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Application of the Lie symmetry approach to an extended Jimbo–Miwa equation in (3+1) dimensions

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Abstract In this article, we study Lie point symmetries, closed-form invariant solutions, and dynamics of exact solitons to an extended (3+1)-dimensional Jimbo–Miwa (JM) equation by employing the Lie symmetry method. Under the resulting symmetries, the extended JM equation is reduced to lower-dimensional equations. We exploit the travelling wave ansatz to determine closed-form invariant solutions of the reduced equations. The physical interpretations of the obtained solutions are exhibited in the forms of single solitons, multi-wave solitons, multiple solitons with parabolic waves, oscillating lump solitons, triply solitons, and double solitons via numerical simulation for adequate choices of the involved arbitrary constants through the mathematical software *Wolfram Mathematica*. These constructed solutions can help us better understand interesting nonlinear complex phenomena and mechanisms.

1 Introduction

Nonlinear partial differential equations (NPDEs) are extensively used to describe complex phenomena in many fields of science, particularly in mathematical physics and fluid mechanics. Travelling waves, solitons, compactons, peakons, shock waves, etc., are important wave solutions of nonlinear evolution equations. Those solutions have be applied successfully in various research areas of applied mathematics, solid-state physics, plasma physics, fluid dynamics, mathematical biology, and chemical kinetics. Closed-form solutions can help us understand mechanics of convoluted physical phenomena and dynamical processes modeled by nonlinear equations better. Moreover, those solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine appropriate parameters in model equations.

There are numerous significant methods [1–6] for obtaining exact solutions of NPDEs. A few direct ansätz have been introduced for the nonlinear Schrödinger equation, particularly for solitons and rogue waves, and a five-dimensional symmetry algebra has also been presented,

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which consists of Lie point symmetries [7]. The Hirota bilinear method has been applied to construction of lump solutions to generalized integrable equations of fourth order, and both nonlinearity and dispersion play important roles in formulating such localized solutions [8–10].

Over the past few years, the classical Lie symmetry technique is the most efficient, robust, and faithful mathematical tool to obtain closed-form invariant solutions to NPDEs. This technique allows us to reduce the number of independent variables in higher dimensions NPDEs by employing the similarity reductions. Moreover, this state-of-the-art methodology is based on invariance possessions under one-parameter Lie group transformations. For this reason, the classical Lie group theoretic technique is a widely used analytical mathematical approach for obtaining closed-form solutions to NPDEs with many applications in various contexts [11–19].

In the recent years, multidimensional integrable systems are one of the major subjects of research in integrable systems. The interesting and integrable (3+1)-dimensional Jimbo–Miwa (JM) equation is given by

$$u_{xxxy} + 3u_x u_{xy} + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0, \tag{1}$$

which is a second member in the Kadomtsev–Petviashvili (KP) hierarchy [20–22], describing certain interesting (3+1)-dimensional waves in mathematical physics. The JM equation behaves differently from typical (2+1)-dimensional integrable equations, for example, the KP equation, particularly since it does not pass the Painlev*é* test [23]. Various forms of solutions of the KP equation have been investigated over a period of time with a variety of helpful techniques, and interaction phenomena of rational, semirational, and abundant lump-type solutions of the (3+1)-dimensional JM equation are obtained in [24]. Specific soliton solutions to the (3+1)-dimensional JM equation are presented by the transformed rational function approach and collectively a B*ä*cklund transformation [25], whereas specific soliton solutions are acquired for the (3+1)-dimensional generalized KP, BKP, and JM equations, in terms of Wronskian determinants [26]. Zhang and Chen [27] discussed a combination of stripe solitons and lump solitons, which produces two different localized excitation phenomena of fusion and fission for a reduced (3+1)-dimensional JM equation, including interaction solutions between kink stripe solitons and rogue waves. Wazwaz [28] used the simplified Hirota's method to derive multiple soliton solutions of distinct physical structures for two extended (3+1)-dimensional JM equations.

In the present article, we would like to investigate the following extended (3+1)-dimensional JM equation:

$$\Delta := u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3(u_{xz} + u_{yz} + u_{zz}) = 0, \tag{2}$$

which includes two additional linear terms u_{yz} and u_{zz}. Here $u(x, y, z, t)$ is the wave amplitude function with three scaled spatial variables x, y, z and the temporal variable t. The extended form could describe more general dispersive waves than standard (3+1)-dimensional JM equation (1). Some novel significant contributions in the discipline of investigating solitary wave solutions, exact rational solutions, and multiple soliton solutions to several extended (3+1)-dimensional JM equations have been discussed in the previous studies [29–32].

In this work, we derive infinitesimal generators, geometric vector fields, commutation relations, and an adjoint table of the considered vectors to above-extended JM equation (2). Besides, we will construct exact invariant solutions and solitonic structures of exact solutions using the Lie group theoretic method via three stages of symmetry reductions in a way that the application-oriented reader can immediately get a practical implementation of the method.

The rest of the paper is organized as follows. In Sect. 2, we outline the method based on the construction of generators of infinitesimal transformations. In Sect. 3, we apply the Lie group theoretic technique to prolonged extended JM equation (2). We mention the Lie point symmetries and interesting symmetry reductions by forming the similarity solutions from the set of Lie algebras. Additionally, we construct vector fields which help us reduce the above-extended (3+1)-dimensional JM equation into some reduced lower-order PDEs and then ODEs. Finally, we derive abundant analytical exact solutions of Eq. (2). In the remaining two sections, we are devoted to the physical/graphical interpretation of the resulting solutions and the conclusion of our results for of the extended JM equation, respectively.

2 Lie symmetry analysis

Initially, we apply Lie symmetry reductions to extended JM equation (2) utilizing the Lie group method of symmetry transformations. We consider a one-parameter Lie group of infinitesimal transformations on ($x_1 = x, x_2 = y, x_3 = z, x_4 = t, u_1 = u$), defined as

$$\begin{aligned}
\tilde{x} &= x + \vartheta \xi^1(x, y, z, t, u) + O(\vartheta^2),\\
\tilde{y} &= y + \vartheta \xi^2(x, y, z, t, u) + O(\vartheta^2),\\
\tilde{z} &= z + \vartheta \xi^3(x, y, z, t, u) + O(\vartheta^2),\\
\tilde{t} &= t + \vartheta \tau(x, y, z, t, u) + O(\vartheta^2),\\
\tilde{u} &= u + \vartheta \eta(x, y, z, t, u) + O(\vartheta^2),
\end{aligned} \tag{3}$$

where ϑ is the continuous group parameter. The general vector field on $\mathbb{R}^3 \times \mathbb{R}$ takes the form

$$\mathbb{V} = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + \tau \partial_t + \eta \partial_u.$$

To obtain the Lie point symmetries to extended JM equation (2), associated vector fields must satisfy the invariant criteria $Pr^{(4)}\mathbb{V}(\Delta) = 0$ whenever $\Delta = 0$, where $Pr^{(4)}$ represents the fourth prolongation of $\mathbb{V}$ which can be written as [3,5]

$$\begin{aligned}
Pr^{(4)}\mathbb{V} =& \mathbb{V} + \eta^x \frac{\partial}{\partial u_x} + \eta^y \frac{\partial}{\partial u_y} + \eta^z \frac{\partial}{\partial u_z} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xxxy} \frac{\partial}{\partial u_{xxxy}} + \eta^{xx} \frac{\partial}{\partial u_{xx}}\\
&+ \eta^{xy} \frac{\partial}{\partial u_{xy}} + \eta^{yt} \frac{\partial}{\partial u_{yt}} + \eta^{xz} \frac{\partial}{\partial u_{xz}} + \eta^{yz} \frac{\partial}{\partial u_{yz}} + \eta^{zz} \frac{\partial}{\partial u_{zz}}.
\end{aligned}$$

Employing the invariance condition $Pr^{(4)}\mathbb{V}$ to extended JM equation (2), we get

$$\eta_{xxxy} + 3\eta_y u_{xx} + 3\eta_{xx} u_y + 3\eta_x u_{xy} + 3\eta_{xy} u_x + 2\eta_{yt} - 3(\eta_{xz} + \eta_{yz} + \eta_{zz}) = 0. \tag{4}$$

Putting the value of expressions η_{xxxy}, η_{xx}, η_{xy}, η_{xz}, η_{yz}, η_{zz}, η_y, etc., [3,5] into (4) provides the following system of determining equations

$$\begin{aligned}
&\xi^1_u = \xi^1_y = 0,\ \xi^1_x = \frac{\tau_t}{3},\ \xi^1_z = \frac{\tau_t}{12} + \frac{\xi^2_y}{4},\\
&\xi^2_u = \xi^2_x = \xi^2_{yy} = \xi^2_{yz} = \xi^2_{zz} = 0,\ \ \xi^2_t = \frac{3\xi^2_z}{2} = 0,\\
&\xi^3_t = -\frac{3}{4}\tau_t + \frac{3}{4}\xi^2_y + 3\xi^2_z,\\
&\tau_x = \tau_y = \tau_z = \tau_u = \tau_{tt} = 0,\\
&\eta_u = \frac{-\tau_t}{3},\ \eta_x = \frac{-\tau_t}{12} - \frac{\xi^2_y}{3} - \frac{\xi^2_z}{3} + \frac{2\xi^1_t}{3},\ \eta_y = -\frac{\tau_t}{2} - \frac{\xi^2_y}{4}.
\end{aligned} \tag{5}$$

On solving these determining equations (5), we get the most general infinitesimals to extended Jimbo–Miwa equation (2) as follows

$$\begin{aligned}
\xi^1 &= \alpha_1(t) + \frac{c_1}{6}(2x - z) + \frac{c_4}{2}z,\\
\xi^2 &= \frac{c_1}{12}(9t - 12y + 6z) + \frac{c_4}{12}(-9t + 24y - 6z) + \frac{c_3}{2}t + \frac{c_3}{3}z + c_6,\\
\xi^3 &= c_3t + c_4z + c_5,\\
\tau &= c_1t + c_2,\\
\eta &= \frac{2}{3}\alpha_1'(t)x + \alpha_2(t)z + \alpha_3(t) + \frac{c_1}{6}(-2x + y - 2u) - \frac{c_3}{3}x - \frac{c_4}{2}y,
\end{aligned} \tag{6}$$

where c_i, $i = 1, \ldots, 6$ are arbitrary parameter constants and α_1, α_2, and α_3 are functions of t. The prime ($'$) denotes the differentiation with respect to its indicated variable throughout the paper. The choices of α_1, α_2, and α_3 need to provide new physically meaningful solutions of Eq. (2). Therefore, we consider $\alpha_1(t) = \frac{c_7}{6}t + c_8$, and $\alpha_3(t) = c_9t + c_{10}$ are linear functions and $\alpha_2(t) = c_{11}t^2 + c_{12}t + c_{13}$ as a quadratic function with arbitrary constants c_i, $i = 1, \ldots, 13$.
In this way, Eq. (6) can be furnished as follows

$$\begin{aligned}
\xi^1 &= \frac{c_7}{6}t + c_8 + \frac{c_1}{6}(2x - z) + \frac{c_4}{2}z,\\
\xi^2 &= \frac{c_1}{12}(9t - 12y + 6z) + \frac{c_4}{12}(-9t + 24y - 6z) + \frac{c_3}{2}t + \frac{c_3}{3}z + c_6,\\
\xi^3 &= c_3t + c_4z + c_5,\\
\tau &= c_1t + c_2,\\
\eta &= \frac{c_7}{9}x + (c_{11}t^2 + c_{12}t + c_{13})z + c_9t + c_{10} + \frac{c_1}{6}(-2x + y - 2u) - \frac{c_3}{3}x - \frac{c_4}{2}y.
\end{aligned} \tag{7}$$

Thus the Lie algebra of infinitesimal symmetries of (2) can be generated with the help of the following vectors:

$$\begin{aligned}
\mathbb{V}_1 &= \frac{1}{6}(2x-z)\frac{\partial}{\partial x} + \frac{1}{12}(9t-12y+6z)\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \frac{1}{6}(-2x+y-2u)\frac{\partial}{\partial u},\\
\mathbb{V}_2 &= \frac{\partial}{\partial t},\ \mathbb{V}_3 = \left(\frac{t}{2}+\frac{z}{3}\right)\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - \frac{x}{3}\frac{\partial}{\partial u},\\
\mathbb{V}_4 &= \frac{z}{2}\frac{\partial}{\partial x} + \frac{1}{12}(-9t+24y-6z)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - \frac{y}{2}\frac{\partial}{\partial u},\\
\mathbb{V}_5 &= \frac{\partial}{\partial z},\ \mathbb{V}_6 = \frac{\partial}{\partial y},\ \mathbb{V}_7 = \frac{t}{6}\frac{\partial}{\partial x} + \frac{x}{9}\frac{\partial}{\partial u},\\
\mathbb{V}_8 &= \frac{\partial}{\partial x},\ \mathbb{V}_9 = t\frac{\partial}{\partial u},\ \mathbb{V}_{10} = \frac{\partial}{\partial u},\ \ \mathbb{V}_{11} = zt^2\frac{\partial}{\partial u},\ \ \mathbb{V}_{12} = zt\frac{\partial}{\partial u},\ \ \mathbb{V}_{13} = z\frac{\partial}{\partial u}.
\end{aligned} \tag{8}$$

2.1 Computation of Lie brackets

We outline a relation between two invariant solutions. The relation holds true if the one can be mapped to the other via a group of transformations generated by a linear combinations of the operators in (8). Those mappings determine the equivalence relation, which provides a partition on the set of all group invariant solutions. Usually, there are a significant number of vector fields of the Lie algebra, obtained by adopting linear combinations of generators $\mathbb{V}_i, i = 1, 2, \ldots, 13$. Subsequently, it is enough to put all similar subalgebras into one class and select a representative from each class.

Commutator Table 1 of the Lie algebra can be obtained from the vector fields $\mathbb{V}_i, i = 1, 2, \ldots, 13$, whose $(i, j)^{th}$ entry is given by $[\mathbb{V}_i, \mathbb{V}_j] = \mathbb{V}_i * \mathbb{V}_j = \mathbb{V}_i \cdot \mathbb{V}_j - \mathbb{V}_j \cdot \mathbb{V}_i$ which is anti-symmetric with its diagonal elements all being zero as we obtain $[\mathbb{V}_\alpha, \mathbb{V}_\beta] = -[\mathbb{V}_\beta, \mathbb{V}_\alpha]$.

2.2 Adjoint representations

To compute adjoint representations of symmetry operators for Eq. (2), we use the Lie series [3,5]

$$Ad(\exp(\vartheta\mathbb{V}_i))\mathbb{V}_j = \sum_{n=0}^{\infty}\frac{\vartheta^n}{n!}(ad\mathbb{V}_i)^n(\mathbb{V}_j) = \mathbb{V}_j - \vartheta[\mathbb{V}_i, \mathbb{V}_j] + \frac{1}{2}\vartheta^2[\mathbb{V}_i, [\mathbb{V}_i, \mathbb{V}_j]] - \cdots. \tag{9}$$

The full adjoint representation table entries are tabulated in Table 2.

With the assistance of Tables 1 and 2 and by carefully applying adjoint maps, we discuss useful linear combinations of vector fields for the considered equation, which are taken as follows:

$$\begin{aligned}
&\text{(i)}\ \ \mathbb{V}_2, && \text{(ii)}\ \ \mathbb{V}_5, && \text{(iii)}\ \ \mathbb{V}_6\\
&\text{(iv)}\ \ \mathbb{V}_7, && \text{(v)}\ \ \mathbb{V}_8, && \text{(vi)}\ \ \mathbb{V}_1+\mathbb{V}_5\\
&\text{(vii)}\ \ \mathbb{V}_2+\mathbb{V}_5, && \text{(viii)}\ \ \mathbb{V}_2+\mathbb{V}_9, && \text{(ix)}\ \ \mathbb{V}_2+\mathbb{V}_5+\mathbb{V}_6\\
&\text{(x)}\ \ \mathbb{V}_2+\mathbb{V}_5+\mathbb{V}_6+\mathbb{V}_8+\mathbb{V}_{11}, && \text{(xi)}\ \ \mathbb{V}_2+\mathbb{V}_5+\mathbb{V}_6+\mathbb{V}_{12}, && \text{(xii)}\ \ \mathbb{V}_2+\mathbb{V}_8+\mathbb{V}_{13}.
\end{aligned} \tag{10}$$

Table 1 Commutator table

*	$\mathbb{V}_1$	$\mathbb{V}_2$	$\mathbb{V}_3$	$\mathbb{V}_4$	$\mathbb{V}_5$	$\mathbb{V}_6$	
$\mathbb{V}_1$	0	$-\frac{3}{4}\mathbb{V}_6 - \mathbb{V}_2$	$\mathbb{V}_3 + \mathbb{V}_7 - \frac{1}{12}\mathbb{V}_9$	$-\frac{1}{4}\mathbb{V}_9$	$\frac{1}{2}(3\mathbb{V}_8 - \mathbb{V}_6)$	$\mathbb{V}_6 - \frac{1}{6}\mathbb{V}_{10}$	
$\mathbb{V}_2$	$\frac{3}{4}\mathbb{V}_6 + \mathbb{V}_2$	0	$\frac{1}{2}\mathbb{V}_6 + \mathbb{V}_5$	$-\frac{3}{4}\mathbb{V}_6$	0	0	
$\mathbb{V}_3$	$-\mathbb{V}_3 - \mathbb{V}_7 + \frac{1}{12}\mathbb{V}_9$	$-\frac{1}{2}\mathbb{V}_6 - \mathbb{V}_5$	0	$-\mathbb{V}_3 - 3\mathbb{V}_7 + \frac{1}{4}\mathbb{V}_9$	$-\frac{1}{3}\mathbb{V}_9$	0	
$\mathbb{V}_4$	$\frac{1}{4}\mathbb{V}_9$	$\frac{3}{4}\mathbb{V}_6$	$\mathbb{V}_3 + 3\mathbb{V}_7 - \frac{1}{4}\mathbb{V}_9$	0	$\frac{1}{2}(-\mathbb{V}_8 + \mathbb{V}_6) - \mathbb{V}_5$	$-2\mathbb{V}_6 + \frac{1}{2}\mathbb{V}_{10}$	
$\mathbb{V}_5$	$-\frac{1}{2}(3\mathbb{V}_8 - \mathbb{V}_6)$	0	$\frac{1}{3}\mathbb{V}_6$	$\frac{1}{2}(\mathbb{V}_8 - \mathbb{V}_6) + \mathbb{V}_5$	0	0	
$\mathbb{V}_6$	$-\mathbb{V}_6 + \frac{1}{6}\mathbb{V}_{10}$	0	0	$2\mathbb{V}_6 - \frac{1}{2}\mathbb{V}_{10}$	0	0	
$\mathbb{V}_7$	$-\frac{2}{3}\mathbb{V}_7 - \frac{1}{18}(\mathbb{V}_9 - \frac{1}{3}\mathbb{V}_{13})$	$-\frac{1}{6}\mathbb{V}_8$	$-\frac{1}{18}\mathbb{V}_9$	$-\frac{1}{18}\mathbb{V}_{13}$	0	0	
$\mathbb{V}_8$	$\frac{1}{3}(\mathbb{V}_8 - \mathbb{V}_{10})$	0	$-\frac{1}{3}\mathbb{V}_{10}$	0	0	0	
$\mathbb{V}_9$	$-\frac{4}{3}\mathbb{V}_9$	$-\mathbb{V}_{10}$	0	0	0	0	
$\mathbb{V}_{10}$	$-\frac{1}{3}\mathbb{V}_{10}$	0	0	0	0	0	
$\mathbb{V}_{11}$	$-\frac{7}{3}\mathbb{V}_{11}$	$-2\mathbb{V}_{12}$	$-t^3\mathbb{V}_{10}$	$-\mathbb{V}_{11}$	$-t^2\mathbb{V}_{10}$	0	
$\mathbb{V}_{12}$	$-\frac{4}{3}\mathbb{V}_{12}$	$-\mathbb{V}_{13}$	$-t^2\mathbb{V}_{10}$	$-\mathbb{V}_{12}$	$-\mathbb{V}_9$	0	
$\mathbb{V}_{13}$	$-\frac{1}{3}\mathbb{V}_{13}$	0	$-\mathbb{V}_9$	$-\mathbb{V}_{13}$	$-\mathbb{V}_{10}$	0	
$\mathbb{V}_1$	$\frac{2}{3}\mathbb{V}_7 + \frac{1}{18}((\mathbb{V}_9 - \frac{1}{3}\mathbb{V}_{11}))$	$\frac{1}{3}(-\mathbb{V}_8 + \mathbb{V}_{10})$	$\frac{4}{3}\mathbb{V}_9$	$\frac{1}{3}\mathbb{V}_{10}$	$\frac{7}{3}\mathbb{V}_{11}$	$\frac{4}{3}\mathbb{V}_{12}$	$\frac{1}{3}\mathbb{V}_{13}$
$\mathbb{V}_2$	$\frac{1}{6}\mathbb{V}_8$	0	$\mathbb{V}_{10}$	0	$2\mathbb{V}_{12}$	$\mathbb{V}_{13}$	0

Table 1 continued

*	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_3$	$\frac{1}{18}\mathbb{V}_9$	$\frac{1}{3}\mathbb{V}_{10}$	0	0	$t^3\mathbb{V}_{10}$	$t^2\mathbb{V}_{10}$	$\mathbb{V}_9$
$\mathbb{V}_4$	$\frac{1}{18}\mathbb{V}_{13}$	0	0	0	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_5$	0	0	0	0	$t^2\mathbb{V}_{10}$	$\mathbb{V}_9$	$\mathbb{V}_{10}$
$\mathbb{V}_6$	0	0	0	0	0	0	0
$\mathbb{V}_7$	0	$-\frac{1}{9}\mathbb{V}_{10}$	0	0	0	0	0
$\mathbb{V}_8$	$\frac{1}{9}\mathbb{V}_{10}$	0	0	0	0	0	0
$\mathbb{V}_9$	0	0	0	0	0	0	0
$\mathbb{V}_{10}$	0	0	0	0	0	0	0
$\mathbb{V}_{11}$	0	0	0	0	0	0	0
$\mathbb{V}_{12}$	0	0	0	0	0	0	0
$\mathbb{V}_{13}$	0	0	0	0	0	0	0

Table 2 Adjoint table

Ad	$\mathbb{V}_1$	$\mathbb{V}_2$	$\mathbb{V}_3$	$\mathbb{V}_4$
$\mathbb{V}_1$	$\mathbb{V}_1$	$e^{\vartheta}\mathbb{V}_2 + e^{\frac{3\vartheta}{4}}\mathbb{V}_6$	$e^{-\vartheta}\mathbb{V}_3$+$e^{-\vartheta}\mathbb{V}_7$+$e^{\frac{-\vartheta}{12}}\mathbb{V}_9$	$\mathbb{V}_4$-$\frac{1}{4}e^{\frac{4\vartheta}{3}}\mathbb{V}_9$
$\mathbb{V}_2$	$e^{-\vartheta}\mathbb{V}_2$-$\frac{3\vartheta}{4}\mathbb{V}_6$	$\mathbb{V}_2$	$\mathbb{V}_3$-$\vartheta(\frac{1}{2}\mathbb{V}_6$+$\mathbb{V}_5)$	$\mathbb{V}_4$+$\frac{3\vartheta}{4}\mathbb{V}_6$
$\mathbb{V}_3$	$\mathbb{V}_1$+$\vartheta(\mathbb{V}_3$+$\mathbb{V}_7$-$\frac{1}{12}\mathbb{V}_9)$+$\frac{\vartheta^2}{36}\mathbb{V}_9$	$\mathbb{V}_2$+$\vartheta(\frac{1}{2}\mathbb{V}_6$+$\mathbb{V}_5)$+$\frac{\vartheta^2}{6}\mathbb{V}_9$	$\mathbb{V}_3$	$\mathbb{V}_4$+$\vartheta(\mathbb{V}_3$+$3\mathbb{V}_7$-$\frac{1}{4}\mathbb{V}_9)$-$\frac{\vartheta^2}{12}\mathbb{V}_9$
$\mathbb{V}_4$	$\mathbb{V}_1$-$\frac{\vartheta}{4}\mathbb{V}_9$	$\mathbb{V}_2$+$(\frac{5\vartheta}{4}$-1-$e^{2\vartheta})\mathbb{V}_6$	$e^{-\vartheta}\mathbb{V}_3$+$(e^{3\vartheta}-1)\mathbb{V}_7$+$(e^{\frac{-\vartheta}{4}}-1)\mathbb{V}_9$	$\mathbb{V}_4$
$\mathbb{V}_5$	$\mathbb{V}_1$+$\frac{\vartheta}{2}(3\mathbb{V}_8$-$\mathbb{V}_6)$	$\mathbb{V}_2$	$\mathbb{V}_3$-$\frac{\vartheta}{3}\mathbb{V}_6$	$\mathbb{V}_4$-$\frac{\vartheta}{2}(\mathbb{V}_8$-$\mathbb{V}_6) + \mathbb{V}_5$
$\mathbb{V}_6$	$\mathbb{V}_1$-$\vartheta(\mathbb{V}_6$+$\frac{1}{6}\mathbb{V}_{10})$	$\mathbb{V}_2$	$\mathbb{V}_3$	$\mathbb{V}_4$-$\vartheta(2\mathbb{V}_6$-$\frac{1}{2}\mathbb{V}_{10})$
$\mathbb{V}_7$	$\mathbb{V}_1$+$\frac{2\vartheta}{3}\mathbb{V}_7$+$\frac{\vartheta}{18}(\mathbb{V}_9$-$\frac{1}{3}\mathbb{V}_{13})$	$\mathbb{V}_2$+$\frac{\vartheta}{6}\mathbb{V}_8$-$\frac{\vartheta^2}{18}\mathbb{V}_{10}$	$\mathbb{V}_3$+$\frac{\vartheta}{18}\mathbb{V}_9$	$\mathbb{V}_4$+$\frac{\vartheta}{18}\mathbb{V}_{13}$
$\mathbb{V}_8$	$\mathbb{V}_1$-$\frac{\vartheta}{3}(\mathbb{V}_8$-$\mathbb{V}_{10})$	$\mathbb{V}_2$	$\mathbb{V}_3$+$\frac{\vartheta}{3}\mathbb{V}_{10}$	$\mathbb{V}_4$
$\mathbb{V}_9$	$\mathbb{V}_1$+$\frac{4\vartheta}{3}\mathbb{V}_9$	$\mathbb{V}_2$+$\vartheta\mathbb{V}_{10}$	$\mathbb{V}_3$	$\mathbb{V}_4$
$\mathbb{V}_{10}$	$\mathbb{V}_1$-$\frac{\vartheta}{3}\mathbb{V}_{10}$	$\mathbb{V}_2$	$\mathbb{V}_3$	$\mathbb{V}_4$
$\mathbb{V}_{11}$	$\mathbb{V}_1$+$\frac{7\vartheta}{3}\mathbb{V}_{11}$	$\mathbb{V}_2 + 2\vartheta\mathbb{V}_{12}$	$\mathbb{V}_3 + t^3\vartheta\mathbb{V}_{10}$	$\mathbb{V}_4 + \vartheta\mathbb{V}_{11}$
$\mathbb{V}_{12}$	$\mathbb{V}_1$+$\frac{4\vartheta}{3}\mathbb{V}_{12}$	$\mathbb{V}_2 + \vartheta\mathbb{V}_{13}$	$\mathbb{V}_3 + t^2\vartheta\mathbb{V}_{10}$	$\mathbb{V}_4 + \vartheta\mathbb{V}_{12}$
$\mathbb{V}_{13}$	$\mathbb{V}_1$+$\frac{\vartheta}{3}\mathbb{V}_{13}$	$\mathbb{V}_2$	$\mathbb{V}_3$+$\vartheta\mathbb{V}_9$	$\mathbb{V}_4$+$\mathbb{V}_{13}$

Table 2 continued

Ad	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_1$	$\mathbb{V}_5$-$\frac{\vartheta}{2}$(3$\mathbb{V}_8$-$\mathbb{V}_6$)	$e^{-\vartheta}\mathbb{V}_6+(e^{\frac{\vartheta}{6}}-1)\mathbb{V}_{10}$	$e^{\frac{-2\vartheta}{3}}\mathbb{V}_7+e^{\frac{-\vartheta}{18}}\mathbb{V}_9+e^{\frac{\vartheta}{54}}\mathbb{V}_{13}$	$e^{\frac{\vartheta}{3}}\mathbb{V}_8$+($e^{\frac{-\vartheta}{3}}$)$\mathbb{V}_{10}$)	$e^{\frac{-4\vartheta}{3}}\mathbb{V}_9$	$e^{\frac{-\vartheta}{3}}\mathbb{V}_{10}$	$\mathbb{V}_{11}e^{\frac{-7\vartheta}{3}}$	$\mathbb{V}_{12}e^{\frac{-4\vartheta}{3}}$	$e^{\frac{-\vartheta}{3}}\mathbb{V}_{13}$
$\mathbb{V}_2$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$-$\frac{\vartheta}{6}\mathbb{V}_8$	$\mathbb{V}_8$	$\mathbb{V}_9$-$\vartheta\mathbb{V}_{10}$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}-2\vartheta\mathbb{V}_{12}$	$\mathbb{V}_{12}-\vartheta\mathbb{V}_{13}$	$\mathbb{V}_{13}$
$\mathbb{V}_3$	$\mathbb{V}_5$+$\frac{\vartheta}{3}\mathbb{V}_9$	$\mathbb{V}_6$	$\mathbb{V}_7$-$\frac{\vartheta}{18}\mathbb{V}_9$	$\mathbb{V}_8$-$\frac{\vartheta}{3}\mathbb{V}_{10}$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}-t^3\vartheta\mathbb{V}_{10}$	$\mathbb{V}_{12}-t^2\vartheta\mathbb{V}_{10}$	$\mathbb{V}_{13}$-$\vartheta\mathbb{V}_9$
$\mathbb{V}_4$	$e^{-\vartheta}\mathbb{V}_5$+$\frac{e^{-\vartheta}}{2}$($\mathbb{V}_6$-$\mathbb{V}_8$)	$e^{2\vartheta}\mathbb{V}_6$-$\frac{\vartheta}{2}\mathbb{V}_{10}$	$\mathbb{V}_7$-$\frac{\vartheta}{18}\mathbb{V}_{13}$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}e^{-\vartheta}$	$\mathbb{V}_{12}e^{-\vartheta}$	$e^{-\vartheta}\mathbb{V}_{13}$
$\mathbb{V}_5$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}-t^2\vartheta\mathbb{V}_{10}$	$\mathbb{V}_{12}-\vartheta\mathbb{V}_9$	$\mathbb{V}_{13}$-$\vartheta\mathbb{V}_{10}$
$\mathbb{V}_6$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_7$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$+$\frac{\vartheta}{9}\mathbb{V}_{10}$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_8$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$-$\frac{\vartheta}{9}\mathbb{V}_{10}$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_9$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_{10}$	$\mathbb{V}_5$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_{11}$	$\mathbb{V}_5+t^2\vartheta\mathbb{V}_{10}$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_{12}$	$\mathbb{V}_5+\vartheta\mathbb{V}_9$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$
$\mathbb{V}_{13}$	$\mathbb{V}_5+\mathbb{V}_{10}$	$\mathbb{V}_6$	$\mathbb{V}_7$	$\mathbb{V}_8$	$\mathbb{V}_9$	$\mathbb{V}_{10}$	$\mathbb{V}_{11}$	$\mathbb{V}_{12}$	$\mathbb{V}_{13}$

3 Symmetry reductions and closed-form solutions

In this section, we obtain numerous closed-form invariant solutions for Eq. (2) utilizing the Lie symmetry technique. Three stages of symmetry reductions will be taken with the aid of invariant (or similarity) functions. To the end, we first solve the associated Lagrange's characteristic system given by

$$\frac{\mathrm{d}x}{\xi^1} = \frac{\mathrm{d}y}{\xi^2} = \frac{\mathrm{d}z}{\xi^3} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\eta}, \tag{11}$$

which leads to similarity functions.

3.1 Vector field $\mathbb{V}_2$

Let us consider

$$\mathbb{V}_2 = \frac{\partial}{\partial t}. \tag{12}$$

The corresponding Lagrange's system is acquired via (11) and (12):

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{0}. \tag{13}$$

The constants of integration provide the invariant functions of Eq. (2):

$$u(x, y, t) = f(X, Y, Z),\ \ X = x,\ \ Y = y,\ \text{ and } Z = z, \tag{14}$$

where X, Y, and Z are three invariant functions, and f is an invariant solution of extended JM equation (2). Substituting Eq. (14) into Eq. (2), we get a reduced equation (a PDE) as follows:

$$f_{XXXY} + 3f_X f_{XY} + 3f_Y f_{XX} - 3(f_{XZ} + f_{YZ} + f_{ZZ}) = 0. \tag{15}$$

We can easily find the two travelling wave solutions for Eq. (15):

$$f(X, Y, Z) = 2\tanh\left(C_2X + C_3Y + \left(-\frac{1}{2}(C_2 + C_3) - A\right)Z + C_1\right)C_2 + C_4,$$
$$f(X, Y, Z) = 2\tanh\left(C_2X + C_3Y + \left(-\frac{1}{2}(C_2 + C_3) + A\right)Z + C_1\right)C_2 + C_4.$$

The corresponding travelling wave solutions to Eq. (2) are given by

$$u(x, y, z, t) = 2\tanh\left(C_2x + C_3y + \left(-\frac{1}{2}(C_2 + C_3) - A\right)z + C_1\right)C_2 + C_4, \tag{16}$$
$$u(x, y, z, t) = 2\tanh\left(C_2x + C_3y + \left(-\frac{1}{2}(C_2 + C_3) + A\right)z + C_1\right)C_2 + C_4, \tag{17}$$

where $A = \sqrt{48C_2^3C_3 + 9C_2^2 + 18C_2C_3 + 9C_3^2}/6$ and C_i, $i = 1, \ldots, 4$ are arbitrary constants.

To obtain more exact solutions of extended JM equation (2), we apply the similarity transformation method (STM) to Eq. (15) and construct a new set of infinitesimal generators

$$\xi_X = \frac{a_1}{3}(X + Z) + a_4,$$
$$\xi_Y = a_1Y + a_3,$$

$$\xi_Z = a_1 Z + a_2,$$
$$\eta_f = -\frac{a_1}{3}(f + X + Y) + a_5 Z + a_6,$$

where ξ_X, ξ_Y, ξ_Z, and η_f denote the generators of infinitesimal transformations with respect to the indicated variables and $a_i, i = 1, \ldots, 6$ are arbitrary constants. To obtain a larger class of solutions of Eq. (2), we study different cases on the parameters a_i.

Case 1(a) Take $a_2 \neq 0$, and all remaining parameters are zero.
Furthermore, the function f can be furnished into a new similarity variable H with new invariant functions r and s:

$$f(X, Y, Z) = H(r, s), \ r = X, \ s = Y. \tag{18}$$

Equation (15) can be rewritten in the new invariant function H as

$$H_{rrrs} + 3H_r H_{rs} + 3H_s H_{rr} = 0. \tag{19}$$

Assume the travelling wave solution $H(r, s) = G(\zeta)$ of Eq. (19), where $\zeta = r - as$ and a is the wave speed. Then the reduced ODE from Eq. (19) is

$$G''''(\zeta) + 6G'G'' = 0. \tag{20}$$

Accordingly, we derive the following exact solutions of Eq. (20):

$$G(\zeta) = \frac{2}{C_1 + \zeta} + C_2,$$
$$G(\zeta) = \frac{-C_1 C_2 \zeta + C_2^2 + 2C_1}{C_1 \zeta - C_2},$$
$$G(\zeta) = 2\,\text{WeierstrassZeta}(C_1 + \zeta, 0, C_2).$$

As a result, we acquire the closed-form solutions of Eq. (2):

$$u(x, y, z, t) = \frac{2}{C_1 + x - ay} + C_2, \tag{21}$$
$$u(x, y, z, t) = \frac{-C_1 C_2 (x - ay) + C_2^2 + 2C_1}{C_1 (x - ay) - C_2}, \tag{22}$$
$$u(x, y, z, t) = 2\,\text{WeierstrassZeta}(C_1 + x - ay, 0, C_2), \tag{23}$$

where C_1 and C_2 are arbitrary constants.
Case 1(b) Take $a_3 \neq 0$, and the rest are zero.
The function $f(X, Y, T)$ can be reformed as

$$f(X, Y, T) = H(r, s), \ r = X, \ s = Z.$$

The new invariant solution $H(r, s)$ with its invariants r and s reduces Eq. (15) to

$$H_{rs} + H_{ss} = 0. \tag{24}$$

The following travelling wave solutions of Eq. (24) can be obtained:

$$H(r, s) = C_1 \exp(s - r),$$
$$H(r, s) = C_3 \tanh^3(C_2(s - r) + C_1) + C_4 \tanh(C_2(s - r) + C_1) + C_5,$$
$$H(r, s) = C_3 \tanh^3(C_2(s - r) + C_1) + C_4 \tanh^2(C_2(s - r) + C_1)$$

$$+\ C_5\tanh(C_2(s-r)+C_1)+C_6.$$

Therefore, three new classes of closed-form solutions of Eq. (2) read

$$u(x,y,z,t)=C_1\exp(z-x), \tag{25}$$

$$u(x,y,z,t)=C_3\tanh^3(C_2(z-x)+C_1)+C_4\tanh(C_2(z-x)+C_1)+C_5, \tag{26}$$

$$u(x,y,z,t)=C_3\tanh^3(C_2(z-x)+C_1)+C_6\tanh^2(C_2(z-x)+C_1)$$
$$+\ C_4\tanh(C_2(z-x)+C_1)+C_5, \tag{27}$$

where $C_i, i=1,\ldots,5$, appearing in the solutions of Eq. (2) are arbitrary constants.

3.2 Vector field $\mathbb{V}_5$

Let us consider

$$\mathbb{V}_5=\frac{\partial}{\partial z}. \tag{28}$$

For this vector field, we directly obtain

$$u(x,y,t)=f(X,Y,Z),\ X=x,\ Y=y,\ \text{and}\ T=t. \tag{29}$$

Using Eqs. (2) and (29), we get a reduced equation

$$f_{XXXY}+3f_Yf_{XX}+3f_Xf_{XY}+2f_{YT}=0. \tag{30}$$

Again, we construct travelling wave solutions to Eq. (30), in which the considered group is a translation group on the (X,Y,T) space:

$$f(X,Y,T)=\frac{C_1}{T^{\frac{1}{3}}}+\frac{X^2}{3T^{\frac{2}{3}}}-\frac{C_2}{X},$$
$$f(X,Y,T)=2C_2\tanh(-2C_2^3T+C_2X+C_3Y+C_1)+C_4.$$

These solutions of Eq. (30) collaborate to obtain the closed-form solutions of Eq. (2):

$$u(x,y,z,t)=\frac{C_1}{t^{\frac{1}{3}}}+\frac{x^2}{3t^{\frac{2}{3}}}-\frac{C_2}{x}, \tag{31}$$

$$u(x,y,z,t)=2C_2\tanh(-2C_2^3t+C_2x+C_3y+C_1)+C_4, \tag{32}$$

where $C_i, i=1,\ldots,4$, are arbitrary constants. Utilizing the Lie group technique, we derive a new set of infinitesimals as

$$\xi_X=\frac{1}{3}\alpha_1'(T)X+\alpha_2(T),$$
$$\xi_Y=\alpha_3(Y),$$
$$\tau_T=\alpha_1(T),$$
$$\eta_f=-\frac{1}{3}\alpha_1'(T)X+\frac{1}{9}\alpha_1^{\prime(T)\prime}X^2+\frac{2}{3}\alpha_2'(T)X+\alpha_4(T).$$

Assume $\alpha_1(T)=aT+b, \alpha_2(T)=\alpha_4(T)=0$ and $\alpha_3(Y)=0$. Thus, we have

$$\frac{\mathrm{d}X}{\frac{aX}{3}}=\frac{\mathrm{d}Y}{0}=\frac{\mathrm{d}dT}{aT+b}=\frac{\mathrm{d}f}{\frac{-af}{3}}. \tag{33}$$

The similarity form H of the solution of Eq. (30) with a pair of new similarity variables r and s is

$$f(X, Y, T) = \frac{H(r, s)}{(aT + b)^{\frac{1}{3}}}, \; r = \frac{X}{(aT + b)^{\frac{1}{3}}} \text{ and } s = Y.$$

Therefore, one obtains

$$H_{rrrs} + 3H_r H_{rs} + 3H_s H_{rr} - \frac{2a}{3}(H_s + H_{rs}) = 0. \tag{34}$$

We solve Eq. (34) to obtain the following solutions:

$$H(r, s) = s + \frac{a}{9}r^2 + C_1 r + C_2, \tag{35}$$

$$H(r, s) = \log s + \frac{a}{9}r^2 + C_1 r + C_2. \tag{36}$$

These solutions of Eq. (34) help us to obtain exact solutions to extended JM equation (2):

$$u(x, y, z, t) = \frac{y}{(at + b)^{\frac{1}{3}}} + \frac{ax^2}{9(at + b)} + C_1 \frac{x}{(at + b)^{\frac{2}{3}}} + \frac{C_2}{(at + b)^{\frac{1}{3}}}, \tag{37}$$

$$u(x, y, z, t) = \frac{\log y}{(at + b)^{\frac{1}{3}}} + \frac{ax^2}{9(at + b)} + C_1 \frac{x}{(at + b)^{\frac{2}{3}}} + \frac{C_2}{(at + b)^{\frac{1}{3}}}, \tag{38}$$

where a, b are arbitrary parameters appearing in the generators of infinitesimal transformations for Eq. (30), and C_1 and C_2 are another pair of arbitrary constants.

3.3 Vector field $\mathbb{V}_6$

Let us consider

$$\mathbb{V}_6 = \frac{\partial}{\partial y}.$$

Characteristic equations (11) for the vector field $\mathbb{V}_6$ reduce Eq. (2) to

$$u(x, y, z, t) = f(X, Z, T), \text{ where } X = x, \quad T = t \text{ and } Z = z. \tag{39}$$

With the help of Eqs. (2) and (39), thus one obtains

$$f_{XZ} + f_{ZZ} = 0. \tag{40}$$

Using (40) and (39) via symbolic computation, we get a desired invariant solution of extended JM equation (2):

$$u(x, y, z, t) = g(x - z, t) + h(x, t), \tag{41}$$

where g, h are arbitrary functions. Some following particular solutions can be worked out as follows:

$$\begin{aligned} u(x, y, z, t) = {} & C_4 \tanh^3(C_3 t - C_2 x + C_2 z + C_1) \\ & + C_5 \tanh(C_3 t - C_2 x + C_2 z + C_1) + C_6, \end{aligned} \tag{42}$$

$$\begin{aligned} u(x, y, z, t) = {} & C_4 \tanh^3(C_3 t - C_2 x + C_2 z + C_1) \\ & + C_7 \tanh^2(C_3 t - C_2 x + C_2 z + C_1) \\ & + C_6 \tanh(C_3 t - C_2 x + C_2 z + C_1) + C_6, \end{aligned} \tag{43}$$

where $C_i, i = 1, \ldots, 6$, are arbitrary constants.

3.4 Vector field $\mathbb{V}_7$

Let us consider

$$\mathbb{V}_7 = \frac{t}{6}\frac{\partial}{\partial x} + \frac{x}{9}\frac{\partial}{\partial u}. \tag{44}$$

The associated characteristic Lagrange's system can be computed by using Eqs. (44) and (11). Solutions of those characteristic equations contain constants of integration, which are known as invariant functions. Thus, we can reduce Eq. (2) to give

$$u(x, y, t) = f(Y, T, Z) + \frac{x^2}{3t}, \text{ where } Y = y,\ Z = z, \text{ and } T = t. \tag{45}$$

The invariant functions in Eq. (45) reduce Eq. (2) into a PDE in f with the new independent variables Y, T, and Z:

$$2f_Y + 2Tf_{YT} - 3T(f_{YZ} + f_{ZZ}) = 0. \tag{46}$$

The following travelling wave solution of Eq. (2) can be found by solving Eq. (46):

$$\begin{aligned} u(x, y, z, t) = {} & \frac{x^2}{3t} + \frac{A}{t}\exp\left(C_1\left(y - \frac{z}{2}\right) + \frac{\sqrt{C_1^2 + 4C_2}}{2C_1}z + \frac{3C_2 t}{2C_1}\right) \\ & + \frac{B}{t}\exp\left(C_1\left(y - \frac{z}{2}\right) - \frac{\sqrt{C_1^2 + 4C_2}}{2C_1}z + \frac{3C_2 t}{2C_1}\right), \end{aligned} \tag{47}$$

where $A = C_1C_2C_4$, $B = C_1C_3C_4$ and C_i's, $i = 1, \ldots, 4$, are arbitrary constants.

3.5 Vector field $\mathbb{V}_8$

Let us consider

$$\mathbb{V}_8 = \frac{\partial}{\partial x}. \tag{48}$$

The vector field $\mathbb{V}_8$ along with Eq. (11) produces a similarity form of solutions of Eq. (2) with new similarity variables:

$$u(x, y, z, t) = f(Y, Z, T), \text{ where } Y = y,\ Z = z, \text{ and } T = t. \tag{49}$$

Using Eqs. (2) and (49), we have

$$2f_{YT} - 3f_{YZ} - 3f_{ZZ} = 0. \tag{50}$$

We solve Eq. (50) via the travelling wave solution method and then obtain the travelling wave solutions of Eq. (2):

$$\begin{aligned} u(x, y, z, t) = {} & C_4\tanh^3\left(\frac{3C_3(C_2 + C_3)t}{2C_2} + C_2y + C_3z + C_1\right) \\ & + C_5\tanh\left(\frac{3C_3(C_2 + C_3)t}{2C_2} + C_2y + C_3z + C_1\right) + C_6, \end{aligned} \tag{51}$$

$$u(x, y, z, t) = C_4\tanh^3\left(\frac{3C_3(C_2 + C_3)t}{2C_2} + C_2y + C_3z + C_1\right)$$

$$+ C_7 \tanh^2\left(\frac{3C_3(C_2+C_3)t}{2C_2} + C_2 y + C_3 z + C_1\right)$$
$$+ C_5 \tanh\left(\frac{3C_3(C_2+C_3)t}{2C_2} + C_2 y + C_3 z + C_1\right) + C_6, \tag{52}$$

where $C_i, i = 1, \ldots, 7$, are arbitrary constants. The profiles of the acquired solutions of the extended JM equation are represented in the following figures. The analysis reveals different types of solutions such as solitons, kinky wave solutions, and interaction solutions.

3.6 Vector field $\mathbb{V}_1 + \mathbb{V}_5$

The related Lagrange's system is interpreted as

$$\frac{\mathrm{d}x}{\frac{1}{6}(2x-z)} = \frac{\mathrm{d}y}{\frac{1}{4}(3t-4y+2z)} = \frac{\mathrm{d}z}{1} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{\frac{1}{6}(-2u-2x+z)}. \tag{53}$$

Equation (53) introduces the following similarity solution

$$u(x,y,z,t) = \frac{U(X,Y,Z)}{\sqrt[3]{t}} + \frac{-16\left(2t^{4/3}X + t(Z+4) + Y\right) - 16t\log(t) + 3t^2}{64t}$$
$$\text{with } X = \frac{x - \frac{1}{2}(\log(t)+Z+3)}{\sqrt[3]{t}}, \quad Y = ty - \left(\frac{3t^2}{8} + \frac{tZ}{2} - \frac{t}{2} + \frac{1}{2}t\log(t)\right) \text{ and}$$
$$Z = z - \log(t). \tag{54}$$

On solving (54) and (2), one obtains

$$9U_{ZZ} + 6U_{YZ} - 6YU_{YY} + 2XU_{XY} - 4U_Y - 9U_XU_{XY} - 9U_YU_{XX} - 3U_{XXXY} = 0. \tag{55}$$

Applying the symmetry transformation method to (55), then the desired infinitesimals are given as follows:

$$\xi_X = a_2, \quad \xi_Y = 0, \xi_Z = a_1, \quad \eta_U = \frac{2}{9}Xa_2 + Za_3 + a_4, \tag{56}$$

where a_i's, $1 \le i \le 4$, are arbitrary constants.
Case (1) Take $a_3 = a_4 = 0$ and all other constants are nonzero.
From Eq. (56), we get the characteristic system as:

$$\frac{\mathrm{d}X}{a_2} = \frac{\mathrm{d}Y}{0} = \frac{\mathrm{d}Z}{a_1} = \frac{\mathrm{d}U}{\frac{2}{9}Xa_2}, \tag{57}$$

which gives the similarity solution

$$U(X,Y,T) = G(P,Q) + \frac{2A_2\left(\frac{A_2Z^2}{2} + PZ\right)}{9} \quad \text{with } P = X - A_2Z, \quad Q = Y. \tag{58}$$

Using (58) and (55), we have the reduced equation

$$2A_2^2 + 6QG_{QQ} - 2(P-3A_2)G_{PQ} + 4G_Q - 9A_2^2G_{PP} + 9G_QG_{PP} + 9G_PG_{PQ} + 3G_{PPPQ} = 0. \tag{59}$$

By utilizing the symmetry transformation method on (59), new infinitesimals are:

$$\eta_G = b_2 + \frac{2}{9}b_1P, \quad \xi_P = b_1, \quad \xi_Q = 0, \tag{60}$$

where b_1 and b_2 are arbitrary constants. The characteristic system for (60) is

$$\frac{\mathrm{d}P}{b_1}=\frac{\mathrm{d}Q}{0}=\frac{\mathrm{d}G}{b_2+\frac{2}{9}b_1P}, \tag{61}$$

which gives the similarity form

$$G(P,Q)=B_2P+\frac{P^2}{9}+H(R) \quad \text{with} \quad R=Q, \tag{62}$$

where $B_2=\frac{b_2}{b_1}$. Using (62) and (59), we get an ODE

$$RH''+H'=0. \tag{63}$$

On solving (63), we have

$$H(R)=\mathscr{K}_1\log(R)+\mathscr{K}_2, \tag{64}$$

where $\mathscr{K}_1$ and $\mathscr{K}_2$ are arbitrary constants. Hence, we obtain

$$\begin{aligned} u(x,y,z,t)=&\frac{-288B_2\sqrt[3]{t}\left(2A_2\sqrt[3]{t}(z-\log(t))-2x+z+3\right)+81t^2-72t(4x+2y-z+3)}{576t}\\ &+\frac{16(-2x+z+3)^2}{576t}+\frac{(\mathscr{K}_2(\log(t(-3t+8y-4z+4))-\log(8))+\mathscr{K}_1)}{t^{1/3}}. \end{aligned} \tag{65}$$

3.7 Vector field $\mathbb{V}_2+\mathbb{V}_9$

The related Lagrange's system is interpreted as

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{t}, \tag{66}$$

which gives

$$u(x,y,z,t)=U(X,Y,Z)+\frac{t^2}{2} \quad \text{with} \quad X=x, \quad Y=y \quad \text{and} \quad Z=z. \tag{67}$$

In view of (67) and (2), one obtains

$$3U_{ZZ}+3U_{YZ}+3U_{XZ}-3U_XU_{XY}-3U_YU_{XX}-U_{XXXY}=0. \tag{68}$$

Utilizing the symmetry transformation method on (68), then new infinitesimals are given as:

$$\begin{aligned} &\xi_X=\frac{a_1}{3}(X+Z)+a_4, \quad \xi_Y=Ya_1+a_3,\\ &\xi_Z=Za_1+a_2, \quad \eta_U=\frac{1}{3}(-X-Y-U)a_1+Za_5+a_6, \end{aligned} \tag{69}$$

where a_i's, $1\le i\le 6$, are arbitrary constants.
Case (1) Take $a_1=a_5=0$ and all other constants are nonzero.
From Eq. (69), we get the characteristic system as:

$$\frac{\mathrm{d}X}{a_4}=\frac{\mathrm{d}Y}{a_3}=\frac{\mathrm{d}Z}{a_2}=\frac{\mathrm{d}U}{a_6}, \tag{70}$$

which produces

$$U(X,Y,T)=G(P,Q)+A_6Z \quad \text{with} \quad P=X-ZA_4, \quad Q=Y-ZA_3, \tag{71}$$

where $A_3=\frac{a_3}{a_2}$, $A_4=\frac{a_4}{a_2}$, and $A_6=\frac{a_6}{a_2}$ are constants.

Using (71) and (68), we have the reduced equation

$$3A_3(A_3-1)G_{QQ}+(3A_3(2A_4-1)-3A_4)G_{PQ}+3(A_4-1)A_4G_{PP} \\ -3G_QG_{PP}-3G_PG_{PQ}-G_{PPPQ}=0. \tag{72}$$

Again, upon applying the Lie symmetry method to equation (72), new infinitesimals are:

$$\eta_G=\frac{1}{3}\Big((4A_3A_4-2A_3-2)P+4QA_4^2-4QA_4-G\Big)b_1+b_4,\ \ \xi_P=\frac{b_1P}{3}+b_3, \xi_Q=b_1Q+b_2, \tag{73}$$

where b_i's, $1\le i\le 3$, are arbitrary constants. The characteristic system for (73) is

$$\frac{\mathrm{d}P}{b_3}=\frac{\mathrm{d}Q}{b_2}=\frac{\mathrm{d}G}{b_4}, \tag{74}$$

which gives the similarity form

$$G(P,Q)=B_4Q+H(R)\quad \text{with}\ \ R=P-B_3Q, \tag{75}$$

where $B_3=\frac{b_3}{b_2}$ and $B_4=\frac{b_4}{b_2}$. Using (75) and (68), we get an ODE

$$B_3H^{(4)}+6B_3H'H''+3LH''(R)=0, \tag{76}$$

where

$$L=A_4\left(-2A_3B_3+B_3-1\right)+A_3B_3\left(\left(A_3-1\right)B_3+1\right)+A_4^2-B_4.$$

On solving (76), we have

$$H(R)=-\frac{LR}{2B_3}+\frac{2}{R}+\mathscr{K}_3\ \ \text{and}\ \ H(R)=\mathscr{K}_4-\frac{LR}{2B_3}, \tag{77}$$

where $\mathscr{K}_3$ and $\mathscr{K}_4$ are arbitrary constants. Accordingly, we derive the following two solutions:

$$u(x,y,z,t)=\frac{L\left(B_3\left(y-A_3z\right)+A_4z-x\right)}{2B_3}+\frac{2}{B_3\left(A_3z-y\right)-A_4z+x} \\ +\ B_4\left(y-A_3z\right)+A_6z+\frac{t^2}{2}+\mathscr{K}_3, \tag{78}$$

$$u(x,y,z,t)=\frac{B_3\left(-A_3z\left(2B_4+L\right)+2A_6z+2B_4y+2\mathscr{K}_4+Ly+t^2\right)+A_4Lz-Lx}{2B_3}. \tag{79}$$

3.8 Vector field $\mathbb{V}_2+\mathbb{V}_5$

The associated characteristic system is

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{0}. \tag{80}$$

Equation (80) yields

$$u(x,y,z,t)=U(X,Y,T)\quad \text{with}\ \ X=x,\ \ Y=y\ \ \text{and}\ \ Z=z-t. \tag{81}$$

On substituting u from (81) into (2), we get a new diminished equation

$$3(U_{XZ}+U_{ZZ})+5U_{YZ}-3U_XU_{XY}-3U_YU_{XX}-U_{XXXY}=0. \tag{82}$$

Again, we apply the Lie symmetry method to Eq. (82), and then new infinitesimals are given as:

$$\xi_X = \frac{a_1}{3}(X+Z) + a_4, \quad \xi_Y = Ya_1 + a_3,$$
$$\xi_Z = Za_1 + a_2, \quad \eta_U = \frac{1}{9}(-5X - 3Y - 3U)a_1 + Za_5 + a_6, \tag{83}$$

where a_i's, $1 \leq i \leq 6$, are arbitrary constants.

Case (1) Take $a_1 = a_5 = 0$ and all other constants are nonzero.

From Eq. (83), we get characteristic system as:

$$\frac{\mathrm{d}X}{a_4} = \frac{\mathrm{d}Y}{a_3} = \frac{\mathrm{d}Z}{a_2} = \frac{\mathrm{d}U}{a_6}, \tag{84}$$

which gives the similarity solution

$$U(X,Y,T) = G(P,Q) + A_6 Z \quad \text{with} \;\; P = X - ZA_4, \;\; Q = Y - ZA_3, \tag{85}$$

where $A_3 = \frac{a_3}{a_2}$, $A_4 = \frac{a_4}{a_2}$, and $A_6 = \frac{a_6}{a_2}$.

Using (85) and (82), we have the reduced equation

$$A_3\,(3A_3 - 5)\,G_{QQ} + 3\,(A_4 - 1)\,A_4 G_{PP} - 3G_Q G_{PP}$$
$$-\,(5A_4 - 3A_3(2A_4 - 1))\,G_{PQ} - 3G_P G_{PQ} - G_{PPPQ} = 0. \tag{86}$$

Again, apply the Lie symmetry method to Eq. (86), and then new infinitesimals are:

$$\eta_G = \frac{1}{3}\left((4A_3A_4 - 2A_3 - 2)P + 4QA_4^2 - 4QA_4 - G - \frac{4p}{3}\right)b_1 + b_4, \xi_P = \frac{b_1 P}{3} + b_3,$$
$$\xi_Q = b_1 Q + b_2, \tag{87}$$

where b_i's, $1 \leq i \leq 4$, are arbitrary constants.

Take $b_1 = 0$ and all other constants are nonzero. The characteristic system for (87) is

$$\frac{\mathrm{d}P}{b_3} = \frac{\mathrm{d}Q}{b_2} = \frac{\mathrm{d}G}{b_4}, \tag{88}$$

that gives the similarity form

$$G(P,Q) = B_4 Q + H(R) \quad \text{with} \;\; R = P - B_3 Q, \tag{89}$$

where $B_4 = \frac{b_4}{b_2}$ and $B_3 = \frac{b_3}{b_2}$. Using (89) and (86), we get an ODE

$$B_3 H^{(4)}(R) + H''(R)\left(6B_3 H'(R)\right) + LH''(R) = 0, \tag{90}$$

where

$$L = A_4\left((5 - 6A_3)\,B_3 - 3\right) + 3A_3^2 B_3^2 + A_3 B_3\,(3 - 5B_3) + 3A_4^2 - 3B_4.$$

On solving (90), we have

$$H(R) = \frac{2}{R} - \frac{LR}{6B_3} + \mathscr{K}_5 \quad \text{and} \quad H(R) = -\frac{LR}{6B_3} + \mathscr{K}_6, \tag{91}$$

where $\mathscr{K}_5$ and $\mathscr{K}_6$ are arbitrary constants. Using back substitution, we acquire the following solutions of extended JM (2):

$$u(x,y,z,t) = \frac{B_3(t-z)\,(A_3\,(6B_4 + L) - 6A_6) + A_4 L(z-t) - Lx}{6B_3}$$

$$+\frac{2}{(A_4-A_3B_3)(t-z)-B_3y+x}+B_4y+\frac{Ly}{6}+\mathscr{K}_5, \tag{92}$$

$$u(x,y,z,t)=\frac{B_3\left((t-z)\left(A_3\left(6B_4+L\right)-6A_6\right)+6B_4y+6\mathscr{K}_6+Ly\right)+A_4L(z-t)-Lx}{6B_3}. \tag{93}$$

3.9 Vector field $\mathbb{V}_2+\mathbb{V}_5+\mathbb{V}_6$

The associated characteristic system is

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{1}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{0}, \tag{94}$$

which yields

$$u(x,y,z,t)=U(X,Y,Z)\quad\text{with}\quad X=x,\quad Y=y-t\quad\text{and}\quad Z=z-t. \tag{95}$$

On substituting u from (95) into (2), one obtains

$$3(U_{XZ}+U_{ZZ})+2U_{YY}+5U_{YZ}-3U_XU_{XY}-3U_YU_{XX}-U_{XXXY}=0. \tag{96}$$

Again, we utilize the Lie symmetry method in the case of equation (96); then, new infinitesimals are given as:

$$\begin{aligned}\xi_X&=\frac{a_1}{3}(X+Z)+a_4,\quad \xi_Y=Ya_1+a_3,\\ \xi_Z&=Za_1+a_2,\quad \eta_U=\frac{1}{3}(-X-Y-U)a_1+Za_5+a_6-\frac{2}{9}a_1X,\end{aligned} \tag{97}$$

where a_i's, $1\le i\le 6$, are arbitrary constants.
Case (i) Take $a_1=a_5=0$ and all other constants are nonzero.
From Eq. (97), we get the characteristic system as:

$$\frac{\mathrm{d}X}{a_4}=\frac{\mathrm{d}Y}{a_3}=\frac{\mathrm{d}Z}{a_2}=\frac{\mathrm{d}U}{a_6}. \tag{98}$$

Thus, we have

$$U(X,Y,T)=G(P,Q)+A_6Z\quad\text{with}\quad P=X-ZA_4,\quad Q=Y-ZA_3, \tag{99}$$

where $A_3=\frac{a_3}{a_2}$, $A_4=\frac{a_4}{a_2}$, and $A_6=\frac{a_6}{a_2}$.

Using (99) and (96), we have the reduced equation

$$\begin{aligned}&(A_3-1)(3A_3-2)G_{QQ}+3\big((A_4-1)A_4-G_Q\big)G_{PP}\\ &-(5A_4+A_3(3-6A_4)+3G_P)G_{PQ}-G_{PPPQ}=0.\end{aligned} \tag{100}$$

Again, apply the Lie symmetry method to Eq. (100), and then new infinitesimals are:

$$\begin{aligned}&\eta_G=\frac{1}{3}\left((4A_3A_4-2A_3-2)P+4QA_4^2-4QA_4-G-\frac{4p}{3}\right)b_1+b_4,\xi_P=\frac{b_1P}{3}+b_3,\\ &\xi_Q=b_1Q+b_2,\end{aligned} \tag{101}$$

where b_i's, $1\le i\le 4$, are arbitrary constants.

Take $b_1=0$ and all other constants are nonzero. The characteristic system for (101) is

$$\frac{\mathrm{d}P}{b_3}=\frac{\mathrm{d}Q}{b_2}=\frac{\mathrm{d}G}{b_4}, \tag{102}$$

which gives the similarity form

$$G(P,Q)=B_4Q+H(R) \quad \text{with } \; R=P-B_3Q, \tag{103}$$

where $B_4=\frac{b_4}{b_2}$ and $B_3=\frac{b_3}{b_2}$. Using (103) and (100), we get an ODE

$$B_3H^{(4)}(R)+H''(R)\left(2B_3kH'(R)\right)+LH''(R)=0, \tag{104}$$

where

$$L=A_4\left(\left(5-6A_3\right)B_3-3\right)+3A_3^2B_3^2+A_3B_3\left(3-5B_3\right)+3A_4^2+2B_3^2-3B_4.$$

On solving (104), we have

$$H(R)=\frac{2}{R}-\frac{LR}{6B_3}+\mathscr{K}_7 \quad \text{and} \quad H(R)=-\frac{LR}{6B_3}+\mathscr{K}_8, \tag{105}$$

where $\mathscr{K}_7$ and $\mathscr{K}_8$ are arbitrary constants. Consequently, we obtain

$$\begin{aligned} u(x,y,z,t)=&-\frac{L\left(B_3\left(A_3(z-t)+t-y\right)+A_4(t-z)+x\right)}{6B_3}\\ &+\frac{2}{B_3\left(A_3(z-t)+t-y\right)+A_4(t-z)+x}\\ &+B_4\left(A_3(t-z)-t+y\right)+A_6(z-t)+\mathscr{K}_7, \end{aligned} \tag{106}$$

$$\begin{aligned} u(x,y,z,t)=&-\frac{L\left(B_3\left(A_3(z-t)+t-y\right)+A_4(t-z)+x\right)}{6B_3}\\ &+B_4\left(A_3(t-z)-t+y\right)+A_6(z-t)+\mathscr{K}_8. \end{aligned} \tag{107}$$

3.10 Vector field $\mathbb{V}_2+\mathbb{V}_5+\mathbb{V}_6+\mathbb{V}_8+\mathbb{V}_{11}$

The associated characteristic system is

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{1}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{z\,t^2}, \tag{108}$$

which yields

$$u(x,y,z,t)=U(X,Y,Z)-\frac{1}{12}t^3(t-4z) \quad \text{with } \; X=x-t, \;\; Y=y-t \;\text{ and }\; Z=z-t. \tag{109}$$

On substituting u from (109) into (2), we obtain

$$3U_{ZZ}+5U_{YZ}+2U_{YY}+3U_{XZ}+2U_{XY}-3U_XU_{XY}-3U_YU_{XX}-U_{XXXY}=0. \tag{110}$$

We can find the travelling wave solution for Eq. (110):

$$\begin{aligned} U(X,Y,Z)=c_5+2c_1\tanh\Big[c_1X+\frac{1}{4}\big(4c_1^3-2c_1-5c_3\\ \pm\sqrt{16c_1^6-16c_1^4-40c_3c_1^3+4c_1^2-4c_3c_1+c_3^2}\big)Y+c_3Z+c_4\Big]. \end{aligned} \tag{111}$$

The corresponding travelling wave solution to Eq. (2) is given by

$$(x, y, z, t) = c_5 + 2c_1 \tanh\Big[c_1(x-t) + \frac{1}{4}\big(4c_1^3 - 2c_1 - 5c_3 \pm \sqrt{16c_1^6 - 16c_1^4 - 40c_3c_1^3 + 4c_1^2 - 4c_3c_1 + c_3^2}\big)(y-t) + c_3(z-t) + c_4\Big] - \frac{1}{12}t^3(t-4z). \tag{112}$$

Let us consider a travelling wave solution $U(X, Y, Z) = H(\zeta)$ for Eq. (110), where $\zeta = \mu X + \nu Y + \lambda Z$. Then the reduced ODE from Eq. (110) is

$$\mu^3\nu H^{(4)} + 6\mu^2\nu H'H'' - (2\nu + 3\lambda)(\mu + \nu + \lambda)H'' = 0. \tag{113}$$

Accordingly, we derive the following exact solutions:

$$H(\zeta) = \frac{(2\,\nu + 3\,\lambda)(\mu + \nu + \lambda)}{6\mu^2\nu}\,\zeta + \mathscr{K}_9,$$
$$H(\zeta) = \frac{2\mu}{\zeta} + \frac{(2\,\nu + 3\,\lambda)(\mu + \nu + \lambda)}{6\mu^2\nu}\,\zeta + \mathscr{K}_9.$$

As a result, we arrive at the invariant solutions of Eq. (2):

$$u(x, y, z, t) = \frac{(2\,\nu + 3\,\lambda)(\mu + \nu + \lambda)}{6\mu^2\nu}\,(\mu(x-t) + \nu(y-t) + \lambda(z-t)) + \mathscr{K}_9, \tag{114}$$

$$u(x, y, z, t) = \frac{(2\,\nu + 3\,\lambda)(\mu + \nu + \lambda)}{6\mu^2\nu}\,(\mu(x-t) + \nu(y-t) + \lambda(z-t)) + \mathscr{K}_9 + \frac{2\mu}{(\mu(x-t) + \nu(y-t) + \lambda(z-t))}. \tag{115}$$

Again, we apply the Lie symmetry method to equation (110), and then new infinitesimals are given as:

$$\xi_X = \frac{a_1}{3}(X + Z) + a_4, \quad \xi_Y = Ya_1 + a_3,$$
$$\xi_Z = Za_1 + a_2, \quad \eta_U = \frac{1}{9}(-X - 3Y - 3U)a_1 + Za_5 + a_6, \tag{116}$$

where a_i's, $1 \le i \le 6$, are arbitrary constants.
Case (i) Take $a_2 \neq 0$ and all other constants are zero.
From Eq. (116), we get the characteristic system as follows:

$$\frac{\mathrm{d}X}{0} = \frac{\mathrm{d}Y}{0} = \frac{\mathrm{d}Z}{a_2} = \frac{\mathrm{d}U}{0}, \tag{117}$$

which gives

$$U(X, Y, T) = G(P, Q) \quad \text{with} \;\; P = X, \quad Q = Y. \tag{118}$$

Using (118) and (110), we have the reduced equation

$$2G_{QQ} + 2G_{PQ} - 3G_PG_{PQ} - 3G_QG_{PP} - G_{PPPQ} = 0. \tag{119}$$

Again, apply the Lie symmetry method to Eq. (119), and then new infinitesimals are:

$$\xi_P = \frac{b_1P}{3} + b_3, \quad \xi_Q = b_1Q + b_2, \quad \eta_G = \frac{4P - 3G}{9}b_1 + b_4, \tag{120}$$

where b_i's, $1 \le i \le 4$, are arbitrary constants.

Take $b_1 \neq 0$ and all other constants are zero. The characteristic system for (120) is

$$\frac{\mathrm{d}P}{\frac{P}{3}} = \frac{\mathrm{d}Q}{Q} = \frac{\mathrm{d}G}{\frac{4P-3G}{9}}, \tag{121}$$

which gives the similarity form

$$G(P,Q) = \frac{H(R)}{\sqrt[3]{Q}} + \frac{2\,R\,\sqrt[3]{Q}}{3} \quad \text{with} \quad R = \frac{P}{\sqrt[3]{Q}}. \tag{122}$$

Using (122) and (119), we get an ODE

$$3RH^{(4)} + 12H^{(3)} + 2R^2H'' + H\left(8 + 9H''\right) + 18H'^2 + 6RH'\left(2 + 3H''\right) = 0. \tag{123}$$

On solving (123), we have

$$H(R) = \frac{\mathscr{K}_{10}}{R} \quad \text{and} \quad H(R) = \frac{\mathscr{K}_{10}}{R} - \frac{2R^2}{9}, \tag{124}$$

where $\mathscr{K}_7$ and $\mathscr{K}_8$ are arbitrary constants. Consequently, we obtain

$$u(x,y,z,t) = \frac{\mathscr{K}_{10}}{x-t} + \frac{1}{12}\left(-t^4 + 4t^3z - 8t + 8x\right), \tag{125}$$

$$u(x,y,z,t) = -\frac{\mathscr{K}_{10}}{t-x} - \frac{1}{12}t^3(t-4z) + \frac{2(t-x)^2}{9(t-y)} + \frac{2(x-t)}{3}. \tag{126}$$

3.11 Vector field $\mathbb{V}_2 + \mathbb{V}_5 + \mathbb{V}_6 + \mathbb{V}_{12}$

The associated characteristic system is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{1} = \frac{dt}{1} = \frac{du}{z\,t}, \tag{127}$$

which yields

$$u(x,y,z,t) = U(X,Y,Z) + \frac{t^2\,z}{2} \quad \text{with} \quad X = x, \quad Y = y - t \quad \text{and} \quad Z = z - t. \tag{128}$$

On substituting u from (128) into (2), we obtain

$$3U_{ZZ} + 5U_{YZ} + 2U_{YY} + 3U_{XZ} - 3U_XU_{XY} - 3U_YU_{XX} - U_{XXXY} = 0. \tag{129}$$

We can find the travelling wave solution for Eq. (129):

$$U(X,Y,Z) = 2c_1\tanh\left(c_1X + \frac{1}{4}\left(4c_1^3 - 5c_3 \pm \sqrt{16c_1^6 - 40c_3c_1^3 - 24c_3c_1 + c_3^2}\right)\right.$$
$$\left.\times\, Y + c_3Z + c_4\right) + c_5. \tag{130}$$

The corresponding travelling wave solution to Eq. (2) is given by

$$u(x,y,z,t) = 2c_1\tanh\left(c_1x + \frac{1}{4}\left(4c_1^3 - 5c_3 \pm \sqrt{16c_1^6 - 40c_3c_1^3 - 24c_3c_1 + c_3^2}\right)(y-t)\right.$$
$$\left. +\, c_3(z-t) + c_4\right) + c_5. \tag{131}$$

Again, we apply the Lie symmetry method to equation (129), and then new infinitesimals are given as:

$$\xi_X = \frac{a_1}{3}(X+Z) + a_4, \quad \xi_Y = Ya_1 + a_3,$$
$$\xi_Z = Za_1 + a_2, \quad \eta_U = \frac{1}{9}(-5X - 3Y - 3U)a_1 + Za_5 + a_6, \tag{132}$$

where a_i's, $1 \le i \le 6$, are arbitrary constants.
Case (1) Take $a_1 = a_5 = 0$ and all other constants are nonzero.
From Eq. (132), we get the characteristic system as:

$$\frac{\mathrm{d}X}{a_4} = \frac{\mathrm{d}Y}{a_3} = \frac{\mathrm{d}Z}{a_2} = \frac{\mathrm{d}U}{a_6}, \tag{133}$$

which gives the similarity solution

$$U(X,Y,T) = G(P,Q) + A_6 Z \quad \text{with} \;\; P = X - ZA_4, \;\; Q = Y - ZA_3, \tag{134}$$

where $A_3 = \frac{a_3}{a_2}$, $A_4 = \frac{a_4}{a_2}$, and $A_6 = \frac{a_6}{a_2}$.
Using (134) and (129), we have the reduced equation

$$G_{PPPQ} + 3G_P G_{PQ} + 3G_Q G_{PP} + (1 - A_3)(3A_3 - 2)G_{QQ} + 3(1 - A_4)A_4 G_{PP} + (5A_4 + 3A_3(1 - 2A_4))G_{PQ} = 0. \tag{135}$$

Again, upon applying the Lie symmetry method to equation (135), new infinitesimals are:

$$\eta_G = \frac{1}{9}\left((6A_4^2 + (12A_3 - 16)A_4 - 6A_3)P - 3G\right)b_1 + b_4,$$
$$\xi_P = \frac{b_1 P}{3} + b_3, \; \xi_Q = b_1 Q + b_2, \tag{136}$$

where b_i's, $1 \le i \le 4$, are arbitrary constants.

Take $b_1 = 0$ and the other constants are nonzero. The characteristic system for (136) is

$$\frac{\mathrm{d}P}{b_3} = \frac{\mathrm{d}Q}{b_2} = \frac{\mathrm{d}G}{b_4}, \tag{137}$$

which gives the similarity form

$$G(P,Q) = B_4 Q + H(R) \quad \text{with} \;\; R = P - B_3 Q, \tag{138}$$

where $B_4 = \frac{b_4}{b_2}$ and $B_3 = \frac{b_3}{b_2}$. Using (138) and (135), we get an ODE

$$B_3 H^{(4)}(R) + H''(R)\left(6B_3 H'(R)\right) + LH''(R) = 0, \tag{139}$$

where

$$L = A_4((6A_3 - 5)B_3 - 3) + A_3 B_3(5B_3 - 3) - 3A_3^2 B_3^2 - 3A_4^2 - 2B_3^2 + 3B_4.$$

On solving (139), we have

$$H(R) = \frac{2}{R} + \frac{LR}{6B_3} + \mathscr{K}_{11} \quad \text{and} \quad H(R) = \frac{LR}{6B_3} + \mathscr{K}_{12}, \tag{140}$$

where $\mathscr{K}_{11}$ and $\mathscr{K}_{12}$ are arbitrary constants. Accordingly, we receive the following solutions:

$$u(x,y,z,t) = \frac{2}{x + A_4(t - z) + B_3(t - y + (z - t)A_3)}$$

$$
\begin{aligned}
&+\frac{L(x+A_4(t-z)+B_3(t-y+(z-t)A_3))}{6B_3}\\
&+\mathscr{K}_{11}+B_4(y-t+(t-z)A_3)\\
&+A_6(z-t)+\frac{t^2z}{2},
\end{aligned} \tag{141}
$$

$$
\begin{aligned}
u(x,y,z,t)=&\frac{L(x+A_4(t-z)+B_3(t-y+(z-t)A_3))}{6B_3}\\
&+\mathscr{K}_{12}+B_4(y-t+(t-z)A_3)\\
&+A_6(z-t)+\frac{t^2z}{2}.
\end{aligned} \tag{142}
$$

3.12 Vector field $\mathbb{V}_2+\mathbb{V}_8+\mathbb{V}_{13}$

The associated characteristic system is

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{z}, \tag{143}$$

which yields

$$u(x,y,z,t)=U(X,Y,Z)+zt \quad \text{with} \quad X=x-t, \quad Y=y \text{ and } Z=z. \tag{144}$$

On substituting u from (144) into (2), we get

$$3(U_{ZZ}+U_{YZ}+U_{XZ})+2U_{XY}-3U_XU_{XY}-3U_YU_{XX}-U_{XXXY}=0. \tag{145}$$

Again, we apply the transformation technique to equation (145), and then new infinitesimals are given as:

$$
\begin{aligned}
&\xi_X=\frac{a_1}{3}(X+Z)+a_4, \quad \xi_Y=Ya_1+a_3,\\
&\xi_Z=Za_1+a_2, \quad \eta_U=\frac{1}{9}(X-3Y-3U)a_1+Za_5+a_6,
\end{aligned} \tag{146}
$$

where a_i's, $1\le i\le 6$, are arbitrary constants.

Case (1) Take $a_2\neq 0$ and all other constants are zero.

From Eq. (146), we get the characteristic system as:

$$\frac{\mathrm{d}X}{0}=\frac{\mathrm{d}Y}{0}=\frac{\mathrm{d}Z}{a_2}=\frac{\mathrm{d}U}{0}, \tag{147}$$

which yields

$$U(X,Y,T)=G(P,Q) \quad \text{with} \quad P=X, \quad Q=Y. \tag{148}$$

Using (148) and (145), we have the reduced equation

$$G_{PPPQ}+3G_PG_{PQ}+3G_QG_{PP}-2G_{PQ}=0. \tag{149}$$

We can find the travelling wave solution for Eq. (145):

$$G(P,Q)=c_1+\sqrt{2}\tanh\left(\frac{P}{\sqrt{2}}\pm(c_2Q+c_3)\right), \tag{150}$$

where c_i, $i=1,\ldots,3$, are arbitrary constants.

The corresponding travelling wave solution to Eq. (2) is given by

$$u(x, y, z, t) = zt + c_1 + \sqrt{2}\tanh\left(\frac{x-t}{\sqrt{2}} \pm (c_2 y + c_3)\right). \tag{151}$$

Upon using the symmetry transformation method on (149), new infinitesimals are:

$$\eta_G = \frac{1}{3}(4P - 3G)b_1 + b_3, \quad \xi_P = b_1 P + b_2, \quad \xi_Q = \phi(Q), \tag{152}$$

where b_i's, $1 \leq i \leq 3$, are arbitrary constants and $\phi(Q)$ is any arbitrary function.

Take $b_1 = 1$ and the other constants as zero. The characteristic system for (152) is

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}Q}{\phi(Q)} = \frac{\mathrm{d}G}{\frac{1}{3}(4P - 3G)}, \tag{153}$$

which provides

$$G(P, Q) = \frac{H(R)}{P} + \frac{2P}{3} \quad \text{with } R = \log(P) - \int \frac{\mathrm{d}Q}{\phi(Q)}. \tag{154}$$

Using (154) and (149), we get an ODE

$$12H'^2 + H'(6 - 9H - 6H'') + H''(3H - 11) + 6H^{(3)} - H^{(4)} = 0. \tag{155}$$

On solving (155), we have

$$H(R) = \frac{2}{3} + \frac{2}{R}. \tag{156}$$

Consequently, we receive the following solutions of extended JM equation (2):

$$u(x, y, z, t) = z\,t + \frac{2(x-t)}{3} + \frac{1}{x-t}\left(\frac{2}{3} + \frac{2}{\log(x-t) - \int \frac{\mathrm{d}y}{\phi(y)}}\right). \tag{157}$$

4 Physical interpretation and discussions

In the present section, we exhibit the physical interpretation of the obtained soliton solutions via Eqs. (21), (22), (23), (47), (65), (78), (92), (106), and (141) with the assistance of numerical simulation by choosing appropriate values to the arbitrary parameter constants C_i's, $i = 1, \ldots, 7$. The value of an arbitrary parameter constant is shown as a random number of 3D-shapes tracing of solutions in such a way that we can get analytically as well as physically meaningful results and record the specific value in the corresponding analysis. The interaction of soliton solutions can be explained by observing the distinct solitonic wave structures of solutions in the above figures.

Figure 1 reflects the evolution wave profile behavior represented via Eqs. (21) and (22) which are the solutions of extended JM equation (2). We have taken appropriate values of the arbitrary constants as $C_1 = 0.002$, $C_2 = 1$, and $a = 1$ for the space range $-5000 \leq x, y \leq$ 50,000. Evolution wave profiles can be observed via Fig. 1a, b shows rational soliton forms.

Figure 2 shows the multi-solitons behavior of the solutions given by Eq. (23) is represented graphically via Fig. 2a when $C_1 = 0.971$, $C_2 = 0.003$, and $a = 14.047$ for the space range $-5 \leq x, y \leq 5$, Fig. 2b for the similar space range $-5 \leq x \leq 5$, $-2 \leq y \leq 2$, and the values of the arbitrary constants and space range are the same as in Fig. 2a.

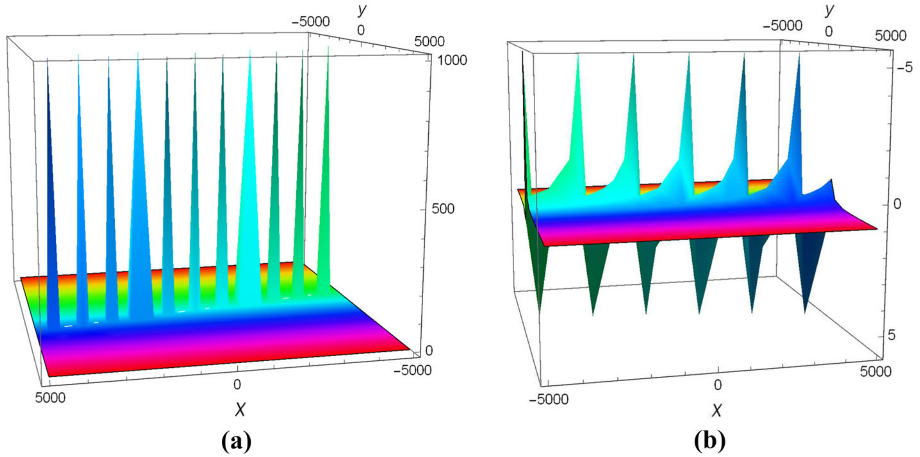


Fig. 1 3D-graphs of evolution profiles for solutions (21) and (22) with involved arbitrary constants as $C_1 = 0.002$, $C_2 = 1$ and $a = 1$ for the space range $-5000 \leq x, y \leq 50000$

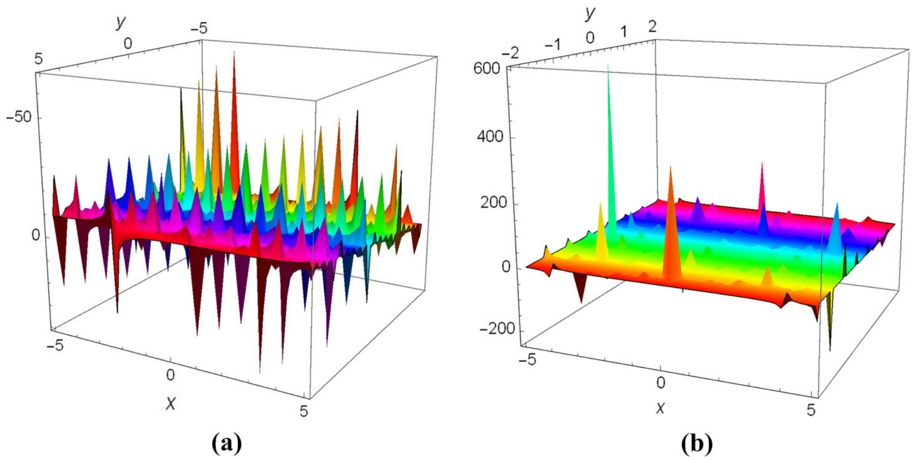


Fig. 2 3D-graphs of multi-solitons shapes for solution (23) with $C_1 = 0.971$, $C_2 = 0.003$, and $a = 14.047$ for the space range $-5 \leq x, y \leq 5$ and $-5 \leq x \leq 5$, $-2 \leq y \leq 2$

Figure 3 shows the physical behavior of the wave profile of solution (47) is observed. This figure shows the doubly soliton wave profile at $y = 1$ and after $y = 11$ profile annihilated into a single soliton wave profile when $C_1 = 1,\ C_2 = 0.01,\ A = 1.0321,\ B = 0.1456$, and $z = -1.4351$ for the space range $-100 \leq x,\ t \leq 100$.

Figure 4 shows the parabolic wave profile of solution (65) is represented in Fig. 4a, b. The arbitrary constants are chosen as $A_2 = 0.15,\ B_2 = 11,\ z = 1$, $\mathscr{K}_1 = 3,\ \mathscr{K}_2 = 3$ and the space range $-650 \leq x \leq 500$, $1 \leq y \leq 100$.

Figure 5 shows the solution given by Eq. (78) represents lump-type soliton profiles with arbitrary constants $L = 1.2,\ A_3 = 0.1, A_4 = 1.11, A_6 = 0.15, B_2 = 0.2, t = 1$, and $\mathscr{K}_3 = 3$ and the space range $-30 \leq x, z \leq 30$. These solitary wave solutions arrive at a balance between the nonlinearity and the dispersion.

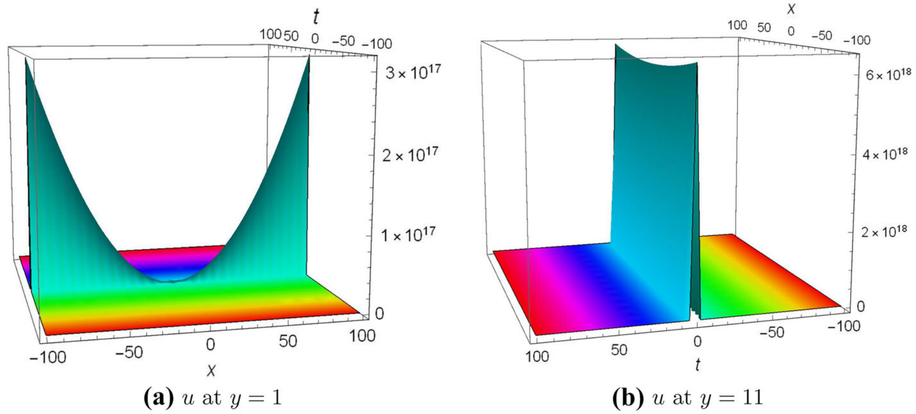


Fig. 3 3D-graphs of doubly solitons and single solitons for solution (47) with $C_1 = 1,\ C_2 = 0.01,\ A = 1.0321,\ B = 0.1456$, and $z = -1.4351$ for the space range $-100 \le x,\ t \le 100$

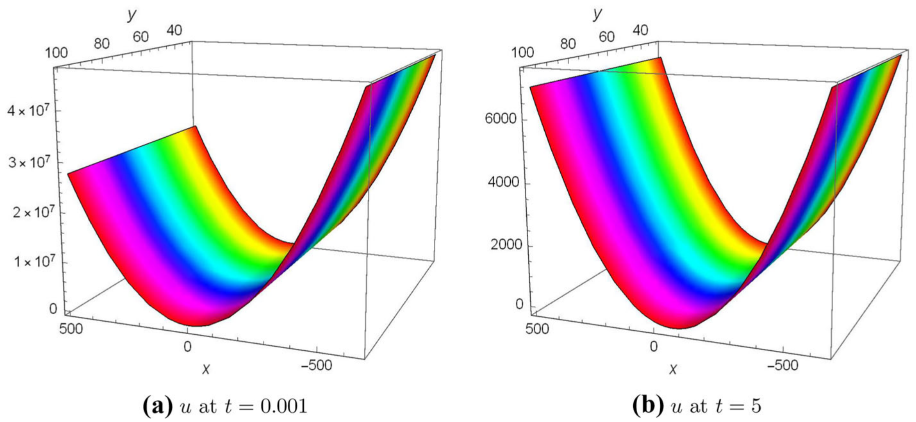


Fig. 4 3D-graphs of parabolic solitons for solution (65) with $A_2 = 0.15,\ B_2 = 11,\ z = 1\,, \mathscr{K}_1 = 3,\ \mathscr{K}_2 = 3$ and the space range $-650 \le x \le 500$, $1 \le y \le 100$

Figure 6 shows the dynamical structure of the solution is shown two solitonic behaviors for Eq. (92) with arbitrary constants $L = 2,\ A_3 = 13, A_4 = 18, A_6 = 1, B_3 = 6, B_4 = 21, z = 1$, and $\mathscr{K}_5 = 3$ and the space range $-30 \le x, y \le 30$.

Figure 7 shows the multi-solitons wave profiles are exhibited graphically for Eq. (106) by suitable values of arbitrary constants $L = 0.003,\ A_3 = 0.5, A_4 = 0.19, A_6 = 0.11, B_3 = 2, B_4 = 0.1, y = 0.006$, and $\mathscr{K}_7 = 13$ and the space range $-30 \le x, z \le 30$.

Figure 8 shows the interactions between parabolic solitons structures with lump-type solitons for solution (141) have been observed in this figure. The arbitrary constants are taken as $L = 0.003,\ A_3 = 1.8,\ A_4 = 18,\ A_6 = 0.2,\ B_3 = 10.07,\ B_4 = 0.3,\ z = 0.42,\ \mathscr{K}_{11} = 1$ for the range $-30 \le x \le 30$ and $-10 \le t \le 10$.

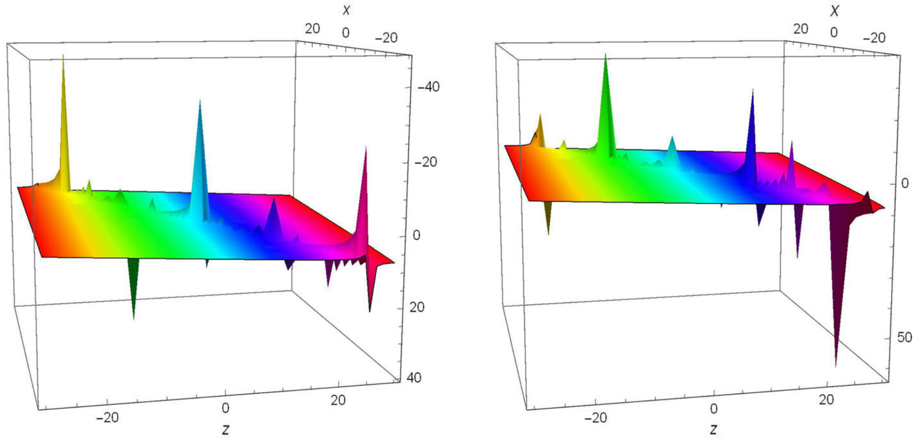


Fig. 5 3D-graphs of lump-form solitons for solution (78) with arbitrary constants $L = 1.2,\ A_3 = 0.1,\ A_4 = 1.11,\ A_6 = 0.15,\ B_2 = 23,\ B_3 = 0.2,\ t = 1\ ,\mathscr{K}_3 = 3$

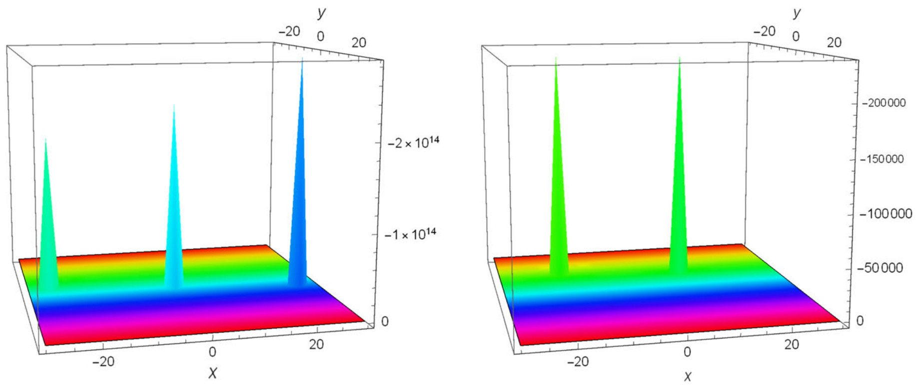
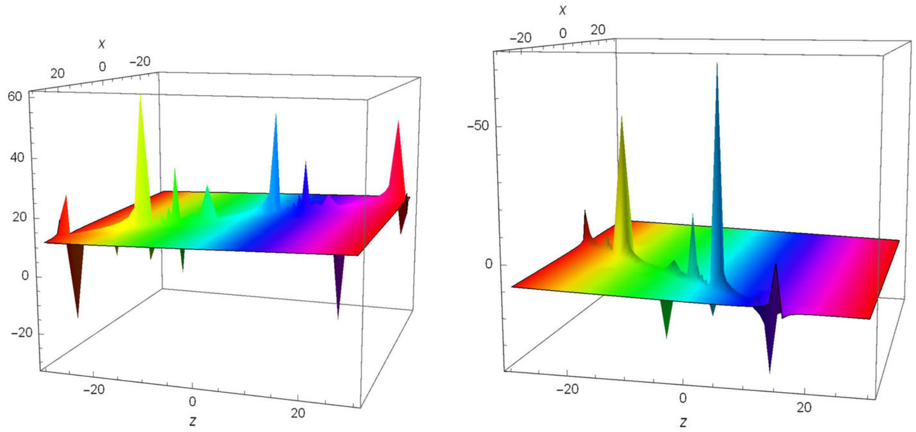


Fig. 6 3D-graphs of triply solitons and double solitons for solution (92) with arbitrary constants $L = 2,\ A_3 = 13,\ A_4 = 18,\ A_6 = 1,\ B_3 = 6,\ B_4 = 21,\ z = 1\ ,\mathscr{K}_5 = 3$

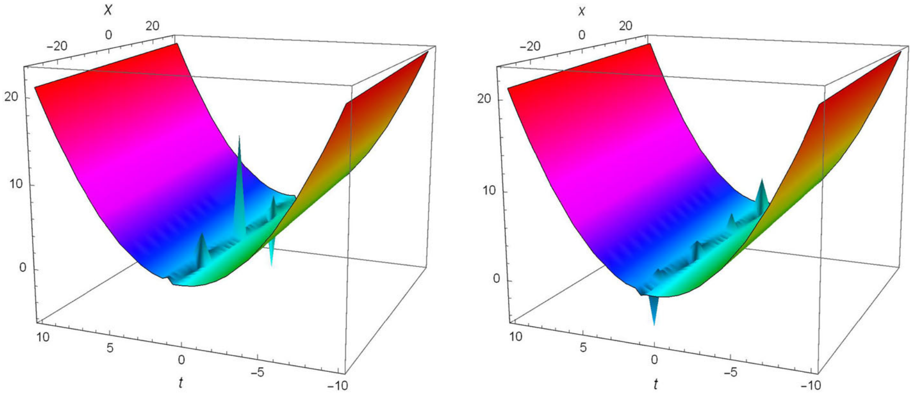
5 Conclusion

In summary, Lie point symmetries and their corresponding similarity reductions have been constructed for an extended (3+1)-dimensional Jimbo–Miwa (JM) equation. Abundant closed-form invariant solutions of the extended JM equation have been successfully derived by employing the Lie group technique. The resulting solutions reflect the dynamics of multiple solitons structures and are compatible with numerical results. It is remarkable to notify that the generated invariant closed-form solutions in this work have not been documented in the previous findings. Furthermore, the wide diversity of features and physical parameters of these constructed solutions are expressed via three-dimensional graphics by using the best choice of the involved constant parameters. This research work is highly suggested in the fields of advanced research and development.



(a) u at $t = 1$ **(b)** u at $t = 15$

Fig. 7 3D-graphs of lump-form solitons for solution (106) with arbitrary constants $L = 0.003,\ A_3 = 0.5,\ A_4 = 0.19,\ A_6 = 0.11,\ B_3 = 2,\ B_4 = 0.1,\ y = 0.06\ , \mathscr{K}_7 = 13$



(a) u at $y = 0.01$ **(b)** u at $y = 1$

Fig. 8 3D-graphs of interactions between lump solitons and parabolic solitons for solution (141) with arbitrary constants $L = 0.003,\ A_3 = 1.8,\ A_4 = 18,\ A_6 = 0.2,\ B_3 = 10.07,\ B_4 = 0.3,\ z = 0.42\ , \mathscr{K}_{11} = 1$

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