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Lie group analysis with the optimal system, generalized invariant solutions, and an enormous variety of different wave profiles for the higher-dimensional modified dispersive water wave system of equations

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Abstract In this work, the method of the Lie group of invariance is used to explore a (2+1)-dimensional modified dispersive water wave (mDWW) system of equations. This system is used to describe dispersive, nonlinear, long gravity waves traveling in two horizontal directions on shallow water. Calculating infinitesimal generators through symbolic computation is accomplished. Employing infinitesimal generators, the vector fields are obtained, and corresponding to the vector fields, the commutator table and the adjoint table are constructed. Furthermore, based on the adjoint table, the one-dimensional optimal system of subalgebras is obtained. With the aid of the optimal system, similarity reductions are performed for different cases. Through similarity reductions, the considered system of nonlinear partial differential equations (PDEs) is converted to a system of ordinary differential equations (ODEs) using Lie symmetry analysis, resulting in closed-form group-invariant solutions. The graphs consist of the periodic solitons. Dromions and peakon excitations are revealed in the graphical representations of the solutions. Using these graphs, mathematicians and physicists can follow complicated physical phenomena more efficiently. The mDWW system is fully integrated and has numerous applications in tidal waves and ocean tsunamis.

1 Introduction

Since most physical phenomena are nonlinear in nature and can be modeled in the form of nonlinear PDEs, exact analytical solutions, different forms of solitons, and traveling wave solutions of nonlinear PDEs may very well characterize a variety of physical phenomena directly implicated in our surroundings. These nonlinear PDEs have various physical applications in a variety of scientific fields, especially in ocean engineering, applied mathematics, quantum mechanics, natural sciences, nonlinear optics, fluid dynamics, hydrodynamics, plasma physics, optical fibers, etc. [1–13]. With the help of the exact solutions, people may provide a better insight about the physical aspects related to the problems. Therefore, finding the explicit form of solutions of nonlinear PDEs is an important and crucial task in solving the nonlinear problems. So many techniques are developed by the researchers in literature to find the exact solutions of these nonlinear PDEs such as extended transformed rational function method [14], Inverse scattering method [15], Darboux transformations, [16], Backlund transformations [17], Jacobi elliptic function method [18, 19], multiple exp-function method [20], generalized unified method [21], and Lie symmetry technique [22–28]. Symmetry is a powerful and effective mathematical approach to conceptualize the nature's laws. The method of Lie group of invariance [29, 30] is one of the most significant and systematic techniques for obtaining the analytic group-invariant solutions of these nonlinear PDEs arising in the different fields of mathematical physics. For more details of the Lie symmetry method, one can go through with the papers [31–39]. The primary objective of this Lie group approach is to find infinitesimal generators that leave the partial differential equation (PDE) invariant. This procedure provides a dependable structure for solving nonlinear differential equations.

In this article, we investigate the enormous variety of invariant solutions to the following system of (2+1)-dimensional modified dispersive water wave (mDWW) equations [40]:

$$u_{yt} + u_{xxy} - 2v_{xx} - 2(uu_y)_x = 0,$$

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$$v_t - v_{xx} - 2(uv)_x = 0. \tag{1}$$

There are two variables involved here: u represents the horizontal velocity of waves and v represents the elevation above the undisturbed surface of the water. (x, y) and t represent the propagation and time scales, respectively. Dispersive nonlinear long gravity waves traveling horizontally on shallow water are described by this mDWW system (1).By using the symmetry constraint [41], one can also obtain system (1) from the Kadomtsev–Petviashvili equation. The explicit form of the solutions of system (1) is quite useful for the coastal and engineers to use the nonlinear dispersive water wave model in coastal and port designing [42, 43]. The system is completely integrable, and thus, it has many applications, while studying the propagation of ocean waves. Wen and Xu [44] studied the (2+1)-dimensional mDWW system and obtained some multiple soliton solutions using Hirota bilinear method. With the Painlevé–Bäcklund transformation and simplified Hirota's method, Wazwaz [45] solved the dispersive long-water wave system to find traveling wave-type solutions.

1.1 Motivation of the work

In the above paragraph, we have discussed much about the different types of the analytic solutions obtained for the mDWW system. Though the mDWW system is studied by many researchers with respect to derive the various explicit form of solutions, a researcher should always keen to analyze and explore the new types of closed-form solutions. Moreover, the closed-form solutions of the mdWW system (1) are extremely useful, especially for civil engineers to apply the nonlinear water wave model for the coastal and harbor designing. Motivated from the literature, the authors have obtained some more exact analytic solutions of the mDWW system via Lie symmetry approach in this work which shows the practical importance of the mDWW system.

The main motivation for completing the task in this paper is to generalize the work of Singh et al. [40]. They used the Lie group method on a (2+1)-dimensional mDWW system to seek very few solutions to the governing system of equations. By taking advantage of the same Lie group for invariance analysis, in this paper, we encountered more generalized invariant solutions with arbitrary functional parameters. The presence of these arbitrary functional parameters increases the importance and significance of this research. The obtained solutions are more generalized than in the author's work of Singh et al. [40]. It should be noted that the generalized invariant solutions of the mDWW system were obtained by selecting some special forms of arbitrary functional parameters such as trigonometric functions, hyperbolic functions, and rational functions.

1.2 Frame-work

The present article is made up into six sections as follows: Sect. 1 is introductory. It comprises the brief literature and the importance of the mDWW system of equations. Section 2 is about the symmetry analysis of the mDWW system. The Lie group technique is performed on system (1) to obtain the generators and the commutator table. Section 3 deals with the adjoint table and one-dimensional optimal system of (1). In Sect. 4, the new closed-form solutions are obtained for the mDWW system. Graphs are also provided to illustrate the obtained solutions in this section, which provided a better idea about the involved nonlinear phenomena. In Sect. 5, the obtained results are discussed in detail. Periodic solitons along with the dromion and peakon profiles are captured in the figures of the obtained solutions. In Sect. 6, a comparison is shown between the results obtained in this paper and the results reported in the literature. Section 7 represents the conclusion of the overall work about this paper. The computational packages Maple 16 and Wolfram Mathematica 13.0 are used for the computational simulation.

2 Lie group analysis

This section deals with the symmetry analysis of the mDWW equations. The Lie group of infinitesimal transformation about the parameter (ϵ) [22] for the mDWW equations (1) is taken as follows:

$$\begin{aligned}
u^\star &= u + \epsilon\,\phi + O(\epsilon^2),\\
v^\star &= v + \epsilon\,\psi + O(\epsilon^2),\\
x^\star &= x + \epsilon\,\xi + O(\epsilon^2),\\
y^\star &= y + \epsilon\,\eta + O(\epsilon^2),\\
t^\star &= t + \epsilon\,\tau + O(\epsilon^2),
\end{aligned} \tag{2}$$

A variable u, v, x, y, and t each has an infinitesimal ϕ, ψ, ξ, η, and τ , respectively. Thus, the vector field, which is associated with these generators, is taken as

$$\mathcal{P} = \xi\,\frac{\partial}{\partial x} + \eta\,\frac{\partial}{\partial y} + \tau\,\frac{\partial}{\partial t} + \phi\,\frac{\partial}{\partial u} + \psi\,\frac{\partial}{\partial v}. \tag{3}$$

Table 1 Commutator table

*	$\mathcal{P}_1(f_1)$	$\mathcal{P}_2(f_2)$	$\mathcal{P}_3(f_3)$
$\mathcal{P}_1(f_1)$	0	0	0
$\mathcal{P}_2(f_2)$	0	0	$\mathcal{P}_3(f_2f_3' - \frac{1}{2}f_2'f_3)$
$\mathcal{P}_3(f_3)$	0	$-\mathcal{P}_3(f_2f_3' - \frac{1}{2}f_2'f_3)$	0

Table 2 Adjoint table

*	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
$\mathcal{P}_1$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
$\mathcal{P}_2$	$\mathcal{P}_1$	$\mathcal{P}_2$	$e^{-\epsilon}\mathcal{P}_3$
$\mathcal{P}_3$	$\mathcal{P}_1$	$\mathcal{P}_2 + \epsilon\mathcal{P}_3$	$\mathcal{P}_3$

The prolongations formulas for the mDWW equations are as follows:

$$
\begin{aligned}
&Pr^3\mathcal{P} = \mathcal{P} + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \psi^x\frac{\partial}{\partial v_x} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}} + ...\\
&Pr^1\mathcal{P} = \mathcal{P} + \phi^y\frac{\partial}{\partial u_y} + \psi^x\frac{\partial}{\partial v_x} + ...
\end{aligned}
\tag{4}
$$

The above expressions are implemented into system (1), and we get

$$
\begin{aligned}
&\phi^{yt} + \phi^{xxy} - 2\psi^{xx} - 2\phi^x u_y - 2\phi^y u_x - 2u\phi^{yx} - 2\phi u_{yx} = 0,\\
&\psi^t - \psi^{xx} - 2\phi v_x - 2u\psi^x - 2u_x\psi - 2\phi^x v = 0.
\end{aligned}
\tag{5}
$$

Substituting the values of ϕ^t, ϕ^x, ϕ^y,ϕ^{xxy}, and ψ^x from [30] into equations (5), the set of differential equations is obtained, and by solving this set of differential equations, we obtain the infinitesimals as follows:

$$
\begin{aligned}
&\xi_x = \frac{x}{2}f_2'(t) + f_3(t), \quad \xi_y = f_1(y), \quad \xi_t = f_2(t),\\
&\eta_u = -\frac{u}{2}f_2'(t) - \frac{x}{4}f_2''(t) - \frac{1}{2}f_3'(t), \quad \eta_v = -\frac{v}{2}f_2'(t) - vf_1'(y),
\end{aligned}
\tag{6}
$$

where $f_1(y)$ is an arbitrary function of y only and $f_2(t)$ and $f_3(t)$ are arbitrary functions of t only. Now, with the help of (6), we obtain the following vector field for the mDWW equations (1), as in [9]

$$
\begin{aligned}
&\mathcal{P}_1(f_1) = f_1(y)\frac{\partial}{\partial y} - vf_1'(y)\frac{\partial}{\partial v}, \qquad \mathcal{P}_2(f_2) = \frac{x}{2}f_2'(t)\frac{\partial}{\partial x} + f_2(t)\frac{\partial}{\partial t} - \frac{u}{2}f_2'(t)\frac{\partial}{\partial u} - \frac{x}{4}f_2''(t)\frac{\partial}{\partial u} - \frac{v}{2}f_2'(t)\frac{\partial}{\partial v},\\
&\mathcal{P}_3(f_3) = f_3(t)\frac{\partial}{\partial x} - \frac{1}{2}f_3'(t)\frac{\partial}{\partial u}.
\end{aligned}
$$

3 Derivation of the optimal system

In this sections, we have obtained the optimal system for the mDWW equations. For obtaining the optimal system of Lie algebra, we determine the invariants for selecting the representative elements. To determine the invariants, we have obtained the following matrix representations of $ad(\mathcal{P}_i)$ by using Tables 1,2:

$$
\begin{aligned}
Ad(exp(\epsilon\mathbf{W}))(\mathcal{P}) &= e^{-\epsilon\mathbf{W}}\mathcal{P}e^{\epsilon\mathbf{W}} = \mathcal{P} - \epsilon[\mathbf{W}, \mathcal{P}] + \frac{1}{2!}\epsilon^2[\mathbf{W}, [\mathbf{W}, \mathcal{P}]] - \dots\\
&= (\alpha_1\mathcal{P}_1 + \cdots + \alpha_n\mathcal{P}_n) - \epsilon[\beta_1\mathcal{P}_1 + \cdots + \beta_n\mathcal{P}_n, \alpha_1\mathcal{P}_1 + \cdots + \alpha_n\mathcal{P}_n] + O(\epsilon^2)\\
&= (\alpha_1\mathcal{P}_1 + \cdots + \alpha_n\mathcal{P}_n) - \epsilon(\Theta_1\mathcal{P}_1 + \cdots + \Theta_n\mathcal{P}_n).
\end{aligned}
\tag{7}
$$

Here, $\Theta = \Theta(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$ is determined using Table 1. The commutator relations are presented in Table 1. On substituting $\mathcal{P} = \sum_{i=1}^{3}\alpha_i\mathcal{P}_i$ and $\mathbf{W} = \sum_{j=1}^{3}\beta_j\mathcal{P}_j$ in equation (12) with

$$
\Theta_1 = 0, \quad \Theta_2 = 0, \quad \Theta_3 = \alpha_3\beta_2 - \alpha_2\beta_3.
\tag{8}
$$

It follows that for any β_j, $1 \le j \le 3$,

$$
\Theta_1\frac{\partial\phi}{\partial\alpha_1} + \Theta_2\frac{\partial\phi}{\partial\alpha_2} + \Theta_3\frac{\partial\phi}{\partial\alpha_3} = 0.
\tag{9}
$$

In the above equation, we get the following system of equations by equating all like powers of β_j:

$$\beta_2 : \alpha_3 \frac{\partial \phi}{\partial \alpha_3} = 0,$$
$$\beta_3 : -\alpha_2 \frac{\partial \phi}{\partial \alpha_3} = 0. \tag{10}$$

By solving the system of equations (10), we get $\phi(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1, \alpha_2)$, where F is an arbitrary invariant function of α_1 and α_2 for the Lie algebra $\mathbb{R}^3$. Thus, the mDWW system (1) has only two invariants.

3.1 Formation of adjoint matrix

$F_i^s : g \to g$ defined as $\mathcal{P} \to Ad(exp(\epsilon_i \mathcal{P}_i))$ is a linear map, for $i = 1, 2, 3$. The matrices A_i^ϵ of $F_i^\epsilon (i = 1, 2, 3)$ with respect to basis $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ are given as follows and defined [10] as

$$A_1^\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2^\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\epsilon_1} \end{pmatrix}, \qquad A_3^\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the following adjoint matrix is obtained by using these three matrices as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon_2 \\ 0 & 0 & e^{-\epsilon_1} \end{pmatrix}. \tag{11}$$

3.2 Optimal system for the mDWW equations

The adjoint transformation system to the RW (1) model is

$$(\alpha_1, \alpha_2, \alpha_3) \cdot A = (\gamma_1, \gamma_2, \gamma_3), \tag{12}$$

where A is the matrix which is expressed and

$$\gamma_1 = \alpha_1, \quad \gamma_2 = \alpha_2, \quad \gamma_3 = \alpha_2 \epsilon_2 + \alpha_3 e^{-\epsilon_1}. \tag{13}$$

Case 1 *When $\alpha_1 = 1$ and $\alpha_2 = c$, where $c \neq \{0, 1\}$ is a real constant. Selecting a representative element $\tilde{\mathcal{P}} = \mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$ and putting $\gamma_1 = 1$, $\gamma_2 = c$, $\gamma_3 = 0$ in equation (13), we obtain the solution as*

$$\epsilon_3 = -\alpha_3. \tag{14}$$

Now, using the action of adjoint maps on $\tilde{\mathcal{P}}$, the coefficients of $\mathcal{P}_3$ from $\tilde{\mathcal{P}}$ will be vanished. Thus, $\tilde{\mathcal{P}} = \mathcal{P}_1 + a_2 \mathcal{P}_2$ is equivalent to $\mathcal{P}_1 + c\mathcal{P}_2 + \alpha_3 \mathcal{P}_3$.

Case 2 *When $\alpha_1 = 1$ and $\alpha_2 = 0$. Selecting a representative element $\tilde{\mathcal{P}} = \mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$, and putting $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$ in equation (13), we obtain the solution as*

$$\epsilon_2 = 0. \tag{15}$$

Now, using the action of adjoint maps on $\tilde{\mathcal{P}}$, the coefficients of $\mathcal{P}_2$ from $\tilde{\mathcal{P}}$ will be vanished. Thus, $\tilde{\mathcal{P}} = \mathcal{P}_1 + a_3 \mathcal{P}_3$ is equivalent to $\mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$.

Case 3 *When $\alpha_1 = 0$ and $\alpha_2 = 1$. Selecting a representative element $\tilde{\mathcal{P}} = \mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$, and putting $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$ in equation (13), we obtain the solution as*

$$\epsilon_1 = 0 \; \epsilon_3 = -\frac{\alpha_2}{\alpha_3}. \tag{16}$$

Now, using the action of adjoint maps on $\tilde{\mathcal{P}}$, the coefficients of $\mathcal{P}_1$ and $\mathcal{P}_3$ from $\tilde{\mathcal{P}}$ will be vanished. Thus, $\tilde{\mathcal{P}} = \alpha_2 \mathcal{P}_2$ is equivalent to $\mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$.

Case 4 *When $\alpha_1 = 0$ and $\alpha_2 = 0$. Selecting a representative element $\tilde{\mathcal{P}} = \mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$, and putting $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$ in equation (13), we obtain the solution as*

$$\epsilon_1 = \epsilon_2 = 0. \tag{17}$$

Now, using the action of adjoint maps on $\tilde{\mathcal{P}}$, the coefficients of $\mathcal{P}_1$ and $\mathcal{P}_2$ from $\tilde{\mathcal{P}}$ will be vanished. Thus, $\tilde{\mathcal{P}} = c\mathcal{P}_3$, $(c \neq 0)$ is equivalent to $\mathcal{P}_1 + \alpha_2 \mathcal{P}_2 + \alpha_3 \mathcal{P}_3$.

Thus, the mDWW system has the following optimal system:

$$\begin{aligned}(i)\ \mathfrak{S}_1 &= \mathcal{P}_1 + \alpha_2\mathcal{P}_2\\ (ii)\ \mathfrak{S}_2 &= \mathcal{P}_1 + \alpha_3\mathcal{P}_3\\ (iii)\ \mathfrak{S}_3 &= \alpha_2\mathcal{P}_2\\ (iv)\ \mathfrak{S}_4 &= c\mathcal{P}_3\end{aligned} \tag{18}$$

4 Generalized invariant solutions

The associated characteristic equation is

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}t}{\tau} = \frac{du}{\phi} = \frac{dv}{\psi}. \tag{19}$$

4.1 Vector field $\mathfrak{S}_1 = \mathcal{P}_1 + \alpha_2\mathcal{P}_2$

The vector field $\mathcal{P}_1 + \alpha_2\mathcal{P}_2$ gives the solution of (19) as follows:

$$\begin{aligned}u(x,y,t) &= \frac{U(X,Y)}{\sqrt{f_2(t)}} - \frac{Xf_2'(t)}{4\sqrt{f_2(t)}},\\ v(x,y,t) &= \frac{V(X,Y)}{\sqrt{f_2(t)}},\end{aligned} \tag{20}$$

with $X = \frac{x}{\sqrt{f_2(t)}}$ and $Y = y - \int \frac{k_1}{\alpha_2 f_2(t)}\,dt$, which represents the similarity invariants, and U and V are the arbitrary functions of X and Y. Inserting (20) into (1), we obtain

$$\begin{aligned}&2\alpha_2 U_Y U_X + 2\alpha_2 U U_{XY} - \alpha_2 U_{XXY} + 2\alpha_2 V_{XX} + k_1 U_{YY} = 0\\ &2\alpha_2 U_X V + 2\alpha_2 U V_X + \alpha_2 V_{XX} + k_1 V_Y = 0.\end{aligned} \tag{21}$$

Now, again using the STM on equation (21), the infinitesimals are determined as:

$$\begin{aligned}&\xi_X = b_1 X + b_2, \quad \xi_Y = 2Yb_1 + b_3,\\ &\eta_U = -b_1 U, \quad \eta_V = -3b_1 V,\end{aligned} \tag{22}$$

where b_1, b_2, and b_3 are the arbitrary constants. Taking $b_2 = 0$ and $b_3 = 0$, (22) gives the following values of U and V:

$$\begin{aligned}U(X,Y) &= \frac{L(\theta)}{\sqrt{Y}},\\ V(X,Y) &= \frac{M(\theta)}{Y^{3/2}},\end{aligned} \tag{23}$$

where $\theta = \frac{X}{\sqrt{Y}}$. Using the values of U and V in (21), we obtain

$$\begin{aligned}&L(\theta)\big(3k_1 - 4\alpha_2(\theta L''(\theta) + 3L'(\theta))\big) + 2\alpha_2\big(3L''(\theta) + 4M''(\theta)\big) + 2\alpha_2\theta\Big(L^{(3)}(\theta) - 2L'(\theta)^2\Big)\\ &+ \theta k_1\big(\theta L''(\theta) + 5L'(\theta)\big) = 0,\\ &M(\theta)\big(3k_1 - 4\alpha_2 L'(\theta)\big) - 2\alpha_2\big(2L(\theta)M'(\theta) + M''(\theta)\big) + \theta k_1 M'(\theta) = 0.\end{aligned} \tag{24}$$

Now, (24) gives the following solution:

$$\begin{aligned}M(\theta) &= \frac{\theta(-2B_1k_1 - 3k_1)}{16a_2} - \frac{\theta^3 k_1^2}{64a_2^2},\\ L(\theta) &= \frac{\theta(3k_1)}{8a_2} + \frac{B_1}{\theta},\end{aligned} \tag{25}$$

$$\begin{aligned}M(\theta) &= \frac{1}{6}\theta(-2B_2B_1 - 3B_2) - \frac{B_2^2\theta^3}{9},\\ L(\theta) &= B_2\theta + \frac{B_1}{\theta},\end{aligned} \tag{26}$$

$$\begin{aligned}M(\theta) &= B_3\theta,\\ L(\theta) &= \frac{B_1}{\theta},\end{aligned} \tag{27}$$

$$M(\theta) = B_3\theta,$$
$$L(\theta) = B_2\theta + \frac{B_1}{\theta} \tag{28}$$

where $B_i's$, $(i = 1, 2, 3)$ are the arbitrary constants. Therefore, the solutions of mDWW equations (1) are obtained as follows:

$$u_1 = \frac{x}{8f_2(t)}\left(\frac{3k_1}{a_2y - k_1\left(\int \frac{1}{f_2(t)}\,dt\right)} - 2f_2'(t)\right) + \frac{B_1}{x},$$
$$v_1 = \frac{a_2k_1x\left(k_1\left(-x^2\right) - 4(2B_1+3)f_2(t)\left(a_2y - k_1\left(\int \frac{1}{f_2(t)}\,dt\right)\right)\right)}{64f_2(t)^2\left(a_2y - k_1\left(\int \frac{1}{f_2(t)}\,dt\right)\right)^3}, \tag{29}$$

$$u_2 = \frac{1}{f_2(t)}\left(\frac{3B_2x}{3y - 8B_2\int \frac{1}{f_2(t)}\,dt} - \frac{1}{4}xf_2'(t)\right) + \frac{B_1}{x},$$
$$v_2 = \frac{3B_2x\left((2B_1+3)f_2(t)\left(3y - 8B_1\int \frac{1}{f_2(t)}\,dt\right) + 2B_2x^2\right)}{2f_2(t)^2\left(3y - 8B_2\int \frac{1}{f_2(t)}\,dt\right)^3}, \tag{30}$$

$$u_3 = \frac{B_1}{x} - \frac{xf_2'(t)}{4f_2(t)},$$
$$v_3 = \frac{B_3x}{y^2f_2(t)}, \tag{31}$$

$$u_4 = \frac{1}{f_2(t)}\left(\frac{B_2x}{y - 2B_2\int \frac{1}{f_2(t)}\,dt} - \frac{1}{4}xf_2'(t)\right) + \frac{B_1}{x},$$
$$v_4 = \frac{B_3x}{f_2(t)\left(y - 2B_2\int \frac{1}{f_2(t)}\,dt\right)^2}. \tag{32}$$

4.1.1 Take $\alpha_2 = 0$, Vector field $\mathfrak{S}_1 = \mathcal{P}_1$

The vector field $\mathcal{P}_1$ provides the solution:

$$u(x, y, t) = U(X, T),$$
$$v(x, y, t) = \frac{V(X, T)}{f_1(y)}. \tag{33}$$

Here, $X = x$ and $T = t$ represents the symmetry invariants, and U and V are the arbitrary functional parameters of X and T. Inserting (33) into (1), we get

$$V_{XX} = 0,$$
$$-2U_XV - 2UV_X + V_T - V_{XX} = 0. \tag{34}$$

Now, integrating Eq. (34) twice, we obtain

$$U(X, T) = \frac{H_1(T)}{H_3(T) + XH_4(T)} + \frac{\frac{1}{2}X^2H_4'(T) + XH_3'(T)}{2(H_3(T) + XH_4(T))},$$
$$V(X, T) = XH_2(T) + H_1(T), \tag{35}$$

where $H_1(T)$, $H_2(T)$, $H_3(T)$, and $H_4(T)$ are arbitrary functional parameters of T only. Using Eqs. (33) and (35), we obtain the following form of the solution of (2+1)-dimensional mDWW system (1):

$$u_5 = \frac{4H_1(t) + x^2H_4'(t) + 2xH_3'(t)}{4xH_4(t) + 4H_3(t)},$$
$$v_5 = \frac{xH_4(t) + H_3(t)}{f_1(y)}. \tag{36}$$

4.2 Vector field $\mathfrak{S}_3 = \alpha_2\mathcal{P}_2$

The vector field $\alpha_2\mathcal{P}_2$ provides the solution as follows:

$$u(x, y, t) = \frac{U(X, Y)}{\sqrt{f_2(t)}} - \frac{Xf_2'(t)}{4\sqrt{f_2(t)}},$$
$$v(x, y, t) = \frac{V(X, Y)}{\sqrt{f_2(t)}}, \tag{37}$$

with $X = \frac{x}{\sqrt{f_2(t)}}$ and $Y = y$ represents similarity invariants, and U and V are the arbitrary functional parameters of X and Y. Substituting (37) into (1), we obtain

$$2(U_Y U_X + UU_{XY} + V_{XX}) - U_{XXY} = 0,$$
$$2U_X V + 2UV_X + V_{XX} = 0. \tag{38}$$

Now, again using the STM on equation (38), the infinitesimals are determined as:

$$\xi_X = b_1X + b_2, \quad \xi_Y = G_1(Y),$$
$$\eta_U = -b_1U, \quad \eta_V = -V(G_1'(Y) + b_1), \tag{39}$$

where b_1 and b_2 are the arbitrary constants and $G_1(Y)$ is arbitrary function. Taking $b_2 = 0$ and $G_1(Y) = Y$, (39) gives the following values of U and V:

$$U(X, Y) = Y^{-b_1}L(\theta),$$
$$V(X, Y) = Y^{-b_1-1}M(\theta), \tag{40}$$

where $\theta = \frac{X}{Y^{b_1}}$. Using the values of U and V in (38), we obtain

$$-b_1\theta L^{(3)}(\theta) + 2b_1\theta L(\theta)L''(\theta) - 3b_1L''(\theta) + 2b_1\theta L'(\theta)^2 + 6b_1L(\theta)L'(\theta) - 2M''(\theta) = 0,$$
$$2M(\theta)L'(\theta) + 2L(\theta)M'(\theta) + M''(\theta) = 0. \tag{41}$$

Now, (41) gives the following solution:

$$M(\theta) = B_5\theta,$$
$$L(\theta) = \frac{B_6}{\theta}, \tag{42}$$
$$M(\theta) = B_4 + B_5\theta,$$
$$L(\theta) = 0, \tag{43}$$

where $B_i's$, $(i = 4, 5, 6)$ are arbitrary constants. Thus, the solution of mDWW system (1) is obtained as follows:

$$u_6 = \frac{B_6}{x} - \frac{xf_2'(t)}{4f_2(t)},$$
$$v_6 = \frac{B_5xy^{-2b_1-1}}{f_2(t)}, \tag{44}$$
$$u_7 = -\frac{xf_2'(t)}{4f_2(t)},$$
$$v_7 = \frac{y^{-2b_1-1}\left(B_4y^{b_1}\sqrt{f_2(t)} + B_5x\right)}{f_2(t)}. \tag{45}$$

4.3 Vector field $\mathfrak{S}_4 = c\mathcal{P}_3$

The vector field $\mathcal{P}_3$ provides the solution:

$$u(x, y, t) = U(Y, T) - \frac{xf_3'(t)}{2f_3(t)},$$
$$v(x, y, t) = V(Y, T), \tag{46}$$

where $Y = y$ and $T = t$ represents the symmetry invariants, and U and V are the arbitrary functional parameters of Y and T. Inserting (46) into (1), we get

$$\frac{f_3'(T)U_Y}{f_3(T)} + U_{YT} = 0,$$

$$\frac{f_3'(T)V}{f_3(T)} + V_T = 0. \tag{47}$$

Equation (47) on integration twice gives

$$U(Y,T) = \int \frac{H_5(Y)}{f_3(T)}\,dY + H_6(T),$$
$$V(Y,T) = \frac{H_5(Y)}{f_3(T)}, \tag{48}$$

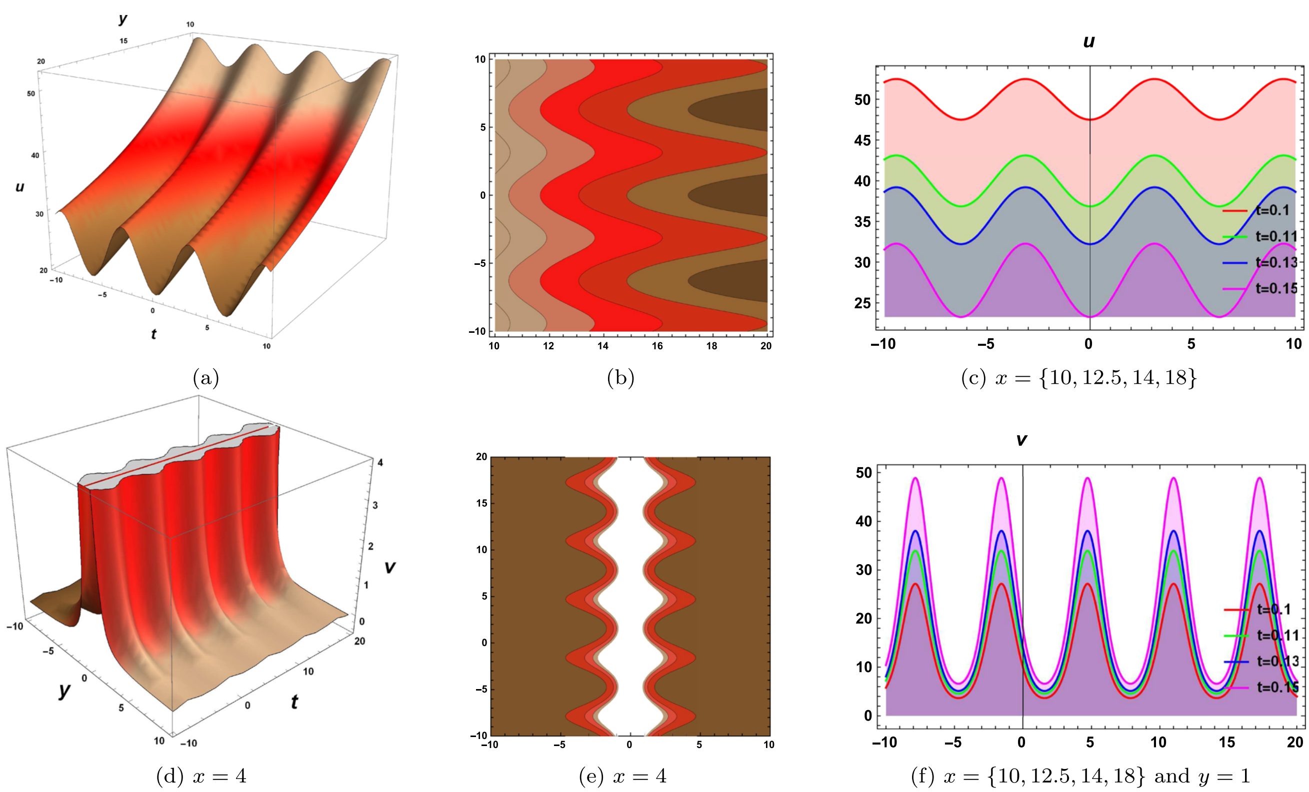
where H_5 is the arbitrary functional parameter of Y only and H_6 is the arbitrary function of T only. Using Eqs. (46) and (48), we obtain the solution of mDWW system (1) as follows:

$$u_8 = \frac{2\int H_5(y)\,dy - x f_3'(t)}{2 f_3(t)} + H_6(t),$$
$$v_8 = \frac{H_5(y)}{f_3(t)}. \tag{49}$$

5 Analysis and discussion of results

In this section, we present the graphic analysis of the obtained closed-form solutions to the (2+1)-dimensional mDWW equations. The graphical analysis presents a broader view of the physical phenomena involved in the mDWW equations. By choosing the appropriate values for the arbitrary parameters, the computational software Wolfram Mathematica constructs several types of two-dimensional (2D) and three-dimensional (3D) profiles that illustrate the mDWW equations' mechanism of physics. The obtained results are presented graphically as follows:

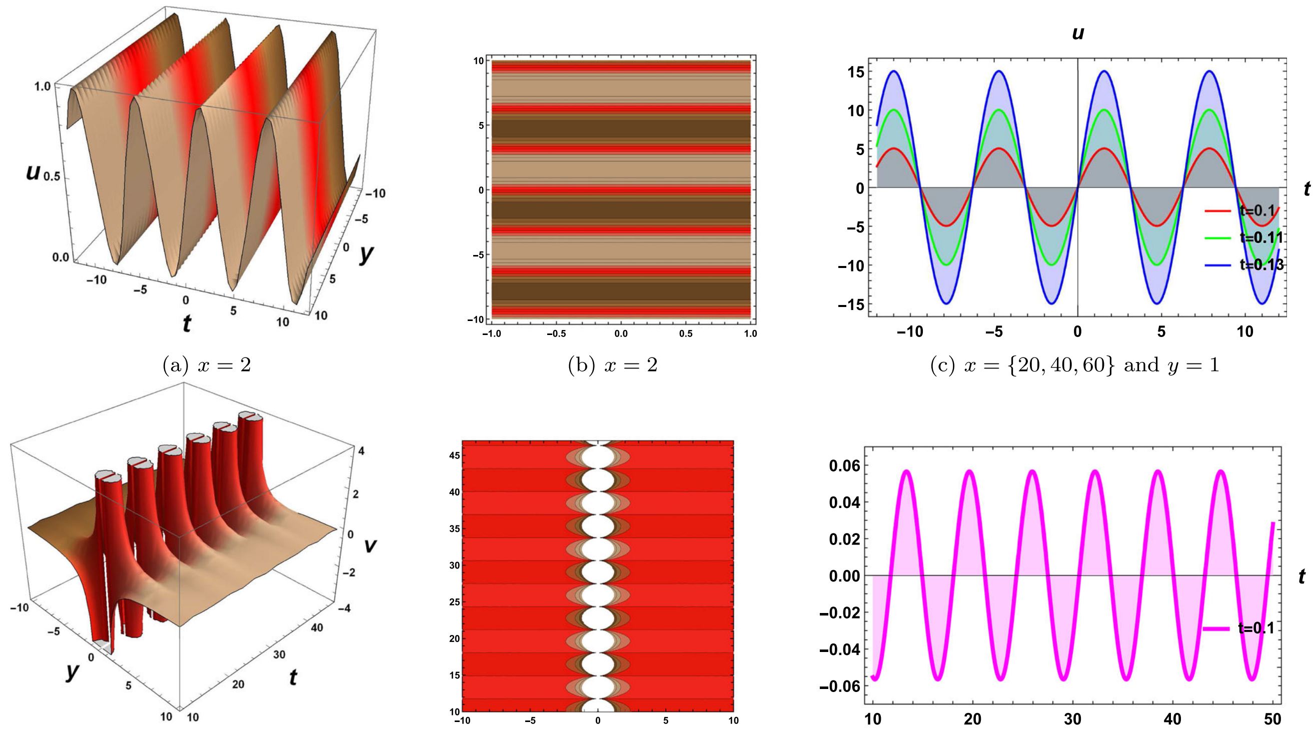
Figure 1 shows the solution obtained in (31) in various forms with additional parameters values $B_1 = 500$, $B_3 = 1$, and $f_2(t) = e^{\sin(t)}$. Here, figure (a) shows multi-dromions in intervals $(-10, 10)$ and $(10, 20)$ for u. Figure (b) represents its contour plot in intervals $(-10, 10)$ and $(10, 20)$. Figure (c) represents its absolute profile for $x = \{10, 12.5, 14, 18\}$ in $(-10, 10)$. Figure (d) represents three-dromions for $x = 4$ in intervals $(-10, 20)$ and $(-10, 10)$ for v. Figure (e) represents its contour plot for $x = 4$ in intervals $(-10, 20)$ and $(-10, 10)$. Figure (f) represents its absolute profile for $x = \{10, 12.5, 14, 18\}$, $y = 1$ in $(-10, 20)$.



(a) (b) (c) $x = \{10, 12.5, 14, 18\}$

(d) $x = 4$ (e) $x = 4$ (f) $x = \{10, 12.5, 14, 18\}$ and $y = 1$

Fig. 1 3D, contour, and absolute plots of the solution listed in (31)



(a) $x = 2$ (b) $x = 2$ (c) $x = \{20, 40, 60\}$ and $y = 1$

(d) $x = 4$ (e) $x = 4$ (f) $x = 4$ and $y = 10$

Fig. 2 3D, contour, and absolute plots of the solution listed in (32)

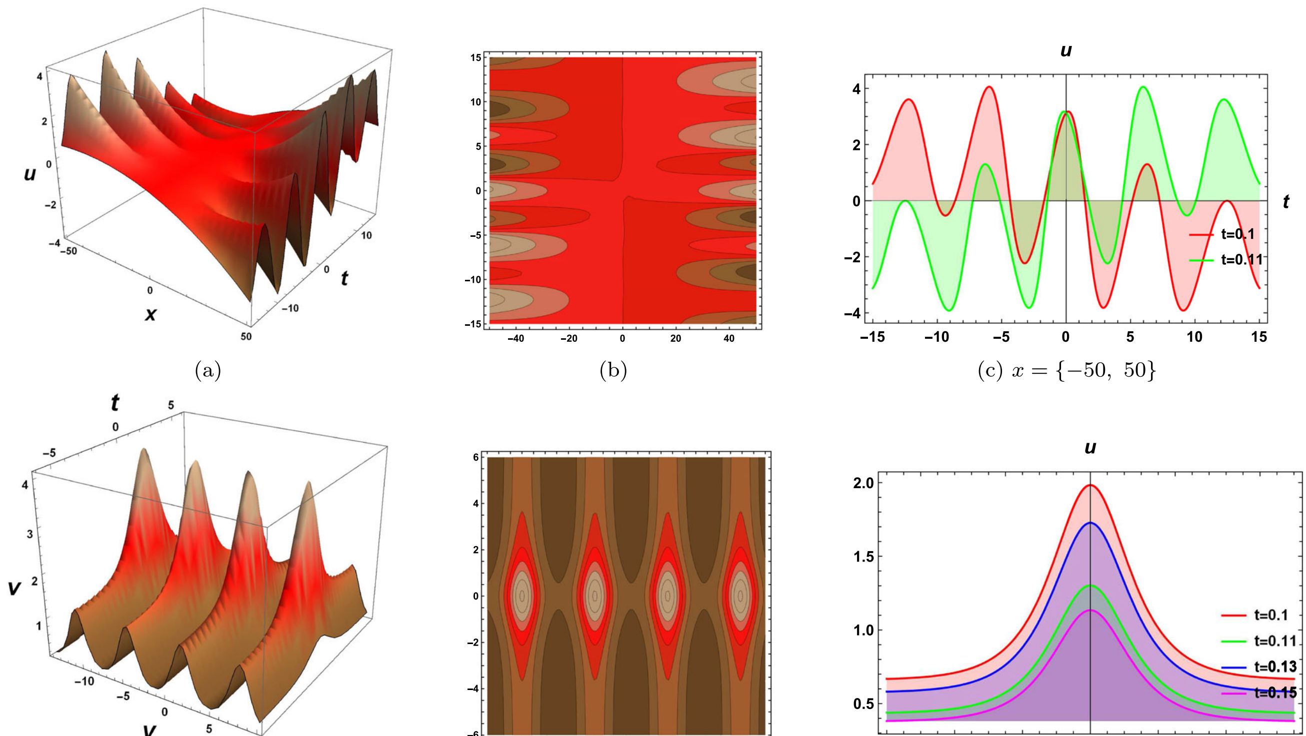
Figure 2 shows the solution obtained in (32) with additional parameters values $B_1 = 1$, $B_2 = 0$, and $f_2(t) = e^{\cos(t)}$, for $x = 2$ in $(-12, 12)$ and $(-10, 10)$. Here, figure (a) represents the periodic waves for u and v. Figure (b) represents its contour plot in intervals $(-10, 10)$ and $(-1, 1)$. Figure (c) represents its absolute profile for $x = \{20, 40, 60\}$, $y = 1$ in $(-12, 12)$. Figure (d) shows periodic solitons with additional parameters value $B_2 = 0.01$, $B_3 = 1$, and $f_2(t) = \frac{1}{\sin(t)+\cos(t)}$ for $x = 4$ in intervals $(10, 47)$ and $(-10, 10)$. Figure (e) represents its contour plot for $x = 4$ in intervals $(10, 47)$ and $(-10, 10)$. Figure (f) represents its absolute profile for $x = 4$, $y = 10$ in $(10, 50)$.

Figure 3 shows the solution obtained in (36) with arbitrary functions $H_1(t) = e^{\sin(t)}$, $H_3(t) = t^2 + 200$, and $H_4(t) = \sin(t)$ in intervals $(-15, 15)$ and $(-50, 50)$. Here, figure (a) represents the periodic waves for u and v. Figure (b) represents its contour plot in intervals $(-15, 15)$ and $(-50, 50)$. Figure (c) represents its absolute profile for $x = \{-50, 50\}$ in $(-15, 15)$. Figure (d) shows periodic solitons with arbitrary functions $f_1(y) = e^{-\sin(y)}$, $H_3(t) = \frac{1}{\cosh(t)}$, and $H_4(t) = 1$ for $x = 0.5$ in intervals $(-6, 6)$ and $(-14, 10)$. And figure (e) represents its contour plot for $x = 0.5$ in intervals $(-6, 6)$ and $(-14, 10)$. Figure (f) represents its absolute profile for $x = 0.5$, $y = \{-6, -3, 3, 6\}$ in $(-6, 6)$.

Figure 4 shows the solution obtained in (45) with arbitrary function $f_2(t) = e^{\sin(t)}$ in intervals $(-4\pi, 4\pi)$ and $(-15, 15)$. Here, figure (a) represents the periodic waves profile for u and v. Figure (b) represents its contour plot in intervals $(-4\pi, 4\pi)$ and $(-15, 15)$. Figure (c) represents its absolute profile for $x = \{5, 7.5, 10\}$ in $(-4\pi, 4\pi)$. Figure (d) shows periodic solitons with $b_1 = 1$, $B_4 = 1$, $B_5 = 1$, and arbitrary functions $f_2(t) = \frac{e^{\sin(t)}}{t}$ for $y = 1$ in intervals $(\pi, 10\pi)$ and $(40, 60)$. Figure (e) represents its contour plot for $y = 1$ in intervals $(\pi, 10\pi)$ and $(40, 60)$. Figure (f) represents its absolute profile for $x = \{10, 20\}$, $y = 1$ in interval $(\pi, 10\pi)$.

Figure 5 shows the solution obtained in (49) with arbitrary functions $H_5(y) = \cos(y)$, $H_6(t) = \sin(t)$, and $f_3(t) = e^{\cos(t)}$ for $x = 0.1$ in intervals $(-4\pi, 4\pi)$ and $(-4\pi, 4\pi)$. Here, figure (a) shows the periodic waves for u and v. Figure (b) represents its contour plot in intervals $(-4\pi, 4\pi)$ and $(-4\pi, 4\pi)$. Figure (c) represents its absolute profile for $x = \{0.1, 0.5\}$ and $y = \{-4\pi, 4\pi\}$ in $(-4\pi, 4\pi)$. Figure (d) shows periodic solitons with arbitrary functions $H_5(y) = \sin(y)$, and $f_3(t) = \cosh(t)$ in intervals $(-2\pi, 2\pi)$ and $(-3\pi, 3\pi)$. And figure (e) represents its contour plot in intervals $(-\frac{3}{2}\pi, \frac{3}{2}\pi)$ and $(-3\pi, 3\pi)$. Figure (f) represents its absolute profile for $t = \{3, 4, 5\}$ in interval $(-3\pi, 3\pi)$.

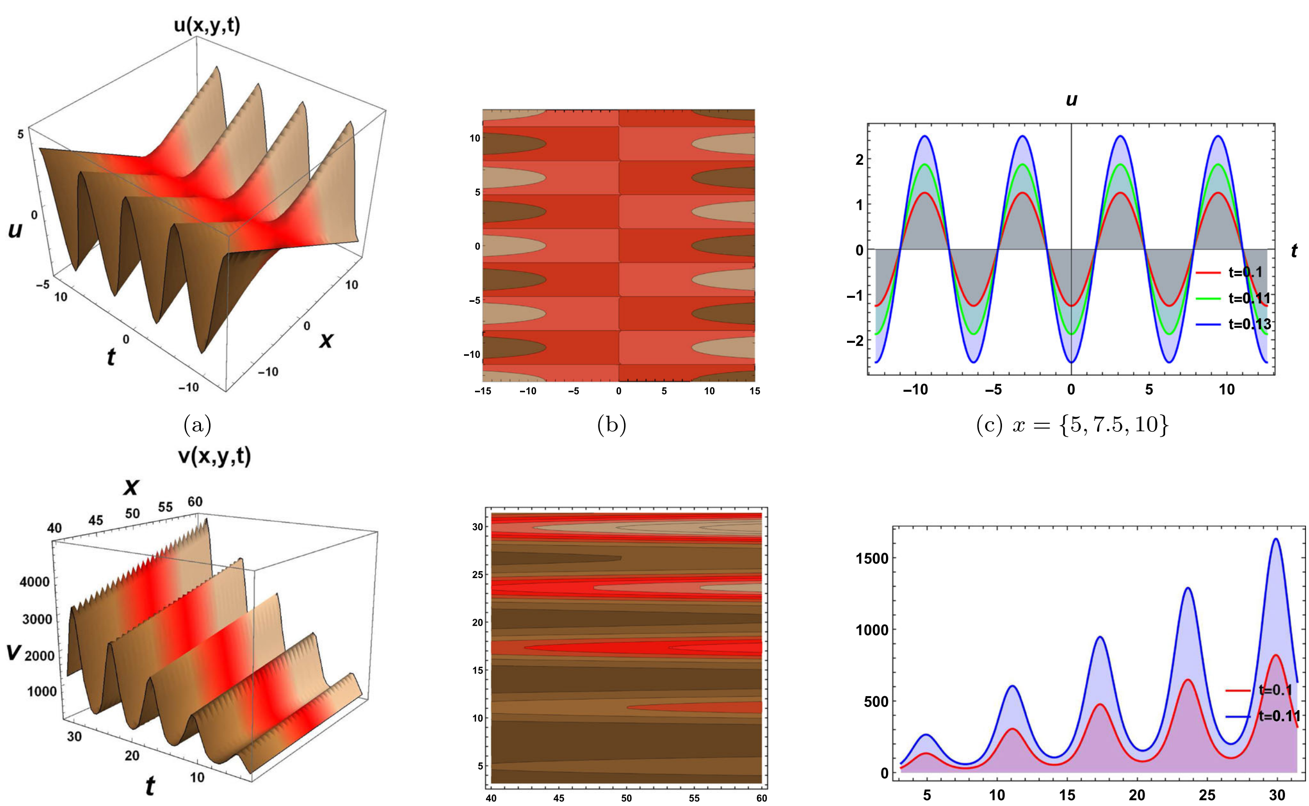
Thus, we can say that the closed-form solutions and their graphical representations presented in this work may be useful to further understand the propagation phenomena of nonlinear and dispersive gravity waves on the surface of the shallow water of uniform depth. It is beneficial to explore more about these periodic soliton solutions, which is of dromion-type structures, and related evolutional properties. These solutions might be useful in future studies to intricate the nature world. Additionally, it should be noted that this work uses Lie group method in a direct, powerful, and computer-literate manner, avoiding complex algebraic calculations.



(a) (b) (c) $x = \{-50,\ 50\}$

(d) $x = 0.5$ (e) $x = 0.5$ (f) $x = 0.5$ and $y = \{-6, -3, 3, 6\}$

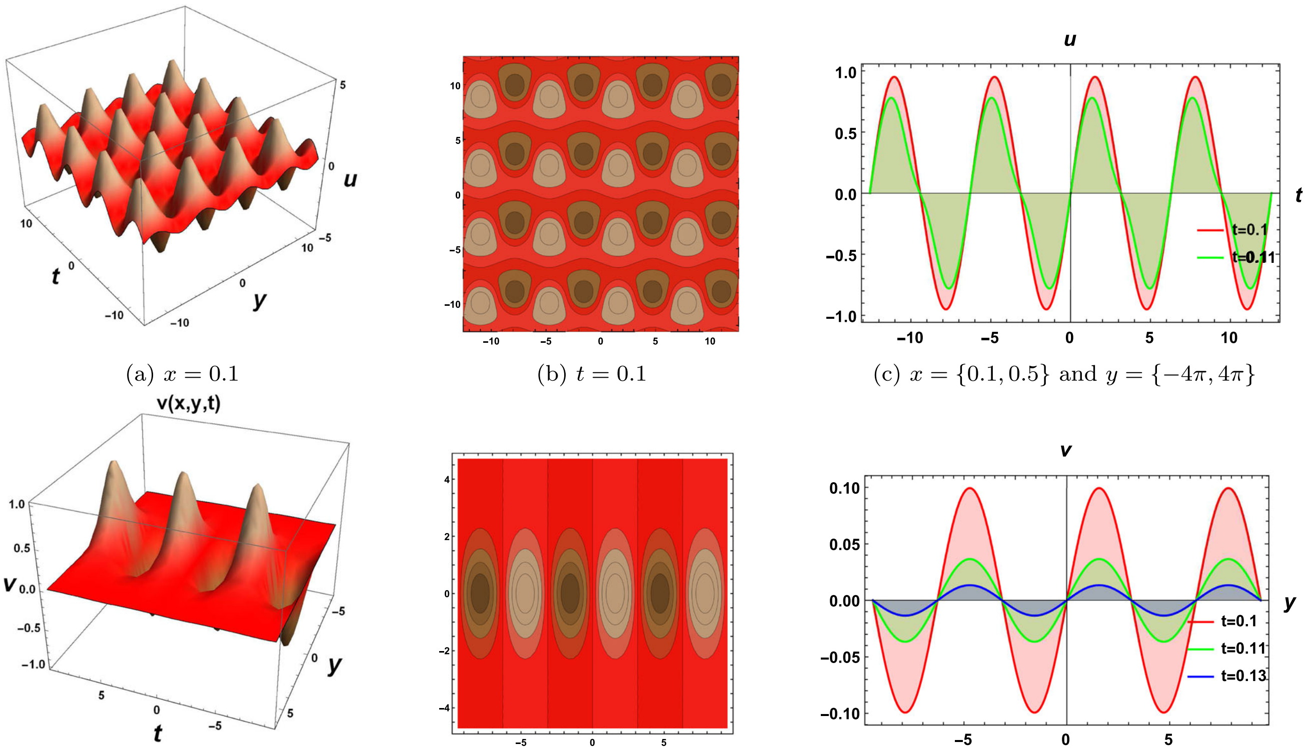
Fig. 3 3D, contour, and absolute plots of the solution listed in (36)



(a) (b) (c) $x = \{5, 7.5, 10\}$

(d) $y = 1$ (e) $y = 1$ (f) $x = \{10, 20\}$ and $y = 1$

Fig. 4 3D, contour, and absolute plots of the solution listed in(45)



(a) $x = 0.1$ (b) $t = 0.1$ (c) $x = \{0.1, 0.5\}$ and $y = \{-4\pi, 4\pi\}$

(d) (e) (f) t={3,4,5}

Fig. 5 3D, contour, and absolute plots of the solution listed in (49)

6 Comparison of the results

In this section, a comparison is shown between the obtained results and the results reported by the other researchers in literature. The comparison is as follows:

- Singh et al. [40] obtained only the commutator table and adjoint table for system (1) by selecting a few constant values of the arbitrary independent functional parameters, whereas in this research, the vector fields and optimal system are generated in the form of independent arbitrary functional parameters, making this study more significant and effective. It implies that we have the option of taking functional parameters in rational, trigonometric, hyperbolic, polynominal, or other forms.
- Singh et al. obtained only a few solutions in [40], by taking some constants into account in place of arbitrary functions appearing in the set of infinitesimal generators. In our article, we obtained a variety of generalized invariant solutions in the form of arbitrary functional parameters, making this research extends richer than previous findings.
- Finally, it is evident that our invariant solutions are more generalized because they are obtained in terms of different functional parameters. Because of the arbitrary functions, the majority of the solutions obtained from this research are generalized in comparison with previous findings.

7 Conclusion

In summary, a (2+1)-dimensional mDWW system of equations was investigated using the Lie group method of invariance. A one-parameter Lie group of transformations is obtained through symmetry analysis, which preserves the invariant nature of system (1). The infinitesimal generators are obtained using this one-parameter Lie group of transformations. The commutator and the adjoint tables are constructed, which are represented in Tables 1 and 2, respectively. Furthermore, an optimal system of subalgebras is established using infinitesimals. This optimal system is spanned by the vector fields $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. System (1) of PDEs is transformed into the system of ODEs using similarity reductions. Then, the exact group-invariant solutions are obtained for these ODEs. The obtained solutions are reported in Eqs. (29), (30), (31), (32), (36), (44), (45), and (49). These solutions are completely new and have never been reported in the literature. These solutions are depicted graphically to analyze the nature of the solutions via 2D, 3D, and contour plots by allocating different parametric preferences. The dromion and peakon excitations are captured in the graphical representation of the obtained solutions. Also, it is found that the solutions are periodic in nature and have periodic solitons. Solitons are caused by the balancing of nonlinear and dispersion terms that appears in the mathematical formulation of the governing system of equations.

The obtained solutions are also plotted to show how the specific varieties of solutions form as the space and time coordinates change. Consequently, we have formed the dynamics of various types of obtained exact solutions, which are suitable, demonstrative, and proficient in comparison with the results, which are presented in the literature and interpret the physical phenomenon. These obtained closed-form solutions may provide a benchmark to compare the numerical results and for the accuracy testing of newly developed numerical techniques. Moreover, the obtained closed-form solutions are extremely valuable and efficient in the fields of nonlinear science and ocean engineering for describing the realistic physical phenomena modeled in the form of a (2+1)-dimensional mDWW system of PDEs, which represents the formation, interaction, and breaking of waves as a result of external effects on the surface of the ocean. Furthermore, we can say that the Lie symmetry approach is a powerful and time-consuming tool for solving many nonlinear systems of partial differential equations.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

References

1. R. Radha, M. Laksmanan, Dromion like structures in the (2+1)-dimensional breaking soliton equation. Phys. Lett. A **197**, 7–12 (1995)
2. E.V. Krishnan, Two classes of new exact solutions to (2+1)-dimensional breaking soliton equation. Commun. Theorit. Phys. **44**(5), 807 (2005)
3. G.Q. Xu, Integrability of a (2+1)-dimensional generalized breaking soliton equation. Appl. Math. Lett. **50**, 16–22 (2015)
4. H.K. Barman, S. Aktar, M.H. Uddin, M.A. Akbar, M.S. Dumitru Baleanu, Osman, Physically significant waves solutions to the Riemann wave equations and the Landau-Ginsburg-Higgs equation. Result Phys. **27**, 104517 (2021)
5. K.H. Spatschek, P.K. Shukla, Nonlinear interaction of magneto-sound waves with whistler turbulence. Radio Sci. **13**(1), 211–214 (1978)
6. B. Ghanbari, M. Inc, A new generalized exponential rational function method to find exact special solutions for the resonance nonlinear Schrödinger equation. The Eur. Phys. J. Plus **133**(4), 142 (2018)
7. J.F. Zhang, J.P. Meng, New localized coherent structures to the (2+ 1)-dimensional breaking soliton equation. Phys. Lett. A **321**(3), 173–178 (2004)
8. S. Zhang, T. Xia, A generalized new auxiliary equation method and its application to the (2+ 1)-dimensional breaking soliton equations. Appl. Math. Comput. **190**(1), 510–516 (2007)
9. M.B. Abd-el-Malek, A.M. Amin, Lie group method for solving viscous barotropic vorticity equation in ocean climate models. Comput. Math. Appl. **75**(4), 1443–1461 (2018)
10. X. Hu, Y. Li, Y. Chen, A direct algorithm of one dimensional optimal system for the group invariant solutions. J Math Phys **56**, 053504 (2015)
11. R. Silambarasan, H.M. Baskonus, R.V. Anand, M. Dinakaran, B. Balusamy, W. Gao, Longitudinal strain waves propagating in an infinitely long cylindrical rod composed of generally incompressible materials and it’s Jacobi elliptic function solutions. Math. Comput. Simulat. **182**, 566–602 (2021)
12. J. Manafian, B.M. Ivatloo, M. Abapour, Lump-type solutions and interaction phenomenon to the (2+1)-dimensional Breaking Soliton equation. Appl. Math. Comput. **356**, 13–41 (2019)
13. S.K. Dhiman, S. Kumar, Different dynamics of invariant solutions to a generalized (3+1)-dimensional Camassa-Holm- Kadomtsev-Petviashvili equation arising in shallow water-waves. J Ocean Eng Sci (2022). https://doi.org/10.1016/j.joes.2022.06.019
14. H. Zhang, W.X. Ma, Extended transformed rational function method and applications to complexiton solutions. Appl Math Comput **230**, 509–515 (2014)
15. M. J. Ablowitz, M. A. Ablowitz, A. Peter, Solitons, nonlinear evolution equations and inverse scattering Cambridge University Press 149 (1991)
16. B. Xia, R. Zhou, Z. Aiao, Darboux transformation and multi-soliton solutions of the Camassa-Holm equation and modified Camassa-Holm equation. J Math Phys **57**(10), 103502 (2016)
17. Y.H. Yin, X. Lu, W.X. Ma, Backlund transformation, exact solutions and diverse interaction phenomena to a (3+1)-dimensional nonlinear evolution equation. Nonlinear Dynam **108**(4), 4181–4194 (2022)
18. A.R. Adem, Solitary and periodic wave solutions of the Majda-Biello system. Modern Phys Lett B **30**(15), 1650237 (2016)
19. A.R. Adem, B. Muatjetjeja, Conservation laws and exact solutions for a 2D Zakharov-Kuznetsov equation. Appl Math Lett **48**, 109–117 (2015)
20. A.R. Adem, The generalized (1+1)-dimensional and (2+1)-dimensional Ito equations: Multiple exp-function algorithm and multiple wave solutions. Comput Math Appl **71**, 1248–1258 (2016)
21. M.S. Osman, D. Baleanu, A.R. Adem, K. Hosseini, M. Mirzazadeh, M. Eslami, Double-wave solutions and Lie symmetry analysis to the (2+1)-dimensional coupled Burgers equations. Chinese J. Phys. **63**, 122–129 (2020)
22. G.W. Bluman, J.D. Cole, *Similarity Methods Different Equat* (Springer, New York, 1974)
23. A. Chauhan, K. Sharma, R. Arora, Lie symmetry analysis, optimal system, and generalized group invariant solutions of the (2+ 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation. Math Methods Appl Sci **43**(15), 8823–8840 (2020)
24. Z.Y. Zhang, C.B. Liu, Leibniz-type rule of variable-order fractional derivative and application to build Lie symmetry framework. Appl Math Comput **430**, 127268 (2022)
25. B. Muatjetjeja, Group classification and conservation laws of the generalized Klein Gordon Fock equation. Int J Modern Phys B **30**(28 & 29), 1640023 (2016)

26. A.R. Adem, X. Lü, Travelling wave solutions of a two-dimensional generalized Sawada-Kotera equation. Nonlinear Dynam **84**, 915–922 (2016)
27. A.R. Adem, B. Muatjetjeja, T.S. Moretlo, An Extended (2+1)-dimensional Coupled Burgers System in Fluid Mechanics: Symmetry Reductions; Kudryashov Method; Conservation Laws. Int J Theoret Phys **62**(2), 1–12 (2023)
28. B. Muatjetjeja, T. Moretlo, A. Adem, Soliton solutions and other analytical solutions of a new (3+1)-dimensional novel KP like equation. Int. J. Nonlinear Anal. Appl **14**(1), 2623–2632 (2023)
29. S. Lie, Theorie der Transformationsgruppen I. Mathematische Annalen **16**, 441–528 (1880)
30. P.J. Olver, *Applications of Lie groups to differential equations* (Springer, New York, 1993)
31. I. Humbu, B. Muatjetjeja, T.G. Motsumi, A.R. Adem, Periodic solutions and symmetry reductions of a generalized Chaffe-Infante equation. Partial Differential Equations in Applied Mathematics **7**, 100497 (2023)
32. S.F. Tian, Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation. Appl Math Lett **100**, 106056 (2020)
33. S. Kumar, S.K. Dhiman, Lie symmetry analysis, optimal system, exact solutions and dynamics of solitons of a (3+1)-dimensional generalised BKP-Boussinesq equation. Pramana - J Phys **96**, 31 (2022)
34. M. Devi, S. Yadav, R. Arora, Optimal system, invariance analysis of fourth-Order nonlinear ablowitz-Kaup-Newell-Segur water wave dynamical equation using lie symmetry approach. Appl Math Comput **404**, 126230 (2021)
35. B. Bira, T.R. Sekhar, D. Zeidan, Application of Lie groups to compressible model of two-phase flows. Physica Scripta **71**(1), 46–56 (2016)
36. S.K. Dhiman, S. Kumar, An optimal system, invariant solutions, conservation laws, and complete classification of Lie group symmetries fora generalized (2+1)-dimensional Davey-Stewartson system of equations for the wave propagation in water of finitedepth. Eur Phys J Plus **138**(3), 195 (2023)
37. S. Kumar, S. Rani, N. Mann, Diverse analytical wave solutions and dynamical behaviors of the new $(2+1)$-dimensional Sakovich equationemerging in fluid dynamics. Eur Phys J Plus **137**(11), 1226 (2023)
38. S. Kumar, S.K. Dhiman, A. Chauhan, Symmetry reductions, generalized solutions and dynamics of wave profiles for the (2+ 1)-dimensional system of Broer-Kaup-Kupershmidt (BKK) equations. Math Comput Simulat **196**, 319–335 (2022)
39. R. Kumar, A. Kumar, Dynamical behavior of similarity solutions of CKOEs with conservation law. Appl Math Comput **422**, 126976 (2022)
40. D. Singh, S. Yadav, R. Arora, A (2+ 1)-dimensional modified dispersive water-wave (MDWW) system: Lie symmetry analysis, optimal system and invariant solutions. Commun Nonlinear Sci Num Simulat **115**, 106786 (2022)
41. S.Y. Lou, X.B. Hu, Infinitely many Lax pairs and symmetry constraints of the KP equation. J Math Phys **38**(12), 6401–6427 (1997)
42. J.X. Fei, C.L. Zheng, Localized excitations in a dispersive long water-wave system via an extended projective approach. Zeitschrift für Naturforschung A **62**(3–4), 140–146 (2007)
43. C.Q. Dai, R.P. Chen, Solitons with fusion and fission properties in the (2+1)-dimensional modified dispersive water-wave system. Zeitschrift für Naturforschung A **61**(7–8), 307–315 (2006)
44. X.Y. Wen, X.G. Xu, Multiple soliton solutions and fusion interaction phenomena for the (2+1)-dimensional modified dispersive water-wave system. Appl Math Comput **219**(14), 7730–7740 (2013)
45. A.M. Wazwaz, Multiple soliton solutions and rational solutions for the (2+1)-dimensional dispersive long water-wave system. Ocean Eng **60**, 95–98 (2013)