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LIE SYMMETRIES, MODULATION INSTABILITY, CONSERVATION LAWS, AND THE DYNAMIC WAVEFORM PATTERNS OF SEVERAL INVARIANT SOLUTIONS TO A (2+1)-DIMENSIONAL HIROTA BILINEAR EQUATION

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Abstract. In this paper, we dealt for the first time with a highly nonlinear (2+1)-dimensional associated Hirota bilinear equation using the Lie symmetry approach and symbolic computation with Mathematica. The primary objective of this paper is to employ the Lie symmetry analysis for the purpose of finding newly generated explicit exact solutions, as well as conservation laws and investigating modulation instability within the context of the (2+1)-dimensional associated Hirota bilinear equation. Equivalence transformations, a commutator table, and an adjoint table are generated using Lie's invariance infinitesimal criterion. Applying the optimal algebra classification, the differential invariants are generated. By utilizing the optimal system and capitalizing on infinitesimal symmetries, we successfully obtain numerous symmetry reductions and invariant solutions through the implementation of the Lie group method. The obtained solutions include the time variable, space variables, arbitrary constants, as well as arbitrary functions. By taking advantage of symbolic computation work with Mathematica, the achieved outcomes are manifested with 3D, 2D, and contour graphics to interpret the physical meaning of the acquired results, which show traveling waves, solitary waves, and periodic wave structures. The solutions exhibit different physical structures concerning the involved parameters. All the attained results are novel and are absolutely distinct from the earlier findings. Moreover, we establish nonlinear self-adjointness and derive conservation laws for the provided equation. Finally, the linear stability analysis of the governing equation is presented to study the modulational instability.

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Abbrevation. ODE: Ordinary Differential Equation
PDE: Partial Differential Equation
NLEEs: Non-linear Evolution Equations
MI: Modulation Instability

1. **Introduction.** Non-linear evolution equations play an increasingly important role due to their applications in physical phenomena, especially in quantum mechanics, plasma physics, population models, hydrodynamics, electron thermal energy, fluid dynamics, optical fibers, and other scientific fields as well [23, 22, 5, 11]. Hence, the exploration of precise solutions for these equations holds significant importance in the examination of diverse nonlinear physical phenomena, making it a thriving research area. For the last few decades, researchers have made significant efforts to study the natural solutions of these equations. A variety of methodologies have been developed by researchers to find the exact analytical solutions of NLEEs, for example, the extended tanh method [26], Hirota's bilinear technique [4], the extended elliptic Jacobian function expansion approach [24], $\left(\frac{G'}{G}\right)$-expansion method [2, 9], Lie symmetry method [14, 13], generalized Kudryashov approach [12, 20], improved auxiliary equation [10], inverse scattering technique [3], Darboux transformation [1] approach, etc. Among these methodologies, the Lie symmetry technique is an efficient approach introduced by Sophus Lie for finding the exact solutions to all types of differential equations.

A new form of Hirota bilinear equation was proposed and studied by Yu and Lü in [6], which reads as

$$(D_tD_y - {D_x}^3D_y - 3{D_x}^2 + 3{D_z}^2)f.f = 0, \tag{1}$$

that is

$$2[(ff_{ty} - f_tf_y) + f_{xxx}f_y + 3f_{xxy}f_x - 3f_{xx}f_{xy} - ff_{xxxy} \tag{2}$$

$$- 3(ff_{xx} - {f_x}^2) + 3(ff_{zz} - {f_z}^2)] = 0. \tag{3}$$

Through the selection of a transformation, as presented in [6],

$$v = 2[lnf(x, y, z, t)]_x, \tag{4}$$

and equation (2) maps into

$$v_{yt} - v_{xxxy} - 3(v_xv_y)_x - 3v_{xx} + 3v_{zz} = 0. \tag{5}$$

The objective of this study is to achieve Lie symmetries, invariant solutions, conservation laws, and explore the modulation instability of the (2+1)-dimensional NLEE [25]

$$u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} - 3\alpha u_{xx} = 0, \tag{6}$$

which is derived by the equation (5) through the transformation

$$v(x, y, t) = \alpha y + u(x, y, t), \quad z = x, \tag{7}$$

where α is an arbitrary constant. Utilizing the linear superposition principle [19], two distinct sets of resonant N-wave solutions for equation (5) were derived. In [15], Lü and Ma also investigated the associated bilinear equation (5) and obtained two categories of lump solutions for the dimensionally reduced equations, primarily by considering the cases where $z = y$ and $z = t$. Necessary and sufficient conditions are provided for the involved parameters in the solutions to ensure analyticity and rational localization. Additionally, the localized characteristics and energy distributions of the lump solutions are investigated. Wang [25] introduced lump solutions for equation (6), obtained through the associated Hirota bilinear equation (5) by employing the binary Bell polynomial [16] and Hirota bilinear approaches. Moreover, the equation (6) has demonstrated integrability through the Lax pair concept, with the derivation of bilinear Bäcklund transformations stemming from the binary Bell polynomial theory.

This particular associated Hirota bilinear equation has been extensively discussed in the literature, owing to its significance across various scientific disciplines. By studying the literature cited above, we are prompted to solve the associated (2+1)-dimensional Hirota bilinear equation (6) by applying the Lie symmetry analysis. In many fields of natural science, symmetry is important, especially in integrable systems, where there can be infinitely many symmetry variations. Motivated by Galois theory, Lie introduced the group properties admitted by the differential equations, which resulted in the exact solutions. He ascertains that apparent techniques of solving ODEs can be incorporated into a systematic theory built on unvarying differential equations. This method focuses on finding a new coordinate system by finding the infinitesimal generators, which makes solving the differential equation easier. Through the identification of symmetries and their corresponding infinitesimal generators, the method facilitates the reduction of non-linear PDEs into new ones with fewer independent variables through the invariance condition. Consequently, the attainment of multiple reductions results in the derivation of numerous solutions. Here, we obtained some new invariant solutions of the associated Hirota bilinear equation (6) by means of the Lie symmetry analysis method. We utilize Lie vectors to construct an optimal system of one-dimensional subalgebras. Furthermore, through the application of the Lie symmetry method, we attain an extensive array of exact solutions for nonlinear PDEs, encompassing similarity solutions, traveling wave solutions, and soliton solutions. Notably, among these solutions, traveling wave solutions, similarity solutions, and other exact solutions hold particular physical significance. Furthermore, we derive conservation laws by applying the novel non-local conservation theorem introduced by Ibragimov in [8]. The linear stability analysis of the associated Hirota bilinear equation is also presented to study the MI. In the past literature, lump-type solutions are constructed, and the integrability of the dimensionally reduced equations are discussed in [6, 25, 15]. As far as our knowledge extends, there have been no previous reports regarding Lie symmetry, modulation instability, or conservation laws pertaining to the governing equation (6).

The primary goals of this article includes:

- To construct novel exact analytical solutions for the governing model, facilitating a deeper comprehension of their nonlinear characteristics.
- Formation of conservation laws to understand the fundamental physical principles of the equation.
- To achieve the modulation instability condition for finding the stability of solutions.
- Analyzing the graphs of the generated solutions offers valuable insights into the underlying dynamics of these solutions.

The structure of this article is as follows: In Section 2, we delve into the derivation of Lie symmetries along with their corresponding Lie groups. Section 3 focuses on the computation of optimal system, achieved through the utilization of the adjoint table presented in Table 2. Analytical invariant solutions for the governing model are attained in Section 4 via a reduction procedure. Section 5 discusses the physical interpretation of obtained results. A comprehensive explanation of conservation laws is detailed in Section 6. Section 7 explores modulation instability through linear stability analysis. Section 8 compares our derived solutions with the previously established results. Finally, the paper is concluded in Section 9 with the summary of findings and closing remarks.

2. Description of the proposed method.

Lie point symmetry is an effective and essential technique for discovering invariant solutions and symmetry characteristics of NLEEs. Here, we employ this methodology to the (2+1)-dimensional associated Hirota bilinear equation (6), from which we derive the corresponding generators featuring four arbitrary functions. Consider a transformation for equation (6) as

$$\begin{aligned} x_a^* &= x_a + \epsilon\xi_a(x_a, u_b) + O(\epsilon^2), \\ u_b^* &= u_b + \epsilon\eta_b(x_a, u_b) + O(\epsilon^2). \end{aligned} \tag{8}$$

Here, ξ_1, ξ_2, ξ_3, and η_1 are infinitesimal generators and $\epsilon << 1$ is a small parameter. Lie symmetry generators for equation (6) are expressed as

$$\mathtt{V} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial u}. \tag{9}$$

For equation (6) to remain invariant under a Lie group, the operator (9) must satisfy the following infinitesimal invariance criterion:

$$Pr^{(4)}\mathtt{V}(\triangle)|_{\triangle=0} = 0. \tag{10}$$

Here, $Pr^{(4)}\mathtt{V}$ represents the fourth prolongations and is determined as [21]

$$\begin{aligned} Pr^{(4)}\mathtt{V}(\triangle) =& \mathtt{V} + \eta_1^x\frac{\partial}{\partial u_x} + \eta_1^y\frac{\partial}{\partial u_y} + \eta_1^t\frac{\partial}{\partial u_t} + 2\eta_1^{xt}\frac{\partial}{\partial u_{xt}} + \eta_1^{yt}\frac{\partial}{\partial u_{yt}} + \alpha u_y\eta_1^{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \alpha u_{xx}\eta_1^y\frac{\partial}{\partial u_y} + \alpha u_x\eta_1^{yx}\frac{\partial}{\partial u_{yx}} + \alpha u_{yx}\eta_1^x\frac{\partial}{\partial u_x} + \eta_1^{xxxy}\frac{\partial}{\partial u_{xxxy}}... \end{aligned}$$

To obtain the determining system, we apply the $Pr^{(4)}\mathtt{V}$ to equation (6) and get

$$\eta_1^{ty} - \eta_1^{xxxy} - 3\eta_1^{xx}u_y - 3u_{xx}\eta_1^y - 3\eta_1^x u_{yx} - 3u_x\eta_1^{yx} - 3\alpha\eta_1^{xx} = 0. \tag{11}$$

By using the above condition, we get the determining equations as

$$\begin{aligned} (\xi_1)_u = 0, (\xi_1)_x = \frac{1}{3}(\xi_3)_t, (\xi_1)_y = 0, \\ (\xi_2)_t = 0, (\xi_2)_x = 0, (\xi_2)_u = 0, \\ (\xi_3)_u = 0, (\xi_3)_x = 0, (\xi_3)_y = 0, \\ (\eta_1)_u = -\frac{1}{3}(\xi_3)_t,\ (\eta_1)_x = -\frac{1}{3}(\xi_1)_t,\ (\eta_1)_y = -\alpha(\xi_2)_y + (\xi_3)_t. \end{aligned} \tag{12}$$

Upon solving the aforementioned equations (12), we acquire

$$\begin{aligned} &\xi_1 = \frac{x}{3}f_1'(t) + f_3(t), \quad \xi_2 = f_2(y), \quad \xi_3 = f_1(t), \\ &\eta_1 = -\frac{x^2}{18}f_1''(t) + \frac{(-6\alpha y - 6u)}{18}f_1'(t) - \frac{x}{3}f_3'(t) - \alpha f_2(y) + f_4(t). \end{aligned} \tag{13}$$

The generators for equation (6) are

$$\mathtt{V} = \mathtt{V}_1 + \mathtt{V}_2 + \mathtt{V}_3 + \mathtt{V}_4. \tag{14}$$

Here,

$$\begin{aligned} \mathtt{V}_1 &= \frac{x}{3} f_1'(t) \frac{\partial}{\partial x} + f_1(t) \frac{\partial}{\partial t} - \frac{x^2}{18} f_1''(t) \frac{\partial}{\partial u} - \frac{1}{3}(\alpha y + u) f_1'(t) \frac{\partial}{\partial u}, \\ \mathtt{V}_2 &= f_2(y) \frac{\partial}{\partial y} - \alpha f_2(y) \frac{\partial}{\partial u}, \\ \mathtt{V}_3 &= f_3(t) \frac{\partial}{\partial x} - \frac{x}{3} f_3'(t) \frac{\partial}{\partial u}, \quad \mathtt{V}_4 = f_4(t) \frac{\partial}{\partial u}. \end{aligned} \tag{15}$$

Table 1. Commutator table for the (2+1)-dimensional associated Hirota bilinear equation (6)

$[\mathtt{V}_i, \mathtt{V}_j]$	$\mathtt{V}_1$	$\mathtt{V}_2$	$\mathtt{V}_3$	$\mathtt{V}_4$
$\mathtt{V}_1$	0	0	$\mathtt{V}_3(f_1 f_3' - \frac{f_3 f_1'}{3})$	$\mathtt{V}_4(f_1 f_4' + \frac{f_4 f_1'}{3})$
$\mathtt{V}_2$	0	0	0	0
$\mathtt{V}_3$	$-\mathtt{V}_3(f_1 f_3' - \frac{f_3 f_1'}{3})$	0	0	0
$\mathtt{V}_4$	$-\mathtt{V}_4(f_1 f_4' + \frac{f_4 f_1'}{3})$	0	0	0

Table 2. Adjoint table for the (2+1)-dimensional associated Hirota bilinear equation (6)

$[\mathtt{V}_i, \mathtt{V}_j]$	$\mathtt{V}_1$	$\mathtt{V}_2$	$\mathtt{V}_3$	$\mathtt{V}_4$
$\mathtt{V}_1$	$\mathtt{V}_1$	$\mathtt{V}_2$	$e^{-\epsilon}\mathtt{V}_3$	$e^{-\epsilon}\mathtt{V}_4$
$\mathtt{V}_2$	$\mathtt{V}_1$	$\mathtt{V}_2$	$\mathtt{V}_3$	$\mathtt{V}_4$
$\mathtt{V}_3$	$\mathtt{V}_1 + \epsilon\mathtt{V}_3$	$\mathtt{V}_2$	$\mathtt{V}_3$	$\mathtt{V}_4$
$\mathtt{V}_4$	$\mathtt{V}_1 + \epsilon\mathtt{V}_4$	$\mathtt{V}_2$	$\mathtt{V}_3$	$\mathtt{V}_4$

As a result of these infinitesimal generators $\mathtt{V}_i (i = 1, 2, 3, 4)$, equation (6) accommodates four Lie groups $\mathcal{G}_i$ of point transformation as follows:

$$\begin{aligned} &\mathcal{G}_1 : (x, y, t, u) \to \left(x + \frac{x}{3} \epsilon f_1'(t), y, t + \epsilon f_1(t), u - \frac{(u + y\alpha)}{3} \epsilon f_1'(t) - \frac{x^2}{18} \epsilon f_1''(t) \right), \\ &\mathcal{G}_2 : (x, y, t, u) \to (x, y + \epsilon f_2(y), t, u - \alpha \epsilon f_2(y)), \\ &\mathcal{G}_3 : (x, y, t, u) \to \left(x + \epsilon f_3(t), y, t, u - \frac{1}{3} x \epsilon f_3'(t) \right), \\ &\mathcal{G}_4 : (x, y, t, u) \to (x, y, t, u + \epsilon f_4(t)). \end{aligned} \tag{16}$$

Using the above group, different solutions $\mathfrak{u}_i$ of equation (6) can be achieved as

$$\begin{aligned} \mathfrak{u}_1 &= \frac{x^2}{18} \epsilon f_1''(t) + \frac{y\alpha}{3} \epsilon f_1'(t) + e^{\frac{\epsilon f_1'(t)}{3}} g_1 \left(x - \frac{x}{3} \epsilon f_1'(t), y, t - \epsilon f_1(t) \right), \\ \mathfrak{u}_2 &= \alpha \epsilon f_2(y) + g_1(x, y - \epsilon f_2(y), t), \\ \mathfrak{u}_3 &= \frac{x}{3} \epsilon f_3'(t) + g_1 (x - \epsilon f_3(t), y, t), \\ \mathfrak{u}_4 &= -\epsilon f_4(t) + g_1(x, y, t). \end{aligned} \tag{17}$$

3. **Optimal system.** The subgroups (of same dimension) of a Lie group can be infinite, and one can obtain group invariant solutions corresponding to any subgroup. Hence, it is necessary to categorize these subgroups based on a specific equivalence relation. Therefore, we proceed to compute the optimal system of one-dimensional subalgebras. This involves an initial search for the invariant function, followed by the determination of the adjoint matrix, and ultimately leading to the categorization of the Lie algebra G, as outlined in [7].

3.1. **Calculation of invariants.** The invariant function Ψ is a real valued function which gives $\Psi(Ad_g(v)) = \Psi(v)$ for all $v \in \mathbb{R}^4$ and $g \in G$. Then, for $\mathtt{V}_i = \sum_{i=1}^{4} \beta_i \mathfrak{v}_i$ and for $\mathtt{V}_j = \sum_{i=1}^{4} \alpha_i \mathfrak{v}_i$, we have

$$\begin{aligned} Ad(e^{(\epsilon \mathtt{V}_i)})(\mathtt{V}_j) &= e^{-\epsilon \mathtt{V}_i} \mathtt{V}_j e^{\epsilon \mathtt{V}_i} \qquad (18) \\ &= \mathtt{V}_j - \epsilon[\mathtt{V}_i, \mathtt{V}_j] + \frac{\epsilon^2}{2!}[\mathtt{V}_i, [\mathtt{V}_i, \mathtt{V}_j]] - \dots \\ &= (\alpha_1 \mathfrak{v}_1 + \dots + \alpha_4 \mathfrak{v}_4) - \epsilon[\beta_1 \mathfrak{v}_1 + \dots + \beta_4 \mathfrak{v}_4, \alpha_1 \mathfrak{v}_1 + \dots + \alpha_4 \mathfrak{v}_4] + O(\epsilon^2) \\ &= (\alpha_1 \mathfrak{v}_1 + \dots + \alpha_4 \mathfrak{v}_4) - \epsilon(\theta_1 \mathfrak{v}_1 + \dots + \theta_4 \mathfrak{v}_4). \qquad (19) \end{aligned}$$

Using commutator Table 1, we receive the following values of $\theta_i's$

$$\theta_1 = 0, \quad \theta_2 = 0, \ \theta_3 = \alpha_1 \beta_3 - \alpha_3 \beta_1, \ \theta_4 = -\beta_1 \alpha_4 + \beta_4 \alpha_1. \qquad (20)$$

To obtain the invariant function Ψ [7], it takes

$$\sum_{i=1}^{4} \theta_i \frac{\partial \phi}{\partial \alpha_i} = 0. \qquad (21)$$

Solving equation (21) for $\beta_i's$, we get the following equations:

$$\begin{aligned} \beta_1 &: -\alpha_3 \frac{\partial \Psi}{\partial \alpha_3} - \alpha_4 \frac{\partial \Psi}{\partial \alpha_4} = 0, \\ \beta_3 &: \alpha_1 \frac{\partial \Psi}{\partial \alpha_3} = 0, \\ \beta_4 &: \alpha_1 \frac{\partial \Psi}{\partial \alpha_4} = 0. \qquad (22) \end{aligned}$$

Upon solving system (22), we get the invariant function as $\Psi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = H(\alpha_1, \alpha_2)$.

3.2. **Construction of adjoint matrix.** Let $\epsilon_i, 1 \le i \le 4$ be real constants and $g = e^{\epsilon_i \mathtt{V}_i}$. Then, the matrices B_i^ϵ of $\mathtt{V}_i, i = 1, 2, 3, 4$ are constructed with the help of Table 2 as defined in [7]

$$B_1^\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\epsilon_1} & 0 \\ 0 & 0 & 0 & e^{-\epsilon_1} \end{pmatrix}, \ B_2^\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3^\epsilon = \begin{pmatrix} 1 & 0 & \epsilon_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ B_4^\epsilon = \begin{pmatrix} 1 & 0 & 0 & \epsilon_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The general transformation matrix is obtained by multiplying the matrices $B = B_1^\epsilon.B_2^\epsilon.B_3^\epsilon.B_4^\epsilon$

$$B = \begin{pmatrix} 1 & 0 & \epsilon_3 & \epsilon_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\epsilon_1} & 0 \\ 0 & 0 & 0 & e^{-\epsilon_1} \end{pmatrix}. \tag{23}$$

3.3. **One-dimensional optimal system for equation (6).** The adjoint transformation equation for (6) is

$$(p_1, p_2, p_3, p_4) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cdot B, \tag{24}$$

where B is defined above. By solving equation (24), we get

$$\begin{aligned} p_1 &= \alpha_1, \\ p_2 &= \alpha_2, \\ p_3 &= \epsilon_3\alpha_1 + \alpha_3 e^{-\epsilon_1}, \\ p_4 &= \epsilon_4\alpha_1 + \alpha_4 e^{-\epsilon_1}, \end{aligned} \tag{25}$$

which must have solutions for $\epsilon_i's$ $1 \leq i \leq 4$. Consider the following cases.
Case 1: $p_1 = 1,\ p_2 = p_3 = p_4 = 0$. By choosing a representative element $\tilde{\mathtt{V}} = \mathtt{V}_1$ and taking $\alpha_1 = 1, \alpha_2 = 0$ in equation (25), we get the solution as

$$\epsilon_1 = 0,\ \ \epsilon_3 = -\alpha_3,\ \ \epsilon_4 = -\alpha_4.$$

Case 2: $p_1 = 1, p_2 = 0, p_3 = 1, p_4 = 1$. By choosing a representative element $\tilde{\mathtt{V}} = \mathtt{V}_1 + \mathtt{V}_3 + \mathtt{V}_4$ and taking $\alpha_1 = 1,\ \alpha_2 = 0$ in equation (25), we get the solution as

$$\epsilon_1 = 0,\ \ \epsilon_3 = 1 - \alpha_3,\ \ \epsilon_4 = 1 - \alpha_4.$$

Similarly, by following the same procedure, we can determine the values of ϵ_i for the remaining members.

Taking into account all combinations, we obtain the following representatives:

$$\{\mathtt{V}_1,\ \ \mathtt{V}_3,\ \ \mathtt{V}_2 + \mathtt{V}_4,\ \ \mathtt{V}_3 + \mathtt{V}_4,\ \ \mathtt{V}_1 + \mathtt{V}_3 + \mathtt{V}_4,\ \ \mathtt{V}_2 + \mathtt{V}_3 + \mathtt{V}_4\}. \tag{26}$$

4. **The essence of proposed algorithm.** This section obtains the reductions for equation (6) via optimal system (26) for which its auxiliary equations are

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{du}{\eta_1}, \tag{27}$$

where $\{\xi_1, \xi_2, \xi_3, \eta_1\}$ are given explicitly in (13).

Case 1. Subalgebra $\mathtt{V}_1 = \frac{x}{3} f_1'(t)\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial t} - \frac{x^2}{18} f_1''(t)\frac{\partial}{\partial u} - \frac{1}{3}(\alpha y + u) f_1'(t)\frac{\partial}{\partial u}$.

The symmetry $\mathtt{V}_1$ yields the characteristic equation

$$\frac{dx}{\frac{x}{3} f_1'(t)} = \frac{dy}{0} = \frac{dt}{f_1(t)} = \frac{du}{-\frac{x^2}{18} f_1''(t) - \frac{1}{3}(\alpha y + u) f_1'(t)}, \tag{28}$$

which provides similarity reduction

$$u(x, y, t) = -\alpha Y - \frac{X^2 f_1'(t)}{18 f_1(t)^{\frac{1}{3}}} + \frac{U(X, Y)}{f_1(t)^{\frac{1}{3}}}, \tag{29}$$

with group invariants as $X = xf_1(t)^{-\frac{1}{3}}$ and $Y = y$. Substituting equation (29) into equation (6) gives rise to

$$3U_XU_{XY} + 3U_YU_{XX} + U_{XXXY} = 0. \tag{30}$$

The reduced PDE (30) furnishes the following generators:

$$\xi_X = c_1X + c_2, \quad \xi_Y = h_1(Y), \quad \eta_U = -c_1U + c_3. \tag{31}$$

Subcase 1. $c_1 = 1, c_2 = 0, c_3 = 0$. Then,

$$U(X,Y) = H(X_1)e^{-\int \frac{1}{h_1(Y)}\,dY}, \tag{32}$$

where $X_1 = Xe^{-\int \frac{1}{h_1(Y)}\,dY}$. Substituting (32) into (30), we get

$$\begin{aligned} &X_1\left(6H'(X_1)H''(X_1) + H^{(4)}(X_1)\right) + 6H'(X_1)^2 \\ &+ 3H(X_1)H''(X_1) + 4H^{(3)}(X_1) = 0. \end{aligned} \tag{33}$$

On solving (33), we get some particular solutions as

$$H(X_1) = \frac{b_1}{X_1}, \quad H(X_1) = \frac{2}{X_1 + b_2}, \tag{34}$$

where b_1 and b_2 are constants. Using back substitution, one can get the solutions of equation (6) as

$$\begin{aligned} u(x,y,t) &= -\alpha y + \frac{b_1}{x} - \frac{x^2 f_1'(t)}{18 f_1(t)}, \\ u(x,y,t) &= -\alpha y + \frac{2}{b_2\sqrt[3]{f_1(t)}e^{\int \frac{1}{h_1(y)}\,dy} + x} - \frac{x^2 f_1'(t)}{18 f_1(t)}. \end{aligned} \tag{35}$$

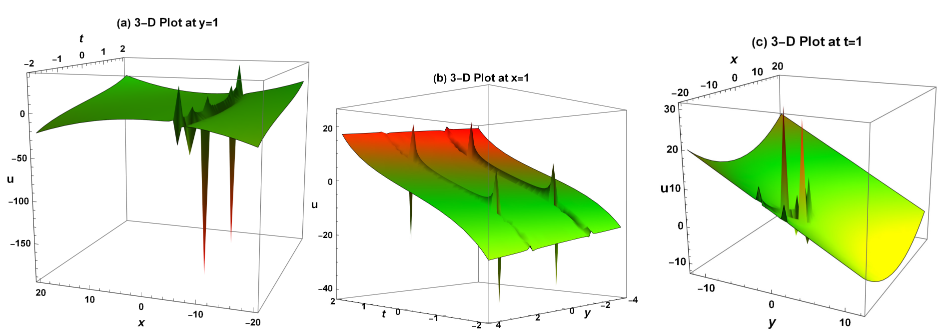


FIGURE 1. The three-dimensional plots depict the wave propagations of the solution given by equation (35).

Subcase 2. $c_1 = 0, c_2 \neq 0, c_3 \neq 0$. Then,

$$U(X,Y) = H(X_1) - \int \frac{c_3}{h_1(Y)}\,dY, \tag{36}$$

where $X_1 = X - \displaystyle\int \frac{c_2}{h_1(Y)}\,dY$. Using equation (36) in equation (30), we get

$$3\left(c_3 + 2c_2H'(X_1)\right)H''(X_1) + c_2H''''(X_1) = 0, \tag{37}$$

which gives

$$H(X_1) = b_3 + b_4 X_1, \quad H(X_1) = b_5 - \frac{c_3}{2c_2} X_1 + \frac{2}{X_1}. \tag{38}$$

Thus, the solutions of (6) are

$$u(x,y,t) = -\alpha y + \frac{b_3 - (b_4 c_2 + c_3)\left(\int \frac{1}{h_1(y)}\,dy\right)}{\sqrt[3]{f_1(t)}} + \frac{b_4 x}{f_1(t)^{2/3}} - \frac{x^2 f_1'(t)}{18 f_1(t)}, \tag{39}$$

$$u(x,y,t) = -\alpha y - \frac{9 f_1(t)^{2/3}\left(c_3\left(\int \frac{1}{h_1(y)}\,dy\right) - 2b_5\right) + \frac{9 c_3 x \sqrt[3]{f_1(t)}}{c_2} + x^2 f_1'(t)}{18 f_1(t)} + \frac{2}{x - c_2 \sqrt[3]{f_1(t)}\left(\int \frac{1}{h_1(y)}\,dy\right)}. \tag{40}$$

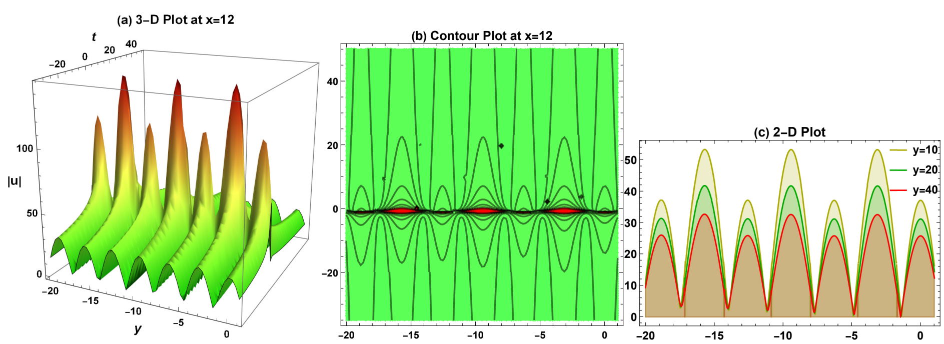


FIGURE 2. The three-dimensional, contour surface, and two-dimensional plots illustrate the wave propagations of the solution given by equation (40).

Furthermore, some additional solutions for equation (30) are discovered as

$$U(X,Y) = h_2(Y) + X h_3(Y) + e^{-6X} h_4(Y) + h_5(X), \tag{41}$$

$$U(X,Y) = \frac{\varphi_0\left(\eta e^{\alpha X + \beta Y} + 1\right)^2 - \varphi_0\left(\eta e^{\alpha X + \beta Y} + 1\right)}{\left(\eta e^{\alpha X + \beta Y} + 1\right)\left(q_0\left(\eta e^{\alpha X + \beta Y} + 1\right) - q_0\right)}, \tag{42}$$

$$U(X,Y) = \frac{\frac{\varphi_0 q_1\left(\eta e^{\alpha X + \beta Y} + 1\right)}{q_0} + \varphi_0\left(\eta e^{\alpha X + \beta Y} + 1\right)^2}{\left(\eta e^{\alpha X + \beta Y} + 1\right)\left(q_0\left(\eta e^{\alpha X + \beta Y} + 1\right) + q_1\right)}. \tag{43}$$

Hence, solutions of (6) are obtained as

$$u(x,y,t) = -\alpha y - \frac{x^2 f_1'(t)}{18 f_1(t)} + \frac{x h_3(y)}{f_1(t)^{2/3}} + \frac{h_4(y) e^{-\frac{6x}{\sqrt[3]{f_1(t)}}} + h_5\left(\frac{x}{\sqrt[3]{f_1(t)}}\right) + h_2(y)}{\sqrt[3]{f_1(t)}}, \tag{44}$$

$$u(x,y,t) = -\alpha y + \frac{\frac{\varphi_0\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}} + \beta y} + 1\right)^2 - \varphi_0\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}} + \beta y} + 1\right)}{\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}} + \beta y} + 1\right)\left(q_0\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}} + \beta y} + 1\right) - q_0\right)} - \frac{x^2 f_1'(t)}{18 f_1(t)^{2/3}}}{\sqrt[3]{f_1(t)}}, \tag{45}$$

$$u(x,y,t) = -\alpha y + \frac{\dfrac{\dfrac{\varphi_0 q_1\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}}+\beta y}+1\right)}{q_0}+\varphi_0\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}}+\beta y}+1\right)^2}{\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}}+\beta y}+1\right)\left(q_0\left(\eta e^{\frac{\alpha x}{\sqrt[3]{f_1(t)}}+\beta y}+1\right)+q_1\right)} - \dfrac{x^2 f_1'(t)}{18 f_1(t)^{2/3}}}{\sqrt[3]{f_1(t)}}. \tag{46}$$

Case 2. Subalgebra $\mathtt{V}_3 = f_3(t)\frac{\partial}{\partial x} - \frac{x}{3}f_3'(t)\frac{\partial}{\partial u}$.

For the vector field $\mathtt{V}_3$, the similarity variable is

$$u(x,y,t) = -\frac{x^2 f_3'(t)}{6 f_3(t)} + U(Y,T), \tag{47}$$

with invariants $Y = y$ and $T = t$. Substituting equation (47) into equation (6), we have

$$\frac{f_3'(T)(\alpha + U_Y)}{f_3(T)} + U_{YT} = 0. \tag{48}$$

The reduced PDE (48) furnishes the following solution

$$U(Y,T) = h_7(T) - \alpha Y + \frac{1}{f_3(T)}\int h_6(Y)dY. \tag{49}$$

Thus, we get the exact solution of (6) as

$$u(x,y,t) = h_7(t) - \alpha y - \frac{x^2 f_3'(t)}{6 f_3(t)} + \frac{1}{f_3(t)}\int h_6(y)dy, \tag{50}$$

where $h_6(y)$ and $h_7(t)$ are arbitrary functions.

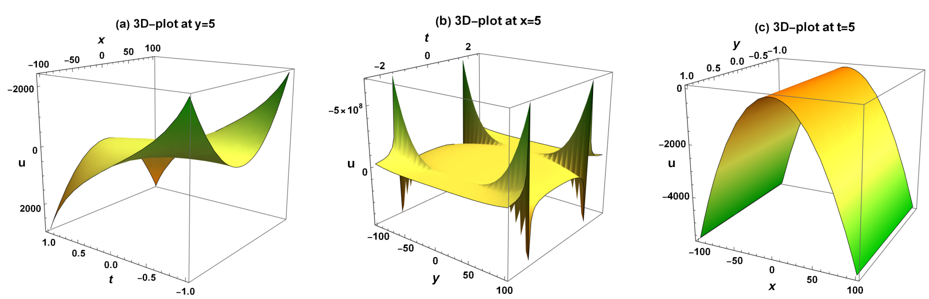


FIGURE 3. The three-dimensional plots depict the wave propagations of the solution given by equation (50).

Case 3. Subalgebra $\mathtt{V}_2 + \mathtt{V}_4 = f_2(y)\frac{\partial}{\partial y} - \alpha f_2(y)\frac{\partial}{\partial u} + f_4(t)\frac{\partial}{\partial u}$.

For the above vector field, the similarity reduction is

$$u(x,y,t) = -\alpha y + \int \frac{f_4(t)}{f_2(y)}\,dy + U(X,T), \tag{51}$$

with similarity variables $X = x$ and $T = t$. The obtained transformation (51) reduces the equation (6) to the PDE

$$f_4'(T) - 3f_4(T)U_{XX} = 0, \tag{52}$$

which after solving, gives

$$U(X,T) = h_8(T) + X h_9(T) + \frac{X^2 f_4'(T)}{6 f_4(T)}. \tag{53}$$

Hence, we obtain the solution of (6) as

$$u(x,y,t) = -\alpha y + h_8(t) + x\, h_9(t) + \frac{x^2 f_4'(t)}{6 f_4(t)} + f_4(t)\left(\int \frac{1}{f_2(y)}\, dy\right). \tag{54}$$

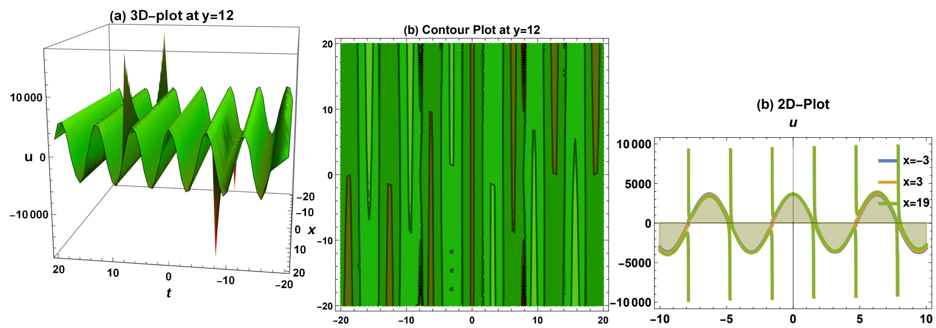


FIGURE 4. The three-dimensional, contour surface, and two-dimensional plots illustrate the wave propagations of the solution given by equation (54).

Case 4. Subalgebra $V_3 + V_4 = f_3(t)\frac{\partial}{\partial x} - \frac{x}{3} f_3'(t)\frac{\partial}{\partial u} + f_4(t)\frac{\partial}{\partial u}$.

For the above vector field, we have

$$u(x,y,t) = \frac{3x f_4(t) - \frac{1}{2}x^2 f_3'(t)}{3 f_3(t)} + U(Y,T), \tag{55}$$

with $Y = y$ and $T = t$. The obtained transformation (55) reduces equation (6) into the PDE

$$f_3'(T)(U_Y + \alpha) + f_3(T) U_{YT} = 0. \tag{56}$$

The reduced PDE (56) furnishes the solution

$$U(Y,T) = -\alpha Y + g_2(T) + \int \frac{g_1(Y)}{f_3(T)}\, dY. \tag{57}$$

Thus, we get

$$u(x,y,t) = -\alpha y + g_2(t) + \frac{\int g_1(y)\, dy}{f_3(t)} + \frac{3x f_4(t) - \frac{1}{2}x^2 f_3'(t)}{3 f_3(t)}. \tag{58}$$

Case 5. Subalgebra $V_1 + V_3 + V_4 = \frac{x}{3} f_1'(t)\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial t} - \frac{x^2}{18} f_1''(t)\frac{\partial}{\partial u} - \frac{1}{3}(\alpha y + u) f_1'(t)\frac{\partial}{\partial u} + f_3(t)\frac{\partial}{\partial x} - \frac{x}{3} f_3'(t)\frac{\partial}{\partial u} + f_4(t)\frac{\partial}{\partial u}$.

The Lagrange system is

$$\frac{dx}{\frac{x}{3} f_1'(t) + f_3(t)} = \frac{dy}{0} = \frac{dt}{f_1(t)} = \frac{du}{-\frac{x^2}{18} f_1''(t) - \frac{1}{3}(\alpha y + u) f_1'(t) - \frac{x}{3} f_3'(t) + f_4(t)}. \tag{59}$$

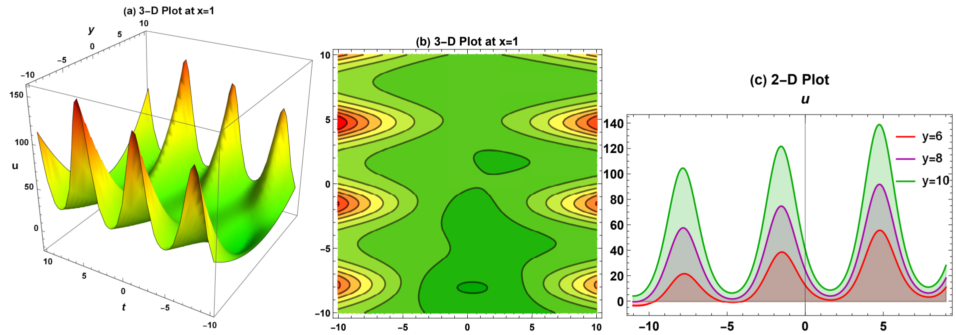


FIGURE 5. The three-dimensional, contour surface, and two-dimensional plots illustrate the wave propagations of the solution given by equation (58).

By selecting $f_3(t) = \frac{a}{3}f_1'(t)$ and $f_4(t) = -\frac{a^2}{18}f_1''(t)$, the above system can be solved accurately, where a is an arbitrary constant.

Therefore, for the above vector field, the invariant form is

$$u(x,y,t) = -\alpha y - \frac{X^2 f_1'(t)}{18\sqrt[3]{f_1(t)}} + \frac{U(X,Y)}{\sqrt[3]{f_1(t)}}, \tag{60}$$

with invariants $X = \frac{x+a}{\sqrt[3]{f_1(t)}}$ and $Y = y$. Inserting equation (60) into (6), we get

$$3U_X U_{XY} + 3U_Y U_{XX} + U_{XXXY} = 0, \tag{61}$$

which can be solved further as equation (30).

Case 6. Subalgebra $\mathtt{V}_2 + \mathtt{V}_3 + \mathtt{V}_4 = f_2(y)\frac{\partial}{\partial y} - \alpha f_2(y)\frac{\partial}{\partial u} + f_3(t)\frac{\partial}{\partial x} - \frac{x}{3}f_3'(t)\frac{\partial}{\partial u} + f_4(t)\frac{\partial}{\partial u}$.

The auxiliary equations are

$$\frac{dx}{f_3(t)} = \frac{dy}{f_2(y)} = \frac{dt}{0} = \frac{du}{-\alpha f_2(y) - \frac{x}{3}f_3'(t) + f_4(t)}. \tag{62}$$

By choosing arbitrary functions $f_3(t) = n_0 t^2, f_4(t) = n_1 t$, and $f_2(y) = n_2 y$, the above system can be solved accurately, where n_0, n_1, and n_2 are constants.

Therefore, for the above vector field, we have

$$u(x,y,t) = -\alpha y - \frac{n_0^2 t^3 \log^2(y)}{3n_2^2} - \frac{2n_0 t(x - \frac{n_0 t^2 \log(y)}{n_2})\log(y)}{3n_2} + \frac{n_1 t\log(y)}{n_2} + U(X,T), \tag{63}$$

with $X = x - \frac{n_0 t^2 \log(y)}{n_2}$ and $T = t$. Substituting (63) into (6), we get

$$-3n_1 + 2n_0 X + 3n_0 T^2 U_{XT} + 9n_1 T U_{XX} - 6n_0 XTU_{XX} - 18n_0 T^2 U_X U_{XX} - 3n_0 T^2 U_{XXXX} = 0. \tag{64}$$

The infinitesimals of this are

$$\xi_X = \frac{(6\varphi_1 T^5 + \varphi_2)X}{3} + T^2\varphi_4 + \frac{\varphi_3}{T^3} - \frac{n_1(6\varphi_1 T^5 + \varphi_2)}{2n_0}, \quad \xi_Y = T(\varphi_1 T^5 + \varphi_2),$$
$$\eta_U = -2\left(\varphi_1 T^5 + \frac{\varphi_2}{6}\right)U - \frac{2T\varphi_4 X}{3} - \frac{5T^4X^2\varphi_1}{3} + \frac{5T^4\varphi_1 n_1 X}{n_0} + \frac{\varphi_3 X}{6T^4} + h_{10}(T). \tag{65}$$

Subcase 1. $\varphi_1 \neq 0$ **with the rest of the** $\varphi_i' s = 0$, **and** $h_{10}(T) = \frac{T^5}{3n_0\varphi_1}$. Then,

$$U(X,T) = \frac{\frac{T^2}{6n_0} - \frac{T(2n_0X-3n_1)^2}{12n_0^2} + \frac{15n_1^2T}{4n_0^2}}{T^2} + \frac{H(X_1)}{T^2}, \tag{66}$$

where $X_1 = \dfrac{2n_0X - 3n_1}{2n_0T^2}$. Substituting (66) into (64), we get

$$\left(6H'(X_1)H''(X_1) + H^{(4)}(X_1)\right) = 0. \tag{67}$$

On solving (67), some particular solutions are obtained as

$$H(X_1) = n_3 + n_4X_1,$$
$$H(X_1) = n_5 + \frac{2}{X_1}, \tag{68}$$

where $n_i,\ 3 \le i \le 5$, are constants. Performing back substitution yields the subsequent solutions

$$u(x,y,t) = \frac{n_4x}{t^4} + \frac{n_1\left(\frac{x}{t} - \frac{3n_4}{2t^4}\right) + \frac{1}{6}}{n_0} - \frac{n_0n_4\log(y)}{n_2t^2} + \frac{n_3}{t^2} + \frac{3n_1^2}{n_0^2t} - \frac{x^2}{3t} - \alpha y. \tag{69}$$

$$u(x,y,t) = \frac{4n_0}{2n_0\left(x - \frac{n_0t^2\log(y)}{n_2}\right) - 3n_1} + \frac{n_5}{t^2} + \frac{\frac{n_1x}{t} + \frac{1}{6}}{n_0} + \frac{3n_1^2}{n_0^2t} - \frac{x^2}{3t} - \alpha y. \tag{70}$$

Subcase 2. $\varphi_2 \neq 0$**, with the rest of the** $\varphi_i' s = 0$. Then,

$$U(X,T) = \frac{H(X_1)}{\sqrt[3]{T}}, \tag{71}$$

where $X_1 = \dfrac{2n_0X - 3n_1}{2n_0\sqrt[3]{T}}$. Substituting (71) into (64), we get the reduced ODE as

$$-4X_1 + 14X_1H''(X_1) + 4H'(X_1) + 36H'(X_1)H''(X_1) + 6H''''(X_1) = 0, \tag{72}$$

which gives

$$H(X_1) = n_6 - \frac{5+\sqrt{29}}{12}X_1^2. \tag{73}$$

Thus, we obtain

$$u(x,y,t) = \frac{2n_0t\log(y)\left(n_0t^2\log(y) - n_2x\right)}{3n_2^2} \tag{74}$$
$$+ \frac{\left(\sqrt{29}-5\right)\left(\frac{2n_0^2t^2\log(y)}{n_2} - 2n_0x + 3n_1\right)^2}{48n_0^2t}$$
$$- \frac{n_0^2t^3\log^2(y)}{3n_2^2} + \frac{n_1t\log(y)}{n_2} + \frac{n_6}{\sqrt[3]{t}} - \alpha y. \tag{75}$$

TABLE 3. Exact solutions of the (2+1)-dimensional associated Hirota bilinear equation (6) by taking various subalgebras

Vector fields	Similarity variable	Exact solutions
$\mathtt{V}_1$	$X = xf_1(t)^{-\frac{1}{3}},\ Y = y$	$u(x,y,t) = -\alpha y + \frac{2}{b_2\sqrt[3]{f_1(t)}e^{\int \frac{1}{h_1(y)}\,dy} + x} - \frac{x^2 f_1'(t)}{18 f_1(t)}.$
$\mathtt{V}_2$	$X = x, T = t$	No solution
$\mathtt{V}_3$	$Y = y, T = t$	$u(x,y,t) = h_6(t) - \alpha y - \frac{x^2 f_3'(t)}{6 f_3(t)} + \frac{1}{f_3(t)}\int h_5(y)dy.$
$\mathtt{V}_2 + \mathtt{V}_4$	$X = x, T = t$	$u(x,y,t) = -\alpha y + h_8(t) + x\,h_9(t) + \frac{x^2 f_4'(t)}{6 f_4(t)} + f_4(t)\left(\int \frac{1}{f_2(y)}\,dy\right).$
$\mathtt{V}_3 + \mathtt{V}_4$	$Y = y,\ T = t$	$u(x,y,t) = -\alpha y + g_2(t) + \frac{\int g_1(y)\,dy}{f_3(t)} + \frac{3x f_4(t) - \frac{1}{2}x^2 f_3'(t)}{3 f_3(t)}.$
$\mathtt{V}_1 + \mathtt{V}_3 + \mathtt{V}_4$	$X = (x+a)f_1(t)^{-\frac{1}{3}},\ Y = y$	$u(x,y,t) = -\alpha y - \frac{(a+x)^2 f_1'(t)}{18 f_1(t)} + \frac{\gamma_2(a+x)}{f_1(t)^{2/3}} + \frac{\gamma_1 - (\gamma_2\gamma_3 + \gamma_4)\left(\int \frac{1}{h_6(y)}\,dy\right)}{\sqrt[3]{f_1(t)}}.$
$\mathtt{V}_2 + \mathtt{V}_3 + \mathtt{V}_4$	$X = x - \frac{n_0 t^2 \log(y)}{n_2},\ T = t$	$u(x,y,t) = \frac{n_4 x}{t^4} + \frac{n_1\left(\frac{x}{t} - \frac{3n_4}{2t^4}\right) + \frac{1}{6}}{n_0} - \frac{n_0 n_4 \log(y)}{n_2 t^2} + \frac{n_3}{t^2} + \frac{3n_1^2}{n_0^2 t} - \frac{x^2}{3t} - \alpha y$ when $\varphi_1 \neq 0$ and rest all $\varphi_i's = 0$.
$\mathtt{V}_2 + \mathtt{V}_3 + \mathtt{V}_4$	$X = x - \frac{n_0 t^2 \log(y)}{n_2},\ T = t$	$u(x,y,t) = \frac{2n_0 t \log(y)\left(n_0 t^2 \log(y) - n_2 x\right)}{3n_2^2} + \frac{(\sqrt{29}-5)\left(\frac{2n_0^2 t^2 \log(y)}{n_2} - 2n_0 x + 3n_1\right)^2}{48 n_0^2 t} - \frac{n_0^2 t^3 \log^2(y)}{3n_2^2} + \frac{n_1 t \log(y)}{n_2} + \frac{n_6}{\sqrt[3]{t}} - \alpha y$ when $\varphi_2 \neq 0$ and rest all $\varphi_i's = 0$.

5. **Physical Interpretation of obtained solutions.** This section holds significance as it discusses the physical characteristics of the presented graphs depicting the reflected wave equations. Graphs serve as visual representations of explicit solutions and are commonly utilized for comparative analysis. Some selected solutions obtained from equations (35), (40), (50), (54), and (58) are subjected to graphical analysis via numerical simulation, yielding various shapes and patterns.

Figure 1 represents the solitary wave structure for solution (35). These profiles are plotted for functions $h_1(y) = y, f_1(t) = \frac{\sin t}{t}$ and parameters $\alpha = 1,\ b_2 = 5$. Plot 1 shows the curve-shaped multi soliton behavior at $y = 1$ within the interval $-20 \leq x \leq 20,\ -2 \leq t \leq 2$, while plot 2 represent the elastic multi solitons for $x = 1$ within the interval $-2 \leq t \leq 2,\ -4 \leq y \leq 4$. Plot 3 demonstrates the curve-shaped multi soliton plotted at $t = 1$ within the interval $-20 \leq x \leq 20,\ -12 \leq y \leq 12$. It was observed that the amplitude of the wave decrease with the decrease in the value of y. The variation in wave amplitude with changing parameters highlights the inherent connection between the solution's free parameters and the wave's profile.

Figure 2 shows the periodic wave structure for solution (40). These profiles are plotted for the functions $h_1(y) = \sin y,\ f_1(t) = t + 1$ and parameters $\alpha = 0.2$, $c_2 = 30,\ c_3 = 200.5i,\ b_5 = 0.02$. Plot 1 shows the absolute plot at $x = 12$ within the interval $-20 \leq y \leq 1,\ -35 \leq t \leq 50$, while plot 2 represents the corresponding contour plot. The two-dimensional profile is plotted for $t = \{10, 20, 40\}$.

Figure 3 shows the periodic wave structure for solution (50). Three different profiles are plotted at the same values of space and time variables. These profiles are plotted for the functions $h_6(y) = y^3, h_7(t) = t, f_3(t) = \cos t$ and parameter $\alpha = 1$. Plot 1 shows the parabolic wave structure at $y = 5$ within the interval $-100 \leq x \leq 100,\ -1 \leq t \leq 1$, while plot 2 represents the periodic wave structure for $x = 5$ within the interval $-2.5 \leq t \leq 2.5,\ -100 \leq y \leq 100$. Plot 3 demonstrates the parabolic wave structure and plotted at $t = 5$ within the interval $-100 \leq x \leq 100$, $-1 \leq y \leq 1$. These parabolic wave profiles have many applications in various fields of sciences, such as in fluid dynamics, optics, acoustics, and radio wave propagation, etc. In fluid dynamics, these are especially relevant in the study of open-channel flow, such as rivers or canals.

Figure 4 represents the periodic solitary wave structure for solution (54). These profiles are plotted for the functions $h_8(t) = t+1, h_9(t) = 2t+1, f_4(t) = \cos t, f_2(y) = y^3$ and parameter $\alpha = 1$. Plot 1 demonstrates the periodic plot at $y = 12$ within the interval $-20 \leq t \leq 1$, $-20 \leq x \leq 20$, while plot 2 represents the corresponding contour plot. Two dimensional profile is plotted for $x = \{-3, 3, 19\}$. Changing the parameters of a wave will result in changes to its amplitude. This illustrates how the wave profile and the free parameters are inextricably linked in the obtained solutions.

Figure 5 demonstrates the periodic wave structure for solution (58). These profiles are plotted for the functions $g_1(y) = y, g_2(t) = \cos t, f_3(t) = e^{\sin t}, f_4(t) = t$ and parameter $\alpha = 1$. Plot 1 shows the periodic behavior at $x = 1$ within the interval $-10 \leq y, t \leq 10$, while plot 2 represents the corresponding contour plot. The two-dimensional profile is plotted at $y = \{6, 8, 10\}$. These periodic wave structures find diverse applications in various fields, including wireless communication, crystallography, acoustics, and more.

6. **Conservation laws.**

Conservation laws of an equation serve as mathematical expressions for understanding the fundamental physical principles. They are very useful for finding the stability and integrability of the systems governed by PDEs [17]. Also, the conserved quantity in conservation laws can be used to construct structure-preserving numerical schemes in the development of numerical methods. Ibragimov [8] proposed the well-known direct and systematic algorithm for constructing the conservation laws of nonlinearly self-adjoint equations symmetries. In this section, we will introduce certain symbols and theorems to get the conservation laws using the adjoint equation of the associated Hirota bilinear equation (6).

Theorem 6.1. *Any symmetry (Lie point, Lie-Backlund and non-local symmetry)*

$$\mathtt{V} = \xi_i(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)})\frac{\partial}{\partial x_i} + \eta_1(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)})\frac{\partial}{\partial u}, \tag{76}$$

of pth order nonlinear PDE

$$P(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}) = 0, \tag{77}$$

having independent variable $x = (x_1, x_2...x_n)$ and dependent variable u

$$u_{(1)} = u_i = D_i(u), \quad u_{(2)} = u_{ij} = D_iD_j(u), \ \ ... \ \ , u_{(p)} = u_{i_1i_2...i_p} = D_{i_1,i_2...i_p}(u),$$

where D_i is defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + ... \tag{78}$$

then u has an adjoint equation

$$P^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, ..., u_{(p)}, v_{(p)}) = 0, \tag{79}$$

with the adjoint operator P^* defined by

$$P^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, ..., u_{(p)}, v_{(p)}) = \frac{\delta \mathtt{L}}{\delta u}, \tag{80}$$

where $\mathtt{L}$ represents the formal Lagrangian provided by

$$\mathtt{L} = v(x)P(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}), \tag{81}$$

and v is the adjoint symmetry [18] of equation (77).

The conserved vector [8] *is*

$$\begin{aligned}\mathtt{V^i} = \xi_i \mathtt{L} + \mathtt{W}\left(\frac{\partial \mathtt{L}}{\partial u_i} - D_j\left(\frac{\partial \mathtt{L}}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial \mathtt{L}}{\partial u_{ijk}}\right) - ...\right) \\ + D_j(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathtt{L}}{\partial u_{ijk}}\right) + ...\right) \\ + D_j D_k(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{ijk}} - ...\right),\end{aligned} \tag{82}$$

with

$$\mathtt{W} = \eta_1 - \xi_i u_j, \quad \mathtt{L} = v(x)P(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}). \tag{83}$$

A conserved vector follows the conservation equation

$$D_t(\mathtt{V^t}) + D_x(\mathtt{V^x}) + D_y(\mathtt{V^y}) = 0, \tag{84}$$

where $\mathtt{V^t}, \mathtt{V^x}, \mathtt{V^y}$ *are functions of* $x, y, t,$ *and* u.

Conservation laws for equation (6). The Lagrangian equation for (6) is

$$\mathtt{L} = v(x, y, t)[u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} - 3\alpha u_{xx}]. \tag{85}$$

From (85), we get

$$\begin{aligned}&\frac{\partial \mathtt{L}}{\partial u} = 0, \quad \frac{\partial \mathtt{L}}{\partial u_x} = -3vu_{xy}, \quad \frac{\partial \mathtt{L}}{\partial u_y} = -3vu_{xx}, \\ &\frac{\partial \mathtt{L}}{\partial u_{xx}} = -3\alpha v - 3vu_y, \quad \frac{\partial \mathtt{L}}{\partial u_{xy}} = -3vu_x, \quad \frac{\partial \mathtt{L}}{\partial u_{ty}} = v, \quad \frac{\partial \mathtt{L}}{\partial u_{xxxy}} = -v.\end{aligned} \tag{86}$$

For equation (6), the adjoint equation takes the form

$$P^* = \frac{\delta \mathtt{L}}{\delta u} = 0, \tag{87}$$

where

$$\frac{\delta \mathtt{L}}{\delta u} = \frac{\partial \mathtt{L}}{\partial u} + \sum_{p=1}^{\infty}(-1)^p D_{i_1,i_2,....,i_p}\frac{\partial \mathtt{L}}{\partial u_{i_1,i_2,....,i_p}}. \tag{88}$$

Using equations (87) and (88), we get the adjoint equation as follows:

$$\begin{aligned}P^* &= \frac{\delta \mathtt{L}}{\delta u} \\ &= 3v_x u_{xy} + 3vu_{xxy} + 3v_y u_{xx} + 3vu_{xxy} \\ &\quad - 3\alpha v_{xx} - 3v_{xx}u_y - 6v_x u_{xy} - 3vu_{xxy} \\ &\quad + v_{ty} - 3v_{xy}u_x - 3v_x u_{xy} - 3vu_{xxy} - 3v_y u_{xx} - v_{xxxy} = 0.\end{aligned} \tag{89}$$

By substituting $v = u$ in equation (89), we get

$$u_{ty} - 3\alpha u_{xx} - 6u_x u_{xy} - 3u_x u_{xy} - 3u_y u_{xx} - u_{xxxy} = 0. \tag{90}$$

Clearly, the given equation is not self-adjoint. The conserved vectors are obtained as in equation (82):

$$\mathtt{V^t} = \xi_t \mathtt{L} + \mathtt{W}\left(\frac{\partial \mathtt{L}}{\partial u_t} - D_x\left(\frac{\partial \mathtt{L}}{\partial u_{tx}}\right) - D_y\left(\frac{\partial \mathtt{L}}{\partial u_{ty}}\right)\right) + D_y(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{ty}}\right), \tag{91}$$

$$\mathtt{V^x} = \xi_x \mathtt{L} + \mathtt{W}\left(\frac{\partial \mathtt{L}}{\partial u_x} - D_x\left(\frac{\partial \mathtt{L}}{\partial u_{xx}}\right) - D_y\left(\frac{\partial \mathtt{L}}{\partial u_{xy}}\right) - D_{xxy}\left(\frac{\partial \mathtt{L}}{\partial u_{xxxy}}\right)\right)$$

$$+ D_x(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{xx}} - D_{xy}\left(\frac{\partial \mathtt{L}}{\partial u_{xxxy}}\right)\right) + D_y(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{xy}}\right)$$
$$+ D_{xxy}(\mathtt{W})\left(\frac{\partial \mathtt{L}}{\partial u_{xxxy}}\right), \tag{92}$$

$$\mathtt{V}^{\mathtt{y}} = \xi_y \mathtt{L} + \mathtt{W}\left(\frac{\partial \mathtt{L}}{\partial u_y}\right). \tag{93}$$

Using the values of equation (86) in equations (91), (92), and (93), we get

$$\mathtt{V}^{\mathtt{t}} = \xi_3 \mathtt{L} - v_y \mathtt{W} + \mathtt{W}_y v, \tag{94}$$

$$\mathtt{V}^{\mathtt{x}} = \xi_1 \mathtt{L} + \mathtt{W}\left(-3vu_{xy} - 3\alpha v_x - 3vu_{xy} - 3v_x u_y + v_{xxy}\right) + \mathtt{W}_x\left(-3\alpha v - 3vu_y + v_{xy}\right)$$
$$+ \mathtt{W}_y\left(-3vu_x\right) + \mathtt{W}_{xxy}\left(-v\right), \tag{95}$$

$$\mathtt{V}^{\mathtt{y}} = \xi_2 \mathtt{L} - 3vu_{xx}\mathtt{W}, \tag{96}$$

where

$$\mathtt{W} = \eta_1 - \xi_1 u_x - \xi_2 u_y - \xi_3 u_t. \tag{97}$$

Now, for the vector field $\mathtt{V}_1 = \frac{x}{3} f_1'(t)\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial t} - \frac{x^2}{18} f_1''(t)\frac{\partial}{\partial u} - \frac{1}{3}(\alpha y + u) f_1'(t)\frac{\partial}{\partial u}$, we have

$$\mathtt{W} = -\frac{x^2}{18} f_1''(t) - \frac{1}{3}(\alpha y + u) f_1'(t) - \frac{x}{3} f_1'(t) u_x - f_1(t) u_t, \tag{98}$$

$$\mathtt{V}^{\mathtt{t}} = v f_1(t)(u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} - 3\alpha u_{xx}) \tag{99}$$
$$+ v_y\left(\frac{x^2}{18} f_1''(t) + \frac{1}{3}(\alpha y + u) f_1'(t) + \frac{x}{3} f_1'(t) u_x + f_1(t) u_t\right) - \frac{1}{3}\alpha f_1'(t) v,$$

$$\mathtt{V}^{\mathtt{x}} = \frac{x}{3} v f_1'(t)(u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} - 3\alpha u_{xx}) \tag{100}$$
$$+ (3vu_{xy} + 3\alpha v_x + 3vu_{xy} + 3v_x u_y - v_{xxy})\left(\frac{x^2}{18} f_1''(t) + \frac{1}{3}(\alpha y + u) f_1'(t) + \frac{x}{3} f_1'(t) u_x + f_1(t) u_t\right)$$
$$+ (3\alpha v + 3vu_y - v_{xy})\left(\frac{x}{9} f_1''(t) + \frac{1}{3} f_1'(t) u_x + \frac{x}{3} f_1'(t) u_{xx}\right) + \alpha f_1'(t)\left(vu_x\right) + \frac{x}{3}\alpha f_1'(t) u_{xxy}\left(v\right),$$

$$\mathtt{V}^{\mathtt{y}} = v f_2(y)(u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} - 3\alpha u_{xx}) \tag{101}$$
$$+ 3vu_{xx}\left(\frac{x^2}{18} f_1''(t) + \frac{1}{3}(\alpha y + u) f_1'(t) + \frac{x}{3} f_1'(t) u_x + f_1(t) u_t\right).$$

Equations (99), (100), and (101) contain an arbitrary adjoint variable $v(x, y, t)$, and therefore produce an infinite number of conservation laws. Additionally, the conservation laws for vector fields $\mathtt{V}_2, \mathtt{V}_3$, and $\mathtt{V}_4$ can be derived in the same way.

7. **Modulation instability.** It is well known that modulational instability is a fundamental and universal process. Various nonlinear phenomena display instability, resulting in the research of steady-state modulation. This phenomenon arises due to the interplay between nonlinear and dispersive effects in the time domain or as a result of diffraction in the spatial domain. In this section of the article, we delve into the modulation instability (MI) of equation (6) through the application of standard linear stability analysis [27]. We begin by assuming the steady-state solution of (6) as

$$u(x, y, t) = (q + \mu\phi(x, y, t))e^{i\psi t}. \tag{102}$$

Here, q and ψ represent arbitrary constants. The evolution of the perturbation $\phi(x,y,t)$ is investigated by the concept of linear stability analysis. Inserting equation (102) into (6) and linearizing, we get

$$\mu\phi_{ty} - \mu\phi_{xxxy} - 3\alpha\mu\phi_{xx} + i\mu\psi\phi_y = 0. \tag{103}$$

Assume the solution of (103) as

$$\phi(x,y,t) = \delta_1 e^{i(k_1x+k_2y-k_3t)} + \delta_2 e^{-i(k_1x+k_2y-k_3t)}, \tag{104}$$

where k_3 is the frequency of perturbation, k_1, k_2 are the normalized wave numbers, and δ_1 and δ_2 are the coefficients of the linear combination. Putting equation (104) into (103), we obtain the following set of two homogeneous equations:

$$\begin{aligned} -\delta_1 k_2\psi\mu + \delta_1 k_2 k_3\mu - \delta_1\mu k_1^3 k_2 + 3\alpha\delta_1\mu k_1^2 &= 0,\\ \delta_2 k_2\psi\mu + \delta_2 k_2 k_3\mu - \delta_2\mu k_1^3 k_2 + 3\alpha\mu\delta_2 k_1^2 &= 0. \end{aligned} \tag{105}$$

By evaluating the determinant, we get the following relation:

$$9\alpha^2k_1^4\mu^2 - 6\alpha k_2k_1^5\mu^2 + 6\alpha k_2k_3k_1^2\mu^2 - k_2^2\mu^2\psi^2 + k_2^2k_1^6\mu^2 - 2k_2^2k_3k_1^3\mu^2 + k_2^2k_3^2\mu^2 = 0. \tag{106}$$

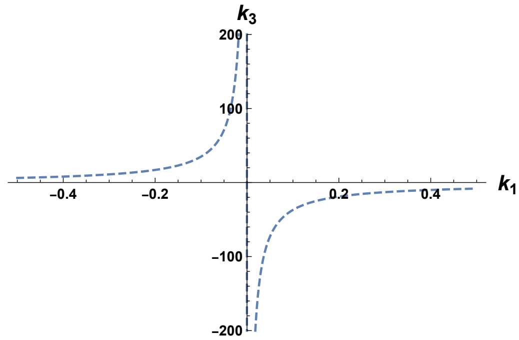
The dispersion relation determines how spatial oscillations e^{k_1x}, e^{k_2y} are linked to time oscillations e^{k_3t} of a wave number. Therefore, on solving equation (106), we acquire the following dispersive relation

$$k_3 = \frac{-3\alpha k_1^2 \mp k_2\psi + k_2k_1^3}{k_2}. \tag{107}$$

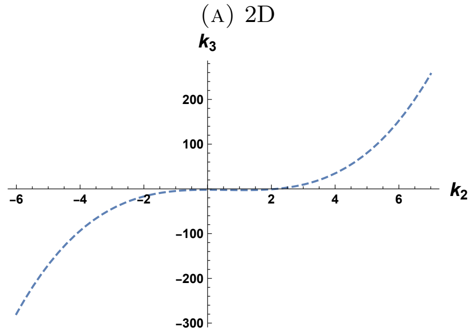
Equation (107) exhibits that the steady-state stability depends on the group velocity, wave numbers and self-phase modulation. Since, for $k_2 \neq 0$, the frequency of perturbation, i.e. k_3, is real for all the values of k_1, this illustrates the stability of the steady state to minor disturbances. This steady state appears to exhibit stability against small perturbations if the wavenumber k_3 possesses a real component.

8. **Discussion and comparison with the previous results.** This section involves a concise comparison of our derived closed-form solutions with prior work by Wang [25], leading to the following conclusions:

1. Wang ([25]) introduced lump solutions for equation (6) using the binary Bell polynomial and Hirota bilinear approaches. Equation (6) has also shown integrability through the Lax pair concept, with bilinear Bäcklund transformations derived from the binary Bell polynomial theory.
2. In this study, we employed the Lie symmetry technique to acquire all solutions of equation (6). However, we applied a systematic approach of Lie symmetry analysis to derive closed-form exact solutions for equation (6), followed by the optimality.
3. All the solutions obtained are completely novel and distinct, not previously reported in existing literature. Furthermore, our exact explicit solutions offer a more comprehensive analysis, incorporating arbitrary functional parameters and other constant parameters.
4. Due to the inclusion of independent arbitrary functional parameters in certain solutions (which allow for the freedom to choose the function), they have become notably valuable and relevant for elucidating nonlinear wave propagation across various domains. This encompasses fields such as nonlinear



(A) 2D



(B) 2D

FIGURE 6. The dispersion relation $k_3 = k_3(k_1, k_2)$ between frequency k_3 and wave number k_1 and k_2 of perturbation.

optics and plasma physics, which are concerned with phenomena related to electromagnetic waves.

5. Moreover, this study stands as the pioneering endeavor in applying conservation laws on equation (6). Additionally, the modulation stability of the selected model is evaluated and visually depicted in Figure 6.

9. **Conclusion.** In this study, we conducted a comprehensive analysis, combining analytical techniques with numerical simulations, to uncover a range of solutions for the associated Hirota bilinear equation (6) by utilizing the Lie symmetry analysis method. The primary objective of this technique is to identify symmetries, reduce the dimensionality of the equation, and discover exact solutions for the given NLEEs. To achieve this, we initiated our analysis by computing a one-dimensional optimal system. Subsequently, we utilized the obtained subalgebras to perform symmetry reductions on equation (6). As a result, we derived a set of reduced ODEs whose solutions provide exact solutions for the governing equation.

The obtained results manifest in various mathematical forms, including polynomials, trigonometric functions, and exponentials, incorporating arbitrary constants and functions of the independent variables, denoted as x, y, and t. It is worth noting that all these solutions satisfy the principal equation and have not been reported previously. To enhance our understanding of these solutions, we visualize them through 3D, 2D, and contour plots (see Figures 1, 2, 3, 4, 5), providing insights into the dynamic behavior of the equation. Our findings reveal diverse types of wave propagation, encompassing traveling waves, solitary waves, and periodic waves. Additionally, we determined the modulation instability condition through linear stability analysis.

Furthermore, we deduced conservation laws for equation (6), encompassing both local and nonlocal conserved vectors. Notably, this work marks the first exploration of invariant solutions using Lie symmetry, conservation laws, and modulation instability for the associated Hirota bilinear equation. The results underscore the precision of our approach in solving NLEEs and its potential utility in elucidating various qualitative aspects of wave phenomena across the domains of mathematical physics, applied mathematics, and engineering sciences.

Conflict of Interest. The authors affirm that they do not have any known conflicts of interest.

Data availability statement. Data sharing is not applicable to this article since no datasets were created or analyzed in the course of this study.

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