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Riemann–Hilbert method for multi-soliton solutions of a fifth-order nonlinear Schrödinger equation

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Abstract

A fifth-order nonlinear Schrödinger equation which describes the one-dimensional anisotropic Heisenberg ferromagnetic spin chain is under investigation in this paper. Starting from the spectral analysis, a matrix Riemann–Hilbert problem is established on the real axis. Then, through solving the resulting matrix Riemann–Hilbert problem under the condition of no reflection, we systematically derive multi-soliton solutions to the fifth-order nonlinear Schrödinger equation. In addition, the localized structures of one-soliton solution are shown vividly via a few plots.

Keywords Fifth-order nonlinear Schrödinger equation · Riemann–Hilbert problem · Soliton solutions

Mathematics Subject Classification 35Q55 · 37K10 · 35C08

1 Introduction

One of the three branches of nonlinear science is the theory of solitons. Due to the fact that investigating exact solutions to nonlinear evolution equations (NLEEs) can provide more insight into interpreting nonlinear phenomena in many fields, such as

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hydrodynamics, plasma dynamics, optical communication and solid state physics, it is of particular significance to seek abundant exact solutions of NLEEs [1–6]. Thus far, a number of efficient approaches have been presented for deriving exact solutions, some of which include the Hirota's bilinear method [7–9], the Darboux transformation method [10–12], the Riemann–Hilbert method [13–22], the KP hierarchy reduction method [23] and the generalized unified method [24]. In recent years, there has been an increasing interest in treating NLEEs via utilizing the Riemann–Hilbert technique. For example, Wang et al. [20] investigated the focusing Kundu–Eckhaus equation through Riemann–Hilbert formulation. Consequently, the bright N-soliton solutions to this equation were gained explicitly. More recently, a matrix Riemann–Hilbert problem was formulated for a six-component system of fourth-order AKNS equations [21], and then multi-soliton solutions to the considered system were worked out.

In this paper, we consider the following fifth-order nonlinear Schrödinger (NLS) equation [25]

$$\begin{aligned}&iq_t+\frac{1}{2}q_{xx}+|q|^2q-i\alpha(q_{xxx}+6|q|^2q_x)+\gamma(q_{xxxx}+6q_x^2q^*+4q|q_x|^2+8|q|^2q_{xx}\\&+2q^2q_{xx}^*+6q|q|^4)-i\delta(q_{xxxxx}+10|q|^2q_{xxx}+30|q|^4q_x+10qq_xq_{xx}^*\\&+10qq_x^*q_{xx}+20q^*q_xq_{xx}+10q_x^2q_x^*)=0,\end{aligned}\tag{1}$$

which is used to describe one-dimensional anisotropic Heisenberg ferromagnetic spin chain. Here q represents a normalized complex amplitude of the optical pulse envelope, the subscripts denote the partial derivatives with respect to the scaled spatial coordinate x and time coordinate t correspondingly, whereas α, γ and δ are respectively the real coefficients of the third-, fourth- and fifth-order terms, and the asterisk signifies the complex conjugation. Actually, Eq. (1) covers many significant nonlinear differential equations, which are given below:

- (i) When $\alpha=\gamma=\delta=0$, Eq. (1) is reduced to the focusing NLS equation [26] describing the wave evolution in different physical systems.
- (ii) When $\alpha\neq 0$ and $\gamma=\delta=0$, Eq. (1) becomes the Hirota equation [27] describing the propagation of a subpicosecond or femtosecond pulse.
- (iii) When $\alpha=\delta=0$ and $\gamma\neq 0$, Eq. (1) is turned into the fourth-order dispersive NLS equation [28] describing the one-dimensional anisotropic Heisenberg ferromagnetic spin chain with the octuple-dipole interaction.
- (iv) When $\alpha=\gamma=0$ and $\delta\neq 0$, Eq. (1) is converted into the fifth-order NLS equation [29] describing the Heisenberg ferromagnetic spin system.

There have been several studies on the fifth-order NLS equation (1) up to now. For instance, the study in [25] presented Lax pair, and exact expressions for the most representative soliton solutions, which involves two-soliton collisions and the degenerate case of two-soliton solution, as well as beating structures composed of two or three solitons, were attained by applying the Darboux scheme. In another study [30], infinitely-many conservation laws for Eq. (1) were constructed on basis of the Lax pair. By use of the Hirota's bilinear method, the one-, two- and three-soliton solutions in analytic forms were generated. In addition, the Akhmediev breathers, Kuznetsov-Ma solitons and rogue wave solutions were explored by using the Darboux transformation method [31].

The current study seeks to find multi-soliton solutions of the fifth-order NLS equation (1) via the Riemann–Hilbert method. The outline of the paper is as follows. In Sect. 2, we build a matrix Riemann–Hilbert problem on the real line by performing the analysis on the given spectral problem. In Sect. 3, based on the obtained Riemann–Hilbert problem in which the jump matrix is takn as the identity matrix, we compute multi-soliton solutions to the considered equation (1). The last section is a brief conclusion.

2 Matrix Riemann–Hilbert problem

The aim of this section is to formulate a matrix Riemann–Hilbert problem. The Lax pair [25] associated with Eq. (1) takes the form

$$\Phi_x = U\Phi, \quad U = i\begin{pmatrix} \zeta & q^* \\ q & -\zeta \end{pmatrix}, \tag{2a}$$

$$\Phi_t = V\Phi, \quad V = \sum_{c=0}^{5} i\zeta^c \begin{pmatrix} A_c & B_c^* \\ B_c & -A_c \end{pmatrix}, \tag{2b}$$

where $\Phi = (\varphi, \psi)^{\mathrm{T}}$ is the spectral function, the symbol T means transpose of the vector, and ζ is a spectral parameter. Moreover,

$$\begin{aligned}
&A_5 = 16\delta, \quad A_4 = -8\gamma, \quad A_3 = -4\alpha - 8\delta|q|^2,\\
&A_2 = 1 + 4\gamma|q|^2 + 4i\delta(q_x^* q - q_x q^*),\\
&A_1 = 2\alpha|q|^2 + 6\delta|q|^4 - 2i\gamma(q_x^* q - q_x q^*) + 2\delta(q_{xx}^* q - |q_x|^2 + q_{xx} q^*),\\
A_0 = &-\frac{1}{2}|q|^2 - 3\gamma|q|^4 - i\alpha(q_x^* q - q_x q^*)\\
&- \gamma(q_{xx}^* q - |q_x|^2 + q_{xx} q^*) - i\delta(q_{xxx}^* q - q_{xx}^* q_x\\
&+ q_{xx} q_x^* - q_{xxx} q^*) - 6i\delta(q_x^* q - q_x q^*)|q|^2,\\
&B_5 = 0, \quad B_4 = 16\delta q, \quad B_3 = -8\gamma q + 8i\delta q_x, \quad B_2 = -4\alpha q - 8\delta|q|^2 q - 4i\gamma q_x - 4\delta q_{xx},\\
&B_1 = q + 4\gamma|q|^2 q - 2i\alpha q_x - 12i\delta|q|^2 q_x + 2\gamma q_{xx} - 2i\delta q_{xxx},\\
B_0 = &\ 2\alpha|q|^2 q + 6\delta|q|^4 q + \frac{1}{2} i q_x + 6i\gamma|q|^2 q_x + \alpha q_{xx}\\
&+ 2\delta q_{xx}^* q^2 + 4\delta|q_x|^2 q + 6\delta q_x^2 q^*\\
&+ 8\delta q_{xx}|q|^2 + i\gamma q_{xxx} + \delta q_{xxxx}.
\end{aligned}$$

For the convenience of analysis, we write the Lax pair (2) as the equivalent form

$$\Phi_x = (i\zeta\sigma + \tilde{U})\Phi, \tag{3a}$$

$$\Phi_t = \left[(16i\delta\zeta^5 - 8i\gamma\zeta^4 - 4i\alpha\zeta^3 + i\zeta^2)\sigma + \tilde{Q}\right]\Phi, \tag{3b}$$

where

$$\sigma=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad Q=\begin{pmatrix}0&q^*\\q&0\end{pmatrix},$$
$$\tilde{Q}=i\begin{pmatrix}A_0&B_0^*\\B_0&-A_0\end{pmatrix}+i\zeta\begin{pmatrix}A_1&B_1^*\\B_1&-A_1\end{pmatrix}+i\zeta^2\begin{pmatrix}0&B_2^*\\B_2&0\end{pmatrix}+i\zeta^3\begin{pmatrix}0&B_3^*\\B_3&0\end{pmatrix}+i\zeta^4\begin{pmatrix}0&B_4^*\\B_4&0\end{pmatrix}$$
$$+i\zeta^5\begin{pmatrix}0&B_5^*\\B_5&0\end{pmatrix}+i\zeta^2\big[4\gamma|q|^2+4i\gamma(q_x^*q-q_xq^*)\big]\sigma-8i\zeta^3\delta|q|^2\sigma.$$

Here we posit that the potential function q in the Lax pair (3) decays to zero sufficiently fast as $x\to\pm\infty$. It can be seen from (3) that when $x\to\pm\infty$, $\Phi\propto e^{i\zeta\sigma x+\left(16i\delta\zeta^5-8i\gamma\zeta^4-4i\alpha\zeta^3+i\zeta^2\right)\sigma t}$. This leads us to introduce the following transformation

$$\Phi=\mu e^{i\zeta\sigma x+\left(16i\delta\zeta^5-8i\gamma\zeta^4-4i\alpha\zeta^3+i\zeta^2\right)\sigma t},$$

based on which the Lax pair (3) becomes

$$\mu_x=i\zeta[\sigma,\mu]+\tilde{U}\mu, \tag{4a}$$

$$\mu_t=(16i\delta\zeta^5-8i\gamma\zeta^4-4i\alpha\zeta^3+i\zeta^2)[\sigma,\mu]+\tilde{Q}\mu, \tag{4b}$$

where $[\sigma,\mu]=\sigma\mu-\mu\sigma$ is the commutator and $\tilde{U}=iQ$.

Now we begin to consider the spectral analysis, for which we merely concentrate on the spectral problem (4a). Because the analysis will take place at a fixed time, the t-dependence will be suppressed. As for (4a), we write its two matrix Jost solutions as a collection of columns

$$\mu_\pm(x,\zeta)=([\mu_\pm]_1,[\mu_\pm]_2)(x,\zeta), \tag{5}$$

obeying the asymptotic conditions

$$\mu_-(x,\zeta)\to I_2,\quad x\to-\infty, \tag{6a}$$

$$\mu_+(x,\zeta)\to I_2,\quad x\to+\infty, \tag{6b}$$

where the subscripts of μ signify which end of the x-axis the boundary conditions are set, and I_2 is the unit matrix of rank 2. The $\mu_\pm(x,\zeta)$ are uniquely determined by the integral equations of Volterra-type

$$\mu_-(x,\zeta)=I_2+\int_{-\infty}^{x}e^{i\zeta\sigma(x-y)}\tilde{U}(y)\mu_-(y,\zeta)e^{-i\zeta\sigma(x-y)}dy, \tag{7a}$$

$$\mu_+(x,\zeta)=I_2-\int_{x}^{+\infty}e^{i\zeta\sigma(x-y)}\tilde{U}(y)\mu_+(y,\zeta)e^{-i\zeta\sigma(x-y)}dy. \tag{7b}$$

The direct analysis on (7) yields that $[\mu_-]_1, [\mu_+]_2$ are analytic for $\zeta \in \mathbb{D}^-$ and continuous for $\zeta \in \mathbb{D}^- \cup \mathbb{R}$, while $[\mu_+]_1, [\mu_-]_2$ are analytic for $\zeta \in \mathbb{D}^+$ and continuous for $\zeta \in \mathbb{D}^+ \cup \mathbb{R}$, where $\mathbb{D}^-$ and $\mathbb{D}^+$ are respectively the lower and upper half ζ-plane.

It is indicated due to the Abel's identity and $\mathrm{tr} Q = 0$ that the determinants of $\mu_\pm$ are independent of x. Evaluating $\det \mu_-$ at $x = -\infty$ and $\det \mu_+$ at $x = +\infty$, we see that $\det \mu_\pm(x, \zeta) = 1$ for $\zeta \in \mathbb{R}$. Since both $\mu_- E$ and $\mu_+ E$ are matrix solutions of the spectral problem (4a), they must be linearly dependent, namely

$$\mu_- E = \mu_+ E S(\zeta), \quad E = e^{i\zeta\sigma x}, \quad S(\zeta) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad \zeta \in \mathbb{R}. \tag{8}$$

Here $S(\zeta)$ is a scattering matrix. It is obvious that $\det S(\zeta) = 1$.

A matrix Riemann–Hilbert problem we are looking for is related to two matrix functions: one is analytic in $\mathbb{D}^+$ and the other is analytic in $\mathbb{D}^-$. Set the analytic function P_1 in $\mathbb{D}^+$ be

$$P_1(x, \zeta) = ([\mu_+]_1, [\mu_-]_2)(x, \zeta). \tag{9}$$

And then, P_1 can be expanded into the asymptotic series at large-ζ

$$P_1(x, \zeta) = P_1^{(0)} + \frac{P_1^{(1)}}{\zeta} + \frac{P_1^{(2)}}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right), \quad \zeta \to \infty. \tag{10}$$

Since P_1 satisfies the spectral problem (4a), inserting (10) into (4a) and equating terms with like powers of ζ directly leads to

$$\begin{aligned} &O(1) : i\left[\sigma, P_1^{(1)}\right] + iQP_1^{(0)} = P_{1x}^{(0)}, \\ &O(\zeta) : i\left[\sigma, P_1^{(0)}\right] = 0. \end{aligned}$$

Hence we have $P_1^{(0)} = I_2$, i.e. $P_1 \to I_2, \zeta \in \mathbb{D}^+ \to \infty$.

We continue to find the analytic counterpart of P_1 in $\mathbb{D}^-$. For this purpose, we partition the inverse matrices of $\mu_\pm$ into rows, that is

$$\mu_\pm^{-1} = \begin{pmatrix} [\mu_\pm^{-1}]^1 \\ [\mu_\pm^{-1}]^2 \end{pmatrix}, \tag{11}$$

which fulfill the adjoint scattering equation related to (4a)

$$K_x = i\zeta[\sigma, K] - K\tilde{U}, \tag{12}$$

and follow the boundary conditions $\mu_\pm^{-1} \to I_2$ as $x \to \pm\infty$. It is apparent from (8) that

$$E^{-1}\mu_-^{-1} = R(\zeta)E^{-1}\mu_+^{-1}, \tag{13}$$

where $R(\zeta) = (r_{lk})_{2\times 2}$ is the inverse matrix of $S(\zeta)$. Hence, the matrix function P_2 which is analytic in $\mathbb{D}^-$ is given as

$$P_2(x, \zeta) = \begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_-^{-1}]^2 \end{pmatrix}(x, \zeta). \tag{14}$$

In the same way as P_1, it turns out that the very large-ζ asymptotic behavior of P_2 is $P_2 \to I_2$ as $\zeta \in \mathbb{D}^- \to \infty$.

Substituting (5) into (8) gives rise to

$$([\mu_-]_1, [\mu_-]_2) = ([\mu_+]_1, [\mu_+]_2)\begin{pmatrix} s_{11} & s_{12}e^{2i\zeta x} \\ s_{21}e^{-2i\zeta x} & s_{22} \end{pmatrix},$$

from which we obtain

$$[\mu_-]_2 = s_{12}e^{2i\zeta x}[\mu_+]_1 + s_{22}[\mu_+]_2.$$

Hence, P_1 is of the form

$$P_1 = ([\mu_+]_1, [\mu_-]_2) = ([\mu_+]_1, [\mu_+]_2)\begin{pmatrix} 1 & s_{12}e^{2i\zeta x} \\ 0 & s_{22} \end{pmatrix}.$$

On the other hand, via carrying (11) into (13), we derive

$$\begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_-^{-1}]^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}e^{2i\zeta x} \\ r_{21}e^{-2i\zeta x} & r_{22} \end{pmatrix}\begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix},$$

from which we can express $[\mu_-^{-1}]^2$ as

$$[\mu_-^{-1}]^2 = r_{21}e^{-2i\zeta x}[\mu_+^{-1}]^1 + r_{22}[\mu_+^{-1}]^2.$$

Subsequently, P_2 is of the form

$$P_2 = \begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_-^{-1}]^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r_{21}e^{-2i\zeta x} & r_{22} \end{pmatrix}\begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix}.$$

Having constructed two matrix functions P_1 and P_2 which are analytic in $\mathbb{D}^+$ and $\mathbb{D}^-$ respectively so far, we are in a position to work out a matrix Riemann–Hilbert problem. Here we denote that the limit of P_1 is P^+ as $\zeta \in \mathbb{D}^+ \to \mathbb{R}$ and the limit of P_2 is P^- as $\zeta \in \mathbb{D}^- \to \mathbb{R}$, based on which a matrix Riemann–Hilbert problem desired can be acquired below

$$P^-(x, \zeta)P^+(x, \zeta) = \begin{pmatrix} 1 & s_{12}e^{2i\zeta x} \\ r_{21}e^{-2i\zeta x} & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}. \tag{15}$$

The canonical normalization conditions are given by

$$P_1(x,\zeta) \to I_2, \quad \zeta \in \mathbb{D}^+ \to \infty,$$
$$P_2(x,\zeta) \to I_2, \quad \zeta \in \mathbb{D}^- \to \infty,$$

and $s_{12}r_{21} + s_{22}r_{22} = 1$.

In what follows, we plan to retrieve the potential function $q(x, t)$. As a matter of fact, expanding P_1 at large-ζ as

$$P_1(\zeta) = I_2 + \frac{P_1^{(1)}}{\zeta} + \frac{P_1^{(2)}}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right), \quad \zeta \to \infty,$$

and then inserting this expansion into (4a), we know

$$Q = -\left[\sigma, P_1^{(1)}\right].$$

Thereupon, the potential function is restructured as

$$q(x,t) = 2(P_1^{(1)})_{21},$$

where $(P_1^{(1)})_{21}$ is the (2,1)-element of $P_1^{(1)}$.

3 Soliton solutions

The previous section has described a matrix Riemann–Hilbert problem for Eq. (1). Next we treat the Riemann–Hilbert problem in the sense of irregularity. The irregularity means that both $\det P_1(\zeta)$ and $\det P_2(\zeta)$ possess some zeros in their analytic domains. By drawing on the definitions of P_1 and P_2 as well as the scattering relation (8), we have

$$\det P_1(\zeta) = s_{22}(\zeta), \quad \det P_2(\zeta) = r_{22}(\zeta),$$

from which we know that $\det P_1(\zeta)$ and $\det P_2(\zeta)$ possess the same zeros as $s_{22}(\zeta)$ and $r_{22}(\zeta)$ respectively, and $r_{22} = (S^{-1})_{22} = s_{11}$.

In view of the above, it is time to discuss the characteristic feature of zeros. Regarding the matrix Q having the symmetry relation $Q^{\clubsuit} = Q$, where the superscript $\clubsuit$ signifies the Hermitian of a matrix, we know

$$\mu_\pm^{\clubsuit}(\zeta^*) = \mu_\pm^{-1}(\zeta). \tag{16}$$

The expressions (9) and (14) can be rewritten as

$$P_1 = \mu_+ H_1 + \mu_- H_2, \tag{17a}$$

$$P_2 = H_1 \mu_+^{-1} + H_2 \mu_-^{-1}, \tag{17b}$$

where $H_1 = \mathrm{diag}(1, 0)$ and $H_2 = \mathrm{diag}(0, 1)$. By taking the Hermitian of (17a) and making use of (16), we find

$$P_1^{\clubsuit}(\zeta^*) = P_2(\zeta), \quad \zeta \in \mathbb{D}^-, \tag{18}$$

and the involution property of the scattering matrix $S^{\clubsuit}(\zeta^*) = S^{-1}(\zeta)$. This evidently leads to

$$s_{22}^*(\zeta^*) = r_{22}(\zeta), \quad \zeta \in \mathbb{D}^-, \tag{19}$$

which suggests that each zero ζ_j of s_{22} leads to each zero ζ_j^* of r_{22} correspondingly. Therefore, we suppose that det P_1 has N simple zeros $\{\zeta_j\}_{j=1}^N$ in $\mathbb{D}^+$ and det P_2 has N simple zeros $\{\hat{\zeta}_j\}_{j=1}^N$ in $\mathbb{D}^-$, where $\hat{\zeta}_j = \zeta_j^*$. These zeros together with the nonzero vectors ν_j and $\hat{\nu}_j$ constitute the full set of the generic discrete data, which meet the equations

$$P_1(\zeta_j)\nu_j = 0, \tag{20a}$$

$$\hat{\nu}_j P_2(\hat{\zeta}_j) = 0, \tag{20b}$$

where ν_j and $\hat{\nu}_j$ denote column vectors and row vectors respectively. Taking the Hermitian of Eq. (20a) and using (18), we see that

$$\hat{\nu}_j = \nu_j^{\clubsuit}, \quad 1 \le j \le N. \tag{21}$$

By differentiations of Eq. (20a) in x and t respectively and use of the Lax pair (4), we get

$$\begin{aligned} &P_1(\zeta_j)\left(\frac{\partial \nu_j}{\partial x} - i\zeta_j\sigma\nu_j\right) = 0, \\ &P_1(\zeta_j)\left(\frac{\partial \nu_j}{\partial t} - (16i\delta\zeta_j^5 - 8i\gamma\zeta_j^4 - 4i\alpha\zeta_j^3 + i\zeta_j^2)\sigma\nu_j\right) = 0, \end{aligned}$$

which generates

$$\nu_j = e^{\left(i\zeta_j x + \left(16i\delta\zeta_j^5 - 8i\gamma\zeta_j^4 - 4i\alpha\zeta_j^3 + i\zeta_j^2\right)t\right)\sigma}\nu_{j0}, \quad 1 \le j \le N,$$

with ν_{j0} being the complex constant vectors. Also according to the relation (21), we have

$$\hat{\nu}_j = \nu_{j0}^{\clubsuit} e^{\left(-i\zeta_j^* x + \left(-16i\delta\zeta_j^{*5} + 8i\gamma\zeta_j^{*4} + 4i\alpha\zeta_j^{*3} - i\zeta_j^{*2}\right)t\right)\sigma}, \quad 1 \le j \le N.$$

Worthy to note that, the Riemann–Hilbert problem (15) we treat corresponds to the reflectionless case. Hence, the solutions [32] for the Riemann–Hilbert problem (15) can be given as

$$P_1(\zeta) = I_2 - \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{\nu_k \hat{\nu}_j (M^{-1})_{kj}}{\zeta - \hat{\zeta}_j}, \tag{22a}$$

$$P_2(\zeta) = I_2 + \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{\nu_k \hat{\nu}_j (M^{-1})_{kj}}{\zeta - \zeta_k}, \tag{22b}$$

where M is a $N \times N$ matrix with entries

$$m_{kj} = \frac{\hat{\nu}_k \nu_j}{\zeta_j - \hat{\zeta}_k}, \quad 1 \le k, j \le N,$$

and $(M^{-1})_{kj}$ means the (k, j)-entry of the inverse matrix of M. From expression (22a), we have

$$P_1^{(1)} = -\sum_{k=1}^{N}\sum_{j=1}^{N} \nu_k \hat{\nu}_j (M^{-1})_{kj}.$$

As a consequence, the expression of general N-soliton solution of the fifth-order NLS equation (1) can be derived as follows

$$q = -2\sum_{k=1}^{N}\sum_{j=1}^{N} \alpha_j^* \beta_k e^{-\theta_k + \theta_j^*} (M^{-1})_{kj}, \tag{23}$$

where

$$m_{kj} = \frac{\alpha_k^* \alpha_j e^{\theta_k^* + \theta_j} + \beta_k^* \beta_j e^{-\theta_k^* - \theta_j}}{\zeta_j - \zeta_k^*}, \quad 1 \le k, j \le N.$$

Here we have set nonzero vectors $\nu_{k0} = (\alpha_k, \beta_k)^{\mathrm{T}}$ and $\theta_k = i\zeta_k x + (16i\delta\zeta_k^5 - 8i\gamma\zeta_k^4 - 4i\alpha\zeta_k^3 + i\zeta_k^2)t$, $(\mathrm{Im}\zeta_k > 0, 1 \le k \le N)$.

In the rest of this section, we write out one- and two-soliton solutions explicitly. For the case of $N = 1$, the one-soliton solution can be readily obtained as

$$q(x,t) = -\frac{2\alpha_1^* \beta_1 (\zeta_1 - \zeta_1^*) e^{-\theta_1 + \theta_1^*}}{|\alpha_1|^2 e^{\theta_1^* + \theta_1} + |\beta_1|^2 e^{-\theta_1^* - \theta_1}}, \tag{24}$$

in which $\theta_1 = i\zeta_1 x + (16i\delta\zeta_1^5 - 8i\gamma\zeta_1^4 - 4i\alpha\zeta_1^3 + i\zeta_1^2)t$. Furthermore, fixing $\beta_1 = 1$ and setting $\zeta_1 = a_1 + ib_1$ as well as $|\alpha_1|^2 = e^{2\xi_1}$, the expression (24) is then turned into

$$q(x,t) = -2i\alpha_1^* b_1 e^{-\xi_1} e^{\theta_1^* - \theta_1} \operatorname{sech}(\theta_1^* + \theta_1 + \xi_1). \tag{25}$$

According to the notation above, we arrive at

$$\begin{aligned}
\theta_1^* + \theta_1 = & -2b_1\big[x + (80\delta a_1^4 - 160\delta a_1^2 b_1^2 + 16\delta b_1^4 - 32\gamma a_1^3 + 32\gamma a_1 b_1^2 - 12\alpha a_1^2 \\
& + 4\alpha b_1^2 + 2a_1)t\big], \\
\theta_1^* - \theta_1 = & -2ia_1 x + (320i\delta a_1^3 b_1^2 - 160i\delta a_1 b_1^4 - 96i\gamma a_1^2 b_1^2 - 24i\alpha a_1 b_1^2 + 16i\gamma a_1^4 \\
& - 32i\delta a_1^5 + 8i\alpha a_1^3 + 16i\gamma b_1^4 + 2ib_1^2 - 2ia_1^2)t.
\end{aligned}$$

Thus, the one-soliton solution (25) can be further written as

$$\begin{aligned}
q(x,t) = & -2i\alpha_1^* b_1 e^{-\xi_1} e^{\theta_1^* - \theta_1} \operatorname{sech}\{2b_1\big[x + (80\delta a_1^4 - 160\delta a_1^2 b_1^2 + 16\delta b_1^4 \\
& - 32\gamma a_1^3 + 32\gamma a_1 b_1^2 - 12\alpha a_1^2 + 4\alpha b_1^2 + 2a_1)t\big] + \xi_1\}.
\end{aligned} \tag{26}$$

From expression (26), it can be seen that the one-soliton solution is of the shape of hyperbolic secant function with peak amplitude

$$\mathcal{A} = 2\big|\alpha_1^*\big| b_1 e^{-\xi_1},$$

and the velocity

$$\mathcal{V} = 80\delta a_1^4 - 160\delta a_1^2 b_1^2 + 16\delta b_1^4 - 32\gamma a_1^3 + 32\gamma a_1 b_1^2 - 12\alpha a_1^2 + 4\alpha b_1^2 + 2a_1$$

relying on both the real part a_1 and the imaginary part b_1 of the spectral parameter ζ_1. The localized structures of the one-soliton solution (26) are depicted in Figs. 1, 2 and 3 with the parameters chosen as $a_1 = 0.2, b_1 = 0.3, \xi_1 = 0, \alpha = 1, \gamma = 1, \delta = 1, \alpha_1 = 1$.

In addition, the two-soliton solution to Eq. (1) can be generated by taking $N = 2$ in the formula (23)

$$q(x,t) = -\frac{2(\alpha_1^*\beta_1 m_{22} e^{-\theta_1 + \theta_1^*} - \alpha_2^*\beta_1 m_{12} e^{-\theta_1 + \theta_2^*} - \alpha_1^*\beta_2 m_{21} e^{-\theta_2 + \theta_1^*} + \alpha_2^*\beta_2 m_{11} e^{-\theta_2 + \theta_2^*})}{m_{11}m_{22} - m_{12}m_{21}}, \tag{27}$$

where

$$\begin{aligned}
m_{11} &= \frac{|\alpha_1|^2 e^{\theta_1^* + \theta_1} + |\beta_1|^2 e^{-\theta_1^* - \theta_1}}{\zeta_1 - \zeta_1^*}, \quad m_{12} = \frac{\alpha_1^*\alpha_2 e^{\theta_1^* + \theta_2} + \beta_1^*\beta_2 e^{-\theta_1^* - \theta_2}}{\zeta_2 - \zeta_1^*}, \\
m_{21} &= \frac{\alpha_2^*\alpha_1 e^{\theta_2^* + \theta_1} + \beta_2^*\beta_1 e^{-\theta_2^* - \theta_1}}{\zeta_1 - \zeta_2^*}, \quad m_{22} = \frac{|\alpha_2|^2 e^{\theta_2^* + \theta_2} + |\beta_2|^2 e^{-\theta_2^* - \theta_2}}{\zeta_2 - \zeta_2^*},
\end{aligned}$$

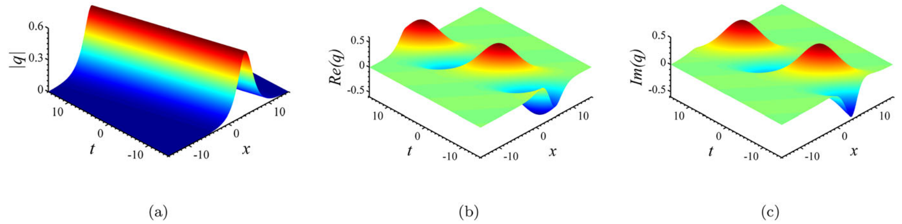

Fig. 1 Plots of one-soliton solution (26) with $a_1 = 0.2, b_1 = 0.3, \xi_1 = 0, \alpha = 1, \gamma = 1, \delta = 1, \alpha_1 = 1$. (**a**) Perspective view of modulus of q; (**b**) Perspective view of real part of q; (**c**) Perspective view of imaginary part of q

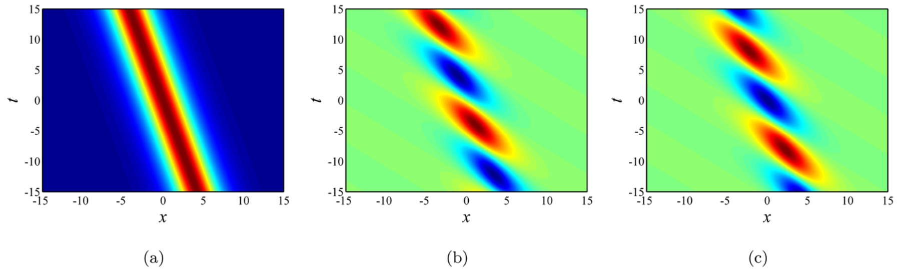

Fig. 2 Plots of one-soliton solution (26) with $a_1 = 0.2, b_1 = 0.3, \xi_1 = 0, \alpha = 1, \gamma = 1, \delta = 1, \alpha_1 = 1$. (**a**) Overhead view of modulus of q; (**b**) Overhead view of real part of q; (**c**) Overhead view of imaginary part of q

and $\theta_1 = i\zeta_1 x + (16i\delta\zeta_1^5 - 8i\gamma\zeta_1^4 - 4i\alpha\zeta_1^3 + i\zeta_1^2)t$, $\theta_2 = i\zeta_2 x + (16i\delta\zeta_2^5 - 8i\gamma\zeta_2^4 - 4i\alpha\zeta_2^3 + i\zeta_2^2)t$, $\zeta_1 = a_1 + ib_1$, $\zeta_2 = a_2 + ib_2$.

If we let $\beta_1 = \beta_2 = 1, \alpha_1 = \alpha_2$ and $|\alpha_1|^2 = e^{2\xi_1}$, then the two-soliton solution (27) has the form

$$q(x,t) = -\frac{2(\alpha_1^* m_{22} e^{-\theta_1+\theta_1^*} - \alpha_2^* m_{12} e^{-\theta_1+\theta_2^*} - \alpha_1^* m_{21} e^{-\theta_2+\theta_1^*} + \alpha_2^* m_{11} e^{-\theta_2+\theta_2^*})}{m_{11}m_{22} - m_{12}m_{21}}, \tag{28}$$

where

$$\begin{aligned}
m_{11} &= -\frac{i}{b_1} e^{\xi_1} \cosh(\theta_1^* + \theta_1 + \xi_1), \\
m_{12} &= \frac{2e^{\xi_1}}{(a_2 - a_1) + i(b_1 + b_2)} \cosh(\theta_1^* + \theta_2 + \xi_1), \\
m_{22} &= -\frac{i}{b_2} e^{\xi_1} \cosh(\theta_2^* + \theta_2 + \xi_1), \\
m_{21} &= \frac{2e^{\xi_1}}{(a_1 - a_2) + i(b_1 + b_2)} \cosh(\theta_2^* + \theta_1 + \xi_1).
\end{aligned}$$

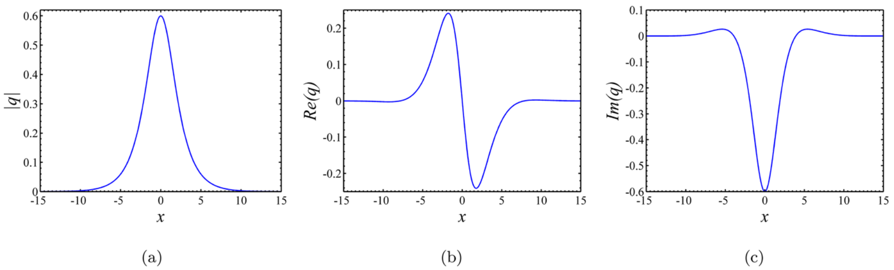

Fig. 3 Plots of one-soliton solution (26) with $a_1 = 0.2, b_1 = 0.3, \xi_1 = 0, \alpha = 1, \gamma = 1, \delta = 1, \alpha_1 = 1, t = 0$. (**a**) x-curve of modulus of q; (**b**) x-curve of real part of q; (**c**) x-curve of imaginary part of q

4 Conclusion

The aim of the paper was to investigate a fifth-order nonlinear Schrödinger equation describing the one-dimensional anisotropic Heisenberg ferromagnetic spin chain via the Riemann–Hilbert method. The spectral analysis was first carried out and a matrix Riemann–Hilbert problem was established. After that, via solving the resulting Riemann–Hilbert problem without reflection, the general multi-soliton solutions to the fifth-order nonlinear Schrödinger equation were attained. Furthermore, by selecting particular values for the involved parameters, a few plots of one-soliton solution were made to display the localized structures.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Yu, F.J.: Dynamics of nonautonomous discrete rogue wave solutions for an Ablowitz–Musslimani equation with PT-symmetric potential. Chaos **27**, 023108 (2017)
2. Yu, F.J., Li, L.: Inverse scattering transformation and soliton stability for a nonlinear Gross–Pitaevskii equation with external potentials. Appl. Math. Lett. **91**, 41–47 (2019)
3. Yu, F.J.: Inverse scattering solutions and dynamics for a nonlocal nonlinear Gross–Pitaevskii equation with PT-symmetric external potentials. Appl. Math. Lett. **92**, 108–114 (2019)
4. Yu, F.J.: A novel non-isospectral hierarchy and soliton wave dynamics for a parity-time-symmetric nonlocal vector nonlinear Gross–Pitaevskii equations. Commun. Nonlinear. Sci. Numer. Simulat. **78**, 104852 (2019)
5. Li, L., Yu, F.J., Duan, C.N.: A generalized nonlocal Gross–Pitaevskii (NGP) equation with an arbitrary time-dependent linear potential. Appl. Math. Lett. **110**, 106584 (2020)
6. Yu, F.J., Fan, R.: Nonstandard bilinearization and interaction phenomenon for PT-symmetric coupled nonlocal nonlinear Schrödinger equations. Appl. Math. Lett. **103**, 106209 (2020)
7. Ma, W.X., Fan, E.G.: Linear superposition principle applying to Hirota bilinear equations. Comput. Math. Appl. **61**, 950–959 (2011)

8. Wazwaz, A.M., El-Tantawy, S.A.: Solving the (3+1)-dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota's method. Nonlinear Dynam. **88**, 3017–3021 (2017)
9. Yue, Y.F., Huang, L.L., Chen, Y.: N-solitons, breathers, lumps and rogue wave solutions to a (3+1)-dimensional nonlinear evolution equation. Comput. Math. Appl. **75**, 2538–2548 (2018)
10. Ling, L.M., Zhao, L.C., Guo, B.L.: Darboux transformation and multi-dark soliton for N-component nonlinear Schrödinger equations. Nonlinearity **28**(9), 3243–3261 (2015)
11. Xu, T., Chen, Y.: Darboux transformation of the coupled nonisospectral Gross–Pitaevskii system and its multi-component generalization. Commun. Nonlinear Sci. Numer. Simulat. **57**, 276–289 (2018)
12. Xu, T., Chen, Y.: Localized waves in three-component coupled nonlinear Schrödinger equation. Chin. Phys. B **25**(9), 090201 (2016)
13. Lenells, J.: The Degasperis–Procesi equation on the half-line. Nonlinear Anal. **76**, 122–139 (2013)
14. Wu, J.P.: Integrability aspects and multi-soliton solutions of a new coupled Gerdjikov–Ivanov derivative nonlinear Schrödinger equation. Nonlinear Dynam. **96**, 789–800 (2019)
15. Zhang, N., Xia, T.C., Hu, B.B.: A Riemann–Hilbert approach to complex Sharma–Tasso–Olver equation on half line. Commun. Theor. Phys. **68**, 580–594 (2017)
16. Kang, Z.Z., Xia, T.C.: Construction of multi-soliton solutions of the N-coupled Hirota equations in an optical fiber. Chin. Phys. Lett. **36**, 110201 (2019)
17. Wang, D.S., Yin, S.J., Tian, Y., Liu, Y.F.: Integrability and bright soliton solutions to the coupled nonlinear Schrödinger equation with higher-order effects. Appl. Math. Comput. **229**, 296–309 (2014)
18. Wu, J.P.: Riemann–Hilbert approach of the Newell-type long-wave-short-wave equation via the temporal-part spectral analysis. Nonlinear Dynam. **98**, 749–760 (2019)
19. Hu, B.B., Xia, T.C., Ma, W.X.: Riemann–Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg–de Vries equation on the half-line. Appl. Math. Comput. **332**, 148–159 (2018)
20. Wang, D.S., Wang, X.L.: Long-time asymptotics and the bright N-soliton solutions of the Kundu–Eckhaus equation via the Riemann–Hilbert approach. Nonlinear Anal. Real World Appl. **41**, 334–361 (2018)
21. Ma, W.X.: Riemann–Hilbert problems of a six-component fourth-order AKNS system and its soliton solutions. Comput. Appl. Math. **37**, 6359–6375 (2018)
22. Ma, W.X.: Riemann–Hilbert problems of a six-component mKdV system and its soliton solutions. Acta Math. Sci. Ser. B **39**(2), 509–523 (2019)
23. Han, Z., Chen, Y., Chen, J.C.: General N-dark soliton solutions of the multi-component Mel'nikov system. J. Phys. Soc. Jpn. **86**, 074005 (2017)
24. Osman, M.S., Wazwaz, A.M.: An efficient algorithm to construct multi-soliton rational solutions of the (2+1)-dimensional KdV equation with variable coefficients. Appl. Math. Comput. **321**, 282–289 (2018)
25. Chowdury, A., Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms. Phys. Rev. E **90**, 032922 (2014)
26. Ablowitz, M.J.: Nonlinear Dispersive Waves. Cambridge University Press, Cambridge (2011)
27. Ankiewicz, A., Soto-Crespo, J.M., Akhmediev, N.: Rogue waves and rational solutions of the Hirota equation. Phys. Rev. E **81**, 046602 (2010)
28. Radha, R., Kumar, V.R.: Explode-decay solitons in the generalized inhomogeneous higher order nonlinear Schrödinger equations. Z. Naturforsch. A **62**, 381–386 (2007)
29. Wang, P.: Conservation laws and solitons for a generalized inhomogeneous fifth-order nonlinear Schrödinger equation from the inhomogeneous Heisenberg ferromagnetic spin system. Eur. Phys. J. D **68**, 1–8 (2014)
30. Chai, J., Tian, B., Zhen, H.L., Sun, W.R.: Conservation laws, bilinear forms and solitons for a fifth-order nonlinear Schrödinger equation for the attosecond pulses in an optical fiber. Ann. Phys. **359**, 371–384 (2015)
31. Sun, W.R., Tian, B., Zhen, H.L., Sun, Y.: Breathers and rogue waves of the fifth-order nonlinear Schrödinger equation in the Heisenberg ferromagnetic spin chain. Nonlinear Dynam. **81**, 725–732 (2015)
32. Yang, J.K.: Nonlinear Waves in Integrable and Nonintegrable Systems. SIAM, Philadelphia (2010)