Chinese Journal of Physics 69 (2021) 1–23

Contents lists available at ScienceDirect

Chinese Journal of Physics

journal homepage: www.sciencedirect.com/journal/chinese-journal-of-physics

Lie symmetries, optimal system and group-invariant solutions of the (3+1)-dimensional generalized KP equation

Sachin Kumar [*,a], Wen-Xiu Ma [b,c,d,e], Amit Kumar [f]

[a] Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India
[b] Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
[c] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
[d] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[e] School of Mathematics, South China University of Technology, Guangzhou 510640, China
[f] Department of Mathematics, Sri Vankateswara College, University of Delhi, Delhi 110021, India

ARTICLE INFO

Keywords:
Group-invariant solutions
Optimal system
Solitons
(3+1)-dimensional generalized KP equation
Lie symmetry method

ABSTRACT

By applying the Lie symmetry method, abundant group-invariant solutions are constructed for a (3+1)-dimensional generalized Kadomtsev-Petviashvili (gKP) equation, which provides a water wave model for long waves of small amplitude with weakly non-linear restoring forces and frequency dispersion. Infinitesimal generators of symmetries, the commutation table of the symmetry Lie algebra, and an optimal system of one-dimensional Lie symmetry sub-algebras are presented. The governing gKP equation is reduced into several nonlinear ordinary differential equations utilizing thrice symmetry reductions. Consequently, abundant group invariant solutions are obtained in the shapes of different dynamical wave structures of solitons, multi-solitons, W-shaped solitons, doubly solitons, kink-type solitons, lump-type solitons interaction between parabolic waves and lump solitons, and annihilation multi-solitons profiles. The physical interpretation of the resulting soliton solutions is illustrated by three dimensional graphical through numerical simulation. The obtained group-invariant solutions involve many arbitrary functions, thereby exhibiting rich physical structures and including the existing solutions in the literature.

1. Introduction

Nonlinear partial differential equations (PDEs) appearing in various physical fields like plasma physics, oceanography, condensed matter physics, fiber optics, fluid dynamics and marine engineering demonstrate a rich range of nonlinear complex phenomena. It is a well-known fact that it is one of the hottest topics in mathematical and physical sciences to seek exact analytic solutions of nonlinear PDEs. In the study on nonlinear PDEs, it has increasing importance to understand the dynamical behavior of solutions through qualitative and quantitative characteristics of nonlinear equations. To further understand nonlinear complex phenomena, with the help of symbolic computation, numerous powerful methods are introduced to investigation of nonlinear PDEs, including the inverse scattering transform [1], Bäcklund and Darboux transformations [2], the Hirota bilinear method [3], homotopy analysis method [4,5], the generalized bilinear method [6,7] and the Lie symmetry method [8].

Lie symmetry analysis is among the most systematic methods to obtain exact closed-form solutions to nonlinear PDEs of

* Corresponding author.
E-mail addresses: sachinambariya@gmail.com (S. Kumar), mawx@cas.usf.edu (W.-X. Ma).

https://doi.org/10.1016/j.cjph.2020.11.013
Received 21 July 2020; Received in revised form 7 November 2020; Accepted 16 November 2020
Available online 27 November 2020

mathematical physics. Over the last few decades, the Lie symmetry method has been applied to many examples in mathematical physics and engineering sciences (see, e.g., [9–18]). Once Lie group analysis is performed for nonlinear equations, symmetry reductions and group-invariant solutions can be obtained. The resulting symmetries can be often used to reduce the number of independent variables of a PDE as well as the order of an ordinary differential equation (ODE). Usually, researchers focus on lower-dimensional models, particularly soliton equations in (1+1)-dimensions and (2+1)-dimensions. There are few studies on higher-dimensional equations due to their complexity. We would like to think one (3+1)-dimensional generalized KP equation in soliton theory and make connections with reduced lower-dimensional equations and present abundant group-invariant solutions.

In the present paper, an attempt will be made to obtain abundant group-invariant (exact closed-form) solutions to a (3+1)-dimensional generalized KP (gKP) equation in the following form,

$$u_{xxxy} + 3\left(u_x u_y\right)_x + u_{tx} + u_{ty} - u_{zz} = 0, \tag{1}$$

where $u = u(x, y, z, t)$ is the amplitude wave function. Eq. (1)appears in a wide range of physical applications. This four dimension gKP equation provides a water wave model for long waves of small amplitude with weakly non-linear restoring forces and frequency dispersion. It is important to exploit group-invariant solutions and compute symmetry reductions for the governing gKP equation. Recently, the (3+1)-dimensional gKP equation was studied and examined by distinct methods inclduing the Hirota bilinear method and the multiple exp-function method [19–21]. Multiple-soliton solutions, rational traveling wave solutions, Wronskian, and Grammian formulations, and one-, two-, and three-wave solutions were presented for this gKP equation, indeed [19–21]. Wazwaz [22] obtained the soliton solutions by applying the simplified Hereman-Nuseir method. Dubrovin et al. [23] studied the dispersive shock wave formation for a few generalized KP equations. Besides, Hashemi et al. [24] obtained the invariant and soliton solutions and also used PainlevȨ analysis to check the integrability of the generalized (KP-MEW) equation. Mohyud-Din et al. [25] constructed exact solutions to the gKP equation by the modified $\tan\left(-\frac{\phi}{2}\right)$ method. Lin et al. [26] obtained resonant multiple wave solutions by using the linear superposition principle [27] to the Hirota bilinear form of a new generalized (3+1)-dimensional KP equation. Qin et al. [28] obtained some exact solutions including breather, rogue, and solitary waves by utilizing a homoclinic breather limit approach. Another generalized (3+1)-dimensional KP equation was studied, and symmetry reductions and some invariant solutions were obtained by using the Lie group method [29]. Saleh and Rashed [30] obtained other similarity reductions and five exact solutions using a combination of the Lie symmetry method and the singular manifold method.

Motivated by those studies in the literature, we would like to construct infinitesimal generators, Lie point symmetries, commutator relations, an optimal system of one-dimensional symmetry sub-algebras, similarity reductions, and abundant group-invariant (exact analytic) solutions for the above (3+1)-dimensional gKP equation. Classical Lie group analysis will be made and used to determine group-invariant solutions, which include multi-solitons, W-shaped solitons, doubly solitons, kink-type solitons, and annihilation of different multi-soliton profiles. Most of the group-invariant solutions involve many arbitrary constants and functions. The arbitrary functions exhibit rich localized characteristics to understand complex physical phenomena. The resulting abundant group-invariant solutions are relevant and beneficial in the propagation of different wave structures of solitons, dynamics of solitons, plasma physics, fluid mechanics, fiber optics, nonlinear wave phenomena, mathematical physics, acoustics, and many other physical sciences.

The article is organized as follows: In Section 2, we obtain infinitesimal generators of symmetries, possible geometric vector fields, the invariance condition, and the commutator relations of the infinitesimal generators of the gKP equation. In Section 3, we obtain the adjoint representation relations and also construct an optimal system of one-dimensional symmetry sub-algebras to the (3+1)-dimensional gKP equation. In Section 4, we obtain abundant group-invariant solutions through twice symmetry reductions. We make analyses and discussions of the group-invariant solutions so obtained in Section 5. We summarize our results and make concluding observations in Section 6.

2. Lie symmetry analysis

The crucial step in the classical Lie symmetry method is to determine symmetry algebras of nonlinear PDEs. In this section, we will obtain to infinitesimal generators of symmetries and geometric vector fields for the gKP Eq. (1). Let us consider a one-parameter Lie group of symmetry transformations:

$$\begin{aligned} x^* &= x + \epsilon\,\xi(x, y, t, u) + O(\epsilon^2), \\ y^* &= y + \epsilon\,\eta(x, y, t, u) + O(\epsilon^2), \\ z^* &= z + \epsilon\,\psi(x, y, t, u) + O(\epsilon^2), \\ t^* &= t + \epsilon\,\tau(x, y, t, u) + O(\epsilon^2), \\ u^* &= u + \epsilon\,\phi(x, y, t, u) + O(\epsilon^2), \end{aligned} \tag{2}$$

with an infinitesimal generator

$$V = \xi\,\partial_x + \eta\,\partial_y + \psi\,\partial_z + \tau\,\partial_t + \phi\,\partial_u. \tag{3}$$

The fourth prolongation of V for the gKP Eq. (1) is given by

$$\begin{aligned}\mathrm{Pr}^{(4)}V &= V+\phi\frac{\partial}{\partial u}+\phi^{x}\frac{\partial}{\partial u_x}+\phi^{y}\frac{\partial}{\partial u_y}+\phi^{xy}\frac{\partial}{\partial u_{xy}}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}\\ &+\phi^{yt}\frac{\partial}{\partial u_{yt}}+\phi^{zz}\frac{\partial}{\partial u_{zz}}+\phi^{xxxy}\frac{\partial}{\partial u_{xxxy}},\end{aligned}$$

and so the invariance condition reads as

$$\phi^{xxxy}+3u_x\phi^{xy}+3u_{xy}\phi^{x}+3u_y\phi^{xx}+3u_{xx}\phi^{y}+\phi^{xt}+\phi^{yt}-\phi^{zz}=0, \tag{4}$$

where

$$\begin{aligned}\phi^{x} &= \mathfrak{D}_x\phi-u_x\mathfrak{D}_x\xi-u_y\mathfrak{D}_x\eta-u_z\mathfrak{D}_x\psi-u_t\mathfrak{D}_x\tau,\\ \phi^{y} &= \mathfrak{D}_y\phi-u_x\mathfrak{D}_y\xi-u_y\mathfrak{D}_y\eta-u_z\mathfrak{D}_y\psi-u_t\mathfrak{D}_y\tau,\\ \phi^{xx} &= \mathfrak{D}_x\phi^{x}-u_{xx}\mathfrak{D}_x\xi-u_{xy}\mathfrak{D}_x\eta-u_{xz}\mathfrak{D}_x\psi-u_{xt}\mathfrak{D}_x\tau,\\ \phi^{xy} &= \mathfrak{D}_y\phi^{x}-u_{xx}\mathfrak{D}_y\xi-u_{xy}\mathfrak{D}_y\eta-u_{xz}\mathfrak{D}_y\psi-u_{xt}\mathfrak{D}_y\tau,\\ \phi^{xt} &= \mathfrak{D}_t\phi^{x}-u_{xx}\mathfrak{D}_t\xi-u_{xy}\mathfrak{D}_t\eta-u_{xz}\mathfrak{D}_t\psi-u_{xt}\mathfrak{D}_t\tau,\\ \phi^{yt} &= \mathfrak{D}_t\phi^{y}-u_{xy}\mathfrak{D}_t\xi-u_{yy}\mathfrak{D}_t\eta-u_{yz}\mathfrak{D}_t\psi-u_{yt}\mathfrak{D}_t\tau,\\ \phi^{zz} &= \mathfrak{D}_z\phi^{z}-u_{zx}\mathfrak{D}_z\xi-u_{zy}\mathfrak{D}_z\eta-u_{zz}\mathfrak{D}_z\psi-u_{zt}\mathfrak{D}_z\tau,\\ \phi^{xxx} &= \mathfrak{D}_x\phi^{xx}-u_{xxx}\mathfrak{D}_x\xi-u_{xxy}\mathfrak{D}_x\eta-u_{xxz}\mathfrak{D}_x\psi-u_{xxt}\mathfrak{D}_x\tau\\ \phi^{xxxy} &= \mathfrak{D}_y\phi^{xxx}-u_{xxxx}\mathfrak{D}_y\xi-u_{xxxy}\mathfrak{D}_y\eta-u_{xxxz}\mathfrak{D}_y\psi-u_{xxxt}\mathfrak{D}_y\tau,\end{aligned} \tag{5}$$

and $\mathfrak{D}_x$, $\mathfrak{D}_y$, $\mathfrak{D}_z$ and $\mathfrak{D}_t$ denote the total differential operators with respect to x, y, z and t [8]. Thus, we have

$$\mathfrak{D}_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+u_{ij}\frac{\partial}{\partial u_j}+\ldots, i=1,2,3,4 \tag{6}$$

where $(x^1,x^2,x^3,x^4)=(x,y,z,t)$. We obtain the determining equations with the help of (4) and (5):

$$\begin{aligned}&\phi_u=-\frac{1}{2}\psi_z,\ \phi_x=\frac{1}{3}\xi_t,\ \phi_y=\frac{1}{3}\xi_t,\ \phi_{zz}=\frac{2}{3}\xi_{tt},\ \xi_u=0,\ \xi_x=\frac{1}{2}\psi_z,\ \xi_y=0,\ \xi_z=\frac{1}{2}\psi_t,\\ &\eta_u=0,\ \eta_x=0,\ \eta_y=\frac{1}{2}\psi_z,\ \eta_t=0,\ \eta_z=\frac{1}{2}\psi_t,\ \psi_u=0,\ \psi_x=0,\ \psi_y=0,\\ &\psi_{tt}=0,\ \psi_{tz}=0,\ \psi_{zz}=0,\ \tau_u=0,\ \tau_x=0,\ \tau_y=0,\ \tau_z=0,\ \tau_t=\frac{3}{2}\psi_z.\end{aligned} \tag{7}$$

The solution of those over-determining equations furnishes the following set of infinitesimals for the gKP (1):

$$\begin{aligned}\xi &= \frac{c_2}{2}x+\frac{c_1}{2}z+f_1(t),\quad \eta=\frac{c_2}{2}y+\frac{c_1}{2}z+c_3,\quad \psi=c_1t+c_2z+c_4,\\ \tau &= \frac{3c_2}{2}t+c_5,\quad \phi=-\frac{c_2}{2}u+\frac{z^2}{3}f_1''(t)+\frac{(x+y)}{3}f_1'(t)+zf_2(t)+f_3(t),\end{aligned} \tag{8}$$

Table 1
Commutator table.

	V_1	V_2	V_3	V_4	V_5	V_6	$V_7(f_2)$	$V_8(f_3)$
V_1	0	$-\frac{1}{2}V_1$	0	$-\frac{1}{2}(V_6+V_3)$	$-V_4$	0	$V_8(tf_2)$	0
V_2	$\frac{1}{2}V_1$	0	$-\frac{1}{2}V_3$	$-V_4$	$-\frac{3}{2}V_5$	$-\frac{1}{2}V_6$	$\frac{3}{2}V_7((tf_2)')$	$\frac{1}{2}V_8(3tf_3'+f_3)$
V_3	0	$\frac{1}{2}V_3$	0	0	0	0	0	0
V_4	$\frac{1}{2}(V_6+V_3)$	V_4	0	0	0	0	$V_8(f_2)$	0
V_5	V_4	$\frac{3}{2}V_5$	0	0	0	0	$V_7(f_2')$	$V_8(f_3')$
V_6	0	$\frac{1}{2}V_6$	0	0	0	0	0	0
$V_7(f_2)$	$-V_8(tf_2)$	$-\frac{3}{2}V_7((tf_2)')$	0	$-V_8(f_2)$	$-V_7(f_2')$	0	0	0
$V_8(f_3)$	0	$-\frac{1}{2}V_8(3tf_3'+f_3)$	0	0	$-V_8(f_3')$	0	0	0

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants, and $f_1(t)$, $f_2(t)$ and $f_3(t)$ are arbitrary functions. we suppose $f_1(t) = c_6$, then infinitesimals generators in Eq. (8) can be recast as

$$
\begin{aligned}
\xi &= \frac{c_2}{2}x + \frac{c_1}{2}z + c_6, \quad \eta = \frac{c_2}{2}y + \frac{c_1}{2}z + c_3, \quad \psi = c_1 t + c_2 z + c_4, \\
\tau &= \frac{3c_2}{2}t + c_5, \quad \phi = -\frac{c_2}{2}u + zf_2(t) + f_3(t),
\end{aligned} \tag{9}
$$

This way, we obtain eight infinitesimal generators corresponding to the five arbitrary constants and three arbitrary functions:

$$
\begin{aligned}
& V_1 = \frac{z}{2}\frac{\partial}{\partial x} + \frac{z}{2}\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}, \quad V_2 = \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{3}{2}t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}, \\
& V_3 = \frac{\partial}{\partial y}, \quad V_4 = \frac{\partial}{\partial z}, \quad V_5 = \frac{\partial}{\partial t}, \quad V_6 = \frac{\partial}{\partial x}, \\
& V_7(f_2) = zf_2(t)\frac{\partial}{\partial u}, \quad V_8(f_3) = f_3(t)\frac{\partial}{\partial u}.
\end{aligned} \tag{10}
$$

The commutator relations of the above infinitesimal generators for the gKP Eq. (1) are shown in Table 1, with the $(i;j)$th entry indicating the Lie bracket $[V_i, V_j] = V_iV_j - V_jV_i$. We observe that Table 1 is skew symmetric with zero diagonal elements. Also, the generators $V_1 \ldots V_8$ are linearly independent.

3. An optimal system of one-dimensional subalgebras

In this section, we obtain the optimal system of one dimensional Lie subalgebra. From the commutator relations between eight infinitesimal generators presented in Table 1, these infinitesimals generators given in Eq. (10) can be furnished as a linear combination of V_i as

$$
V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_5 V_5 + \alpha_6 V_6 + \alpha_7 V_7 + \alpha_8 V_8. \tag{11}
$$

Now, we obtain the adjoint representation relations as shown in Table 2. By using the Olver technique [8], the adjoint representation relations of the (3+1)-dimensional generalized KP equation, in Table 2 are computed via symbolic computation for the commutator relations of those vector fields.

3.1. Construction of invariants

To obtain the optimal system of Lie algebra $\mathbb{R}^8$, it is necessary to construct the invariant for the selection of representative elements. Then, from Table 1, we get the desired following matrix representations of $ad(V_i)$:

Table 2
Adjoint representation table.

Ad	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	V_1	$V_2 + \frac{\epsilon}{2}V_1$	V_3	$V_4 + \frac{\epsilon}{2}(V_3 + V_6)$	$V_5 + \epsilon V_4$	V_6	$V_7 - \epsilon V_8$	V_8
V_2	$e^{-\frac{\epsilon}{2}}V_1$	V_2	$e^{\frac{\epsilon}{2}}V_3$	$e^{\epsilon}V_4$	$e^{\frac{3\epsilon}{2}}V_5$	$e^{\frac{\epsilon}{2}}V_6$	$e^{-\frac{3\epsilon}{2}}V_7$	$e^{-\frac{\epsilon}{2}}V_8$
V_3	V_1	$V_2 - \frac{1}{2}\epsilon V_3$	V_3	V_4	V_5	V_6	V_7	V_8
V_4	$V_1 - \frac{\epsilon}{2}(V_3 + V_6)$	$V_2 - \epsilon V_4$	V_3	V_4	V_5	V_6	$V_7 - \epsilon V_8$	V_8
V_5	$V_1 - \epsilon V_4$	$V_2 - \frac{3\epsilon}{2}V_5$	V_3	V_4	V_5	V_6	$e^{-\epsilon}V_7$	$e^{-\epsilon}V_8$
V_6	V_1	$V_2 - \frac{\epsilon}{2}V_6$	V_3	V_4	V_5	V_6	V_7	V_8
V_7	$V_1 + \epsilon V_8$	$V_2 + \frac{3\epsilon}{2}V_7$	V_3	$V_4 + \epsilon V_8$	$V_5 + \epsilon V_7$	V_6	V_7	V_8
V_8	V_1	$V_2 + \frac{\epsilon}{2}V_8$	V_3	V_4	$V_5 + \epsilon V_8$	V_6	V_7	V_8

$$\begin{aligned} Ad(exp(\epsilon\mathbf{W}))(\mathbf{V}) &= e^{-\epsilon\mathbf{W}}\mathbf{V}e^{\epsilon\mathbf{W}} = \mathbf{V} - \epsilon[\mathbf{W},\mathbf{V}] + \frac{1}{2!}\epsilon^2[\mathbf{W},[\mathbf{W},\mathbf{V}]] - \ldots \\ &= (a_1V_1 + \ldots + a_nV_n) - \epsilon[b_1V_1 + \ldots + b_nV_n, a_1V_1 + \ldots + a_nV_n] + O(\epsilon^2) \\ &= (a_1V_1 + \ldots + a_nV_n) - \epsilon(\Theta_1V_1 + \ldots + \Theta_nV_n) \end{aligned} \tag{12}$$

where $\Theta = \Theta(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ are obtained with the help of from the commutator table. The commutator relations of the eight-dimensional Lie algebra are shown in Table 1. On substituting $\mathbf{V} = \sum_{i=1}^{8}\alpha_iV_i$ and $\mathbf{W} = \sum_{j=1}^{8}\beta_jV_j$ in Eq. (12)with

$$\begin{aligned} \Theta_1 &= -\frac{1}{2}\alpha_2\beta_1 + \frac{1}{2}\alpha_1\beta_2, \quad \Theta_2 = 0, \quad \Theta_3 = -\frac{1}{2}\alpha_4\beta_1 - \frac{1}{2}\alpha_3\beta_2 + \frac{1}{2}\alpha_2\beta_3 + \frac{1}{2}\alpha_1\beta_4, \\ \Theta_4 &= -\alpha_5\beta_1 - \alpha_4\beta_2 + \alpha_2\beta_4 + \alpha_1\beta_5, \quad \Theta_5 = -\frac{3}{2}\alpha_5\beta_2 + \frac{3}{2}\alpha_2\beta_5, \\ \Theta_6 &= -\frac{1}{2}\alpha_4\beta_1 - \frac{1}{2}\alpha_6\beta_2 + \frac{1}{2}\alpha_1\beta_4 + \frac{1}{2}\alpha_2\beta_6, \quad \Theta_7 = \frac{3}{2}\alpha_7\beta_2 + \alpha_7\beta_5 - \frac{3}{2}\alpha_2\beta_7 - \alpha_5\beta_7, \\ \Theta_8 &= \alpha_7\beta_1 + \frac{1}{2}\alpha_8\beta_2 + \alpha_7\beta_4 + \alpha_8\beta_5 - \alpha_1\beta_7 - \alpha_4\beta_7 - \frac{1}{2}\alpha_2\beta_8 - \alpha_5\beta_8, \end{aligned} \tag{13}$$

For any $\beta_j, 1 \le j \le 8$, it requires

$$\Theta_1\frac{\partial\phi}{\partial\alpha_1} + \Theta_2\frac{\partial\phi}{\partial\alpha_2} + \Theta_3\frac{\partial\phi}{\partial\alpha_3} + \Theta_4\frac{\partial\phi}{\partial\alpha_4} + \Theta_5\frac{\partial\phi}{\partial\alpha_5} + \Theta_6\frac{\partial\phi}{\partial\alpha_6} + \Theta_7\frac{\partial\phi}{\partial\alpha_7} + \Theta_8\frac{\partial\phi}{\partial\alpha_8} = 0. \tag{14}$$

On equating the coefficients of all same powers of β_j in above equation, then we get the desired eight differential equations about $\phi(\alpha_1, \alpha_2, \ldots, \alpha_8)$ as

$$\begin{aligned} \beta_1 &: -\frac{\alpha_2}{2}\frac{\partial\phi}{\partial\alpha_1} - \frac{\alpha_4}{2}\frac{\partial\phi}{\partial\alpha_3} - \alpha_5\frac{\partial\phi}{\partial\alpha_4} - \frac{\alpha_4}{2}\frac{\partial\phi}{\partial\alpha_6} + \alpha_7\frac{\partial\phi}{\partial\alpha_8} = 0, \\ \beta_2 &: \frac{\alpha_1}{2}\frac{\partial\phi}{\partial\alpha_1} - \frac{\alpha_3}{2}\frac{\partial\phi}{\partial\alpha_3} - \alpha_4\frac{\partial\phi}{\partial\alpha_4} - \frac{3\alpha_5}{2}\frac{\partial\phi}{\partial\alpha_5} - \frac{\alpha_6}{2}\frac{\partial\phi}{\partial\alpha_6} + \frac{3\alpha_7}{2}\frac{\partial\phi}{\partial\alpha_7} + \frac{\alpha_8}{2}\frac{\partial\phi}{\partial\alpha_8} = 0, \\ \beta_3 &: \frac{\alpha_2}{2}\frac{\partial\phi}{\partial\alpha_3} = 0, \quad \beta_4 : \frac{\alpha_1}{2}\frac{\partial\phi}{\partial\alpha_3} + \alpha_2\frac{\partial\phi}{\partial\alpha_4} + \frac{\alpha_1}{2}\frac{\partial\phi}{\partial\alpha_6} + \alpha_7\frac{\partial\phi}{\partial\alpha_8} = 0, \\ \beta_5 &: \alpha_1\frac{\partial\phi}{\partial\alpha_4} + \frac{3\alpha_2}{2}\frac{\partial\phi}{\partial\alpha_5} + \alpha_7\frac{\partial\phi}{\partial\alpha_7} + \alpha_8\frac{\partial\phi}{\partial\alpha_8} = 0, \quad \beta_6 : \frac{\alpha_2}{2}\frac{\partial\phi}{\partial\alpha_6} = 0, \\ \beta_7 &: \left(-\frac{3\alpha_2}{2} - \alpha_5\right)\frac{\partial\phi}{\partial\alpha_7} + (-\alpha_1 - \alpha_4)\frac{\partial\phi}{\partial\alpha_8} = 0, \quad \beta_8 : \left(-\frac{\alpha_2}{2} - \alpha_5\right)\frac{\partial\phi}{\partial\alpha_8} = 0, \end{aligned} \tag{15}$$

By solving the system of Eqs. (15), then we can obtain $\phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8) = F(\alpha_2)$ which is known as the general invariant function of Lie algebra $\mathbb{R}^8$, where F is an arbitrary function of α_2. Thus, the generalized KP equation has only one basic invariant.

3.2. Calculation of the adjoint transformation matrix

For $F_i^s : g \to g$ defined by $V \to Ad(exp(\epsilon_iV_i).V)$ is a linear map, for $i = 1, 2, \ldots, 8$. The matrix A_i^ϵ of $F_i^\epsilon, i = 1, 2, \ldots, 8$ with respect to basis $\{V_1, \ldots, V_8\}$ are given below and defined [31] as

$$A_1^\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon_1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_1}{2} & 1 & 0 & \frac{\epsilon_1}{2} & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\epsilon_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2^\epsilon = \begin{pmatrix} e^{-\frac{\epsilon_2}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{\epsilon_2}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\epsilon_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{3\epsilon_2}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_2}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3\epsilon_2}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\epsilon_2}{2}} \end{pmatrix},$$

$$A_3^\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\epsilon_3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_4^\epsilon = \begin{pmatrix} 1 & 0 & 0 & -\frac{\epsilon_4}{2} & 0 & -\frac{\epsilon_4}{2} & 0 & 0 \\ 0 & 1 & 0 & -\epsilon_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\epsilon_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly, we can find the other matrices $A_5^\epsilon,\ A_6^\epsilon,\ A_7^\epsilon$, and A_8^ϵ.

Therefore, global adjoint matrix can be obtained using these eight matrices as

$$A = \begin{pmatrix} e^{-\frac{\epsilon_2}{2}} & 0 & 0 & A_{14} & 0 & -\frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_4 & 0 & A_{18} \\ \frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_1 & 1 & -\frac{\epsilon_3}{2} & A_{24} & -\frac{3}{2}\epsilon_5 & A_{26} & A_{27} & A_{28} \\ 0 & 0 & e^{\frac{\epsilon_2}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}e^{\frac{\epsilon_2}{2}}\epsilon_1 & e^{\epsilon_2} & 0 & e^{\frac{3\epsilon_2}{2}}\epsilon_7 & 0 & e^{\epsilon_2}\epsilon_7 \\ 0 & 0 & 0 & e^{\epsilon_2}\epsilon_1 & e^{\frac{3\epsilon_2}{2}} & 0 & e^{\frac{3\epsilon_2}{2}}\epsilon_7 & A_{58} \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_2}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3\epsilon_2}{2}-\epsilon_5} & A_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\epsilon_2}{2}-\epsilon_5} \end{pmatrix} \tag{16}$$

where

$$\begin{aligned} A_{14} &= -\frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_4 - e^{-\frac{\epsilon_2}{2}}\epsilon_5, \quad A_{18} = e^{-\frac{\epsilon_2}{2}}\epsilon_7 + \left(-\frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_4 - e^{-\frac{\epsilon_2}{2}}\epsilon_5\right)\epsilon_7, \\ A_{24} &= -\frac{1}{4}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_4 - \epsilon_4 - \frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_5, \quad A_{26} = -\frac{1}{4}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_4 - \frac{\epsilon_6}{2}, \quad A_{27} = \frac{3\epsilon_7}{2} - \frac{3\epsilon_5\epsilon_7}{2} \\ A_{28} &= \frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_7 - \left(\frac{1}{4}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_4 + \epsilon_4 + \frac{1}{2}e^{-\frac{\epsilon_2}{2}}\epsilon_1\epsilon_5\right)\epsilon_7 + \frac{\epsilon_8}{2} - \frac{3}{2}\epsilon_5\epsilon_8, \\ A_{58} &= e^{\epsilon_2}\epsilon_1\epsilon_7 + e^{\frac{3\epsilon_2}{2}}\epsilon_8, \quad A_{78} = -e^{-\epsilon_5}\left(e^{-\frac{\epsilon_2}{2}}\epsilon_1 + e^{-\frac{1}{2}(3\epsilon_2)}\epsilon_4\right). \end{aligned}$$

3.3. One-dimensional optimal system for the gKP equation

The adjoint transformation equation to the generalized KP equation is

$$(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\gamma_8) = (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8)\cdot A \tag{17}$$

where A is global matrix which is expressed above.

$$\begin{aligned} \gamma_1 &= \alpha_1 + \frac{\alpha_2\epsilon_1}{2}, \quad \gamma_2 = \alpha_2, \quad \gamma_3 = \alpha_3 - \frac{\alpha_2\epsilon_3}{2} + \frac{\alpha_4\epsilon_1}{2}, \\ \gamma_4 &= \alpha_4 + \alpha_5\epsilon_1 - \frac{\alpha_1\epsilon_4}{2} + \alpha_2\left(-\frac{\epsilon_1\epsilon_4}{4} - \epsilon_4\right), \quad \gamma_5 = \alpha_5, \\ \gamma_6 &= \alpha_6 + \frac{\alpha_4\epsilon_1}{2} - \frac{\alpha_1\epsilon_4}{2} + \alpha_2\left(-\frac{\epsilon_1\epsilon_4}{4} - \frac{\epsilon_6}{2}\right), \quad \gamma_7 = \alpha_7 + \alpha_5\epsilon_7 + \frac{3\alpha_2\epsilon_7}{2}, \\ \gamma_8 &= \alpha_1\left(\epsilon_7 - \frac{\epsilon_4\epsilon_7}{2}\right) + \alpha_2\left(\epsilon_7\left(-\frac{\epsilon_1\epsilon_4}{4} - \epsilon_4\right) + \frac{\epsilon_1\epsilon_7}{2} + \epsilon_8\right) + \alpha_4\epsilon_7 + \alpha_5(\epsilon_1\epsilon_7 + \epsilon_8) \\ &\quad + \alpha_7(-\epsilon_1 - \epsilon_4) + \alpha_8, \end{aligned} \tag{18}$$

which must have solutions for ϵ_i's for $i = 1, 2, \ldots, 8$ (assuming $\epsilon_2 = \epsilon_5 = 0$).

Case 1: For $\alpha_1 = 1$ representative element $\tilde{V} = V_1$. By substituting $\gamma_1 = 1$ into Eq. (18) we get

$$\epsilon_1 \to 0,\ \epsilon_7 = -\alpha_8 \tag{19}$$

Case 2: For $\alpha_1 = \alpha_5 = 1$ representative element $\tilde{V} = V_1 + V_5$. By substituting $\gamma_1 = \gamma_5 = 1$ into Eq. (18) we get

$$\epsilon_1 \to \alpha_6,\ \epsilon_8 \to 3\alpha_6\alpha_7 + \alpha_7 - \alpha_8,\ \epsilon_7 = -\alpha_7,\ \epsilon_4 = 2\alpha_6 \tag{20}$$

Case 3: Select a representative element $\tilde{V} = V_1 + V_2 + V_5$.

Substituting $\gamma_1 = \gamma_2 = \gamma_5 = 1, \gamma_i = 0, 2 \le i \le 8$ and $\alpha_1 = 1$ into Eq. (18), we obtain the solution

$$\epsilon_1 \to 0,\ \epsilon_3 \to 2\alpha_3,\ \epsilon_4 \to \frac{2\alpha_4}{3},\ \epsilon_6 \to -\frac{2}{3}(\alpha_4 - 3\alpha_6),\ \epsilon_7 \to -\frac{2}{5}\alpha_7,\ \epsilon_8 \to \frac{1}{30}(10\alpha_4\alpha_7 + 6\alpha_7 - 15\alpha_8) \tag{21}$$

It means, all the $V_1 + \alpha_3 V_3 + \ldots + \alpha_8 V_8$ are equivalent $V_1 + V_2 + V_5$.

Case 4: $\alpha_2 = 1$

Select a representative element $\tilde{V} = V_2$. Substituting $\gamma_1 = 0, \gamma_2 = 1, \gamma_i = 0, 3 \le i \le 8$ and $\alpha_2 = 1$ into Eq. (18), we obtain the solution

$$\begin{aligned} &\epsilon_1 \to -2\alpha_1,\ \epsilon_3 \to 2(\alpha_3 - \alpha_1\alpha_4),\ \epsilon_4 \to \alpha_4,\ \epsilon_6 \to -2(\alpha_1\alpha_4 - \alpha_6),\ \epsilon_7 \to -\frac{2}{3}\alpha_7, \\ &\epsilon_8 \to -2\alpha_1\alpha_7 + \alpha_4\alpha_7 - \alpha_8 \end{aligned} \tag{22}$$

It means, all the $\alpha_1 V_1 + V_2 + \alpha_3 V_3 + \ldots + \alpha_8 V_8$ are equivalent V_2g.

Similarly, we can find the value of $\epsilon_i's$ for other members of optimal system.

In summary, one-dimensional optimal system of sub-algebras for the governing equation is obtained as follows

$$\begin{array}{llll} (i) & \chi_1 = V_1, & (ii) & \chi_2 = V_2, \\ (iii) & \chi_3 = V_3 + V_4, & (iv) & \chi_4 = V_4 + V_5, \\ (v) & \chi_5 = V_3 + V_5, & (vi) & \chi_6 = V_5 + V_8, \\ (vii) & \chi_7 = V_3 + V_4 + V_5 + V_7 + V_8, & (viii) & \chi_8 = V_3 + V_4 + V_5 + V_6 + V_7 + V_8, \\ (ix) & \chi_9 = V_1 + V_5, & (x) & \chi_{10} = V_2 + V_5, \\ (xi) & \chi_{11} = V_2 + V_5 + V_7, & (xii) & \chi_{12} = V_1 + V_2 + V_5, \\ (xiii) & \chi_{13} = V_3 + V_4 + V_5 + V_8. & & \end{array} \tag{23}$$

4. Group-invariant solutions

In this section, we will construct various symmetry reductions and the corresponding group-invariant solutions of the (3+1)-dimensional gKP equation (1) using the above optimal system. Similarity variables and similarity solutions associated with any vector field $V = \xi\,\partial_x + \eta\,\partial_y + \psi\,\partial_z + \tau\,\partial_t + \phi\,\partial_u$ in (23) can be accomplished by its Lagrange's system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\psi} = \frac{dt}{\tau} = \frac{du}{\phi}. \tag{24}$$

4.1. Symmetry reduction with $\chi_1 = V_1$

The Eq. (24) becomes

$$\frac{dx}{\frac{z}{2}} = \frac{dy}{\frac{z}{2}} = \frac{dz}{t} = \frac{dt}{0} = \frac{du}{0}. \tag{25}$$

Eq. (25) gives the following similarity solution

$$u(x, y, z, t) = \mathbf{U}(\mathcal{X}, \mathcal{Y}, \mathcal{T}) \quad \text{with } \mathcal{X} = x - \frac{z^2}{4t},\ \mathcal{Y} = y - \frac{z^2}{4t},\ \text{and } \mathcal{T} = t. \tag{26}$$

Substituting the similarity solution u from (26) into (1), we obtain a new reduction equation with the independent variables $\mathcal{X}, \mathcal{Y}$, and $\mathcal{T}$ as follows

$$\mathbf{U}_{\mathcal{X}} + \mathbf{U}_{\mathcal{Y}} + 2\mathcal{T}(\mathbf{U}_{\mathcal{X}\mathcal{T}} + \mathbf{U}_{\mathcal{Y}\mathcal{T}} + 3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{X}\mathcal{Y}} + 3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{X}\mathcal{X}} + \mathbf{U}_{\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{Y}}) = 0. \tag{27}$$

Again, utilizing the Lie symmetry method on Eq. (27), then we get the desired infinitesimals

$$\xi_{\mathcal{X}} = \frac{a_1}{3}\mathcal{X} + a_3\sqrt{\mathcal{T}} + a_4, \quad \xi_{\mathcal{Y}} = \frac{a_1}{3}\mathcal{Y} + a_2,$$
$$\xi_{\mathcal{T}} = a_1\mathcal{T}, \quad \eta_{\mathbf{U}} = -\frac{a_1}{3}\mathbf{U} + \frac{a_3}{6\sqrt{\mathcal{T}}}(\mathcal{X}+\mathcal{Y}) + g(\mathcal{T}), \tag{28}$$

where a_i's $(1 \le i \le 4)$ are arbitrary constants and $g(\mathcal{T})$ is an arbitrary function of $\mathcal{T}$. The Lagrange's equation for (28) is

$$\frac{d\mathcal{X}}{\frac{a_1}{3}\mathcal{X} + a_3\sqrt{\mathcal{T}} + a_4} = \frac{d\mathcal{Y}}{\frac{a_1}{3}\mathcal{Y} + a_2} = \frac{d\mathcal{T}}{a_1\mathcal{T}} = \frac{d\mathbf{U}}{-\frac{a_1}{3}\mathbf{U} + \frac{a_3}{6\sqrt{\mathcal{T}}}(\mathcal{X}+\mathcal{Y}) + g(\mathcal{T})}, \tag{29}$$

which gives the similarity form

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{T}) = \frac{F(R,S)}{\mathcal{T}^{\frac{1}{3}}} + \frac{a_3\left(a_1(\mathcal{X}+\mathcal{Y}) + 6(a_2+a_4) - 3a_3\sqrt{\mathcal{T}}\right)}{a_1^2\sqrt{\mathcal{T}}} + \frac{1}{a_1\mathcal{T}^{\frac{1}{3}}}\int\frac{g(\mathcal{T})}{\mathcal{T}^{2/3}}\,d\mathcal{T} \tag{30}$$

with $R = \frac{a_1\mathcal{X}+3a_4-6a_3\sqrt{\mathcal{T}}}{a_1\mathcal{T}^{\frac{1}{3}}}$ and $S = \frac{a_1\mathcal{Y}+3a_2}{a_1\mathcal{T}^{\frac{1}{3}}}$. Using (30) and (27), we get the following reduced equation

$$6F_{RRRS} + F_{RS}(18F_R - 2S - 2R) + F_{RR}(18F_S - 2R) - 2SF_{SS} - F_R - F_S = 0. \tag{31}$$

On simplifying (31), we assume that

$$F(R,S) = H(w), \qquad \text{where} \quad w = \alpha_1 R + \alpha_2 S, \tag{32}$$

where α_1 and α_2 are arbitrary constants. Substituting (32) into (31), we obtain

$$6\alpha_1^3\alpha_2 H^{(4)} + H'\left(36\alpha_1^2\alpha_2 H'' - (\alpha_1+\alpha_2)\right) - 2(\alpha_1+\alpha_2)wH'' = 0. \tag{33}$$

On solving (33), we have

$$H(w) = \beta_1 + \frac{w^2(\alpha_1+\alpha_2)}{24\alpha_1^2\alpha_2} \quad \text{and} \quad H(w) = \beta_2 + \frac{2\alpha_1}{w} + \frac{w^2(\alpha_1+\alpha_2)}{24\alpha_1^2\alpha_2}, \tag{34}$$

where β_1 and β_2 are arbitrary constants. Using back substitution, thus group-invariant solutions are

$$u(x,y,z,t) = \frac{\beta_1}{t^{\frac{1}{3}}} + \frac{1}{a_1 t^{\frac{1}{3}}}\int\frac{g(t)}{t^{\frac{2}{3}}}\,dt + \frac{a_3}{a_1^2\sqrt{t}}\left(a_1\left(x+y-\frac{z^2}{2t}\right) + 6(a_2+a_4) - 3a_3\sqrt{t}\right)$$
$$+\frac{\alpha_1+\alpha_2}{24a_1\alpha_1^2\alpha_2 t^{\frac{2}{3}}}\left[\alpha_2\left(a_1\left(y-\frac{z^2}{4t}\right)+3a_2\right) + \alpha_1\left(a_1\left(x-\frac{z^2}{4t}\right) - 6a_3\sqrt{t} + 3a_4\right)\right]^2, \tag{35}$$

$$u(x,y,z,t) = \frac{\beta_2}{t^{\frac{1}{3}}} + \frac{1}{a_1 t^{\frac{1}{3}}}\int\frac{g(t)}{t^{\frac{2}{3}}}\,dt + \frac{a_3}{a_1^2\sqrt{t}}\left(a_1\left(x+y-\frac{z^2}{2t}\right) + 6(a_2+a_4) - 3a_3\sqrt{t}\right)$$
$$+\frac{\alpha_1+\alpha_2}{24a_1\alpha_1^2\alpha_2 t^{\frac{2}{3}}}\left[\alpha_2\left(a_1\left(y-\frac{z^2}{4t}\right)+3a_2\right) + \alpha_1\left(a_1\left(x-\frac{z^2}{4t}\right) - 6a_3\sqrt{t} + 3a_4\right)\right]^2$$
$$+\frac{2\alpha_1 a_1}{\alpha_1\left(a_1\left(x-\frac{z^2}{4t}\right) - 6a_3\sqrt{t} + 3a_4\right) + \alpha_2\left(a_1\left(y-\frac{z^2}{4t}\right)+3a_2\right)}. \tag{36}$$

4.2. Symmetry reduction with $\chi_2 = V_2$

χ_2 reduces Eq. (24), then we get the desired PDE

$$3\mathbf{U}_{\mathcal{XXXY}} + 9\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}} + 9\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}} - \mathcal{X}(\mathbf{U}_{\mathcal{XX}} + \mathbf{U}_{\mathcal{XY}}) - \mathcal{Y}(\mathbf{U}_{\mathcal{YY}} + \mathbf{U}_{\mathcal{XY}})$$
$$-2\mathcal{Z}(\mathbf{U}_{\mathcal{XZ}} + \mathbf{U}_{\mathcal{YZ}}) - 3\mathbf{U}_{\mathcal{ZZ}} - 2\mathbf{U}_{\mathcal{X}} - 2\mathbf{U}_{\mathcal{Y}} = 0. \tag{37}$$

through group-invariant form

$$u(x,y,z,t) = \frac{\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})}{t^{\frac{1}{3}}}, \quad \text{with} \quad \mathcal{X} = \frac{x}{t^{\frac{1}{3}}}, \quad \mathcal{Y} = \frac{y}{t^{\frac{1}{3}}}, \quad \text{and} \quad \mathcal{Z} = \frac{z}{t^{\frac{2}{3}}}. \tag{38}$$

Employing Lie symmetry method on (37), then we obtain the infinitesimal generators $\xi_{\mathcal{X}}, \xi_{\mathcal{Y}}, \xi_{\mathcal{Z}}$ and $\eta_{\mathbf{U}}$ as follows:

$$\begin{aligned}\xi_{\mathcal{X}} &= \frac{a_1}{3}\mathcal{Z}^2 + a_2, \quad \xi_{\mathcal{Y}} = 2a_1\mathcal{Y}, \quad \xi_{\mathcal{Z}} = a_1\mathcal{Z}, \quad \text{and} \\ \eta_{\mathbf{U}} &= \frac{a_1}{9}\left(\frac{\mathcal{Z}^4}{3} - (\mathcal{X}+\mathcal{Y})\mathcal{Z}^2 + 2\mathcal{X}\mathcal{Y}\right) + \frac{a_2}{9}\left((\mathcal{X}+\mathcal{Y}) - \frac{2}{3}\mathcal{Z}^2\right) + a_3\mathcal{Z} + a_4.\end{aligned} \tag{39}$$

where a_i's $(1 \le i \le 4)$ are arbitrary constants.

4.2.1. For $a_2 = 0$ in (39).

The associated characteristic equation is

$$\frac{d\mathcal{X}}{\frac{a_1}{3}\mathcal{Z}^2} = \frac{d\mathcal{Y}}{2a_1\mathcal{Y}} = \frac{d\mathcal{Z}}{a_1\mathcal{Z}} = \frac{d\mathbf{U}}{\frac{a_1}{9}\left(\frac{\mathcal{Z}^4}{3} - (\mathcal{X}+\mathcal{Y})\mathcal{Z}^2 + 2\mathcal{X}\mathcal{Y}\right) + a_3\mathcal{Z} + a_4}, \tag{40}$$

the similarity form is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = F(R,S) + \frac{1}{216}\left(24\mathcal{X}\mathcal{Y} - 4\mathcal{Z}^2(3\mathcal{X}+2\mathcal{Y}) + 3\mathcal{Z}^4\right) + \frac{1}{a_1}(a_3\mathcal{Z} + a_4\log(\mathcal{Z})) \tag{41}$$

with $R = \mathcal{X} - \frac{\mathcal{Z}^2}{6}$ and $S = \frac{\mathcal{Y}}{\mathcal{Z}^2}$. Substituting (41) into (37), we get

$$3F_{RRRS} + 3S(1-4S)F_{SS} + (2 - 18S + 9F_{RR})F_S + (9F_R - R)F_{RS} + \frac{3a_4}{a_1} = 0, \tag{42}$$

For which the new infinitesimals ξ_R, ξ_S, and η_F are given by

$$\xi_R = b_1, \quad \xi_S = 0, \quad \eta_F = \frac{b_1 R}{9} + b_2. \tag{43}$$

For simplification (43), the characteristic equation is

$$\frac{dR}{b_1} = \frac{dS}{0} = \frac{dF}{\frac{b_1R}{9} + b_2}, \tag{44}$$

which yields

$$F(R,S) = H(w) + \frac{b_2 R}{b_1} + \frac{R^2}{18}, \quad \text{with} \quad w = S. \tag{45}$$

Substituting $F(R,S)$ from (45) into (42), we obtain a second order ODE as follows:

$$w(4w-1)H''(w) + (6w-1)H'(w) - \frac{a_4}{a_1} = 0. \tag{46}$$

Solving (46), we have

$$H(w) = \beta_4 - 2\beta_3\tanh^{-1}\left(\sqrt{1-4w}\right) + \frac{a_4}{2a_1}\left(\log\left(1-\sqrt{1-4w}\right) + \log\left(1+\sqrt{1-4w}\right)\right), \tag{47}$$

where β_3 and β_4 are constants of integration.

Hence, thus group-invariant solution is

$$\begin{aligned}u_3(x,y,t) &= \frac{a_3 z}{a_1 t} + \frac{a_4}{2a_1 t^{\frac{1}{3}}}\left(\log\left[1-\sqrt{1-\frac{4ty}{z^2}}\right] + \log\left[1+\sqrt{1-\frac{4ty}{z^2}}\right]\right) + \frac{a_4}{a_1 t^{\frac{1}{3}}}\log\left[\frac{z}{t^{\frac{2}{3}}}\right] \\ &+\left(x - \frac{z^2}{6t}\right)\left(\frac{b_2}{b_1 t^{\frac{2}{3}}} + \frac{1}{18t}\left(x - \frac{z^2}{6t}\right) - \frac{z^2}{18t^2}\left(1 - \frac{2ty}{z^2}\right)\right) + \frac{z^4}{216t^3}\left(1 - \frac{4ty}{z^2}\right) \\ &+\frac{1}{t^{\frac{1}{3}}}\left(\beta_4 - 2\beta_3\tanh^{-1}\left(\sqrt{1-\frac{4ty}{z^2}}\right)\right).\end{aligned} \tag{48}$$

4.3. Symmetry reduction with $\chi_3 = V_3 + V_4$

χ_3 reduces Eq. (24), then we obtain

$$\mathbf{U}_{\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{Y}} + 3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{X}\mathcal{Y}} + 3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{X}\mathcal{X}} + \mathbf{U}_{\mathcal{X}\mathcal{T}} + \mathbf{U}_{\mathcal{Y}\mathcal{T}} - \mathbf{U}_{\mathcal{Y}\mathcal{Y}} = 0, \tag{49}$$

through group-invariant form

$$u(x,y,z,t)=\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{T}) \text{ with similarity variables } \mathcal{X}=x, \quad \mathcal{Y}=y-z, \quad \mathcal{T}=t. \tag{50}$$

The solution of (49) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{T})=c_1+2c_2\tanh\left(c_4+c_2\mathcal{X}+c_3\mathcal{Y}+\frac{c_3\left(c_3-4c_2^3\right)\mathcal{T}}{c_2+c_3}\right), \tag{51}$$

where c_1, c_2, c_3 and c_4 are constants of integration.

The group-invariant (exact-analytic) solution is

$$u(x,y,z,t)=c_1+2c_2\tanh\left(c_4+c_2x+c_3(y-z)+\frac{c_3\left(c_3-4c_2^3\right)t}{c_2+c_3}\right). \tag{52}$$

On solving (49), we assume that

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{T})=H(w), \qquad \text{where} \quad w=a\mathcal{X}+b\mathcal{Y}+c\mathcal{T}, \tag{53}$$

where a, b and c are arbitrary constants, to obtain new solutions of (1). Then, substituting (53) into (49), we get a reduced ODE as follows:

$$a^3bH^{(4)}(w)+\left(c(a+b)-b^2\right)H''(w)+6a^2bH'(w)H''(w)=0. \tag{54}$$

The primitives of the above equation are

$$H(w)=\beta_5+\frac{\left(b^2-ac-bc\right)}{6a^2b}w, \quad \text{and} \quad H(w)=\beta_6+\frac{2a}{w}+\frac{\left(b^2-ac-bc\right)}{6a^2b}w, \tag{55}$$

where β_5 and β_6 are constants of integration. Hence, we obtain exact closed-form solutions

$$u(x,y,z,t)=\beta_5+\frac{\left(b^2-ac-bc\right)}{6a^2b}(ax+b(y-z)+ct), \tag{56}$$

$$u(x,y,z,t)=\beta_6+\frac{2a}{(ax+b(y-z)+ct)}+\frac{\left(b^2-ac-bc\right)}{6a^2b}(ax+b(y-z)+ct). \tag{57}$$

4.4. Symmetry reduction with $\chi_4=V_4+V_5$

χ_4 reduces Eq. (24), then we get the desired PDE

$$\mathbf{U}_{\mathcal{XXXY}}+3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}}+3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}}-\mathbf{U}_{\mathcal{XZ}}-\mathbf{U}_{\mathcal{YZ}}-\mathbf{U}_{\mathcal{ZZ}}=0, \tag{58}$$

through the similarity form

$$u(x,y,z,t)=\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) \text{ with similarity variables } \mathcal{X}=x, \quad \mathcal{Y}=y, \quad \mathcal{Z}=z-t. \tag{59}$$

The general solution of (58) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=c_2+2c_1\tanh\left(c_4+\mathcal{X}c_1+\mathcal{Z}c_3+\frac{c_3(c_1+c_3)\mathcal{Y}}{4c_1^3-c_3}\right), \tag{60}$$

where c_1, c_2, c_3 and c_4 are constants of integration. Therefore, one obtains

$$u(x,y,z,t)=c_2+2c_1\tanh\left(c_4+c_1x+c_3(z-t)+\frac{c_3(c_1+c_3)y}{4c_1^3-c_3}\right). \tag{61}$$

Furthermore, we assume that

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=H(w), \qquad \text{where} \quad w=\alpha\mathcal{X}+\beta\mathcal{Y}+\gamma\mathcal{Z}, \tag{62}$$

where α, β and γ are arbitrary constants. Substituting (62) into (58), we get a reduced ODE:

$$\alpha^3\beta H^{(4)}+6\alpha^2\beta H'H''-\gamma(\alpha+\beta+\gamma)H''=0, \tag{63}$$

which produces the following solutions

$$H(w) = \beta_7 + \frac{\gamma(\alpha+\beta+\gamma)}{6\alpha^2\beta}w, \quad \text{and} \quad H(w) = \beta_7 + \frac{2\alpha}{w} + \frac{\gamma(\alpha+\beta+\gamma)}{6\alpha^2\beta}w, \tag{64}$$

where β_7 is a constant. Hence, group-invariant solutions are

$$u(x,y,z,t) = \beta_7 + \frac{\gamma(\alpha+\beta+\gamma)}{6\alpha^2\beta}(\alpha x + \beta y + \gamma(z-t)), \tag{65}$$

$$u(x,y,z,t) = \beta_7 + \frac{2\alpha}{(\alpha x + \beta y + \gamma(z-t))} + \frac{\gamma(\alpha+\beta+\gamma)}{6\alpha^2\beta}(\alpha x + \beta y + \gamma(z-t)). \tag{66}$$

4.5. Symmetry reduction with $\chi_5 = V_3 + V_5$

χ_5 reduces Eq. (24), then we obtain a reduced PDE

$$\mathbf{U}_{\mathcal{XXXY}} + 3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}} + 3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}} - \mathbf{U}_{\mathcal{XY}} - \mathbf{U}_{\mathcal{YY}} - \mathbf{U}_{\mathcal{ZZ}} = 0, \tag{67}$$

through group-invariant form

$$u(x,y,z,t) = \mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) \text{ with similarity variables } \mathcal{X} = x, \quad \mathcal{Y} = y - t, \quad \mathcal{Z} = z. \tag{68}$$

The general solution of (67) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = c_2 + 2c_1\tanh\left(c_4 + c_1\mathcal{X} + c_3\mathcal{Z} + \frac{1}{2}\left(4c_1^3 - c_1 \pm \sqrt{16c_1^6 - 8c_1^4 + c_1^2 - 4c_3^2}\right)\mathcal{Y}\right), \tag{69}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Thus, the resulting solution of the gKP (1) is given by

$$u(x,y,z,t) = c_2 + 2c_1\tanh\left(c_4 + c_1 x + c_3 z + \frac{1}{2}\left(4c_1^3 - c_1 \pm \sqrt{16c_1^6 - 8c_1^4 + c_1^2 - 4c_3^2}\right)(y-t)\right). \tag{70}$$

Furthermore, let

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = H(w), \quad \text{where} \quad w = a\mathcal{X} + b\mathcal{Y} + c\mathcal{Z}, \tag{71}$$

where a, b and c are arbitrary constants.

Substituting (71) into (67), we obtain a reduced ODE:

$$a^3bH^{(4)} - \left(b(a+b) + c^2 - 6a^2bH'\right)H'' = 0. \tag{72}$$

Its primitives are

$$H(w) = \beta_8 + \frac{ab+b^2+c^2}{6a^2b}w, \quad \text{and} \quad H(w) = \beta_8 + \frac{2a}{w} + \frac{ab+b^2+c^2}{6a^2b}w, \tag{73}$$

where β_8 is a constant. Consequently, the group-invariant solutions are

$$u(x,y,z,t) = \beta_8 + \frac{\left(ab+b^2+c^2\right)(ax + b(y-t) + cz)}{6a^2b}, \tag{74}$$

$$u(x,y,z,t) = \beta_8 + \frac{\left(ab+b^2+c^2\right)(ax + b(y-t) + cz)}{6a^2b} + \frac{2a}{ax + b(y-t) + cz}. \tag{75}$$

4.6. Subalgebra $\chi_6 = V_5 + V_8$

χ_6 reduces Eq. (24), then we obtain a reduced PDE

$$\mathbf{U}_{\mathcal{XXXY}} + 3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}} + 3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}} - \mathbf{U}_{\mathcal{ZZ}} = 0. \tag{76}$$

through group-invariant form

$$u(x,y,z,t) = \mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) + \int f_3(t)\,dt \quad \text{with} \quad \mathcal{X} = x, \quad \mathcal{Y} = y, \quad \mathcal{Z} = z. \tag{77}$$

Thus, the solution of (76) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = c_1 + 2c_2\tanh\left(c_2\mathcal{X} + \frac{c_3^2\mathcal{Y}}{4c_2^3} + c_3\mathcal{Z} + c_4\right). \tag{78}$$

where $c_1,\ c_2,\ c_3$, and c_4 be the arbitrary constants. Thus, the general solution of gKP Eq. (1) as

$$u(x,y,z,t) = c_1 + 2c_2\tanh\left(c_2x + \frac{c_3^2y}{4c_2^3} + c_3z + c_4\right) + \int f_3(t)\,dt. \tag{79}$$

Furthermore, let

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = H(w), \qquad \text{where} \quad w = a\mathcal{X} + b\mathcal{Y} + c\mathcal{Z}. \tag{80}$$

where a, b, and c are arbitrary constant for finding the new solution of Eq. (1). Then, substituting Eq. (80) in Eq. (76), we get reduced ordinary differential equation as

$$a^3bH^{(4)} + H''\left(6a^2bH' - c^2\right) = 0 \tag{81}$$

we obtain

$$H(w) = \beta_9 + \frac{c^2w}{6a^2b} \quad \text{and} \quad H(w) = \beta_9 + \frac{c^2w}{6a^2b} + \frac{2a}{w}. \tag{82}$$

where β_9 is an arbitrary constant. Thus, group-invariant solutions are

$$u(x,y,z,t) = \beta_9 + \frac{c^2(ax+by+cz)}{6a^2b} + \int f_3(t)\,dt. \tag{83}$$

$$u(x,y,z,t) = \beta_9 + \frac{2a}{ax+by+cz} + \frac{c^2(ax+by+cz)}{6a^2b} + \int f_3(t)\,dt. \tag{84}$$

Again, to find few more solution of generalized KP equation, we utilize the Lie group transformation method to obtain infinitesimal generators of Eq. (76) which provides

$$\begin{aligned} &\xi_{\mathcal{X}} = \frac{1}{3}(2a_1 - a_3)\mathcal{X} + a_5, \quad \xi_{\mathcal{Y}} = a_3\mathcal{Y} + a_4, \quad \xi_{\mathcal{Z}} = a_1\mathcal{Z} + a_2, \\ &\eta_{\mathbf{U}} = \frac{1}{3}(a_3 - 2a_1)\mathbf{U} + a_6\mathcal{Z} + a_7. \end{aligned} \tag{85}$$

where $c_i's, 1 \le i \le 7$ are arbitrary constants.

Thus, associated Lagrange's system is

$$\frac{d\mathcal{X}}{\frac{1}{3}(2a_1 - a_3)\mathcal{X} + a_5} = \frac{d\mathcal{Y}}{a_3\mathcal{Y} + a_4} = \frac{d\mathcal{Z}}{a_1\mathcal{Z} + a_2} = \frac{d\mathbf{U}}{\frac{1}{3}(a_3 - 2a_1)\mathbf{U} + a_6\mathcal{Z} + a_7} \tag{86}$$

4.6.1. For $a_1 \neq 0$ and all others constant are zero.

In this case, the characteristic Eq. (86) become

$$\frac{d\mathcal{X}}{\frac{2}{3}a_1\mathcal{X}} = \frac{d\mathcal{Y}}{0} = \frac{d\mathcal{Z}}{a_1\mathcal{Z}} = \frac{d\mathbf{U}}{-\frac{2}{3}a_1\mathbf{U}} \tag{87}$$

The similarity form solution of (87) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \frac{F(R,S)}{\mathcal{Z}^{2/3}} \quad \text{with} \quad R = \frac{\mathcal{X}}{\mathcal{Z}^{2/3}}, \quad S = \mathcal{Y}. \tag{88}$$

Putting value of $\mathbf{U}$ in (76), we obtain

$$F_{RRRS} + 3F_RF_{RS} + 3F_SF_{RR} - 2RF_R - \frac{4}{9}R^2F_{RR} - \frac{10}{9}F = 0. \tag{89}$$

Again apply the Lie similarity method on (1+1) nonlinear PDEs, we get

$$\xi_R = -\frac{b_1}{3}R, \qquad \xi_S = b_1S + b_2, \qquad \eta_F = \frac{b_1}{3}F. \tag{90}$$

Hence, using Eq. (90), we obtain associated characteristic equation

$$\frac{dR}{-\frac{b_1}{3}R} = \frac{dS}{b_1S+b_2} = \frac{dF}{\frac{b_1}{3}F}. \tag{91}$$

By solving Eq. (91), we obtain a similarity form

$$F(R,S) = \sqrt[3]{b_1S+b_2}H(w), \quad \text{with} \quad w = R\sqrt[3]{b_1S+b_2}. \tag{92}$$

Thus, we get desired ODEs as

$$H(9b_1H''-10)+3\left(6b_1H'^2+b_1\left(wH^{(4)}+4H^{(3)}\right)+6wH'(b_1H''-1)\right)-4w^2H''=0 \tag{93}$$

By solving Eq. (93), we get the solution

$$H(w) = \frac{\beta_{10}}{w}+\frac{w^2}{3b_1}. \tag{94}$$

where β_{10} is an arbitrary constant. Consequently, exact analytic solution is

$$u(x,y,z,t) = \frac{\beta_{10}}{x}+\frac{x^2}{3z^2}\left(y+\frac{b_2}{b_1}\right)+\int f_3(t)\,dt. \tag{95}$$

4.6.2. For $a_3 = 2a_1$.

In this case, the characteristic Eq. (86) become

$$\frac{d\mathcal{X}}{a_5} = \frac{d\mathcal{Y}}{2a_1\mathcal{Y}+a_4} = \frac{d\mathcal{Z}}{a_1\mathcal{Z}+a_2} = \frac{d\mathbf{U}}{a_6\mathcal{Z}+a_7} \tag{96}$$

The similarity form solution of (96) is

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = F(R,S)+\frac{a_6}{a_1}\mathcal{Z}+\frac{1}{a_1^2}(a_1a_7-a_2a_6)\log(a_1\mathcal{Z}+a_2). \tag{97}$$

with similarity variables $R = \mathcal{X}-\frac{a_5}{a_1}\log(a_1\mathcal{Z}+a_2)$ and $S = \frac{2a_1\mathcal{Y}+a_4}{(a_1\mathcal{Z}+a_2)^2}$. Putting the value of U in (76), we obtain a new nonlinear partial differential equation as

$$\begin{aligned} F_{RRRS}+3F_RF_{RS}+3F_SF_{RR}-2a_1S^2F_{SS}-\frac{a_5^2}{2a_1}F_{RR}-2a_5SF_{RS}-3a_1SF_s-\frac{a_5}{2}F_R \\ +\frac{1}{a_1^2}(a_1a_7-a_2a_6)=0. \end{aligned} \tag{98}$$

Again apply Lie similarity method on nonlinear (1+1) PDE equation

$$\xi_R = b_1, \quad \xi_S = 0, \quad \eta_F = b_2. \tag{99}$$

Hence, using Eq. (99), we have

$$\frac{dR}{b_1} = \frac{dS}{0} = \frac{dF}{b_2}. \tag{100}$$

which yields

$$F(R,S) = H(w)+\frac{b_2}{b_1}R, \quad \text{where} \quad w = S. \tag{101}$$

we obtain reduced ODE as

$$2a_1w^2H''+3a_1wH'+\frac{a_5b_2}{2b_1}-\frac{1}{2a_1}(a_1a_7-a_2a_6)=0. \tag{102}$$

By solving Eq. (102), we get general solution

$$H(w) = c_2-\frac{2c_1}{\sqrt{w}}+\frac{1}{2a_1^2b_1}(a_1a_7b_1-a_2a_6b_1-a_1a_5b_2)\log(w). \tag{103}$$

where c_1 and c_2 are arbitrary constants. Thus, group-invariant solution is

$$\begin{aligned} u(x,y,z,t) &= c_2 - 2c_1\frac{(a_1 z + a_2)}{\sqrt{2a_1 y + a_4}} - \frac{1}{2a_1^2 b_1}(a_2 a_6 b_1 + a_1(a_5 b_2 - a_7 b_1))\log\left(\frac{2a_1 y + a_4}{(a_1 z + a_2)^2}\right) \\ &+ \frac{a_6}{a_1} z + \frac{b_2}{a_1 b_1}(a_1 x - a_5 \log(a_1 z + a_2)) + \frac{1}{a_1^2}(a_1 a_7 - a_2 a_6)\log(a_1 z + a_2) + \int f_3(t)\,dt. \end{aligned} \tag{104}$$

4.7. Symmetry reduction with $\chi_7 = V_3 + V_4 + V_5 + V_7 + V_8$

χ_7 reduces Eq. (24), then we obtain a reduced PDE

$$\mathbf{U}_{\mathcal{XXXY}} + 3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}} + 3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}} - \mathbf{U}_{\mathcal{XY}} - \mathbf{U}_{\mathcal{XZ}} - \mathbf{U}_{\mathcal{YZ}} - \mathbf{U}_{\mathcal{YY}} - \mathbf{U}_{\mathcal{ZZ}} = 0, \tag{105}$$

through similarity form

$$u(x,y,z,t) = \mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) + \int (zf_2(t) + f_3(t))\,dt \tag{106}$$

with $\mathcal{X} = x,\ \mathcal{Y} = y - t$ and $\mathcal{Z} = z - t$.

Thus, the solution of (105)

$$\begin{aligned} &\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = c_2 \\ &+2c_1\tanh\left(c_4 + c_1\mathcal{X} + c_3\mathcal{Z} + \frac{\mathcal{Y}}{2}\left(4c_1^3 - c_1 - c_3 \pm \sqrt{\left(-4c_1^3 + c_1 + c_3\right)^2 - 4\left(c_3^2 + c_1 c_3\right)}\right)\right), \end{aligned} \tag{107}$$

where $c_1,\ c_2,\ c_3$ and c_4 are arbitrary constants. Hence, we obtain the resulting solution of the gKP (1):

$$\begin{aligned} &u(x,y,z,t) = 2c_1\tanh(c_4 + c_1 x + c_3(z-t) \\ &+\frac{y-t}{2}\left(4c_1^3 - c_1 - c_3 \pm \sqrt{\left(-4c_1^3 + c_1 + c_3\right)^2 - 4\left(c_3^2 + c_1 c_3\right)}\ \right)\Bigg) + c_2 + \int (zf_2(t) + f_3(t))\,dt. \end{aligned} \tag{108}$$

Furthermore, let

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = H(w), \qquad \text{where the wave transformation} \quad w = \alpha\mathcal{X} + \beta\mathcal{Y} + \gamma\mathcal{Z}, \tag{109}$$

where α, β and γ are arbitrary constants.

Substituting (109) in (105), then we obtain an ODE as follows:

$$\alpha^3\beta H^{(4)} - H''\left(-6\alpha^2\beta H' + \gamma(\alpha+\beta) + \beta(\alpha+\beta) + \gamma^2\right) = 0. \tag{110}$$

Its primitives are

$$\begin{aligned} H(w) &= \beta_{11} + \frac{w}{6\alpha^2\beta}\left(\alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma + \gamma^2\right), \\ \text{and} \quad H(w) &= \beta_{11} + \frac{2\alpha}{w} + \frac{w}{6\alpha^2\beta}\left(\alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma + \gamma^2\right), \end{aligned} \tag{111}$$

where β_{11} is an arbitrary constant.

Using back substitution,thus group-invariant solutions are

$$u(x,y,z,t) = \beta_{11} + \frac{\alpha x + \beta(y-t) + \gamma(z-t)}{6\alpha^2\beta}\left(\alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma + \gamma^2\right) + \int (zf_2(t) + f_3(t))\,dt. \tag{112}$$

$$\begin{aligned} u(x,y,z,t) &= \beta_{11} + \frac{2\alpha}{\alpha x + \beta(y-t) + \gamma(z-t)} + \int (zf_2(t) + f_3(t))\,dt \\ &+ \frac{\alpha x + \beta(y-t) + \gamma(z-t)}{6\alpha^2\beta}\left(\alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma + \gamma^2\right). \end{aligned} \tag{113}$$

Again, utilizing group-theoretic method to obtain infinitesimal generators of (105)

$$\begin{aligned} &\xi_{\mathcal{X}} = \frac{a_1}{3}(\mathcal{X} + \mathcal{Z}) + a_4, \quad \xi_{\mathcal{Y}} = a_1\mathcal{Y} + a_3, \quad \xi_{\mathcal{Z}} = a_1\mathcal{Z} + a_2, \\ &\qquad\qquad \eta_{\mathbf{U}} = \frac{a_1}{9}(\mathcal{X} - \mathcal{Y} - 3\mathbf{U}) + a_5\mathcal{Z} + a_6, \end{aligned} \tag{114}$$

where a_i's, $(1 \le i \le 6)$ are arbitrary constants.

Subsequently, associated Lagrange's system as follows

$$\frac{dX}{\frac{a_1}{3}(X+Z)+a_4}=\frac{dY}{a_1Y+a_3}=\frac{dZ}{a_1Z+a_2}=\frac{d\mathbf{U}}{\frac{a_1}{9}(X-Y-3\mathbf{U})+a_5Z+a_6}. \tag{115}$$

4.7.1. For $a_2 \neq 0$ and all others constant are zero.

Using Eq. (115), then we obtain

$$F_{RRRS}+3F_RF_{RS}+3F_SF_{RR}-F_{RS}-F_{SS}=0. \tag{116}$$

through

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=F(R,S),\quad \text{with}\quad R=\mathcal{X},\quad S=\mathcal{Y}. \tag{117}$$

Further, we can get the desired infinitesimals of (116) as

$$\xi_R=\frac{b_1}{3}R+b_3,\qquad \xi_S=b_1S+b_2,\qquad \eta_F=\frac{b_1}{9}(2R-3F)+b_4, \tag{118}$$

where b_i's $(1\leq i\leq 4)$ are arbitrary constants. Then, the associated characteristic equation reads

$$\frac{dR}{\frac{b_1}{3}R+b_3}=\frac{dS}{b_1S+b_2}=\frac{dF}{\frac{b_1}{9}(2R-3F)+b_4}, \tag{119}$$

which produces

$$F(R,S)=\frac{H(w)}{(b_1S+b_2)^{\frac{1}{3}}}+\frac{1}{3}w(b_1S+b_2)^{\frac{1}{3}}+\frac{3b_4-2b_3}{b_1},\quad \text{where}\quad w=\frac{1}{(b_1S+b_2)^{\frac{1}{3}}}\left(R+\frac{3b_3}{b_1}\right). \tag{120}$$

Thus, one obtains the reduced ODE as follows:

$$3wH^{(4)}+12H^{(3)}+H''\left(b_1w^2+9H+18wH'\right)+6b_1wH'+18(H')^2+4b_1H=0. \tag{121}$$

The primitive of (121) is

$$H(w)=-\frac{b_1}{9}w^2. \tag{122}$$

Thus, group-invariant solution is

$$\begin{aligned} u(x,y,z,t) &=\frac{1}{b_1}(3b_4-2b_3)+\frac{1}{3}\left(x+\frac{3b_3}{b_1}\right)-\frac{b_1\left(x+\frac{3b_3}{b_1}\right)^2}{9(b_1(y-t)+b_2)}\\ &+\int(zf_2(t)+f_3(t))\,dt. \end{aligned} \tag{123}$$

4.7.2. For $a_3 \neq 0$ and all others constant are zero.

Using Eq. (115), then we obtain

$$F_{RS}+F_{SS}=0, \tag{124}$$

through

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=F(R,S),\quad \text{with}\quad R=\mathcal{X},\ \text{and}\ S=\mathcal{Z}. \tag{125}$$

General solution of (124) is

$$F(R,S)=g(R-S)+h(R), \tag{126}$$

where g and h are arbitrary functions. Thus, group-invariant solution is

$$u(x,y,z,t)=g(x+t-z)+h(x)+\int(zf_2(t)+f_3(t))\,dt. \tag{127}$$

4.8. Symmetry reduction with $\chi_8 = V_3+V_4+V_5+V_6+V_7+V_8$

χ_8 reduces Eq. (24), then we obtain a reduced PDE

$$\mathbf{U}_{\mathcal{XXXY}}+3\mathbf{U}_{\mathcal{X}}\mathbf{U}_{\mathcal{XY}}+3\mathbf{U}_{\mathcal{Y}}\mathbf{U}_{\mathcal{XX}}-2\mathbf{U}_{\mathcal{XY}}-\mathbf{U}_{\mathcal{XZ}}-\mathbf{U}_{\mathcal{YZ}}-\mathbf{U}_{\mathcal{XX}}-\mathbf{U}_{\mathcal{YY}}-\mathbf{U}_{\mathcal{ZZ}}=0. \tag{128}$$

through group-invariant form

$$u(x,y,z,t)=\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})+(z+a_0)\int f_2(t)\,dt \tag{129}$$

with similarity variables $\mathcal{X}=x-t,\ \mathcal{Y}=y-t$ and $\mathcal{Z}=z-t$.

Furthermore, let

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=H(w),\qquad \text{where}\quad w=\alpha\mathcal{X}+\beta\mathcal{Y}+\gamma\mathcal{Z}, \tag{130}$$

where α,β and γ are arbitrary constants, to obtain new solutions of (1). Then, substituting (130) into (128), we get a reduced ODE:

$$\alpha^3\beta H^{(4)}+6\alpha^2\beta H'H''-\big(\gamma(\alpha+\beta+\gamma)+(\alpha+\beta)^2\big)H''=0. \tag{131}$$

The primitives are

$$\begin{aligned} H(w)&=\beta_{12}+\beta_{13}w,\\ H(w)&=\beta_{12}+\frac{w}{6\alpha^2\beta}\big(\gamma(\alpha+\beta+\gamma)+(\alpha+\beta)^2\big),\\ \text{and}\quad H(w)&=\beta_{12}+\frac{2\alpha}{w}+\frac{w}{6\alpha^2\beta}\big(\gamma(\alpha+\beta+\gamma)+(\alpha+\beta)^2\big),\end{aligned} \tag{132}$$

where β_{12} and β_{13} are arbitrary constants. Hence, group-invariant solutions are

$$u(x,y,z,t)=\beta_{12}+\beta_{13}(\alpha(x-t)+\beta(y-t)+\gamma(z-t))+(z+a_0)\int f_2(t)\,dt, \tag{133}$$

$$\begin{aligned} u(x,y,z,t)&=\beta_{12}+\frac{\alpha(x-t)+\beta(y-t)+\gamma(z-t)}{6\alpha^2\beta}\big(\gamma(\alpha+\beta+\gamma)+(\alpha+\beta)^2\big)\\ &+(z+a_0)\int f_2(t)\,dt,\end{aligned} \tag{134}$$

$$\begin{aligned} u(x,y,z,t)&=\beta_{12}+\frac{\alpha(x-t)+\beta(y-t)+\gamma(z-t)}{6\alpha^2\beta}\big(\gamma(\alpha+\beta+\gamma)+(\alpha+\beta)^2\big)\\ &+\frac{2\alpha}{\alpha(x-t)+\beta(y-t)+\gamma(z-t)}+(z+a_0)\int f_2(t)\,dt.\end{aligned} \tag{135}$$

Utilizing the group-theoretic method, we get the desired infinitesimals of (128)

$$\begin{aligned} &\xi_{\mathcal{X}}=\frac{a_1}{3}(\mathcal{X}+\mathcal{Z})+a_4,\quad \xi_{\mathcal{Y}}=a_1\mathcal{Y}+a_3,\quad \xi_{\mathcal{Z}}=a_1\mathcal{Z}+a_2,\\ &\qquad\qquad \eta_{\mathbf{U}}=\frac{a_1}{3}(\mathcal{X}+\mathcal{Y}-\mathbf{U})+a_5\mathcal{Z}+a_6,\end{aligned} \tag{136}$$

where a_i's, $(1\le i\le 6)$ are arbitrary constants. Thus, we have

$$\frac{d\mathcal{X}}{\frac{a_1}{3}(\mathcal{X}+\mathcal{Z})+a_4}=\frac{d\mathcal{Y}}{a_1\mathcal{Y}+a_3}=\frac{d\mathcal{Z}}{a_1\mathcal{Z}+a_2}=\frac{d\mathbf{U}}{\frac{a_1}{3}(\mathcal{X}+\mathcal{Y}-\mathbf{U})+a_5\mathcal{Z}+a_6}. \tag{137}$$

4.8.1. For $a_2\neq 0$ and all others constant are zero.

Using Eq. (137), then we obtain

$$F_{RRRS}+3F_RF_{RS}+3F_SF_{RR}-F_{RR}-2F_{RS}-F_{SS}=0. \tag{138}$$

through

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=F(R,S),\quad \text{with}\quad R=\mathcal{X}\quad S=\mathcal{Y}. \tag{139}$$

Further, we can get the desired infinitesimals of (138) as

$$\xi_R=\frac{b_1}{3}R+b_3,\qquad \xi_S=b_1S+b_2,\qquad \eta_F=\frac{b_1}{9}(2R-3F)+b_4, \tag{140}$$

where b_i's $(1\le i\le 4)$ are arbitrary constants. Then, the associated characteristic equation reads

$$\frac{dR}{\frac{b_1}{3}R+b_3}=\frac{dS}{b_1S+b_2}=\frac{dF}{\frac{4b_1}{9}(4R+4S-3F)+b_4}, \tag{141}$$

which produces

$$F(R,S)=\frac{H(w)}{(b_1S+b_2)^{\frac{1}{3}}}+\frac{1}{3}\left(S+2w(b_1S+b_2)^{\frac{1}{3}}\right)+\frac{3b_4-4b_3-b_2}{b_1}, \quad \text{where} \quad w=\frac{(Rb_1+3b_3)}{b_1(b_1S+b_2)^{\frac{1}{3}}}. \tag{142}$$

Thus, one obtains the reduced ODE as follows:

$$3wH^{(4)}+12H^{(3)}+6wH^{'}(b_1+3H'')+b_1H(4b_1+9H'')+w^2H''+18(H^{'})^2=0. \tag{143}$$

The primitive of (143) is

$$H(w)=-\frac{b_1}{9}w^2. \tag{144}$$

Thus, group-invariant solution is

$$\begin{aligned} u(x,y,z,t) \quad &= \frac{1}{b_1}(3b_4-4b_3-b_2)+\frac{1}{3}(y-t)+\frac{2}{3}\left(\left(x-t+\frac{3b_3}{b_1}\right)-\frac{b_1\left(x-t+\frac{3b_3}{b_1}\right)^2}{9(b_1(y-t)+b_2)}\right. \\ &+(z+a_0)\int f_2(t)\,dt. \end{aligned} \tag{145}$$

4.8.2. For $a_3 \neq 0$ and all others constant are zero.

Using Eq. (137), then we obtain

$$F_{RR}+F_{RS}+F_{SS}=0, \tag{146}$$

through similarity form

$$\mathbf{U}(\mathcal{X},\mathcal{Y},\mathcal{Z})=F(R,S), \quad \text{with} \quad R=\mathcal{X}, \text{ and } S=\mathcal{Z}. \tag{147}$$

Furthermore, let

$$F(R,S)=H(w), \quad \text{where} \quad w=\alpha R+\beta S, \tag{148}$$

Table 3
Some more group-invariant solutions.

Symmetry reduction	Group-invariant solutions
$\chi_9 = V_1 + V_5$	$u(x,y,t) = \beta_{16}+x+z+\frac{t^3}{6}-\frac{t^2}{2}-\frac{tz}{2}+\frac{(a_5-a_3)}{a_2}\left(y+\frac{t^3}{6}-\frac{tz}{2}\right)$
$\chi_{10} = V_2 + V_5$	$u(x,y,t) = \frac{x(x+2y)}{6(2+3t)}-\frac{z^2(2x+y)}{3(2+3t)^2}+\frac{5z^4}{12(2+3t)^3}+\frac{b_2(2x(2+3t)-z^2)}{2b_1(2+3t)^{\frac{5}{3}}}+\frac{1}{(2+3t)^{\frac{1}{3}}}\left(\beta_{18}-\frac{2\beta_{17}}{\sqrt{3}}\tanh^{-1}(\zeta_1)\right)$
$\chi_{11} = V_2 + V_5 + V_7$	$u(x,y,z,t) = \frac{x(x+2y)}{6(2+3t)}-\frac{z^2(2x+y)}{3(2+3t)^2}+\frac{5z^4}{12(2+3t)^3}+\frac{b_2(2x(2+3t)-z^2)}{2b_1(2+3t)^{\frac{5}{3}}}+\frac{2z}{(2+3t)^{\frac{1}{3}}}\int\frac{f_2(t)}{(2+3t)^{\frac{2}{3}}}dt$ $+\frac{1}{(2+3t)^{\frac{1}{3}}}\left(\beta_{20}-\frac{2\beta_{19}}{\sqrt{3}}\tanh^{-1}(\zeta_1)\right)$
$\chi_{12} = V_1 + V_2 + V_5$	$u(x,y,t) = \frac{(z(z+2t)-2t(3x+t+4)-4(x+1))(6t^2+t(8-6y-6z)-4y+3z^2-8z+4)}{12(3t+2)^3}-\frac{(y+t-z+2)(z-2(t+1))^2}{6(2+3t)^2}$ $+\frac{(z(2t+z)-2t(t+3x+4)-4(x+1))^2}{24(3t+2)^3}+\frac{b_2(z(2t+z)-2t(t+3x+4)-4(x+1))^2}{2b_1(3t+2)^{\frac{5}{3}}}$ $+\frac{1}{(2+3t)^{\frac{1}{3}}}\left(\beta_{22}-\frac{2\beta_{21}}{\sqrt{3}}\tanh^{-1}(\zeta_2)\right)+\frac{(z-2(t+1))^4}{8(3t+2)^3}$
$\chi_{13} = V_3 + V_4 + V_5 + V_8$	$u(x,y,z,t) = \beta_{23}+\frac{\alpha x+\beta(y-t)+\gamma(z-t)}{6\alpha^2\beta}(\alpha\beta+\alpha\gamma+\beta^2+\beta\gamma+\gamma^2)+\int f_3(t)\,dt$
	$u(x,y,z,t) = \beta_{24}+\frac{\alpha x+\beta(y-t)+\gamma(z-t)}{6\alpha^2\beta}(\alpha\beta+\alpha\gamma+\beta^2+\beta\gamma+\gamma^2)+\int f_3(t)\,dt+\frac{2\alpha}{\alpha x+\beta(y-t)+\gamma(z-t)}$
	$u(x,y,z,t) = 2c_1\tanh\left(c_4+c_1x+c_3(z-t)+\frac{y-t}{2}(4c_1^3-c_1-c_3\pm\sqrt{(-4c_1^3+c_1+c_3)^2-4(c_3^2+c_1c_3)}\,)\right)c_2$ $+\int f_3(t)\,dt$
	Where $\zeta_1=\sqrt{1-\frac{4(2+3t)y}{3z^2}}$ and $\zeta_2=\sqrt{1-\frac{4(3t+2)(y+t-z+2)}{3(z-2(t+1))^2}}$ and $\beta_i, (16 \leq i \leq 24)$ are arbitrary constants.

where α and β are arbitrary constant. Substituting (148) into (146), we get a reduce ODE

$$\left(\alpha^2+\alpha\beta+\beta^2\right)H''=0, \tag{149}$$

which gives

$$H(w)=\beta_{14}+w\beta_{15} \tag{150}$$

where β_{14} and β_{15} are arbitrary constants. Thus, the corresponding resulting solution is

$$u(x,y,z,t)=\beta_{14}+\beta_{15}(\alpha(x-t)+\beta(z-t))+(z+a_0)\int f_2(t)\,dt. \tag{151}$$

In the same manner, we can obtain some more group-invariant solutions corresponding to another members of optimal system (23), which are provided in Table 3.

5. Analysis and discussions

The geometrical representation of the resulting group-solutions of the (3+1)-dimensional gKP equation is described graphically in this section. The physical phenomena of those obtained exact closed-form solutions can be seen more clearly through graphical evaluation. The obtained solutions of the gKP equation include W-shaped solitons, elastic multi-solitons, doubly solitons, kink waves, lump-type solitons, stationary waves, and annihilation of different dynamical structures. Those solutions contain several sets of arbitrary constants and functions, which exhibit diverse dynamical structures of multiple solitons through their numerical simulation. The dynamical structures determined by the Figs. 1–9, in 3D-plots are made by Mathematica. A summary of the profiles of the solutions follows.

Fig. 1 : The W-shaped soliton reveals from $t=0.30$ to $t=1.50$ for solution (36). The dynamical structure shows the annihilation of the soliton profile with different values of t. Initially, we have observed W-shaped solitons at $t=1$, which is changed into a parabolic wave profile later on, and as time reaches over 500, the soliton drops its nonlinearity and results into a stationary profile. The values of arbitrary constants and function were chosen as $\alpha_1=3,\alpha_2=23,a_1=7,a_2=16,a_3=3.1,a_4=1,b_0=1.2,b_1=0.02,\beta_2=321,y=277.10$, and $g(t)=\frac{t^{\frac{2}{3}}(2t+b_0)}{\sec(t^2+b_0t+b_1)}$, respectively.

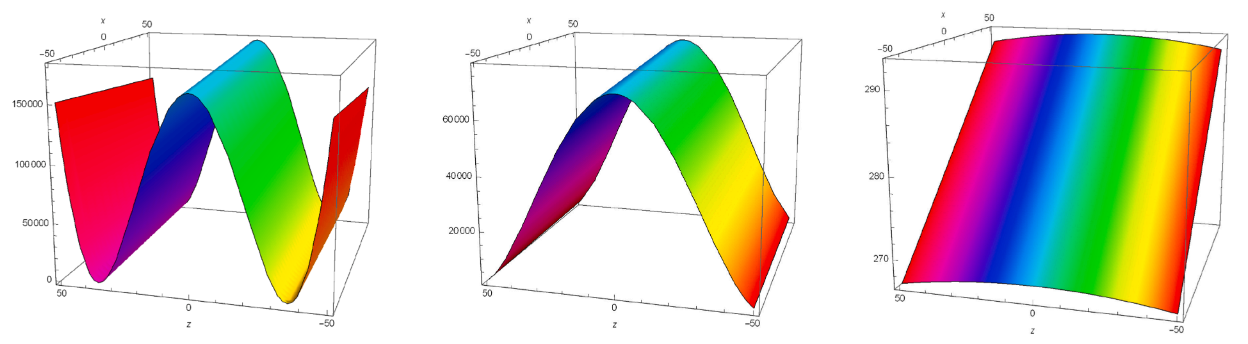
Fig. 2 : Annihilation of a multi-soliton has been observed for Eq. (57) as time increases. The multisoliton form is transformed into a stable wave profile after $t=263$. For the most beneficial representation of the graph, we pick the arbitrary choice of constants $a=3,b=2,c=1,\beta_6=10$ and $z=3$. During the wave propagation, we observe that the amplitude, the velocity, and the shape of the soliton remains invariant.

Fig. 3 : Annihilation of a kink wave profile into a stationary wave profile for Eq. (61) is noticed in this figure via different 3D plots. For the best dynamical behavior, we specify arbitrary constants as $c_1=3,c_2=0.222,c_3=0.70,c_4=0.05$, and $z=3$.

Fig. 4 : In this figure, the elastic behavior of a multi-soliton profile is observed from Eq. (66) by using a suitable selection of arbitrary constants $\alpha=3,\beta=4,\gamma=0.07,\beta_7=10$, and $z=1$. The multi-soliton wave profile is transformed into a single soliton wave profile at $t=960$ which converts into a stationary wave profile after $t=1160$. It is mentioned that the solitons do not transform their sizes and shapes when they are associated with each other and reveal a completely elastic nature.

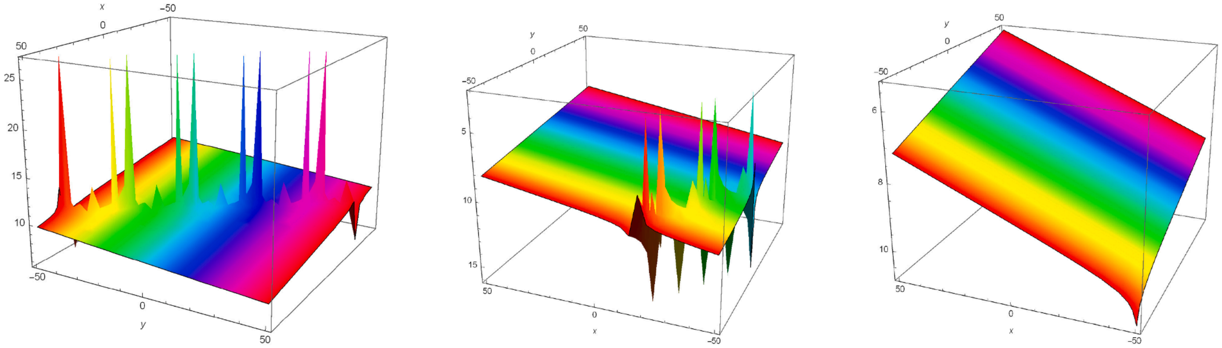
Fig. 5 : The multi-soliton wave solutions are exhibited graphically for Eq. (75) by suitable parametric values $a=13,b=4,c=1,\beta_8=10$, and $z=11$. Annihilation of multi-soliton wave profiles is observed by different 3D plots with the variation of time. As time moves across $t=77$, a dispersion impact dominates over nonlinearity and results in a straight strip.

Fig. 6 : The dynamical behavior of the solution (84) reflects the different lump-type solitons by adopting the adequate choice of



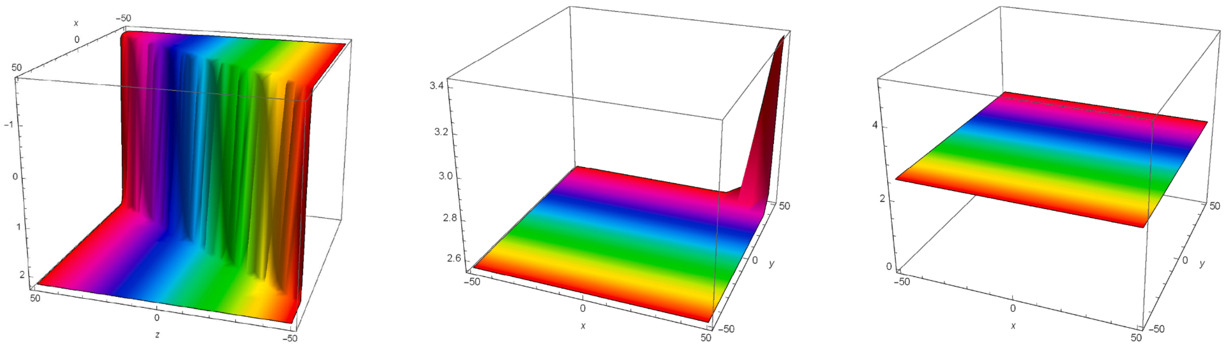
(a) $t=1.30$ (b) $t=3$ (c) $t=877$

Fig. 1. 3D-shapes of annihilation of a W-shaped soliton solution profile for (36) with $\alpha_1=3,\alpha_2=23,a_1=7,a_2=16,a_3=3.1,a_4=1,b_0=1.2,b_1=0.02,\beta_2=321,y=277.10$ and $g(t)=\frac{t^{\frac{2}{3}}(2t+b_0)}{\sec(t^2+b_0t+b_1)}$.



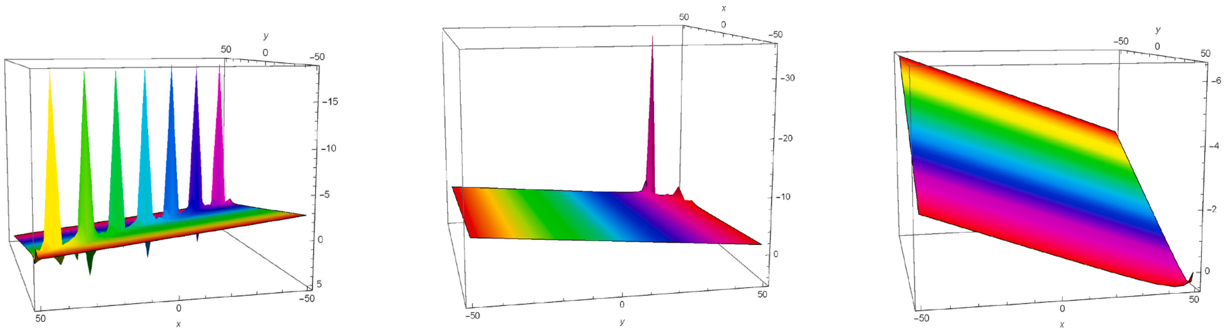
(a) $t = 10$ (b) $t = 157$ (c) $t = 263$

Fig. 2. 3D-shapes of annihilation of a single-soliton solution profile for (57) with $a = 3, b = 2, c = 1, \beta_6 = 10$ and $z = 3$.



(a) $t = 0.1$ (b) $t = 118$ (c) $t = 125$

Fig. 3. 3D-shapes of annihilation of a kink-wave profile for (61) with $c_1 = 3, c_2 = 0.222, c_3 = 0.70, c_4 = 0.5$ and $z = 3$.



(a) $t = 0.2$ (b) $t = 960$ (c) $t = 1160$

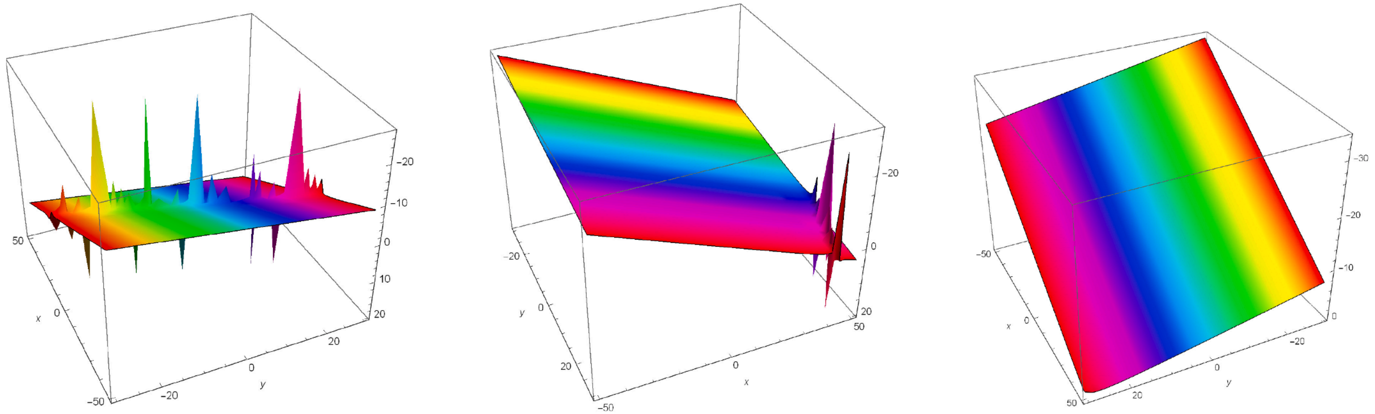
Fig. 4. 3D-shapes of annihilation of multi-soliton profiles for (66) with $\alpha = 3, \beta = 2, \gamma = 0.07, \beta_7 = 10$ and $z = 1$.

arbitrary function $f_3(t) = \frac{1}{t^2}$ and constants $a = 11, b = 3, c = 0.11, \beta_9 = 10, t = 13$.

Fig. 7 : The dynamical behavior of the solution (113) reflects the interaction between a lump type soliton and a parabolic wave profile by adopting the adequate choice of arbitrary function $f_2(t) = f_3(t) = t$ and constants $\alpha = 7, \beta = 3.32, \gamma = 3.3, \beta_{11} = 10, z = 0.019$. These solitons arrive at a balance between dispersion and nonlinearity.

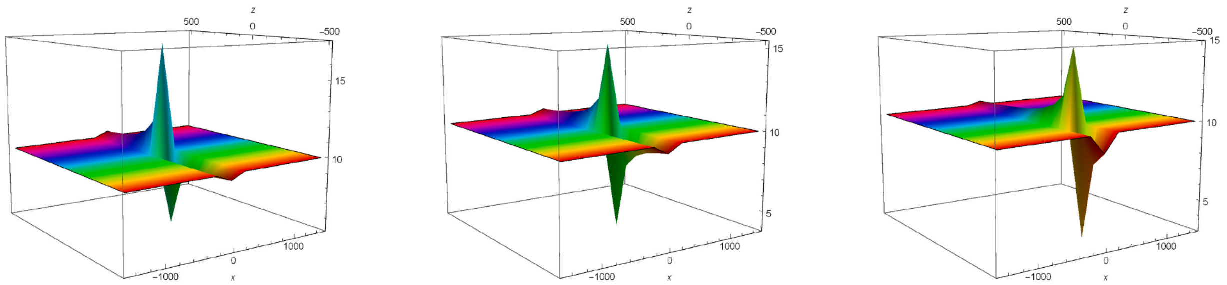
Fig. 8 : In this figure, the physical behavior of the solution is shown in Eq. (127) for functions $f_2(t) = t,\ f_3(t) = a_0 t,\ h(x) = \sin(x)$, and $g = \tan(x + t - z)$ and constants $a_0 = 10$. The annihilation in the solution wave profile is recorded with time, which results give a stationary profile.

Fig. 9 : The solution given by Eq. (123) represents multi-soliton profiles. The annihilation of a multi-soliton for u is confirmed with different values of t. At $t = 0.01$, a doubly soliton profile appears, and after $t = 2$, it is converted into a multi-soliton wave profile, which becomes a stationary profile at $t = 69$ with the function $f_2(t) = t$ and the constants $a_0 = 10,\ b_1 = 77, b_2 = 6, b_3 = 3, b_4 = 11, z =$



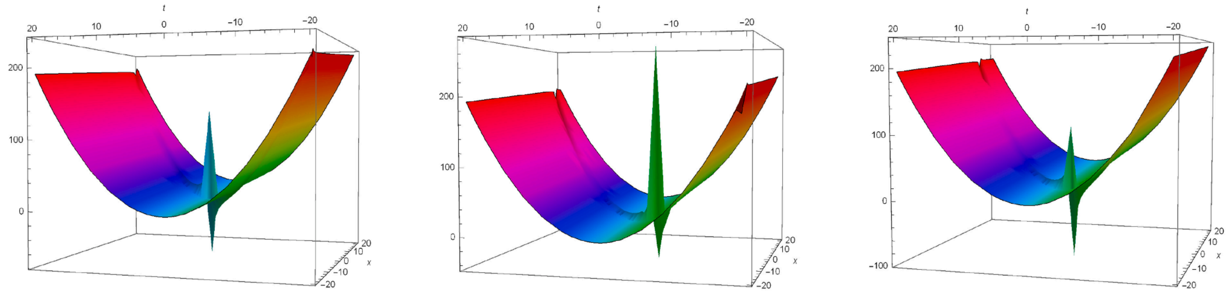
(a) $t = 1$ (b) $t = 60$ (c) $t = 77$

Fig. 5. 3D-shapes of annihilation of an elastic multi-soliton solution for (75) with $a = 13, b = 4, c = 1, \beta_8 = 10$, and $z = 11$.



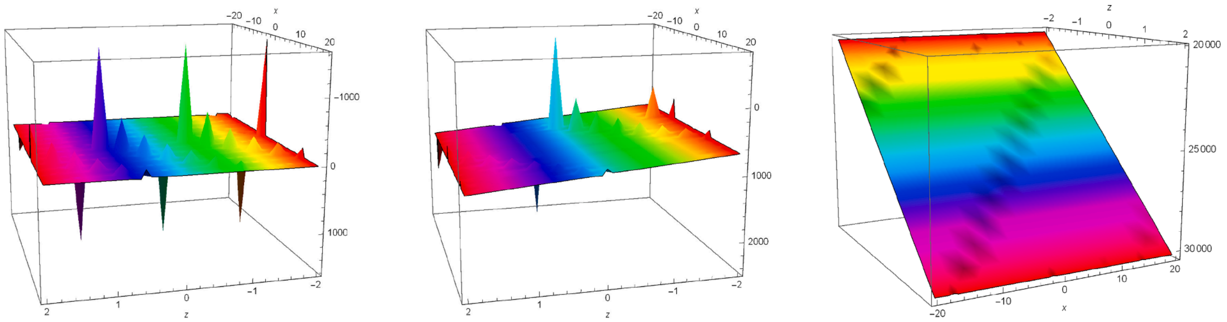
(a) $y = 1$ (b) $y = 4$ (c) $y = 12$

Fig. 6. 3D-shapes of different lump-type soliton solutions for (84) with $a = 11, b = 3, c = 0.11, \beta_9 = 10, t = 13$, and $f_3(t) = \frac{1}{t^2}$.



(a) $y = 0.013$ (b) $y = 3.50$ (c) $y = 12.40$

Fig. 7. 3D-shapes of interaction between parabolic wave and lump-type soliton solution for (113) and treating arbitrary constants $\alpha = 7, \beta = 3.32, \gamma = 3.3, \beta_{11} = 10, z = 0.019$ and $f_2(t) = t,\ f_3(t) = t$.



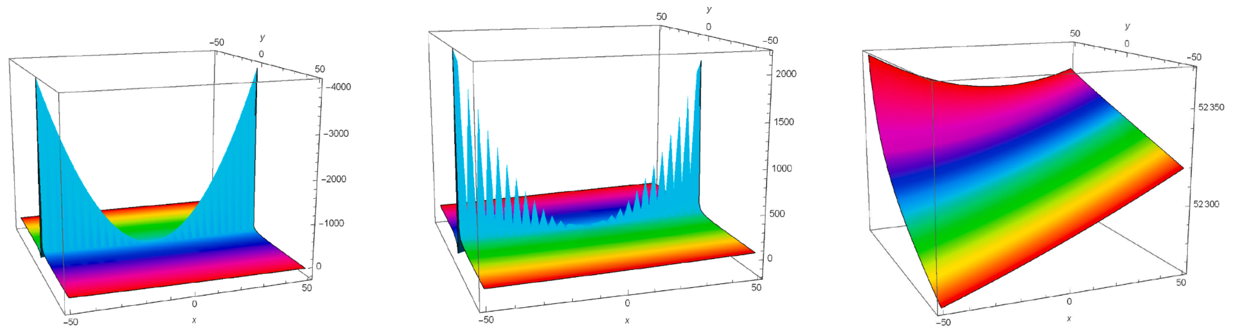
(a) $t = 1$ (b) $t = 21$ (c) $t = 71$

Fig. 8. 3D-shapes of annihilation of a multi-solitons solution for (127) with $a_0 = 1$ as functions $f_1(t) = t,\ h(x) = \sin(x)$, and $g(x + t - z) = \tan(x + t - z)$.

12 for numerical simulation.

6. Conclusion

In summary, we studied and discussed group invariance properties of the (3+1)-dimensional generalized Kadomtsev-Petviashvili (gKP) equation using a group-theoretic method and attained various Lie symmetry reductions. The infinitesimal generators, Lie symmetries, an optimal system of one-dimensional subalgebras, and similarity solutions were presented. The gKP equation was transformed into a number of interesting nonlinear ordinary differential equations and group-invariant solutions were obtained, which involve many arbitrary constants and functions. Also, the dynamical characteristics of the gKP equation were examined through group-invariant solutions, and consequently, different dynamical wave structures of multi-solitons, W-shaped solitons, doubly solitons, kink-type solitons, lump-type solitons and annihilation of multi-solitons were demonstrated via the produced solutions in the analysis and discussions section that makes this current study more trustworthy. Furthermore, the obtained results are entirely new and different compared to the works [29,30].



(a) $t = 0.01$ (b) $t = 2$ (c) $t = 69$

Fig. 9. 3D-shapes of annihilation of a doubly soliton solution for (123) with $b_1 = 77, b_2 = 6, b_3 = 3, b_4 = 11, z = 12$, and $f_1(t) = t$.

The study also shows that the Lie symmetry method is highly powerful, beneficial and effective, and we can definitely implement on other nonlinear evolution equations appearing in physical and engineering sciences. On the other hand, lump solutions exist with higher-order dispersion relations [32] and even for linear wave equations [33]. Such solutions involve many arbitrary constants [34], and therefore, it should be interesting to see how lump solutions could be transformed within the Lie symmetry formulation and explore new characteristics that lump solutions possess.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The third author, Amit Kumar gratefully acknowledges the financial support by the Sri Venkateswara College, University of Delhi, India.

References

[1] C.S. Gardner, J.M. Greene, M.D. Kruskai, R.M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. 19 (1967) 1095–1097.
[2] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin, New York, 1991.
[3] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[4] S.J. Liao. The proposed homotopy analysis technique for the solution of nonlinear problems, Shanghai Jiao Tong University, 1992. Ph.D. thesis.
[5] M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, Numer. Meth. Part. Differ. Equ. 26 (2010) 448–479.
[6] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011) 140–144.
[7] W.X. Ma, Bilinear equations and resonant solutions characterized by bell polynomials, Rep. Math. Phys. 72 (2013) 41–56.
[8] P.J. Olver, Applications of Lie groups to differential equations. Graduate Texts in Mathematics second ed., 107, Springer-Verlag, New York, 1993.
[9] S. Kumar, D. Kumar, Solitary wave solutions of (3+1)-dimensional extended Zakharov-Kuznetsov equation by lie symmetry approach, Comput. Math. Appl. 77 (2019) 2096–2113.
[10] S. Kumar, A. Kumar, Lie symmetry reductions and group invariant solutions of (2+1)-dimensional modified veronese web equation, Nonlinear Dyn. 98 (2019) 1891–1903.
[11] S. Kumar, D. Kumar, A.M. Wazwaz, Group invariant solutions of (3+1)-dimensional generalized b-type kadomstsev petviashvili equation using optimal system of lie subalgebra, Phys. Scr. 94 (2019), 065204.
[12] S. Kumar, A. Kumar, H. Kharbanda, Lie symmetry analysis and generalized invariant solutions of (2+1)-dimensional dispersive long wave (DLW) equations, Phys. Scr. 95 (2020), 065207.
[13] S. Kumar, D. Kumar, A.M. Wazwaz, A. Kumar, Group invariant solutions of (2+1)-dimensional rddym equation using optimal system of lie subalgebra, Phys. Scr. 94 (2019), 115202.
[14] D. Kumar, S. Kumar, Some new periodic solitary wave solutions of (3+1)-dimensional generalized shallow water wave equation by lie symmetry approach, Comput. Math. Appl. 78 (2019) 857–877.
[15] A. Paliathanasis, Lie symmetry analysis and one-dimensional optimal system for the generalized 2 + 1 kadomtsev-petviashvili equation, Phys. Scr. 95 (2020). Article 055223
[16] F. Güngör, P. Winternitz, Generalized Kadomtsev-Petviashvili equation with an infinite-dimensional symmetry algebra, J. Math. Anal. Appl. 276 (1) (2002) 314–328.
[17] M.R. Ali, W.X. Ma, New exact solutions of Bratu Gelfand model in two dimensions using lie symmetry analysis, Chin. J. Phys. 65 (2020) 198–206.
[18] M.S. Osman, D. Baleanu, A.R. Adem, K. Hosseini, M. Mirzazadeh, M. Eslami, Double-wave solutions and lie symmetry analysis to the (2 + 1)-dimensional coupled burgers equations, Chinese J. Phys. 63 (2020) 122–129.
[19] W.X. Ma, A. Abdeljabbar, A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation, Appl. Math. Lett. 12 (2012) 1500–1504.
[20] W.X. Ma, A. Abdeljabbar, M.G. Asaad, Wronskian and Grammian solutions to a (3+1)-dimensional generalized KP equation, Appl. Math. Comput. 217 (2011) 10016–10023.
[21] W.X. Ma, Z.N. Zhu, Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm, Appl. Math. Comput. 218 (2012) 11871–11879.
[22] A.M. Wazwaz, Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation, Commun Nonlinear Sci Numer Simul. 17 (2012) 491–495.
[23] B. Dubrovin, T. Grava, C. Klein, On critical behaviour in generalized Kadomtsev-Petviashvili equations, Phys. Nonlinear Phenom. 333 (2016) 157–170.

[24] M.S. Hashemi, A. Haji-Badali, F. Alizadeh, D. Baleanu, Integrability, invariant and soliton solutions of generalized Kadomtsev-Petviashvili modified equal width equation, Optik 139 (2017) 20–30.
[25] S.T. Mohyud-Din, A. Irshad, N. Ahmed, U. Khan, Exact solutions of (3+1)-dimensional generalized KP equation arising in physics, Results Phys. 7 (2017) 3901–3909.
[26] F.H. Lin, S.T. Chen, Q.X. Qu, J.P. Wang, X.W. Zhou, X. L., Resonant multiple wave solutions to a new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation: linear superposition principle, Appl. Math. Lett. 78 (2018) 112–117.
[27] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[28] C.Y. Qin, S.F. Tian, X.B. Wang, T.T. Zhang, On breather waves, rogue waves and solitary waves to a generalized (2+1)-dimensional Camassa-Holm-Kadomtsev-Petviashvili equation, Commun. Nonlinear Sci. Numer. Simul. 62 (2018) 378–385.
[29] V. Jaduan, S. Kumar, Symmetry analysis and invariant solutions of (3+1)-dimensional kadomtsev-petviashvili equation, Int. J. Geom. Methods Mod. Phys. 15 (2018), 1850125.
[30] R. Saleh, A.S. Rashed, New exact solutions of (3+1)-dimensional generalized Kadomtsev-Petviashvili equation using a combination of lie symmetry and singular manifold methods, Math. Meth. Appl. Sci. 43 (2020) 2045–2055.
[31] X. Hu, Y. Li, Y. Chen, A direct algorithm of one-dimensional optimal system for the group invariant solutions, J. Math. Phys. 56 (2015), 053504.
[32] W.X. Ma, L.Q. Zhang, Lump solutions with higher-order rational dispersion relations, Pramana - J. Phys. 94 (2020), 0043.
[33] W.X. Ma, Lump and interaction solutions to linear PDEs in 2+1 dimensions via symbolic computation, Mod. Phys. Lett. B 33 (2019), 1950457.
[34] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differ. Equ. 264 (2018) 2633–2659.