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A study on resonant multi-soliton solutions to the (2+1)-dimensional Hirota–Satsuma–Ito equations via the linear superposition principle

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ABSTRACT

In this paper, the existence and non-existence of resonant multi-soliton solutions to two different (2+1)-dimensional Hirota–Satsuma–Ito (HSI) equations are explored. After applying the linear superposition principle we generate resonant multi-soliton solutions to the first HSI equation which appeared in the theory of shallow water wave. The conditions of real resonant multi-soliton solutions are revealed. The presented resonant multi-soliton solutions exhibit the inelastic collision phenomenon among the involved solitary waves. Particularly, upon choosing appropriate parameters, we demonstrate the characteristics of inelastic interactions among the multi-front kink waves both graphically and theoretically. Moreover, non-existence of resonant multi-soliton solution is considered for the generalized HSI equation via the linear superposition principle.

1. Introduction

In the past decades, integrable nonlinear evolution equations (NLEEs) have various applications in nonlinear fields. It is well-known that the existence of multi-wave solutions is an important feature of integrable NLEEs, which play the key role in sciences. Such solutions are used to demonstrate the elastic and inelastic interactions among solitary traveling waves in the physical phenomena modeled by NLEEs in the real world [1–9,11–29]. A variety of powerful methods used for seeking multi-wave solutions are developed, including Hirota's method [2,6,7,12,21–25,27], Bäcklund transformation [4], the Inverse Scattering

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transformation (IST) [1], Darboux transformation (DT) [8], the multiple exp-function method [3,18] and so on [5,9–11,13–17,19,20,26,28,29].

However, after surveying the existing literature, it is clearly seen that resonant multi-soliton solutions are rarely used to simulate studies on inelastic collisions of multi-wave solutions. In the last decade, the linear superposition principle has been widely applied to resonant multi-soliton solutions. The linear superposition principle developed in [17,20] paves a direct way of constructing multi-wave solutions with free phase shifts to Hirota bilinear equations. A few detailed algorithms of the linear superposition principle can be found in [9–11,13,17,20,28]. In the next section, we briefly explain the fundamental steps involved in the linear superposition principle.

Recently, an NLEE presented by Hirota et al. [6,7], namely the (2 + 1)-dimensional Hirota–Satsuma equation has attracted a lot of attention [14–16,19,26,29]. Under some certain logarithm transformation, the $(2+1)$-dimensional Hirota–Satsuma equation is transformed and presented as

$$u_{xxxt} + 3(u_x u_t)_x + u_{yt} + \alpha u_{xx} = 0, \tag{1.1}$$

where α is a non-zero arbitrary constant. Eq. (1.1) is called the Hirota–Satsuma–Ito (HSI) equation [15,26]. It has been applied in the theory of the shallow water wave [7] and appears in the Jimbo–Miwa classification [11,24].

Here, we briefly mention the latest studies related to Eq. (1.1). Liu et al. [14] generated the N-soliton solutions, and Zhou et al. [29] presented lump solutions and lump-soliton solutions. Besides, the reference [19] extended Eq. (1.1) by adding three terms

$$u_{xxxt} + 3(u_x u_t)_x + \delta_1 u_{yt} + \delta_2 u_{xx} + \delta_3 u_{xy} + \delta_4 u_{xt} + \delta_5 u_{yy} = 0, \tag{1.2}$$

where δ_{1-5} are variable coefficients, and named it as a generalized HSI equation.

After surveying related studies, it is clear to see that all related results merely show the elastic interactions of solitary traveling waves. Hence, one of primary tasks in this study is to determine if there exist resonant multi-soliton solutions to the HSI equation (1.1) and to the generalized HSI equation (1.2). The obtained solutions will be used to exhibit inelastic interactions of solitary traveling waves. The linear superposition principle is employed to help achieve our results.

In what follows, we firstly apply the linear superposition principle to the HSI equation (1.1) to generate its resonant multi-soliton solutions. The obtained solutions and the used dispersion relations have distinct physical structures, compared to the previous work [14]. Secondly, we reveal that there is no resonant multi-soliton solution to the generalized HSI equation (1.2). This kind of special cases will be elaborated in detail in Section 3. The results simultaneously show the power of the linear superposition principle. Furthermore, to gain an explicit insight into resonant multi-soliton solutions to the HSI equation (1.1), we briefly present the results reported by Liu et al. [14]. The summary will be given on the results obtained via the linear superposition principle.

2. The linear superposition principle

In the first step, one conjectures the transformation, such as $u = (\ln f)_x$, and transforms the equation under consideration into a Hirota bilinear equation

$$P(D_x, D_y, \ldots, D_t) f \cdot f = 0, \tag{2.1}$$

where P is a polynomial and satisfies

$$P(0, 0, \ldots, 0) = 0. \tag{2.2}$$

It is notable that $D_{x,y,\ldots,t}$ are Hirota's bilinear differential operators [6,22].

Consider N wave variables as

$$\eta_i = k_i x + l_i y + \cdots + c_i t, \quad 1 \le i \le N, \tag{2.3}$$

where k_i, l_i, c_i are constants determined later, and construct N exponential wave functions as

$$f_i = e^{\eta_i}, \quad 1 \le i \le N. \tag{2.4}$$

Then, the second step is to consider the N-wave testing function

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N, \tag{2.5}$$

where ε_i, $1 \le i \le N$ are non-zero arbitrary constants.

It is notable the linear character will play the main key to the linear superposition principle for constructing N exponential waves e^{η_i}, $1 \le i \le N$.

Now, upon using Eqs. (2.2-5) and solving the Hirota bilinear equation (2.1) if the following condition is satisfied

$$P(k_i - k_j, l_i - l_j \ldots, c_i - c_j) = 0, \quad 1 \le i < j \le N. \tag{2.6}$$

Solving a system of nonlinear algebraic equations on the related wave numbers k_i, l_i, c_i left from Eq. (2.6) gives N exponential wave functions. Hence, resonant multi-soliton wave solutions could be obtained this way.

Hereby, it is to be noted that solving Eq. (2.6) is much more complicated in cases of high-dimensional and higher-order equations. Moreover, it is extremely difficult to find exact solutions to variable-coefficient versions of Eq. (2.6).

Interestingly, we discover a shortcut to overcome the demerit, no matter to high-dimensional, high-order and variable-coefficient versions. In [17,20] we can find that the wave related numbers satisfy Eq. (2.6) whose powers also satisfy the corresponding dispersion relation. In other words, exact forms of wave related numbers satisfying Eq. (2.6) can be directly conjectured from the corresponding dispersion relation. Thence, we can say that the shortcut is to construct the furnished wave related numbers

$$\begin{aligned} k_i &= k_i, \\ l_i &= a k_i^g, \\ c_i &= b k_i^h, \end{aligned} \tag{2.7}$$

where g, h are powers of k_i and a, b are real constants to be determined later. It is to be noted that Eq. (2.7) is admitted by the associated dispersion relation.

After substituting Eq. (2.7) into (2.6) and determining the values of a, b, the required resonant multi-soliton wave solution can be accordingly constructed by

$$u = (\ln f)_x = (\ln(\sum_{i=1}^{N} \varepsilon_i e^{\eta_i}))_x, \tag{2.8}$$

where

$$\eta_i = k_i x + a k_i^g y + \cdots + b k_i^h t. \tag{2.9}$$

The algorithm of application will be demonstrated in detail in the following section.

3. Resonant multi-soliton solutions

3.1. The $(2+1)$-dimensional H irota–S atsuma–Ito equation

In this section, in order to gain the explicit insight into resonant multi-soliton solutions, the results in [14] are briefly presented as follows.

Via the logarithmic transformation

$$u = 2(\ln f)_x, \tag{3.1}$$

the HSI equation (1.1) is transformed into the Hirota bilinear form as

$$(D_x^3 D_t + D_y D_t + \alpha D_x^2) f \cdot f = 0. \tag{3.2}$$

The 2-wave function solution reads as

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}, \tag{3.3}$$

where

$$\begin{aligned} \theta_i &= k_i x + l_i y - \frac{\alpha k_i^2}{k_i^3 + l_i} t, \\ a_{12} &= -\frac{(k_1-k_2)^3(c_1-c_2) + \alpha(k_1-k_2)^2 + (l_1-l_2)(c_1-c_2)}{(k_1+k_2)^3(c_1+c_2) + \alpha(k_1+k_2)^2 + (l_1+l_2)(c_1+c_2)}. \end{aligned} \tag{3.4}$$

Moreover, the 3-wave function solution is given by

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3}, \tag{3.5}$$

where $a_{123} = a_{12}a_{23}a_{13}$, and a_{13}, a_{23} are similar functions of wave numbers k_i, l_i, and c_i are omitted for simplicity. The formula for a_{ij} is constructed as

$$a_{ij} = -\frac{(k_i-k_j)^3(c_i-c_j) + \alpha(k_i-k_j)^2 + (l_i-l_j)(c_i-c_j)}{(k_i+k_j)^3(c_i+c_j) + \alpha(k_i+k_j)^2 + (l_i+l_j)(c_i+c_j)}, 1 \le i < j \le N. \tag{3.6}$$

Clearly, the phase shifts a_{ij} are tedious. In the following, the resonant multi-soliton solution without phase shifts will be determined via the linear superposition principle.

Firstly, applying the bilinear equation (3.2) and Eq. (2.6) gives the N-wave condition as

$$(k_i-k_j)^3(c_i-c_j) + (l_i-l_j)(c_i-c_j) + \alpha(k_i-k_j)^2 = 0, 1 \le i < j \le N. \tag{3.7}$$

An important step to handle Eq. (3.7) via the linear superposition principle is to introduce weights of dependent variables, which will be demonstrated as follows.

By balancing the linear terms of Eq. (1.1), the dispersion relation is easily determined as $c = \frac{-\alpha k^2}{k^3+l}$. Thus, based on the extracted formula the exact wave numbers can be conjectured as

$$\begin{aligned} k_i &= k_i, \\ l_i &= a k_i^3, \\ c_i &= b k_i^{-1}. \end{aligned} \tag{3.8}$$

Then, substituting Eq. (3.8) into (3.7) and solving the resulting nonlinear algebraic equations yields

$$\begin{aligned} \theta_i &= k_i x + l_i y + c_i t, \\ l_i &= -k_i^3, \\ c_i &= \frac{-\alpha}{3} k_i^{-1}, \end{aligned} \tag{3.9}$$

where k_i is a non-zero arbitrary constant.

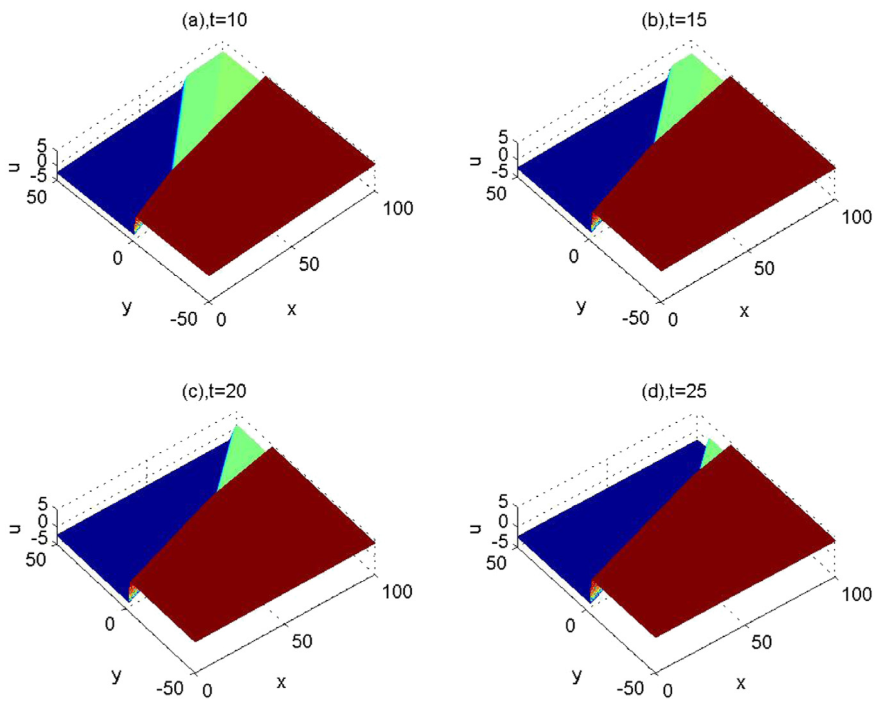

Fig. 1. The traveling 2-kink waves by (3.10) with $k_1 = -1.2$, $k_2 = 0.4$, $k_3 = 2$, $\alpha = 3$, $\varepsilon_i = 1$.

It is noted that Eq. (3.9) makes the phase shift (3.6) vanish. However, Eq. (3.9) is not admitted by $c = \frac{-\alpha k^2}{k^3+l}$. This kind of special cases has also been pointed out in [17,20].

Finally, upon using Eqs. (2.5), (3.1) and (3.9), we obtain a general resonant multi-soliton solution to the HSI equation (1.1), presented as follows:

$$u = 2\frac{f_x}{f} = 2\frac{\sum_{i=1}^{N} k_i \varepsilon_i e^{k_i x - k_i^3 y - \frac{\alpha}{3} k_i^{-1} t}}{\sum_{i=1}^{N} \varepsilon_i e^{k_i x - k_i^3 y - \frac{\alpha}{3} k_i^{-1} t}}. \tag{3.10}$$

It is clear to see that variable coefficient α of the dissipative term u_{xx} influences the wave speed directly. Moreover, the multi-soliton solution form (3.10) and the dispersion relation (3.9) are completely different from the existing ones (3.3) and (3.4). Without loss of generality, we plot the 2-kink and 3-kink waves via Eq. (3.10) in the case of $\alpha = 3$, $\varepsilon_i = 1$, as shown in Figs. 1–4. The inelastic mechanism is elaborated graphically and theoretically in the following paragraph.

Figs. 1 and 2 present the propagation of the traveling 2-kink wave. In Fig. 1 with the specified values $k_1 = -1.2$, $k_2 = 0.4$, $k_3 = 2$ we can see the propagation of wave fusion. As t increases, the lower kink is admitted by the higher one and then keeps the shape and speed without changing. Oppositely, by setting a different sign to k_1 gives the propagation of wave fission as shown in Fig. 2. It presents the traveling kink wave splits into two different kinks whose speeds, shapes and traveling directions are different. Fig. 3 with $k_1 = -0.5$, $k_2 = 0.9$, $k_3 = 1.2$, $k_4 = 1.8$ shows the fusion of the traveling 3-kink wave. At $t = 10$, the first kink overtakes the second kink, and then is admitted by the second one. At $t = 30$, the second kink also overtakes the third kink and is admitted by the third one. Finally, we can clearly see three different waves fuse into a single wave and then keeps the shape and speed without changing. Naturally, setting a different sign to k_1 changes the propagation of wave fusion to fission as shown in Fig. 4.

In summary, it can be found that with the wave numbers k_i chosen as different conditions, $k_1k_2k_3 < 0$ and $k_1k_2k_3k_4 < 0$, the 2- and 3-kink waves behave the overtaking coalescence, as shown in Figs. 1 and 3,

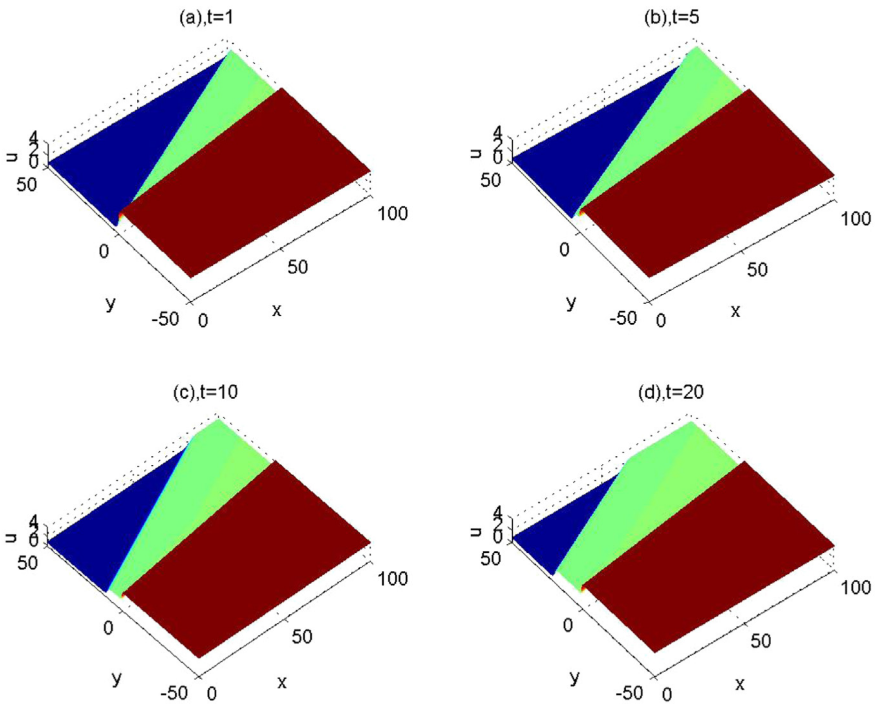

Fig. 2. The traveling 2-kink waves by (3.10) with $k_1 = 1.2,\ k_2 = 0.4,\ k_3 = 2,\ \alpha = 3,\ \varepsilon_i = 1$.

respectively. Oppositely, with the wave numbers k_i chosen as $k_1k_2k_3 > 0$ and $k_1k_2k_3k_4 > 0$, the 2- and 3-kink waves behave fission, as shown in Figs. 2 and 4. The results indicate the number and traveling direction of kink waves after inelastic interactions can be controlled. Compared to the previously reported literature, we can say that they are attributed to the dispersion relation form. In Eq. (3.4), the wave numbers k_i, l_i are independent and c_i is a function of k_i, l_i. Therefore, Eqs. (3.3) and (3.4) give an elastic interaction. However, it is not the case in Eq. (3.9) where l_i, c_i are determined by k_i, leading to an inelastic interaction.

So far, the resonant multi-soliton solution (3.10) has shown different versions of inelastic interactions of multi-front waves by specifying the parameters $k_i, \varepsilon_i, \alpha$. The above results reveal that the propagation of wave coalescence/fission can be performed by setting a different sign to k_i. The resultant solution and figures may provide significant supplements to the existing literature.

3.2. The generalized (2 + 1)-dimensional Hirota–Satsuma–Ito equation

Using the transformation (3.1), one transforms Eq. (1.2) into a Hirota bilinear form

$$(D_x^3 D_t + \delta_1 D_y D_t + \delta_2 D_x^2 + \delta_3 D_x D_y + \delta_4 D_x D_t + \delta_5 D_y^2) f \cdot f = 0. \tag{3.11}$$

Applying Eq. (2.6) gives the condition of existence of resonant N-wave solutions:

$$\begin{aligned}&(k_i - k_j)^3(c_i - c_j) + \delta_1(l_i - l_j)(c_i - c_j) + \delta_2(k_i - k_j)^2 + \delta_3(k_i - k_j)(l_i - l_j)\\&+\delta_4(k_i - k_j)(c_i - c_j) + \delta_5(l_i - l_j)^2 = 0, 1 \le i < j \le N.\end{aligned} \tag{3.12}$$

Clearly, compared with Eq. (3.7), it is much more difficult to generate exact wave numbers to Eq. (3.12) involving arbitrary coefficients by using the linear superposition principle. So, Eq. (3.12) is a good example to demonstrate the applicableness of the linear superposition principle.

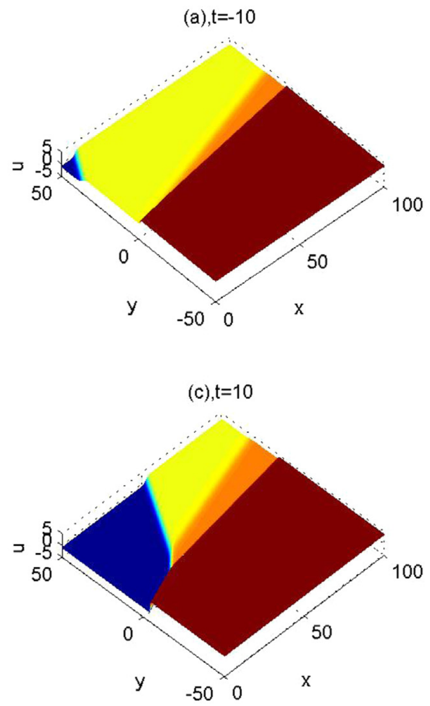

Fig. 3. The traveling 3-kink waves by (3.10) with $k_1 = -0.5$, $k_2 = 0.9$, $k_3 = 1.2$, $k_4 = 1.8$, $\alpha = 3$, $\varepsilon_i = 1$.

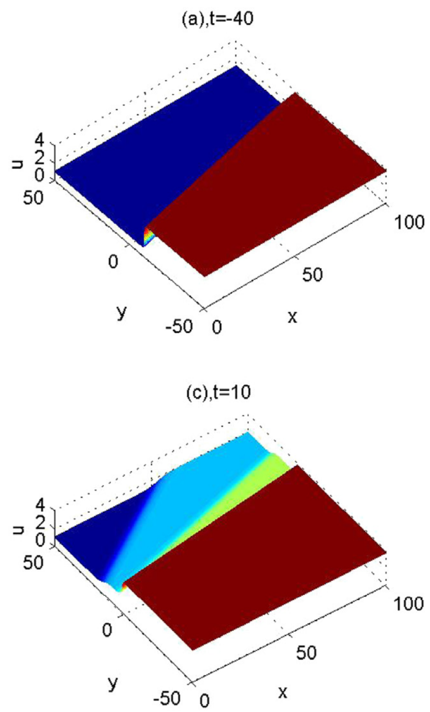

Fig. 4. The traveling 3-kink waves by (3.10) with $k_1 = 0.5$, $k_2 = 0.9$, $k_3 = 1.2$, $k_4 = 1.8$, $\alpha = 3$, $\varepsilon_i = 1$.

By balancing the linear terms of Eq. (1.2) gives the dispersion relation

$$c = \frac{-\delta_2 k^2 + \delta_3 kl + \delta_5 l^2}{k^3 + \delta_1 l + \delta_4 k}. \tag{3.13}$$

Based on Eq. (3.13), one conjectures that wave numbers can be determined as the following two cases.

Case 1

$$\begin{aligned} k_i &= k_i, \\ l_i &= ak_i, \\ c_i &= bk_i^{-1}. \end{aligned} \tag{3.14}$$

Case 2

$$\begin{aligned} k_i &= k_i, \\ l_i &= ak_i^2, \\ c_i &= bk_i. \end{aligned} \tag{3.15}$$

Proceeding as before, substituting Eqs. (3.14) and (3.15) into (3.12) shows that the resulting algebraic equations are not solvable. This means that there does not exist resonant multi-soliton solutions to the generalized HSI equation (1.2) under the two above relations of wave numbers. However, Eq. (1.2) had passed the Painlevé test and the three-soliton test, along with its integrability been shown in the previous literature [5,19]. Therefore, for Eq. (1.2), we hereby declare the phase shifts that come with the multi-soliton solutions must equal to zero if and only if $k_i - k_j = 0$, such as the phase shifts to the KdV equation [6,22] given by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{3.16}$$

It is worth mentioning that no theory exists to guarantee the applicableness of the linear superposition principle of exponential waves or the existence of resonant multi-soliton solutions to integrable equations.

4. Conclusions

We have studied the (2 + 1)-dimensional HSI equation (1.1) and the generalized HSI equation (1.2) by using the linear superposition principle. The resonant multi-soliton solution without phase shifts was precisely determined for Eq. (1.1), which is surviving only with satisfying Eq. (3.9). Furthermore, the propagations of 2- and 3-kink waves were demonstrated in Figs. 1–4, which show cases of inelastic interactions in the HSI equation (1.1). Comparing to the results reported by Liu et al. [14] reveals that the dispersion relation (3.9) plays the key role in the inelastic interaction mechanism. Moreover, through a direct computation, we failed to present any resonant multi-soliton solution to the generalized HSI equation (1.2). The reason leading to failure was clearly illustrated.

Our examples show that the linear superposition principle is powerful and useful in making the reliable judgment on the existence of resonant multi-soliton solutions. It is also well-known that many NLEEs have resonant phenomena and the related results have been employed in maritime security and coastal engineering. Thus, it is expected that our work would be helpful in making further applications in the corresponding fields.

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