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ORIGINAL ARTICLE

Equations of motion for zeros of orthogonal polynomials related to the Toda lattices

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Abstract We study the motion of zeros of time dependent orthogonal polynomials. We also relate the tau function of a Toda type lattice to the discriminant of the corresponding orthogonal polynomials which is then related to the free energy of an electrostatic equilibrium problem.

1. Preliminaries

The purpose of this paper is twofold. First we study the motion of zeros of time dependent orthogonal polynomials. Second we relate the tau function of a Toda type lattice to the discriminant of the corresponding orthogonal polynomials which is then related to the free energy of an electrostatic equilibrium problem.

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Both questions will be addressed in Section 3. In order to explain our results we state general prerequisites in the rest of Section 1 while Section 2 is devoted to exploring the connection between orthogonal polynomials and Toda lattices.

Given a probability measure μ one constructs a unique set of orthonormal polynomials $\{p_{n(x)}\}$ with positive leading terms. These polynomials satisfy a three term recurrence relation (Szegő et al., 1975).

$$xp_n(x) = a_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0 \tag{1.1}$$

with

$$a_0 p_{-1}(x) := 0, \quad p_0(x) := 1. \tag{1.2}$$

Let μ be absolutely continuous, supported on $[a, b]$ with

$$\mu'(x) = w(x) = e^{-v(x)} \tag{1.3}$$

and assume that v is differentiable and the integrals

$$\int_R \frac{v'(x) - v'(y)}{x - y} y^n w(y)\, dy, \tag{1.4}$$

exist for $n = 0, 1, \ldots$ Then the polynomials $\{p_n(x)\}$ satisfy the differential recurrence relation (Chen Y and Ismail, 1997)

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \quad n > 0, \tag{1.5}$$

where

$$\frac{A_n(x)}{a_n} := \left[\frac{w(y)p_n^2(y)}{y - x}\right]_a^b + \int_{\mathbb{R}} \frac{v'(x) - v'(y)}{x - y} p_n^2(y) w(y) dy. \tag{1.6}$$

When the condition $w(x)x^n \to 0$ as $x \to \pm\infty$ is not satisfied the right-hand side of (1.5) will have boundary terms (Chen Y and Ismail, 1997).

Recall that the discriminant of a polynomial f is defined by:

$$D(f) = \gamma^{2n-2} \prod_{1 \leqslant j < k \leqslant n} (x_j - x_k)^2, \quad \text{when } f = \gamma \prod_{j=1}^{n} (x - x_j). \tag{1.7}$$

Ismail (1998) proved that the discriminant of p_n is given by:

$$D(p_n) = \left[\prod_{j=1}^{n} \frac{A_n(x_{n,j})}{a_n}\right] \prod_{k=1}^{n} a_k^{2k+2-2n}, \tag{1.8}$$

see also Chapter 3 in Ismail (2005). This extends results of Stieltjes and Hilbert from Jacobi polynomials to general orthogonal polynomials.

Next we state the electrostatic equilibrium problem of a Coulomb gas in one-dimension. Consider n movable unit charged particles in an electric field with potential v. The particles interact according to the logarithmic potential $-2e_1e_2 \ln|x - y|$. In addition the presence of the particles in the external field creates an additional field whose potential is $\ln A_n(x)/a_n$, where A_n arise from v

via (1.3) and (1.6) and the assumptions in (1.4) hold. Ismail (2000) proved that when $v(x)$ and $v(x) + \ln A_n(x)/a_n$ are convex functions the movable charges will reach a unique electrostatic equilibrium position where the particles are located at the zeros of $p_n(x)$. He also gave a closed form expression for the energy at equilibrium. This is reproduced in Ismail (2005).

The partition function associated with the weight function w in (1.3) is

$$Z_n := \int_{\mathbb{R}^n} \prod_{1\leqslant j<k\leqslant n} (x_j - x_k)^2 \exp\left(-\sum_{j=1}^{n} v(x_j)\right) dx_1 dx_2 \ldots dx_n, \tag{1.9}$$

while the tau function is

$$\tau := Z_n/n!. \tag{1.10}$$

The partition and tau functions are functions of the parameters in v.

2. Orthogonal polynomials and Toda lattice

Let $\mathbf{t} = (t_1, t_2, \ldots, t_M)$ with $M \geqslant 1$ and $x \in \mathbb{R}$. The t_js are time parameters. In this section we allow v to depend on these time parameters in a specified manner, hence the recursion coefficients a_n and α_n depend on $\mathbf{t}$ and will be denoted by $a_n(\mathbf{t})$ and $\alpha_n(\mathbf{t})$, respectively.

We consider the following time-dependent measure $d\nu(x,\mathbf{t})$ over an interval $K \subseteq \mathbb{R}$ as

$$d\nu(x,\mathbf{t}) = \frac{1}{\zeta(\mathbf{t})} \exp\left(-\sum_{l=1}^{M} t_l x^l\right) d\mu(x), \quad \zeta(\mathbf{t}) = \int_K \exp\left(-\sum_{l=1}^{M} t_l x^l\right) d\mu(x). \tag{2.1}$$

Assume that $\{p_n(x,\mathbf{t})\}_0^\infty$ are orthonormal polynomials in x and $\{P_n(x,\mathbf{t})\}_0^\infty$ are the corresponding monic polynomials in x orthogonal with respect to $\nu(x,\mathbf{t})$ over K, where p_n and P_n have exact degree n. Let

$$\beta_n(\mathbf{t}) = a_n^2(\mathbf{t}). \tag{2.2}$$

From (1.1) and (1.2) it follows that

$$P_n(x,\mathbf{t}) = \left[\prod_{i=1}^{n} a_i(\mathbf{t})\right] p_n(x,\mathbf{t}), \quad \int_K P_n^2(x,\mathbf{t}) d\nu(x,\mathbf{t}) = \prod_{i=1}^{n} \beta_i(\mathbf{t}). \tag{2.3}$$

The recursion relations for P_n and p_n are (Andrews et al., 1999):

$$xP_n(x,\mathbf{t}) = P_{n+1}(x,\mathbf{t}) + \alpha_n(\mathbf{t}) P_n(x,\mathbf{t}) + \beta_n(\mathbf{t}) P_{n-1}(x,\mathbf{t}), \quad n \geqslant 0 \tag{2.4}$$

and

$$xp_n(x,\mathbf{t}) = a_{n+1}(\mathbf{t}) p_{n+1}(x,\mathbf{t}) + \alpha_n(\mathbf{t}) p_n(x,\mathbf{t}) + a_n(\mathbf{t}) p_{n-1}(x,\mathbf{t}), \quad n \geqslant 0 \tag{2.5}$$

respectively, where, $n \geqslant 1, \beta_0(\mathbf{t}) = a_0(\mathbf{t}) = 0$ and $P_0(x,\mathbf{t}) = p_0(x,\mathbf{t}) = 1$.

Let Q be the semi-infinite matrix

$$Q = Q(\mathbf{t}) := \begin{bmatrix} \alpha_0(\mathbf{t}) & 1 & & & & & 0 \\ \beta_1(\mathbf{t}) & \alpha_1(\mathbf{t}) & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \beta_n(\mathbf{t}) & \alpha_n(\mathbf{t}) & 1 & & \\ 0 & & & \ddots & \ddots & \ddots & \end{bmatrix}. \tag{2.6}$$

The coefficients $\alpha_n(\mathbf{t})$ and $\beta_n(\mathbf{t})$ in (2.4) satisfy the systems of differential–difference equations, see Deift et al. (1999), or Toda et al. (1989).

$$\dot{Q} = \frac{\partial}{\partial t_l} Q = [Q, (Q^l)_+], \quad 1 \leqslant l \leqslant M, \tag{2.7}$$

where "dot" denotes the partial derivative withe respect to the time variable t_l, A_+ means to select the upper triangle part (including entries in the diagonal) of a matrix A.

All systems of equations in (2.7) are called the Toda lattices, since the first one with t_1 is exactly the Toda lattice (Toda et al., 1989). The first two Toda lattices in the hierarchy (2.7) are

$$\frac{\partial}{\partial t_1}\alpha_n(\mathbf{t}) = \beta_n(\mathbf{t}) - \beta_{n+1}(\mathbf{t}), \quad \frac{\partial}{\partial t_1}\beta_n(\mathbf{t}) = \beta_n(\mathbf{t})(\alpha_{n-1}(\mathbf{t}) - \alpha_n(\mathbf{t})) \tag{2.8}$$

and

$$\begin{aligned} &\frac{\partial}{\partial t_2}\alpha_n(\mathbf{t}) = \beta_n(\mathbf{t})(\alpha_{n-1}(\mathbf{t}) + \alpha_n(\mathbf{t})) - \beta_{n+1}(\mathbf{t})(\alpha_n(\mathbf{t}) + \alpha_{n+1}(\mathbf{t})), \\ &\frac{\partial}{\partial t_2}\beta_n(\mathbf{t}) = \beta_n(\mathbf{t})(\alpha_{n-1}^2(\mathbf{t}) - \alpha_n^2(\mathbf{t}) - \beta_{n+1}(\mathbf{t})), \end{aligned} \tag{2.9}$$

where $n \geqslant 0$ and $\alpha_{-1}(\mathbf{t}) = 0$. The above time-dependent orthogonal polynomials play an important role in the study of the Toda lattices (Adler and van Moerbeke, 1995; Chen et al., 1998).

Another important fact is that the orthonormal polynomials $\{p_n(x,\mathbf{t})\}_{n=0}^{\infty}$ satisfy the differential–difference relations

$$\frac{\partial}{\partial x}p_n(x,\mathbf{t}) = A_n(x,\mathbf{t})p_{n-1}(x,\mathbf{t}) - B_n(x,\mathbf{t})p_n(x,\mathbf{t}), \quad n \geqslant 1, \tag{2.10}$$

provided that the function $v(x)$ determined by

$$d\mu(x) = \mathrm{e}^{-v(x)}\,dx \tag{2.11}$$

is twice continuously differentiable for all x in K. Moreover, if $v(x,\mathbf{t})$ has finite moments of all orders on $K = (a,b) \subseteq \mathbb{R}$ (when $d\mu(x)$ has unbounded support) and $v(a^+,\mathbf{t}) = v(b^-,\mathbf{t}) = 0$, then A_n and B_n can be computed by Chen Y and Ismail (1997):

$$
\begin{aligned}
A_n(x,\mathbf{t}) &= a_n(\mathbf{t})\int_a^b \frac{v'(x)-v'(y)+\sum_{s=1}^{M} st_s(x^{s-1}-y^{s-1})}{x-y}p_n^2(y,\mathbf{t})dv(x,\mathbf{t}),\\
B_n(x,\mathbf{t}) &= a_n(\mathbf{t})\int_a^b \frac{v'(x)-v'(y)+\sum_{s=1}^{M} st_s(x^{s-1}-y^{s-1})}{x-y}p_n(y,\mathbf{t})p_{n-1}(y,\mathbf{t})dv(x,\mathbf{t}),
\end{aligned}
\tag{2.12}
$$

where v' denotes the derivative of v with respect to x. Here we assumed that those two integrals exist.

3. Equations of motion for zeros

Note that the three term recurrence relation (2.4) can be rewritten as

$$
xP(x,\mathbf{t}) = Q(\mathbf{t})P(x,\mathbf{t}), \quad P(x,\mathbf{t}) = (P_0(x,\mathbf{t}), P_1(x,\mathbf{t}), \ldots, P_n(x,\mathbf{t}), \ldots)^T,
\tag{3.1}
$$

where the semi-infinite matrix Q is defined by (2.6). It then follows that

$$
x^l P_n(x,\mathbf{t}) = \sum_{i=\max\{0,n-l\}}^{n+l} r_{n,i}^{(l)}(\mathbf{t})P_i(x,\mathbf{t}), \quad l \geqslant 1,
\tag{3.2}
$$

where all $r_{n,i}^{(l)}(\mathbf{t})$ are functions of $\alpha_j(\mathbf{t})$ and $\beta_j(\mathbf{t})$, $0 \leqslant j \leqslant n+l-1$ satisfying $\left(r_{n,i}^{(l)}\right) = Q^l$. For example, when $l = 2$, we have

$$
\begin{aligned}
x^2P_n(x,\mathbf{t}) = P_{n+2}(x,\mathbf{t}) &+ [\alpha_{n+1}(\mathbf{t}) + \alpha_n(\mathbf{t})]P_{n+1} + [\beta_{n+1}(\mathbf{t}) + \alpha_n^2(\mathbf{t})\\
&+ \beta_n(\mathbf{t})]P_n(x,\mathbf{t}) + \beta_n(\mathbf{t})(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t}))P_{n-1}(x,\mathbf{t})\\
&+ \beta_n(\mathbf{t})\beta_{n-1}(\mathbf{t})P_{n-2}(x,\mathbf{t})
\end{aligned}
\tag{3.3}
$$

and thus

$$
\begin{cases}
r_{n,n-2}^{(l)}(\mathbf{t}) = \beta_n(\mathbf{t})\beta_{n-1}(\mathbf{t}), \quad r_{n,n-1}^{(l)}(\mathbf{t}) = \beta_n(\mathbf{t})(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t})),\\
r_{n,n}^{(l)}(\mathbf{t}) = \beta_{n+1}(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + \beta_n(\mathbf{t}),\\
r_{n,n+1}^{(l)}(\mathbf{t}) = \alpha_{n+1}(\mathbf{t}) + \alpha_n(\mathbf{t}), \quad r_{n,n+2}^{(l)}(\mathbf{t}) = 1.
\end{cases}
$$

Let us recall that the dot denotes the partial derivative with respect to $t_l (1 \leqslant l \leqslant M)$. For each $1 \leqslant l \leqslant M$, assume that

$$
\dot{P}_n(x,\mathbf{t}) = \frac{\partial}{\partial t_l}P_n(x,\mathbf{t}) = \sum_{i=0}^{n-1} c_{n,i}^{(l)}(\mathbf{t})P_i(x,\mathbf{t}),
\tag{3.4}
$$

where $c_{n,i}^{(l)}(\mathbf{t})$ are functions to be determined. The orthogonality relations imply that

$$\int_K P_n(x,\mathbf{t})P_i(x,\mathbf{t})\ \exp\left(-\sum_{l=1}^{M} t_l x^l\right) d\mu(x) = 0, \quad 0 \leqslant i \leqslant n-1.$$

Differentiating this with respect to t_l, we obtain

$$\int_K [\dot{P}_n(x,\mathbf{t})P_i(x,\mathbf{t}) + P_n(x,\mathbf{t})\dot{P}_i(x,\mathbf{t})] \exp\left(-\sum_{l=1}^{M} t_l x^l\right) d\mu(x)$$
$$- \int_K x^l P_n(x,\mathbf{t})P_i(x,\mathbf{t}) \exp\left(-\sum_{l=1}^{M} t_l x^l\right) d\mu(x) = 0, \quad 0 \leqslant i \leqslant n-1,$$

which leads to that

$$\int_{\mathbb{R}} \dot{P}_n(x,\mathbf{t})P_i(x,\mathbf{t}) dv(x,\mathbf{t}) - \int_K x^l P_n(x,\mathbf{t})P_i(x,\mathbf{t}) dv(x,\mathbf{t}) = 0, \quad 0 \leqslant i \leqslant n-1.$$

Using the two expressions (3.2) and (3.4) for $x^l P_n(x,\mathbf{t})$ and $\dot{P}_n(x,\mathbf{t})$ and the orthogonality relations of $\{P_n(x,\mathbf{t})\}_{n=0}^{\infty}$, it then follows that

$$\begin{cases} c_{n,i}^{(l)}(\mathbf{t}) = r_{n,i}^{(l)}(\mathbf{t}), & \max\{0, n-l\} \leqslant i \leqslant n-1, \\ c_{n,i}^{(l)}(\mathbf{t}) = 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

If we denote by $(Q^l)_{SL}$ the strictly lower triangle part of the semi-infinite matrix Q^l, the above equality is equivalent to

$$\left(c_{n,i}^{(l)}(\mathbf{t})\right) = (Q^l(\mathbf{t}))_{SL}. \tag{3.6}$$

Therefore, we have

$$\dot{P}_n(x,\mathbf{t}) = \frac{\partial P_n(x,\mathbf{t})}{\partial t_l} = \sum_{i=\max\{0,n-l\}}^{n-1} r_{n,i}^{(l)}(\mathbf{t}) P_i(x,\mathbf{t}), \tag{3.7}$$

which can be written concisely as

$$\dot{P}(x,\mathbf{t}) = (Q^l(\mathbf{t}))_{SL} P(x,\mathbf{t}). \tag{3.8}$$

Let us now start to derive the equations of motion for the zeros of the orthogonal polynomials $P_n(x,\mathbf{t})$. Let $x_{n,k}(\mathbf{t})$, $1 \leqslant k \leqslant n$, be n roots of $P_n(x,\mathbf{t})$, that is

$$P_n(x_{n,k}(\mathbf{t}),\mathbf{t}) = 0, \quad 1 \leqslant k \leqslant n. \tag{3.9}$$

Differentiating the above equation with respect to t_l, we obtain

$$\frac{\partial}{\partial x} P_n(x,\mathbf{t})\bigg|_{x=x_{n,k}(\mathbf{t})} \frac{\partial x_{n,k}(\mathbf{t})}{\partial t_l} + \dot{P}_n(x_{n,k}(\mathbf{t}),\mathbf{t}) = 0.$$

Therefore, using (3.7), it follows that

$$\frac{\partial x_{n,k}(\mathbf{t})}{\partial t_l} = -\frac{\dot{P}_n(x_{n,k}(\mathbf{t}),\mathbf{t})}{\frac{\partial}{\partial x}P_n(x,\mathbf{t})|_{x=x_{n,k}(\mathbf{t})}} = -\frac{\sum_{i=\max\{0,n-l\}}^{n-1} r_{n,i}^{(l)}(\mathbf{t})P_i(x_{n,k}(\mathbf{t}),\mathbf{t})}{\frac{\partial}{\partial x}P_n(x,\mathbf{t})|_{x=x_{n,k}(\mathbf{t})}}, \tag{3.10}$$

$1 \leqslant k \leqslant n$, where $1 \leqslant l \leqslant M$. This is the set of equations of motion which describe the zeros of the time-dependent orthogonal polynomials $P_n(x,\mathbf{t}), n \geqslant 0$.

Furthermore, using (2.10) and (2.3), we can get

$$\frac{\partial}{\partial x}P_n(x,\mathbf{t})|_{x=x_{n,k}(\mathbf{t})} = a_n(\mathbf{t})A_n(x_{n,k}(\mathbf{t}),\mathbf{t})P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t}).$$

On the other hand, using the three term recurrence relation (2.4), we have

$$\begin{bmatrix} P_{n-2}(x_{n,k}(\mathbf{t}),\mathbf{t}) \\ P_{n-3}(x_{n,k}(\mathbf{t}),\mathbf{t}) \\ \vdots \\ P_1(x_{n,k}(\mathbf{t}),\mathbf{t}) \end{bmatrix} = R \begin{bmatrix} P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t}) \\ P_{n-2}(x_{n,k}(\mathbf{t}),\mathbf{t}) \\ \vdots \\ P_2(x_{n,k}(\mathbf{t}),\mathbf{t}) \end{bmatrix}, \tag{3.11}$$

where the square matrix R of order $n-2$ is given by:

$$R = \begin{bmatrix} \frac{x_{n,k}(\mathbf{t})-\alpha_{n-1}(\mathbf{t})}{\beta_{n-1}(\mathbf{t})} & & & 0 \\ -\frac{1}{\beta_{n-2}(\mathbf{t})} & \frac{x_{n,k}(\mathbf{t})-\alpha_{n-2}(\mathbf{t})}{\beta_{n-2}(\mathbf{t})} & & \\ & \ddots & \ddots & \\ 0 & & -\frac{1}{\beta_2(\mathbf{t})} & \frac{x_{n,k}(\mathbf{t})-\alpha_2(\mathbf{t})}{\beta_2(\mathbf{t})} \end{bmatrix}. \tag{3.12}$$

It then follows that

$$P_{n-i}(x_{n,k}(\mathbf{t}),\mathbf{t}) = \varphi_{n,i}(x_{n,k}(\mathbf{t}),\mathbf{t})P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t}), \quad 1 \leqslant i \leqslant n-1, \tag{3.13}$$

where $\varphi_{n,1}(x_{n,k}(\mathbf{t}),\mathbf{t}) = 1$ and $\varphi_{n,i}(x_{n,k}(\mathbf{t}),\mathbf{t}), 2 \leqslant i \leqslant n-1$, are recursively determined by using the formula (3.11). Therefore, we can now express (3.10) as

$$\frac{\partial x_{n,k}(\mathbf{t})}{\partial t_l} = -\frac{\sum_{i=1}^{n-\max\{0,n-l\}} \varphi_{n,i}(x_{n,k}(\mathbf{t}),\mathbf{t})r_{n,n-i}^{(l)}(\mathbf{t})}{a_n(\mathbf{t})A_n(x_{n,k}(\mathbf{t}),\mathbf{t})}, \quad 1 \leqslant k \leqslant n. \tag{3.14}$$

This will hold, provided that $v(x)$ is twice continuously differentiable in K.

In particular, when $l = 1$, noting that

$$\dot{P}_n(x,\mathbf{t}) = \beta_n(\mathbf{t})P_{n-1}(x,\mathbf{t}), \quad n \geqslant 1,$$

we have

$$\begin{cases} \frac{\partial x_{1,1}(\mathbf{t})}{\partial t_1} = \frac{\partial \alpha_0(\mathbf{t})}{\partial t_1} = -\beta_1(\mathbf{t}), \\ \frac{\partial x_{n,k}(\mathbf{t})}{\partial t_1} = -\frac{\beta_n(\mathbf{t})P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t})}{a_n(\mathbf{t})A_n(x_{n,k},\mathbf{t})P_{n-1}((x_{n,k}(\mathbf{t}),\mathbf{t})} = -\frac{a_n(\mathbf{t})}{A_n(x_{n,k}(\mathbf{t}),\mathbf{t})}, \end{cases} \tag{3.15}$$

where $n \geqslant 2$ and $1 \leqslant k \leqslant n$. Then using (2.12), we arrive at

$$\begin{cases} \frac{\partial x_{1,1}(\mathbf{t})}{\partial t_1} = -\beta_1(\mathbf{t}), \\ \frac{dx_{n,k}(\mathbf{t})}{dt_1} = -\frac{1}{\int_K \left[\frac{v'(x)-v'(y)}{x-y} + \frac{\sum_{s=1}^{M} st_s(x^{s-1}-y^{s-1})}{x-y}\right] p_n^2(y,\mathbf{t})dv(y,\mathbf{t})}, \end{cases} \tag{3.16}$$

where $n \geqslant 2$, $1 \leqslant k \leqslant n$, $d\mu(x) = \mathrm{e}^{-v(x)}dx$, and p_n is the nth orthonormal polynomial.

When $l = 2$, note that

$$\begin{cases} \dot{P}_1(x,\mathbf{t}) = \beta_1(\mathbf{t})(\alpha_1(\mathbf{t}) + \alpha_0(\mathbf{t})), \\ \dot{P}_n(x,\mathbf{t}) = \beta_n(\mathbf{t})\beta_{n-1}(\mathbf{t})P_{n-2}(x,\mathbf{t}) + \beta_n(\mathbf{t})(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t}))P_{n-1}(x,\mathbf{t}), \end{cases} \tag{3.17}$$

where $n \geqslant 2$. So for $n \geqslant 2$, we can compute that

$$\begin{aligned} \frac{\partial x_{n,k}(\mathbf{t})}{\partial t_2} &= -\frac{a_n(\mathbf{t})\beta_{n-1}(\mathbf{t})P_{n-2}(x_{n,k}(\mathbf{t}),\mathbf{t})}{A_n(x_{n,k}(\mathbf{t}),\mathbf{t})P_{n-1}((x_{n,k}(\mathbf{t}),\mathbf{t})} - \frac{a_n(\mathbf{t})(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t}))}{A_n(x_{n,k}(\mathbf{t}),\mathbf{t})} \\ &= -\frac{a_n(\mathbf{t})}{A_n(x_{n,k}(\mathbf{t}),\mathbf{t})}\left[(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t})) + \beta_{n-1}(\mathbf{t})\frac{P_{n-2}(x_{n,k}(\mathbf{t}),\mathbf{t})}{P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t})}\right] \\ &= -\frac{a_n(\mathbf{t})}{A_n(x_{n,k}(\mathbf{t}),\mathbf{t})}\left[(\alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t})) + \beta_{n-1}(\mathbf{t})\frac{x_{n,k}(\mathbf{t}) - \alpha_{n-1}(\mathbf{t})}{\beta_{n-1}(\mathbf{t})}\right] \\ &= -\frac{\alpha_n(\mathbf{t}) + x_{n,k}(\mathbf{t})}{\int_K \left[\frac{v'(x)-v'(y)}{x-y} + \frac{\sum_{s=1}^{M} st_s(x^{s-1}-y^{s-1})}{x-y}\right] p_n^2(y,\mathbf{t})dv(y,\mathbf{t})}. \end{aligned}$$

In the last step above, we have used

$$\frac{P_{n-2}(x_{n,k}(\mathbf{t}),\mathbf{t})}{P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t})} = \frac{x_{n,k}(\mathbf{t}) - \alpha_{n-1}(\mathbf{t})}{\beta_{n-1}(\mathbf{t})},$$

which is a consequence of the three term recurrence relation (2.4). Thus, we arrive at

$$\begin{cases} \frac{\partial x_{1,1}(\mathbf{t})}{\partial t_2} = -\beta_1(\mathbf{t})(\alpha_1(\mathbf{t}) + \alpha_0(\mathbf{t})), \\ \frac{\partial x_{n,k}(\mathbf{t})}{\partial t_2} = -\frac{\alpha_n(\mathbf{t}) + x_{n,k}(\mathbf{t})}{\int_K [\frac{v\prime(x)-v\prime(y)}{x-y} + \frac{\sum_{s=1}^{M} st_s(x^{s-1}-y^{s-1})}{x-y}] p_n^2(y,\mathbf{t})dv(y,\mathbf{t})}, \end{cases} \tag{3.18}$$

where $n \geqslant 2$ and $1 \leqslant k \leqslant n$.

Example 1. Let $M = 1$ and thus $\mathbf{t} = t_1 \equiv t$. Choose $K = \mathbb{R}$ and $v(x) = x^4$. Then

$$\begin{aligned}\frac{A_n(x,\mathbf{t})}{a_n(\mathbf{t})} &= \int_{-\infty}^{\infty} \frac{v'(x) - v'(y)}{x - y} p_n^2(y,\mathbf{t}) dv(y,\mathbf{t}) \\ &= 4\int_{-\infty}^{\infty} (x^2 + xy + y^2) p_n^2(y,\mathbf{t}) dv(y,\mathbf{t}) \\ &= 4[x^2 + a_{n+1}^2(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + a_n^2(\mathbf{t}) + x\alpha_n(\mathbf{t})]\end{aligned}$$

$$\begin{aligned}\frac{B_n(x,\mathbf{t})}{a_n(\mathbf{t})} &= \int_{-\infty}^{\infty} \frac{v'(x) - v'(y)}{x - y} p_n(y,\mathbf{t}) p_{n-1}(y,\mathbf{t}) dv(y,\mathbf{t}) \\ &= 4\int_{-\infty}^{\infty} (x^2 + xy + y^2) p_n(y,\mathbf{t}) p_{n-1}(y,\mathbf{t}) dv(y,\mathbf{t}) \\ &= 4a_n(\mathbf{t})[x + \alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t})].\end{aligned}$$

Therefore, the zeros $x(\mathbf{t}) = x_{n,k}(\mathbf{t}), 1 \leqslant k \leqslant n$, of $P_n(x,\mathbf{t})$ satisfy the following differential equation:

$$\frac{dx(\mathbf{t})}{dt} = -\frac{1}{4[x^2(\mathbf{t}) + a_{n+1}^2(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + a_n^2(\mathbf{t}) + x(\mathbf{t})\alpha_n(\mathbf{t})]}. \tag{3.19}$$

Example 2. Let $M = 2$ and thus $\mathbf{t} = (t_1, t_2)$. Consider the same choice of $K = \mathbb{R}$ and $v(x) = x^4$ as in Example 1. Obviously

$$\begin{aligned}\frac{A_n(x,\mathbf{t})}{a_n(\mathbf{t})} &= 4[x^2 + a_{n+1}^2(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + a_n^2(\mathbf{t}) + x\alpha_n(\mathbf{t})] + 2t_2, \\ \frac{B_n(x,\mathbf{t})}{a_n(\mathbf{t})} &= 4a_n(\mathbf{t})[x + \alpha_n(\mathbf{t}) + \alpha_{n-1}(\mathbf{t})].\end{aligned}$$

If $n = 1$, there is only one zero $\alpha_0(\mathbf{t})$ of $P_1(x,\mathbf{t}) = x - \alpha_0(\mathbf{t})$. Obviously, from (2.8) and (2.9), we have

$$\frac{\partial x_{1,1}(\mathbf{t})}{\partial t_1} = \frac{\partial \alpha_0(\mathbf{t})}{\partial t_1} = -\beta_1(\mathbf{t}), \quad \frac{\partial x_{1,1}(\mathbf{t})}{\partial t_2} = \frac{\partial \alpha_0(\mathbf{t})}{\partial t_2} = -\beta_1(\mathbf{t})(\alpha_1(\mathbf{t}) + \alpha_0(\mathbf{t})). \tag{3.20}$$

If $n \geqslant 2$, then the zeros $x(\mathbf{t}) = x_{n,k}(\mathbf{t})$, $1 \leqslant k \leqslant n$, of $P_n(x,\mathbf{t})$ satisfy the following set of differential equations:

$$\begin{aligned}\frac{\partial x(\mathbf{t})}{\partial t_1} &= -\frac{1}{4[x^2(\mathbf{t}) + a_{n+1}^2(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + a_n^2(\mathbf{t}) + x(\mathbf{t})\alpha_n(\mathbf{t})] + 2t_2}, \\ \frac{\partial x(\mathbf{t})}{\partial t_2} &= -\frac{\alpha_n(\mathbf{t}) + x(\mathbf{t})}{4\{[x^2(\mathbf{t}) + a_{n+1}^2(\mathbf{t}) + \alpha_n^2(\mathbf{t}) + a_n^2(\mathbf{t}) + x(\mathbf{t})\alpha_n(\mathbf{t})] + 2t_2\}}.\end{aligned} \tag{3.21}$$

We also point out that starting from the zeros, we can have the following representation for solutions to the Toda lattices:

$$\begin{cases} \alpha_n(\mathbf{t}) = \frac{x_{n-1,k}(\mathbf{t})P_n(x_{n-1,k}(\mathbf{t}),\mathbf{t}) - P_{n+1}(x_{n-1,k}(\mathbf{t}),\mathbf{t})}{P_n(x_{n-1,k}(\mathbf{t}),\mathbf{t})}, \\ \beta_n(\mathbf{t}) = -\frac{P_{n+1}(x_{n,k}(\mathbf{t}),\mathbf{t})}{P_{n-1}(x_{n,k}(\mathbf{t}),\mathbf{t})}, \end{cases} \tag{3.22}$$

where k can be any integer between 1 and M. So the zeros can provide direct information on the corresponding solutions to the Toda lattices. However, there are other solutions to the Toda lattices which are not generated from the orthogonal polynomials (see Maruno K et al., 2004; Ma and You, 2004 for examples in the case of the Toda lattice).

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