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Novel soliton hierarchies of Levi type and their bi-Hamiltonian structures

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ABSTRACT

We present two new matrix spectral problems, and construct the corresponding soliton hierarchies of Levi type with the aid of symbolic computation by Maple. Bi-Hamiltonian structures of the resulting soliton hierarchies are obtained by means of the trace identity. Thus it is shown that these soliton hierarchies are Liouville integrable.

1. Introduction

It is a very important topic to seek for new soliton hierarchies generated from certain matrix spectral problems in mathematical physics [1–37] because soliton hierarchies often possess bi-Hamiltonian structures which guarantee the existence of hereditary recursion operators [32] and Liouville integrability [33]. The so-called trace identity [12,16], or more generally the variational identity [27], provides a basic tool for generating Hamiltonian structures of soliton hierarchies. Typical examples of soliton hierarchies, which fit into the zero curvature formulation, include the Korteweg-de Vries hierarchy, the Ablowitz–Kaup–Newell–Segur hierarchy, the Kaup–Newell hierarchy, the Wadati–Konno–Ichikawa hierarchy, the Boiti–Pempinelli–Tu hierarchy, and the Levi hierarchy.

In Refs.[6,7,16,28], the Levi hierarchy associated with the following spatial spectral problem

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -p \\ -1 & \lambda - q \end{bmatrix}\phi, \qquad u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{1.1}$$

has been considered, with λ denoting the spectral parameter. Inspired by the form of the spectral matrix U in (1.1), we would like to construct two new spatial spectral problems of Levi type by introducing modifications to (1.1):

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -\lambda p \\ -\lambda & \lambda^2 - q \end{bmatrix}, \tag{1.2}$$

and

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -\lambda p \\ -\lambda & \lambda - q \end{bmatrix}. \tag{1.3}$$

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The remainder of this paper is organized as follows. In Sections 2 and 3, we construct the corresponding soliton hierarchies of Levi type with bi-Hamiltonian structures from the matrix spectral problems (1.2) and (1.3), respectively. In our analysis, we will use the software Maple to deal with some complicated computations. The last section is devoted to a few conclusions and discussions.

2. Soliton hierarchy of Levi type associated with (1.2)

In this section, we will construct a new soliton hierarchy of Levi type from the spatial spectral problem (1.2). Let W be of the following form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{2.1}$$

and we solve the stationary zero curvature equation $W_x = [U, W]$, which becomes

$$\begin{aligned} a_x &= \lambda(b - pc), \\ b_x &= -\lambda^2 b + 2\lambda pa + qb, \\ c_x &= \lambda^2 c - 2\lambda a - qc. \end{aligned} \tag{2.2}$$

According to the relations of the spectral parameter λ in these Eq. (2.2), we take the following Laurent series expansions

$$a = \sum_{k=0}^{\infty} a_k \lambda^{-2k}, \qquad b = \sum_{k=0}^{\infty} b_k \lambda^{-2k-1}, \qquad c = \sum_{k=0}^{\infty} c_k \lambda^{-2k-1}, \tag{2.3}$$

namely,

$$W = \sum_{k=0}^{\infty} W_k \lambda^{-2k}, \qquad W_k = \begin{bmatrix} a_k & b_k\lambda^{-1} \\ c_k\lambda^{-1} & -a_k \end{bmatrix}.$$

Substituting (2.3) into (2.2), we arrive at

$$\begin{aligned} & b_0 - 2pa_0 = 0, \\ & c_0 - 2a_0 = 0, \\ & a_{kx} = b_k - pc_k, \\ & b_{k+1} = -b_{kx} + qb_k + 2pa_{k+1}, \\ & c_{k+1} = c_{kx} + qc_k + 2a_{k+1}, \qquad k \geqslant 0. \end{aligned} \tag{2.4}$$

To determine the sequence of $\{a_k, b_k, c_k, k \geqslant 0\}$ recursively, we need rewrite $a_{k+1,x}$ as

$$a_{k+1,x} = b_{k+1} - pc_{k+1} = \left(-b_{kx} + qb_k + 2pa_{k+1}\right) - p(c_{kx} + qc_k + 2a_{k+1}) = -b_{kx} - pc_{kx} + qb_k - pqc_k. \tag{2.5}$$

Thus the last two equations in (2.4) are transformed to

$$\begin{aligned} & b_{k+1} = -b_{kx} + qb_k + 2p\partial^{-1}(-b_{kx} - pc_{kx} + qb_k - pqc_k), \\ & c_{k+1} = c_{kx} + qc_k + 2\partial^{-1}(-b_{kx} - pc_{kx} + qb_k - pqc_k), \qquad k \geqslant 0. \end{aligned}$$

Then we take the initial values

$$a_0 = 1, \qquad b_0 = 2p, \qquad c_0 = 2, \tag{2.6}$$

and impose the integration conditions

$$a_k|_{u=0} = b_k|_{u=0} = c_k|_{u=0} = 0, \qquad k \geqslant 1, \tag{2.7}$$

to guarantee the uniqueness of the sequence. So, by using the symbolic computation software Maple, the first few sets can be computed as follows:

$$\begin{aligned} & a_1 = -2p, \qquad b_1 = -2p_x - 4p^2 + 2pq, \qquad c_1 = -4p + 2q; \\ & a_2 = 2p_x + 6p^2 - 4pq, \\ & b_2 = 2p_{xx} - 2pq_x - 4p_xq + 12pp_x + 12p^3 - 12p^2q + 2pq^2, \\ & c_2 = 2q_x + 12p^2 - 12pq + 2q^2; \\ & a_3 = -2p_{xx} - 12pp_x + 6p_xq - 20p^3 + 24p^2q - 6pq^2, \\ & b_3 = -2p_{xxx} - 60p_xp^2 + 48pp_xq - 6p_xq^2 + 12p^2q_x - 6pqq_x - 16pp_{xx} + 6p_{xx}q \\ & \qquad - 12p_x^2 + 6p_xq_x + 2pq_{xx} - 40p^4 + 60p^3q - 24p^2q^2 + 2pq^3, \\ & c_3 = -4p_{xx} + 2q_{xx} + 6qq_x - 12pq_x - 40p^3 + 60p^2q - 24pq^2 + 2q^3. \end{aligned}$$

We point out that the localness of the above functions is not an accident, and in fact, all the functions $\{a_k, b_k, c_k, k \geqslant 0\}$ are local. First from $W_x = [U, W]$, we have

$$\frac{d}{dx}\mathrm{tr}\left(W^2\right) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U,W]) = 0,$$

and so, due to $\mathrm{tr}(W^2) = 2(a^2 + bc)$, we can obtain

$$a^2 + bc = (a^2 + bc)|_{u=0} = 1,$$

the last step of which follows from the initial data (2.6). Then, by using the Laurent expansions (2.3), a balance of coefficients of λ^k for each $k \geqslant 3$ tells that

$$a_{k+1} = -\frac{1}{2}\left[\sum_{\substack{i+j=k+1\\ i,j\geqslant 1}} a_i a_j + \sum_{i+j=k}(b_i c_j)\right], \qquad k \geqslant 3. \tag{2.8}$$

Based on the recursion relations of the last two equations in (2.4) and the above Eq. (2.8), an application of the mathematical induction finally shows that all functions $\{a_k, b_k, c_k, k \geqslant 4\}$ are differential rational functions in u, and so, they are all local.

Now, taking

$$V^{[n]} = \lambda(\lambda^{2n+1}W)_+ + \Delta_n = \lambda(\lambda^{2n+1}W)_+ + \begin{bmatrix} 0 & 0 \\ 0 & -f_n \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^n a_k \lambda^{2(n-k)+2} & \sum_{k=0}^n b_k \lambda^{2(n-k)+1} \\ \sum_{k=0}^n c_k \lambda^{2(n-k)+1} & -\sum_{k=0}^n a_k \lambda^{2(n-k)+2} - f_n \end{bmatrix},$$

where P_+ denotes the polynomial part of P in λ, the zero curvature equations for $n \geqslant 0$

$$U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0 \tag{2.9}$$

give

$$\begin{aligned} f_n &= -c_{nx} - qc_n, \\ p_{t_n} &= -b_{nx} + pf_n + qb_n = -b_{nx} - p(c_{nx} + qc_n) + qb_n = a_{n+1,x}, \\ q_{t_n} &= f_{nx} = -c_{nxx} - (qc_n)_x = -c_{n+1,x} + 2a_{n+1,x}, \qquad n \geqslant 0. \end{aligned}$$

Consequently, we have obtained a new soliton hierarchy of Levi type

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} a_{n+1,x} \\ -c_{n+1,x} + 2a_{n+1,x} \end{bmatrix}, \qquad n \geqslant 0. \tag{2.10}$$

The first and second nonlinear examples in this hierarchy (2.10) are as follows:

$$\begin{aligned} p_{t_1} &= 2p_{xx} - 4pq_x - 4p_x q + 12pp_x, \\ q_{t_1} &= 4p_{xx} - 2q_{xx} + 4p_x q + 4pq_x - 4qq_x, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} p_{t_2} &= -2p_{xxx} - 60p_x p^2 + 48pqp_x - 6p_x q^2 + 24p^2 q_x - 12pqq_x - 12pp_{xx} + 6p_{xx}q - 12p_x^2 + 6p_x q_x, \\ q_{t_2} &= -2q_{xxx} - 24pqp_x + 12p_x q^2 - 12p^2 q_x + 24pqq_x - 6q^2 q_x - 24pp_{xx} + 12p_{xx}q - 24p_x^2 \\ &\quad + 24p_x q_x + 12pq_{xx} - 6qq_{xx} - 6q_x^2. \end{aligned} \tag{2.12}$$

By setting $p = 0$, the first nonlinear example (2.11) reduces to the well-known Burgers equation $q_{t_1} = -2(q_{xx} + 2qq_x)$. For $q = 2p$, the second one reduces to mKdV equation $p_{t_2} = -2(p_{xxx} - 6p^2 p_x)$.

Next we consider Hamiltonian structures via the following trace identity [12,16],

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \qquad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\mathrm{tr}\left(W^2\right)\right|. \tag{2.13}$$

It is easy to see

$$\begin{aligned} \frac{\partial U}{\partial \lambda} &= \begin{bmatrix} 0 & -p \\ -1 & 2\lambda \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -2\lambda a - pc - b, \\ \frac{\partial U}{\partial p} &= \begin{bmatrix} 0 & -\lambda \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -\lambda c, \\ \frac{\partial U}{\partial q} &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = a. \end{aligned}$$

Now the corresponding trace identity becomes

$$\frac{\delta}{\delta u}\int(-2\lambda a - pc - b)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix}-\lambda c\\ a\end{bmatrix}.$$

Balancing coefficients of each power of λ in the above equality, we have

$$\frac{\delta}{\delta u}\int(-2a_{n+1} - pc_n - b_n)\mathrm{d}x = (\gamma - 2n)\begin{bmatrix}-c_n\\ a_n\end{bmatrix}, \qquad n \geqslant 0.$$

Checking a particular case with $n = 1$ yields $\gamma = 0$, and thus we obtain

$$\frac{\delta}{\delta u}\int\left(\frac{2a_{n+2} + pc_{n+1} + b_{n+1}}{2n+2}\right)\mathrm{d}x = \begin{bmatrix}-c_{n+1}\\ a_{n+1}\end{bmatrix}, \qquad n \geqslant 0.$$

Consequently, we obtain the following Hamiltonian structure for the soliton hierarchy (2.10)

$$\begin{bmatrix}p\\ q\end{bmatrix}_{t_n} = K_n = \begin{bmatrix}a_{n+1,x}\\ -c_{n+1,x} + 2a_{n+1,x}\end{bmatrix} = J\frac{\delta\mathcal{H}_n}{\delta u}, \qquad n \geqslant 1, \tag{2.14}$$

with the Hamiltonian operator

$$J = \begin{bmatrix}0 & \partial\\ \partial & 2\partial\end{bmatrix}$$

and the Hamiltonian functionals

$$\mathcal{H}_0 = \int(-2p + q)\mathrm{d}x,$$
$$\mathcal{H}_n = \int\left(\frac{2a_{n+2} + pc_{n+1} + b_{n+1}}{2n+2}\right)\mathrm{d}x, \qquad n \geqslant 1.$$

It is now a direct computation that all members in (2.10) are bi-Hamiltonian. Firstly, by means of

$$\begin{aligned}-c_{n+1} &= -c_{nx} - qc_n - 2a_{n+1}\\ &= -c_{nx} - qc_n - 2\partial^{-1}(-b_{nx} - pc_{nx} + qb_n - pqc_n)\\ &= -c_{nx} - qc_n + 2(a_{nx} + pc_n) + 2\partial^{-1}p\partial c_n - 2\partial^{-1}q(a_{nx} + pc_n) + 2\partial^{-1}pqc_n\\ &= (\partial + q - 2p - 2\partial^{-1}p\partial)(-c_n) + (2\partial - 2\partial^{-1}q\partial)a_n,\end{aligned}$$

$$\begin{aligned}a_{n+1} &= \partial^{-1}(-b_{nx} - pc_{nx} + qb_n - pqc_n)\\ &= (p + \partial^{-1}p\partial)(-c_n) + (-\partial + \partial^{-1}q\partial)a_n,\end{aligned}$$

we have

$$\Psi = \begin{bmatrix}\partial + q - 2p - 2\partial^{-1}p\partial & 2\partial - 2\partial^{-1}q\partial\\ p + \partial^{-1}p\partial & -\partial + \partial^{-1}q\partial\end{bmatrix}$$

satisfying $\begin{bmatrix}-c_{n+1}\\ a_{n+1}\end{bmatrix} = \Psi\begin{bmatrix}-c_n\\ a_n\end{bmatrix}$. Then we arrive at

$$u_{t_n} = K_n = J\frac{\delta\mathcal{H}_n}{\delta u} = M\frac{\delta\mathcal{H}_{n-1}}{\delta u}, \qquad n \geqslant 1, \tag{2.15}$$

where the second Hamiltonian operator M is given by

$$M = J\Psi = \begin{bmatrix}\partial p + p\partial & -\partial^2 + q\partial\\ \partial^2 + \partial q & 0\end{bmatrix}.$$

Thus, it is direct to see that the new soliton hierarchy of Levi type (2.10) is Liouville integrable [33], i.e., it possesses infinitely many independent commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra of symmetries,

$$[K_l, K_m] = K_l'(u)[K_m] - K_m'(u)[K_l] = 0, \qquad l, m \geqslant 0 \tag{2.16}$$

and the Abelian algebras of conserved functionals,

$$\{\mathcal{H}_l, \mathcal{H}_m\}_J = \int\left(\frac{\delta\mathcal{H}_l}{\delta u}\right)^T J\frac{\delta\mathcal{H}_m}{\delta u}\mathrm{d}x = 0, \qquad l, m \geqslant 0 \tag{2.17}$$

and

$$\{\mathcal{H}_l, \mathcal{H}_m\}_M = \int\left(\frac{\delta\mathcal{H}_l}{\delta u}\right)^T M\frac{\delta\mathcal{H}_m}{\delta u}\mathrm{d}x = 0, \qquad l, m \geqslant 0. \tag{2.18}$$

We point out that the novel hierarchy (2.10) is among soliton hierarchies resulting from Hamiltonian pairs directly [38].

3. Soliton hierarchy of Levi type associated with (1.3)

Here we will construct the new soliton hierarchy of Levi type from the spatial spectral problem (1.3). Let W be of the form (2.1), and then the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{aligned} a_x &= \lambda(b - pc), \\ b_x &= -\lambda b + 2\lambda pa + qb, \\ c_x &= \lambda c - 2\lambda a - qc. \end{aligned} \tag{3.1}$$

Further, taking the Laurent series expansions,

$$a = \sum_{k=0}^{\infty} a_k\lambda^{-k}, \qquad b = \sum_{k=0}^{\infty} b_k\lambda^{-k}, \qquad c = \sum_{k=0}^{\infty} c_k\lambda^{-k}, \tag{3.2}$$

namely,

$$W = \sum_{k=0}^{\infty} W_k\lambda^{-k}, \qquad W_k = \begin{bmatrix} a_k & b_k \\ c_k & -a_k \end{bmatrix},$$

we arrive at

$$\begin{aligned} & b_0 - pc_0 = 0, \\ & c_0 - 2a_0 = 0, \\ & b_0 - 2pa_0 = 0, \\ & a_{kx} = b_{k+1} - pc_{k+1}, \\ & b_{k+1} = -b_{kx} + qb_k + 2pa_{k+1}, \\ & c_{k+1} = c_{kx} + qc_k + 2a_{k+1}, \qquad k \geqslant 0. \end{aligned} \tag{3.3}$$

To determine the sequence of $\{a_k, b_k, c_k, k \geqslant 0\}$ recursively, we need to represent a_{k+1} by $\{b_k, c_k\}$, using the last two equations in (3.3). In order to achieve this purpose, we rewrite a_{kx} as

$$a_{kx} = b_{k+1} - pc_{k+1} = \left(-b_{kx} + qb_k + 2pa_{k+1}\right) - p(c_{kx} + qc_k + 2a_{k+1}) = -b_{kx} - pc_{kx} + qb_k - pqc_k, \qquad k \geqslant 0. \tag{3.4}$$

Thus for each $k \geqslant 0$, we can change $a_{k+1,x}$ to

$$\begin{aligned} a_{k+1,x} &= -b_{k+1,x} - pc_{k+1,x} + qb_{k+1} - pqc_{k+1} \\ &= -\left[-b_{kxx} + (qb_k)_x + 2(pa_{k+1})_x\right] - p\left[c_{kxx} + (qc_k)_x + 2a_{k+1,x}\right] + q(-b_{kx} + qb_k + 2pa_{k+1}) - pq(c_{kx} + qc_k + 2a_{k+1}). \end{aligned}$$

From this equation, we can have

$$(4p+1)a_{k+1,x} + 2p_x a_{k+1} = b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k,$$

namely,

$$\left(\sqrt{4p+1}\,a_{k+1}\right)_x = \frac{1}{\sqrt{4p+1}}\left[b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k\right].$$

It means that we arrive at

$$\begin{aligned} a_{k+1} = \frac{1}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}\Big[& b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x \\ & - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k\Big], \qquad k \geqslant 0. \end{aligned} \tag{3.5}$$

So by means of this relation (3.5) and the last two equations of (3.3), we can compute $\{a_k, b_k, c_k, k \geqslant 1\}$ recursively from a given initial values $\{b_0, c_0, a_0\}$.

Let us take the initial values

$$a_0 = \frac{1}{\sqrt{4p+1}}, \qquad b_0 = \frac{2p}{\sqrt{4p+1}}, \qquad c_0 = \frac{2}{\sqrt{4p+1}}, \tag{3.6}$$

which are determined by the initial conditions (3.3) and (3.4) with $k = 0$. To guarantee the uniqueness of $\{a_k, b_k, c_k\}$, we also need to impose the integration conditions

$$a_k|_{u=0} = b_k|_{u=0} = c_k|_{u=0} = 0, \qquad k \geqslant 1.$$

Then, by means of the symbolic computation software Maple, the first few sets can be computed as follows:

$$
\begin{aligned}
a_1 &= \frac{2p_x - 4pq}{(4p+1)^{3/2}}, \qquad b_1 = \frac{2pq - 2p_x}{(4p+1)^{3/2}}, \qquad c_1 = \frac{2q}{(4p+1)^{3/2}};\\
a_2 &= \frac{-2}{(4p+1)^{7/2}}\left(4pp_{xx} + p_{xx} - 12pqp_x - 5p_x^2 - 3p_xq + 12p^2q^2 + 3pq^2\right),\\
b_2 &= \frac{2}{(4p+1)^{7/2}}\left(8p^2p_{xx} + 6pp_{xx} + p_{xx} - 16p^3q_x + 16p^2qp_x - 8p^2q_x - 4pqp_x\right.\\
&\quad \left. -14pp_x^2 - pq_x - 2qp_x - 6p_x^2 - 8p^3q^2 + 2p^2q^2 + pq^2\right),\\
c_2 &= \frac{-2}{(4p+1)^{7/2}}\left(8pp_{xx} + 2p_{xx} - 16p^2q_x - 8pq_x - 10p_x^2 - q_x + 8p^2q^2 - 2pq^2 - q^2\right);\\
a_3 &= \frac{-2}{(4p+1)^{9/2}}\left(-16p^3q^3 + 12p^2q^3 + 24p^2q^2p_x + 32p^3q_{xx} + 4pq^3 - 18pq^2p_x - 16p^2qp_{xx}\right.\\
&\quad + 20pqp_x^2 - 48p^2p_xq_x + 16p^2q_{xx} - 6q^2p_x + 12pqp_{xx} - 30p_x^2q - 4pp_xq_x - 16p^2p_{xxx}\\
&\quad \left. +80pp_xp_{xx} - 70p_x^3 + 2pq_{xx} + 4p_{xx}q + 2p_xq_x - 8pp_{xxx} + 20p_xp_{xx} - p_{xxx}\right),\\
b_3 &= \frac{2}{(4p+1)^{9/2}}\left(-48p^3qq_x - 24p^2qq_x - 3pqq_x + 8p^2qp_{xx} + 14pqp_{xx} + 80pp_xp_{xx} + 72p^2q^2p_x\right.\\
&\quad + 6pq^2p_x + 14pp_xq_x + 8p^2p_xq_x - 50pqp_x^2 - p_{xxx} + 16p^3q_{xx} + 8p^2q_{xx} - 16p^2p_{xxx} + pq_{xx}\\
&\quad \left. +3p_{xx}q - 8pp_{xxx} + 20p_xp_{xx} - 24p^3q^3 - 2p^2q^3 + pq^3 - 30p_x^2q + 3p_xq_x - 3p_xq^2 - 70p_x^3\right),\\
c_3 &= \frac{2}{(4p+1)^{9/2}}\left(-24p^2q^3 - 2pq^3 + 48p^2qq_x + 16p^2q_{xx} + q^3 + 24pqq_x - 40pqp_{xx} + 70p_x^2q\right.\\
&\quad \left. -40pp_xq_x + 8pq_{xx} + 3qq_x - 10p_{xx}q - 10p_xq_x + q_{xx}\right).
\end{aligned}
$$

To prove the localness of $\{a_k, b_k, c_k, k \geqslant 4\}$, we have

$$
a_{k+1} = \frac{b_k - pc_{kx} - pqc_k - qb_k}{4p+1} - \frac{1}{2\sqrt{4p+1}}\sum_{\substack{i+j=k+1\\ i,j\geqslant 1}}\left(a_ia_j + b_ic_j\right), \qquad k \geqslant 3. \tag{3.7}
$$

Based on this recursion relation (3.7) and the last two recursion relations in (3.3), an application of the mathematical induction finally shows that all functions $\{a_k, b_k, c_k, k \geqslant 4\}$ are rational differential polynomials in u, namely, they are all local.

Now, taking

$$
V^{[n]} = \lambda^2(\lambda^nW)_+ + \Delta_n = \lambda^2(\lambda^nW)_+ + \begin{bmatrix} 0 & \lambda h_n \\ \lambda g_n & -f_n \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^n a_k\lambda^{n-k+2} & \sum_{k=0}^n b_k\lambda^{n-k+2} + \lambda h_n \\ \sum_{k=0}^n c_k\lambda^{n-k+2} + \lambda g_n & -\sum_{k=0}^n a_k\lambda^{n-k+2} - f_n \end{bmatrix},
$$

the zero curvature equations $U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0, n \geqslant 0$, give

$$
\begin{aligned}
g_n &= c_{nx} + qc_n,\\
h_n &= -b_{nx} + qb_n,\\
f_n &= -g_{nx} - qg_n\\
&= -c_{nxx} - (qc_n)_x - q(c_{nx} + qc_n)\\
&= -c_{n+1,x} + 2a_{n+1,x} - qc_{nx} - q^2c_n\\
&= -c_{n+2} + 2a_{n+2} + 2a_{n+1,x} + 2qa_{n+1}\\
&= -c_{n+2} - 2pc_{n+2} - 2\partial^{-1}p\partial c_{n+2} + 2\partial^{-1}q\partial a_{n+1} + 2qa_{n+1},\\
p_{t_n} &= -h_{nx} + qh_n + pf_n,\\
q_{t_n} &= f_{nx}, \qquad n \geqslant 0.
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
p_{t_n} &= -h_{nx} + qh_n + pf_n\\
&= -h_{nx} + qh_n + p(-g_{nx} - qg_n)\\
&= b_{nxx} - (qb_n)_x + q(-b_{nx} + qb_n) - p[c_{nxx} + (qc_n)_x] - pq(c_{nx} + qc_n)\\
&= a_{n+1,x} + 2(pa_{n+1})_x + 2pa_{n+1,x}\\
&= \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x, \qquad n \geqslant 0,
\end{aligned} \tag{3.8}
$$

$$\begin{aligned} q_{t_n} &= f_{nx} \\ &= \partial(-c_{n+2} - 2pc_{n+2} - 2\partial^{-1}p\partial c_{n+2} + 2\partial^{-1}q\partial a_{n+1} + 2qa_{n+1}) \\ &= (-\partial - 2\partial p - 2p\partial)c_{n+2} + (2q\partial + 2\partial q)a_{n+1} \\ &= -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1}, \qquad n \geqslant 0. \end{aligned} \tag{3.9}$$

Consequently, we have obtained a new soliton hierarchy of WKI type

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x \\ -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1} \end{bmatrix}, \qquad n \geqslant 0. \tag{3.10}$$

The first example in this soliton hierarchy (3.10) is as follows:

$$\begin{aligned} p_{t_0} &= \frac{2}{(4p+1)^{3/2}}(4pp_{xx} + p_{xx} - 8p^2q_x - 2pq_x - 2p_xq - 4p_x^2), \\ q_{t_0} &= \frac{2}{(4p+1)^{7/2}}(32p^2p_{xx} + 16pp_{xxx} + 2p_{xxx} - 2qq_x - 128p^3qq_x - 96p^2qq_x - 24pqq_x \\ &\quad + 64p^2qp_{xx} + 32pqp_{xx} - 144pp_xp_{xx} + 32p^2q^2p_x + 16pq^2p_x + 48pp_xq_x + 96p^2p_xq_x \\ &\quad - 96pqp_x^2 - q_{xx} - 64p^3q_{xx} - 48p^2q_{xx} - 12pq_{xx} + 4p_{xx}q - 36p_xp_{xx} - 24p_x^2q + 6p_xq_x + 2p_xq^2 + 120p_x^3). \end{aligned}$$

Through a similar computation, we have

$$\frac{\delta}{\delta u}\int\left(\frac{b_{n+2} + pc_{n+2} + a_{n+2}}{n+1}\right)dx = \begin{bmatrix} -c_{n+2} \\ a_{n+1} \end{bmatrix}, \qquad n \geqslant 0,$$

via the trace identity (2.13). Thus we obtain the following Hamiltonian structures for the soliton hierarchy (3.10):

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x \\ -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1} \end{bmatrix} = J\begin{bmatrix} -c_{n+2} \\ a_{n+1} \end{bmatrix} = J\frac{\delta\mathcal{H}_{n+1}}{\delta u}, \qquad n \geqslant 0, \tag{3.11}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & \sqrt{4p+1}\partial\sqrt{4p+1} \\ \sqrt{4p+1}\partial\sqrt{4p+1} & 2q\partial + 2\partial q \end{bmatrix} \tag{3.12}$$

and the Hamiltonian functionals

$$\begin{aligned} \mathcal{H}_0 &= \int\left(\frac{2q}{1+\sqrt{4p+1}} - \frac{4pq}{\sqrt{4p+1}(1+\sqrt{4p+1})^2}\right)dx, \\ \mathcal{H}_{n+1} &= \int\left(\frac{b_{n+2} + pc_{n+2} + a_{n+2}}{n+1}\right)dx, \qquad n \geqslant 0. \end{aligned} \tag{3.13}$$

It is now a direct computation that all members in (3.10) are bi-Hamiltonian. By means of

$$\begin{aligned} -c_{n+2} &= -c_{n+1,x} - qc_{n+1} - 2a_{n+2} \\ &= -c_{n+1,x} - qc_{n+1} - 2\frac{1}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}[b_{n+1,xx} - pc_{n+1,xx} - (qb_{n+1})_x \\ &\quad - p(qc_{n+1})_x - qb_{n+1,x} + q^2b_{n+1} - pqc_{n+1,x} - pq^2c_{n+1}] \\ &= \left[\partial + q - \frac{2}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}(p\partial^2 - \partial^2p + p\partial q + q\partial p + pq\partial + \partial pq)\right](-c_{n+1}) \\ &\quad + \left[\frac{2}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}(-\partial^3 + \partial q\partial + q\partial^2 - q^2\partial)\right]a_n \\ &\equiv \Psi_{11}(-c_{n+1}) + \Psi_{12}a_n, \qquad n \geqslant 0, \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \partial^{-1}(-b_{n+1,x} - pc_{n+1,x} + qb_{n+1} - pqc_{n+1}) \\ &= (-1 + \partial^{-1}q)(a_{nx} + pc_{n+1}) - \partial^{-1}p\partial c_{n+1} - \partial^{-1}pqc_{n+1} \\ &= (p + \partial^{-1}p\partial)(-c_{n+1}) + (-\partial + \partial^{-1}q\partial)a_n \\ &\equiv \Psi_{21}(-c_{n+1}) + \Psi_{22}a_n, \qquad n \geqslant 0, \end{aligned}$$

we have

$$
\begin{aligned}
&M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}=J\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix},\\
&M_{11}=\partial p+p\partial+2\partial p^2+2\partial p\partial^{-1}p\partial+2p\partial p+2p^2\partial^2,\\
&M_{12}=-\partial^2+q\partial-2\partial p\partial+2\partial p\partial^{-1}q\partial-2p\partial^2+2pq\partial,\\
&M_{21}=\partial^2+\partial q+2\partial p\partial+2\partial q\partial^{-1}p\partial+2\partial^2 p+2\partial pq,\\
&M_{22}=-2\partial^3+2\partial q\partial^{-1}q\partial.
\end{aligned} \tag{3.14}
$$

Thus, we arrive at the bi-Hamiltonian structures:

$$
u_{t_n}=K_n=J\frac{\delta\mathcal{H}_{n+1}}{\delta u}=M\frac{\delta\mathcal{H}_n}{\delta u},\qquad n\geqslant 0, \tag{3.15}
$$

where the second Hamiltonian operator M is given by (3.14). So, it is now direct to see that the new soliton hierarchy of Levi type (3.10) is Liouville integrable [33], i.e., it possesses infinitely many independent commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra of symmetries (2.16) and the Abelian algebras of conserved functionals (2.17) and (2.18).

4. Conclusions and discussions

There are many soliton hierarchies obtained by means of matrix spectral problems based on the Lie algebras such as $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$ [1–35]. In this paper, we have introduced the two new matrix spectral problems (1.2), (1.3) and the two novel soliton hierarchies of Levi type (2.15), (3.15) have been derived, together with their bi-Hamiltonian structures. We uses the software Maple to deal with some complicated computations.

In a subsequent study, we will consider some further problems: nonlinearization with Bargmann symmetry constraints, constructing these soliton hierarchies (2.15), (3.15) with self-consistent sources, constructing multi-component integrable couplings based on certain non-semisimple Lie algebras, seeking for Darboux transformations and exact solutions of these hierarchies, and classifying multi-component integrable systems with certain Lie algebras.

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References

[1] lax PD. Integrals of nonlinear equations of evolution and solitary waves. Commun. Pure Appl. Math. 1968;21:467–90.
[2] Ablowitz MJ, Kaup DJ, Newell AC, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. Stud. Appl. Math. 1974;53:249–315.
[3] Kaup DJ, Newell A. An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys. 1978;19:798–801.
[4] Wadati M, Konno K, Ichikawa YH. New integrable nonlinear evolution equations. J. Phys. Soc. Jpn. 1979;47:1698–700.
[5] Fokas AS, Fuchssteiner B. The hierarchy of the Benjamin–Ono equation. Phys. Lett. A 1981;86:341–5.
[6] Levi D. Nonlinear differential difference equations as Bäcklund transformations. J. Phys. A: Math. Gen. 1981;14:1083–98.
[7] Levi D, Sym A, Wojciechowski S. A hierarchy of coupled Korteweg-de Vries equations and the normalisation conditions of the Hilbert–Riemann problem. J. Phys. A: Math. Gen. 1983;16:2423–32.
[8] Boiti M, Pempinelli F, Tu GZ. The nonlinear evolution equations related to the Wadati–Konno–Ichikawa spectral problem. Prog. Theor. Phys. 1983;69:48–64.
[9] Boiti M, Pempinelli F, Tu GZ. Canonical structure of soliton equations via isospectral eigenvalue problems. Nuovo Cimento B 1984;79:231–65.
[10] Antonowicz M, Fordy AP. Coupled KdV equations with multi-Hamiltonian structures. Physica D 1987;28:345–57.
[11] Antonowicz M, Fordy AP. Coupled Harry Dym equations with multi-Hamiltonian structures. J. Phys. A: Math. Gen. 1988;21:L269–75.
[12] Tu GZ. On Liouville integrability of zero-curvature equations and the Yang hierarchy. J. Phys. A: Math. Gen. 1989;22:2375–92.
[13] Tu GZ. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J. Math. Phys. 1989;30:330–8.
[14] Tu GZ. A trace identity and its applications to the theory of discrete integrable systems. J. Phys. A: Math. Gen. 1990;23:3903–22.
[15] Tu GZ, Andrushkiw RI, Huang XC. A trace identity and its application to integrable systems of 1 + 2 dimensions. J. Math. Phys. 1991;32:1900–7.
[16] W.X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, Chin. J. Contemp. Math. 13 (1992) 79-89; Chin. Ann. Math. Ser. A 13 (1992) 115-123.
[17] Ma WX. A hierarchy of coupled Burgers systems possessing a hereditary structure. J. Phys. A: Math. Gen. 1993;26:L1169–74.
[18] Ma WX, Fuchssteiner B, Oevel W. A three-by-three matrix spectral problem for AKNS hierarchy and its binary nonlinearization. Phys. A 1996;233:331–54.
[19] Ma WX, Zhou ZX. Coupled integrable systems associated with a polynomial spectral problem and their Virasoro symmetry algebras. Prog. Theor. Phys. 1996;96:449–57.
[20] Ma WX, Li KS. Virasoro symmetry algebra of Dirac soliton hierarchy. Inverse Prob. 1996;12:L25–31.
[21] Ma WX, Pavlov M. Extending Hamiltonian operators to get bi-Hamiltonian coupled KdV systems. Phys. Lett. A 1998;246:511–22.
[22] Ma WX, Zhou RG. A coupled AKNS–Kaup–Newell soliton hierarchy. J. Math. Phys. 1999;40:4419–28.
[23] Xu XX. A generalized Wadati–Konno–Ichikawa hierarchy and new finite-dimensional integrable systems. Phys. Lett. A 2002;301:250–62.

[24] Xu XX. A generalized Wadati–Konno–Ichikawa hierarchy and its binary nonlinearization by symmetry constraints. Chaos Solitons Fractals 2003;15:475–86.
[25] Guo FK, Zhang YF. The quadratic-form identity for constructing the Hamiltonian structure of integrable systems. J. Phys. A: Math. Gen. 2005;38:8537–48.
[26] Ma WX. Integrable couplings of vector AKNS soliton equations. J. Math. Phys. 2005;46:033507. 19pp.
[27] Ma WX, Chen M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. J. Phys. A: Math. Gen. 2006;39:10787–801.
[28] Tam Honwah, Zhang YF. Spectral radius analysis of matrices and the associated with integrable systems. Int. J. Modern Phys. B 2009;23:4855–79.
[29] Baldwin DE, Hereman W. A symbolic algorithm for computing recursion operators of nonlinear partial differential equations. Int. J. Comput. Math. 2010;87:1094–119.
[30] Tam Hon-Wah, Zhang YF. An integrable system and associated integrable models as well as Hamiltonian structures. J. Math. Phys. 2012;53:103508. p. 25.
[31] Zhu XY, Zhang DJ. Lie algebras and Hamiltonian structures of multi-component Ablowitz–Kaup–Newell–Segur hierarchy. J. Math. Phys. 2013;54:053508. p. 13.
[32] Olver PJ. Applications of Lie groups to differential equations. GTM 107. Verlag, New York: Springer; 1986.
[33] Ma WX. An integrable counterpart of the D-AKNS soliton hierarchy from $\mathrm{so}(3,\mathbb{R})$. Phys. Lett. A 2014;378:1717–20.
[34] Ma WX. A soliton hierarchy associated with $\mathrm{so}(3,\mathbb{R})$. Appl. Math. Comput. 2013;220:117–22.
[35] Ma WX. A spectral problem based on $\mathrm{so}(3,\mathbb{R})$ and its associated commuting soliton equations. J. Math. Phys. 2013;54:103509. 8pp.
[36] W.X. Ma, S. Manukure, H.C. Zheng, A counterpart of the WKI soliton hierarchy associated with $\mathrm{so}(3,\mathbb{R})$, to appear in Zeitschrift fuer Naturforschung A, arXiv:1405.1089.
[37] W.X. Ma, C.G. Shi, E.A. Appiah, C.X. Li, S.F. Shen, An integrable generalization of the Kaup–Newell soliton hierarchy, to appear in Physica Scripta.
[38] Ma WX. A kind of Hamiltonian operators, hereditary symmetries and integrable systems. Appl. Math. J. Chin. Univ. Ser. A 1993;8:28–35.