3 + 1)$ -dimensional waves in nonlinear science [20–25]. It is worth to point out that one extension to KP hierarchy, called B-type KP (BKP) hierarchy, is obtained by replacing  $u_{xxxx}$  with  $u_{xxxy}$ . Naturally, the BKP hierarchy also possesses the integrable structures and various versions of the BKP equation have been proposed in the literature [23–44].

In the last decade, a standard  $(3 + 1)$ -dimensional BKP equation [26,32] was widely investigated and read

$$u_{zt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0. \quad (1)$$

As already known using the transformation  $u = 2(\ln f)_x$  transforms Equation (1) into the Hirota bilinear form

$$(D_z D_t - D_x^3 D_y + 3D_x^2) f \cdot f = 0. \quad (2)$$

Moreover, various versions of the extended form to Equation (2) had been presented and investigated [35,36,43,44] such as

$$(D_x^3 D_y + D_x D_t + D_y D_t + D_z D_t - D_x^2 - D_y^2 - D_z^2) f \cdot f = 0, \quad (3)$$

$$(D_y D_t - D_x^3 D_z + 3D_y^2) f \cdot f = 0, \quad (4)$$

$$(D_x D_t - D_x^3 D_y + 3D_z^2) f \cdot f = 0, \quad (5)$$

$$(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2) f \cdot f = 0. \quad (6)$$

The associated works of the above equations have profoundly attracted our attention. For example, Lan et al. [23] applied the Bäcklund transformation and Hirota method to obtain the multi-soliton solutions. Wazwaz [25,26] established the multi-soliton solutions by using the simplified form of the Hirota method. Darvishi et al. [27] extracted the multi-soliton solutions by invoking the multiple exp-function method. Lin et al. [31] and [32] derived the resonant multi-soliton solutions by using the linear superposition principle. In addition, Wazwaz [39] generated a new integrable equation by combining the recursion operator of the modified KdV equation and the sense of the negative-order recursion operator, which enjoys a variety of solutions including multi-soliton solutions; Sun et al. [40] derived a BKP-like equation by combining the bilinear forms of KP and Boussinesq equations, respectively. Gao et al. [43] presented the resonant behavior of multiple wave solutions to Equation (6). Mabrouk and Rashed [44] generated versions of wave solutions to Equation (6) by three distinct methods.

The major motivation of this study is to establish new NLEEs arising from Equation (2). New equations are examined by the simplified linear superposition principle [38,45,46], which gives the reliable judgments to the existence of the new NLEEs by the generated resonant multi-soliton solutions. The objectives of this work are twofold. First, two extensions of Equation (2) are constructed and simultaneously examined by the simplified linear superposition principle. The conditions for the determination of the resonant multi-soliton solutions are revealed. Second, the relevant physical features are shown where the obtained solutions and dispersion relations are constructed with distinct physical structures for each equation. And the propagations of inelastic interactions to traveling solitary waves are investigated both theoretically and graphically.

## 2. The simplified linear superposition principle

The linear superposition principle [32,33,37] is powerful to seek the resonant multi-soliton solutions. In our latest works [38,45,46], in order to improve efficiency and reduce the complexity of calculations the simplified version has been presented. Hereby, the fundamental steps involved in the simplified linear superposition principle are carefully illustrated as follows.

In the first step, the transformation is conjectured, such as  $u = (\ln f)_x$ , and used to transform the considered equation into a Hirota bilinear equation

$$P(D_x, D_y, \dots, D_t) f \cdot f = 0, \tag{7}$$

where  $P$  is a polynomial and satisfies

$$P(0, 0, \dots, 0) = 0. \tag{8}$$

It is notable that  $D_{x,y,\dots,t}$  are Hirota's bilinear differential operators [7,12].

Then, consider  $N$  wave variables as

$$\eta_i = k_i x + l_i y + \dots + \omega_i t, 1 \leq i \leq N, \tag{9}$$

where  $k_i, l_i, \omega_i$  are constants which are going to be determined later.

The second step is to construct  $N$  exponential wave functions as

$$f_i = e^{\eta_i}, 1 \leq i \leq N, \tag{10}$$

and consider the  $N$ -wave testing function

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \dots + \varepsilon_N f_N, \tag{11}$$

where  $\varepsilon_i, 1 \leq i \leq N$  are non-zero arbitrary constants.

It is notable the linear character will play the main key to the linear superposition principle for constructing  $N$  exponential waves  $e^{\eta_i}, 1 \leq i \leq N$ . Now, upon using Equations (8–11) and solving the Hirota bilinear Equation (7) if the following condition is satisfied

$$P(k_i - k_j, l_i - l_j, \dots, \omega_i - \omega_j) = 0, 1 \leq i < j \leq N. \tag{12}$$

Solving a family of nonlinear algebraic equations on the related wave numbers  $k_i, l_i, \omega_i$  left from Equation (12) gives  $N$  exponential wave functions. Hence, the exact resonant multi-soliton wave solutions could be obtained this way. Herein, it is worth to mention that solving Equation (12) is much more complicated in cases of high-dimensional and high-order equations. Inspiring, a shortcut to overcome the demerit is discovered, no matter to high-dimensional or high-order. In [32,33] one can find that the wave-related numbers can be directly constructed via the form of the dispersion relation. Thence, the tedious wave-related numbers can be directly furnished as

$$\begin{aligned} k_i &= k_i, \\ l_i &= a k_i^g, \\ \omega_i &= b k_i^h, \end{aligned} \tag{13}$$

where  $g, h$  are powers of  $k_i$  and  $a, b$  are real constants to be determined later. Re-emphasizing that Equation (13) is conjectured based on the form of the dispersion relation.

After substituting Equation (13) into (12) and determining the values of  $a, b$ , the required resonant multi-soliton wave solution is naturally constructed as

$$u = (\ln f)_x = \left( \ln \left( \sum_{i=1}^N \varepsilon_i e^{\eta_i} \right) \right)_x, \tag{14}$$

$$\eta_i = k_i x + a k_i^g y + \dots + b k_i^h t. \tag{15}$$

The algorithms of applications will be demonstrated in detail in the following section.

### 3. Two new (3 + 1)-dimensional NLEEs

It is well known that variable NLEEs can provide much more information than their constant-coefficient counterparts in all physical fields. Thus, in order to search versions of resonant phenomena as much as we can, Equation (2) is reconsidered as

$$(\alpha D_z D_t - \beta D_x^3 D_y + \gamma D_x^2) f \cdot f = 0, \tag{16}$$

where  $\alpha, \beta, \gamma$  are variable parameters. Based on Equation (16) two different equations will be given as follows.

#### 3.1. The new equation with three time derivative terms

Adding two extra terms  $D_x D_t, D_y D_t$  to Equation (16) reads

$$(\delta D_x D_t + \mu D_y D_t + \alpha D_z D_t - \beta D_x^3 D_y + \gamma D_x^2) f \cdot f = 0, \tag{17}$$

where  $\delta, \mu$  are arbitrary parameters. It is easy to see that using the transformation  $u = 2(\ln f)_x$  reverses Equation (17) to

$$(\delta u_{xt} + \mu u_{yt} + \alpha u_{zt}) - \beta u_{xxx} - 3(u_x u_y)_x + \gamma u_{xx} = 0. \tag{18}$$

Comparing Equation (3), it is clear to find that Equations (3) and (17) are established with distinct physical structures. Most of all, Equation (18) will be reliable and meaningful if and only if (17) gives the multivariate polynomials which satisfy the property (12) and guarantee the implement of the simplified linear superposition principle for exponential wave solutions.

So, substituting Equation (9) into (17) gives

$$\delta(k_i - k_j)(\omega_i - \omega_j) + \mu(l_i - l_j)(\omega_i - \omega_j) + \alpha(m_i - m_j)(\omega_i - \omega_j) - \beta(k_i - k_j)^3(l_i - l_j) + \gamma(k_i - k_j)^2 = 0. \tag{19}$$

Now, based on the dispersion relation given by Equation (18) and via Equation (13) the required exact wave-related numbers are constructed as

$$\begin{aligned} l_i &= a k_i^{-1}, \\ m_i &= \frac{-\delta}{\alpha} k_i, \\ \omega_i &= b k_i^3. \end{aligned} \tag{20}$$

Then, substituting Equation (20) into (19) and solving yields

$$\begin{aligned}
 a &= \frac{\gamma}{3\beta}, \\
 b &= \frac{\beta}{\mu} \\
 l_i &= \frac{\gamma}{3\beta} k_i^{-1}, \\
 m_i &= \frac{-\delta}{\alpha} k_i, \\
 \omega_i &= \frac{\beta}{\mu} k_i^3,
 \end{aligned} \tag{21}$$

where  $k_i$  is an arbitrary constant. It is to be noted that  $\omega_i = (\beta/\mu)k_i^3$  is not admitted by the dispersion relation  $\omega = (\beta k^3 l - \gamma k^2 / \delta k + \mu l + \alpha m)$  given by Equation (18). This kind of special cases has been mentioned in [32,33,45].

Upon using Equations (11), (15), (21) and  $u = 2(\ln f)_x$ , the generalized resonant multi-soliton solution is constructed as

$$u = 2 \frac{\sum_{i=1}^N k_i \varepsilon_i e^{k_i x + (\gamma/3\beta) k_i^{-1} y - (\delta/\alpha) k_i z + (\beta/\mu) k_i^3 t}}{\sum_{i=1}^N \varepsilon_i e^{k_i x + (\gamma/3\beta) k_i^{-1} y - (\delta/\alpha) k_i z + (\beta/\mu) k_i^3 t}}. \tag{22}$$

The existence of Equation (22) not only gives resonant  $N$ -soliton solutions but also simultaneously indicates the justifiability of Equations (17) and (18). Furthermore, by specifying values to free parameters  $\alpha, \beta, \gamma, \delta, \mu$  the solution (22) enjoys a lot of versions of inelastic interactions. Without loss of generality, the propagations of traveling 2-wave and 3-wave by the solution (22) with  $\alpha = \beta = \delta = \mu = 1, \gamma = 3$  are presented in Figures 1 and 2.

### 3.2. The new equation with three space dissipative terms

Adding  $D_y D_x, D_z D_x$  to Equation (16) and proceeding as before gives the extension

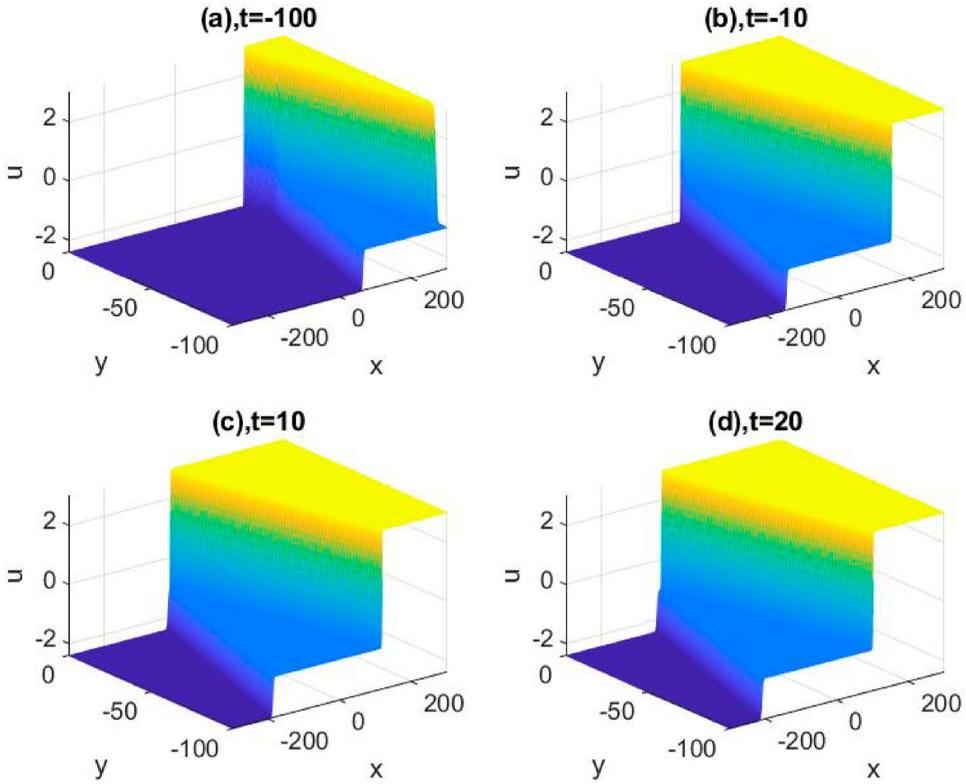
$$(\alpha D_z D_t - \beta D_x^3 D_y + \gamma (D_x^2 + D_x D_y + D_x D_z)) f \cdot f = 0. \tag{23}$$

Comparing Equations (4) and (5), it is clear to find that  $D_x D_t, D_y D_t, D_y D_y, D_z D_z$  are replaced by  $D_z D_t, D_y D_x, D_z D_x$  in Equation (23), respectively. Then, using Equations (12) and (13) to handle (23) yields

$$\begin{aligned}
 &\alpha (m_i - m_j) (\omega_i - \omega_j) - \beta (k_i - k_j)^3 (l_i - l_j) + \gamma (k_i - k_j)^2 + \gamma (l_i - l_j) (k_i - k_j) \\
 &+ \gamma (m_i - m_j) (k_i - k_j) = 0,
 \end{aligned} \tag{24}$$

and

$$\begin{aligned}
 l_i &= a k_i^{-1}, \\
 m_i &= -a k_i^{-1} \\
 \omega_i &= b k_i^3.
 \end{aligned} \tag{25}$$

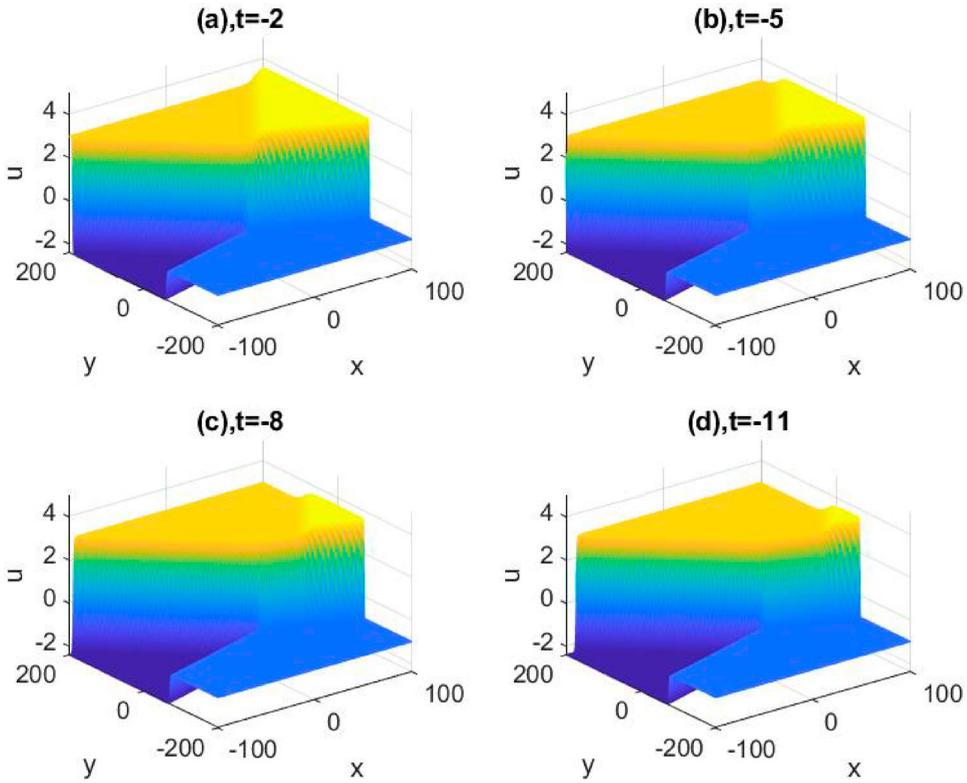


**Figure 1.** The fission of traveling 2-kink waves by Equation (22) with  $k_1 = -0.5, k_2 = -1.2, k_3 = 1.5, z = \varepsilon_j = 1$ .

Via Equation (25), Equation (24) is easily solved and the exact wave-related number is obtained as

$$\begin{aligned}
 a &= \frac{\gamma}{3\beta}, \\
 b &= \frac{-\beta}{\alpha}, \\
 l_i &= \frac{\gamma}{3\beta} k_i^{-1}, \\
 m_i &= -\frac{\gamma}{3\beta} k_i^{-1} \\
 \omega_i &= \frac{-\beta}{\alpha} k_i^3,
 \end{aligned} \tag{26}$$

where  $k_i$  is an arbitrary constant. It is to be noted that  $\omega_i = (-\beta/\alpha)k_i^3$  is not admitted by the dispersion relation  $\omega = (\beta k^3 l - \gamma(k^2 + kl + km))/\alpha m$  given by Equation (23). This kind of special cases has been mentioned in [32,33,45]. Via Equations (11), (15) and (26), the



**Figure 2.** The interaction of traveling 3-kink waves by Equation (22) with  $k_1 = -0.5, k_2 = -1.2, k_3 = 1.5, k_4 = 1.8, z = \varepsilon_j = 1$ .

generalized resonant multi-soliton solution is established as

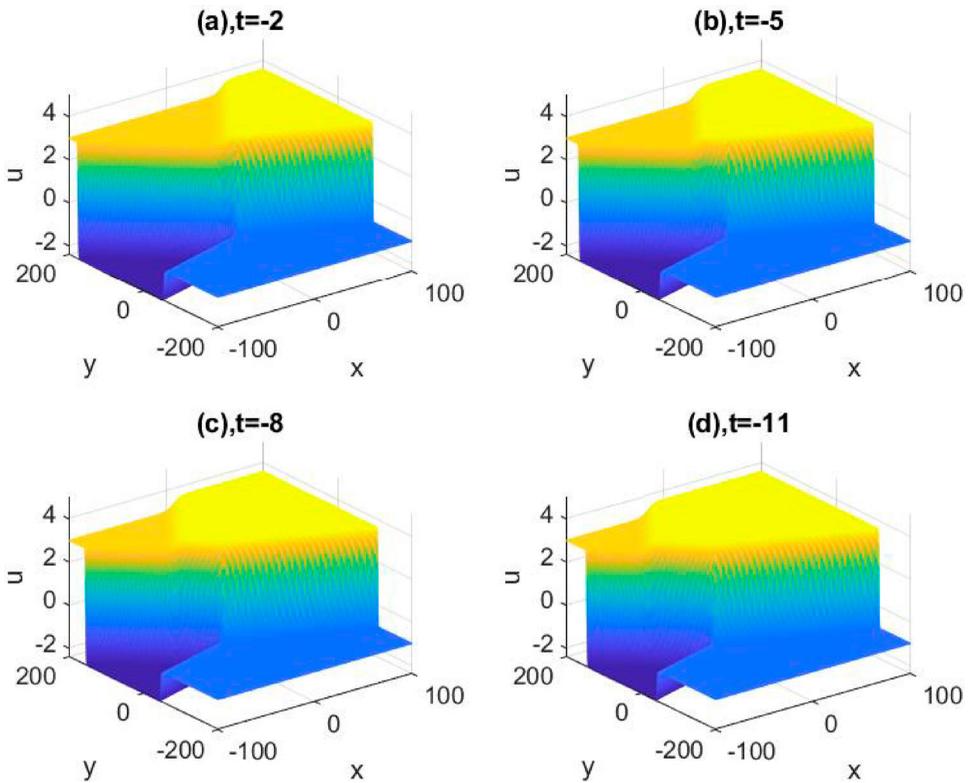
$$u = 2 \frac{\sum_{i=1}^N k_i \varepsilon_i e^{k_i x + (\gamma/3\beta) k_i^{-1} y - (\gamma/3\beta) k_i^{-1} z - (\beta/\alpha) k_i^3 t}}{\sum_{i=1}^N \varepsilon_i e^{k_i x + (\gamma/3\beta) k_i^{-1} y - (\gamma/3\beta) k_i^{-1} z - (\beta/\alpha) k_i^3 t}}. \tag{27}$$

By specifying values to free parameters the solution (27) enjoys a lot of versions of inelastic interactions. Without loss of generality, the traveling 3-wave by the solution (27) with  $\alpha = \beta = 1, \gamma = 3$  is presented in Figures 3. Naturally, using the transformation  $u = 2(\ln f)_x$  reverses Equation (23) to

$$\alpha u_{zt} - \beta u_{xxx} - 3(u_x u_y)_x + \gamma(u_{xx} + u_{yx} + u_{zx}) = 0. \tag{28}$$

So far, the explicit resonant multi-soliton solutions (22) and (27) are successfully generated. Meanwhile, the existence of solutions (22) and (27) indicates the justifiability of the NLEEs (18) and (28), which retains the order and the dimension of the standard BKP Equation (1) without changing. The dynamics of resonant 2-kink and 3-kink waves are demonstrated in Figures 1–3.

To the author’s best knowledge, the NLEEs (18) and (28) and the generalized solutions (22) and (27) are not reported in the previous literature, which are remarkably new results.



**Figure 3.** The interaction of traveling 3-kink waves by Equation (27) with  $k_1 = -0.5, k_2 = -1.2, k_3 = 1.5, k_4 = 1.8, z = \varepsilon_j = 1$ .

Naturally, they are helpful to describe new physical phenomena of nonlinear science to the real world.

### 3.3. Discussions

All obtained solutions are checked via Maple 13. The graphics of 2-kink and 3-kink waves are shown with the aid of MATLAB (R2018b) and elaborated as follows.

- (i) The dispersion relations and solution forms of Equations (18) and (28) are not only different to each other but also completely different from the ones reported in [23–32,35,36,40–44].
- (ii) As mentioned above the wave number of frequency term is not admitted by the dispersion relation for each newly derived equation. This kind of special cases is worth studied further.
- (iii) The resonant phenomena are performed as shown in Figures 1–3. As shown in Figure1, the amplitude of both traveling kink waves does not change as interacting, and then the small kink is dispersed from the tall one and keeps the speed without changing. Figures 2 and 3 with  $\alpha = \beta = \delta = \mu = 1, \gamma = 3$  show the inelastic interactions of the traveling 3-kink waves for the solutions (22) and (27) which gives

different types of propagations, respectively. In Figure 2, all kink waves move to the right and keep the shape and speed without changing. Meanwhile, the top kink is the fastest wave. In Figure 3, the top kink is also the fastest wave and moves to the left whose direction is opposite to others. To sum up, the solutions (22) and (27) can perform various versions of inelastic interactions because the wave numbers  $k_i, k_j$  are independent which plays a key role in the inelastic mechanism [37,45].

- (iv) After carefully examining it is found that both Equations (18) and (28) do not pass the Painlevé test. The result is similar to the Zakharov–Kuznetsov equation which does not pass the inverse scattering transform test [47,48] and the Jimbo-Miwa equation which does not pass any of the normal integrability tests [34], but both enjoy the abundant multi-soliton solutions.
- (v) Some important physical characters of Equations (18) and (28) are summarized: 1. Based on Equations (21) and (26) the key constraint to the existence of resonant solutions is all of variable parameters  $\alpha, \beta, \gamma, \delta, \mu$  cannot be equal to zero. It means  $u_{xt}$  and  $u_{yt}$  are interdependent at Equation (18). And  $u_{xy}$  and  $u_{xz}$  influence the dispersion relation to Equation (28). 2. Before inelastic interactions the speed of each travelling wave  $f_i$  can be specified by  $\omega_i = \frac{\beta}{\mu} k_i^3$  to Equation (18) and  $\omega_i = \frac{-\beta}{\alpha} k_i^3$  to Equation (28), respectively. 3. The value of wave number  $k_i$  not only affects the wave speed and amplitude but also influences the angle of oblique collision to resonant multi-soliton waves [46]. 4. By specifying values to free parameters  $\alpha, \beta, \gamma, \delta, \mu$  various versions of KP-like equations and the corresponding resonant multi-soliton solutions are easily generated.

## 4. Conclusions

To summarize, new KP-like Equations (18) and (28) are formally presented via reverse engineering and successfully examined via the simplified linear superposition principle. The results show that the resonant multi-kink solutions enjoy a variety of inelastic interactions. The solutions assume the form of 2-kink and 3-kink and the shape-changing nature of the solutions is explored, as shown in Figures 1–3. To the best of our knowledge, the generated Equations (18) and (28), solutions and simulating figures have not been reported before. The results confirm that the simplified linear superposition principle suggests a promising and robust mathematical tool to seek resonant multi-soliton solutions by utilizing the Hirota bilinear equations. It is hoped that the examined systems (18) and (28) could be considered as wider ramifications in KP hierarchy and useful in practical situations in physical and engineering sciences.

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No potential conflict of interest was reported by the author(s).

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