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Dynamics of soliton and mixed lump-soliton waves to a generalized Bogoyavlensky-Konopelchenko equation

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Abstract

We study dynamics of soliton waves, lump solutions and interaction solutions to a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation, which possesses a Hirota bilinear form. Multi-soliton solutions, one-M-lump solutions, and physical interactions between solitons and 1-M-lump solutions are presented. By using a positive quadratic function, lump solutions and their interaction solutions with kink and solitary waves are also generated. To show dynamical properties and physical behaviors of the resulting solutions, 3D-plots and contour plots at different times are made and analyzed.

1. Introduction

One of the fundamental problems in the theory of differential equations is the Cauchy problem. The problem requires to find a solution to a differential equation that satisfies what is known as the initial values. As classified [1, 2], Laplace's method [3] is used to solve linear ordinary differential equations and the Fourier transform method [4] to find the solution of linear partial differential equations, and in the modern soliton theory, the isomonodromic transform method and the inverse scattering transform method [5, 6] are systematical methods that have been developed to find solutions of nonlinear ordinary and partial differential equations, respectively.

Only the simplest differential equations, usually constant-coefficient and linear, can be explicitly resolved. It is extremely hard to pinpoint exact solutions for nonlinear differential equations. For this purpose, recently some analytic approaches have been reported to construct exact solutions to partial differential equations, for example, the RiemannHilbert method [7], the $\left(\frac{1}{G'}\right)$-expansion method [8, 9], the Taylor expansion approach [10], the Lie symmetry analysis [11, 12], the $\left(m+\frac{G'}{G}\right)$-expansion method [13, 14], a transformed rational function method [15], the multiple exp-function method [16, 17], Darboux transformation method [18, 19], the modified auxiliary expansion method [20], the dressing method [21], the Bernoulli sub-equation function method [22–25], the extended sinh-Gordon method [26, 27], the $\left(\frac{G'}{G^2}\right)$-expansion method [28], the generalized Kudryashov method [29], the Hirota bilinear method [30–32], the Hirota bilinear system and Pfaffian method [33], and the modified extended direct algebraic method [34].

The (2+1)-dimensional Bogoyavlensky-Konopelchenko (BK) equation reads

$$u_{tx}+\alpha(6u_xu_{xx}+u_{xxxx})+\beta(u_{xxxy}+3u_xu_{xy}+3u_{xx}u_y)=0, \tag{1.1}$$

which was introduced as a (2+1)-dimensional version of the KdV equation in [35] and used to describe the interaction of a long wave propagating along the x-axis and a Riemann wave propagating along the y-axis [36]. In

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this paper, we study the following (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko (gBK) equation

$$u_{tx} + \alpha(6u_x u_{xx} + u_{xxxx}) + \beta(u_{xxxy} + 3u_x u_{xy} + 3u_{xx}u_y) \\ +\gamma_1 u_{xx} + \gamma_2 u_{xy} + \gamma_3 u_{yy} = 0, \tag{1.2}$$

which is a generalization of equation (1.1). The earlier paper [1] has studied lower-order lumps to the suggested equation. In [2], a few classes of exact and explicit solutions have been reported from different ansätze on solution forms, for example, traveling wave, 2-wave solutions, and polynomial solutions. To investigate equation (1.2), we use the link between u and f:

$$u = 2(\ln f(x, y, t))_x, \tag{1.3}$$

where

$$f_1 = 1 + e^{k_1(x+l_1y+w_1t)}, \tag{1.4}$$

$$f_2 = 1 + e^{\Omega_1} + e^{\Omega_2} + e^{\Omega_1+\Omega_2+A_{12}}, \tag{1.5}$$

$$f_3 = 1 + e^{\Omega_1} + e^{\Omega_2} + e^{\Omega_3} + e^{\Omega_1+\Omega_2+A_{12}} + e^{\Omega_1+\Omega_3+A_{13}} \\ + e^{\Omega_2+\Omega_3+A_{23}} + e^{\Omega_1+\Omega_2+\Omega_3+A_{123}}, \tag{1.6}$$

$$f_4 = 1 + e^{\Omega_1} + e^{\Omega_2} + e^{\Omega_3} + e^{\Omega_4} + e^{\Omega_1+\Omega_2+A_{12}} + e^{\Omega_1+\Omega_3+A_{13}} + e^{\Omega_1+\Omega_4+A_{14}} + e^{\Omega_2+\Omega_3+A_{23}} \\ + e^{\Omega_2+\Omega_4+A_{24}} + e^{\Omega_3+\Omega_4+A_{34}} + e^{\Omega_1+\Omega_2+\Omega_3+A_{123}} + e^{\Omega_1+\Omega_2+\Omega_4+A_{124}} + e^{\Omega_2+\Omega_3+\Omega_4+A_{234}} \\ + e^{\Omega_1+\Omega_2+\Omega_3+\Omega_4+A_{1234}}. \tag{1.7}$$

Insert equation (1.3) into equation (1.2), we get

$$2f(\gamma_3 f_{yy} + f_{xt} + \gamma_2 f_{xy} + \gamma_1 f_{xx} + \beta f_{xxxy} + \alpha f_{xxxx}) - 2\gamma_3 f_y^2 - 2f_y(\gamma_2 f_x + \beta f_{xxx}) \\ - 2(f_t f_x + \gamma_1 f_x^2 - 3f_{xx}(\beta f_{xy} + \alpha f_{xx}) + f_x(3\beta f_{xxy} + 4\alpha f_{xxx})) = 0. \tag{1.8}$$

The logarithmic variable transformation (1.3) is also a characteristic one in establishing Bell polynomial theories of soliton equations [37]. Equation (1.8) can be rewritten as a Hirota bilinear form as follows

$$(D_x D_t + \alpha D_x^4 + \beta D_x^3 D_y + \gamma_1 D_x^2 + \gamma_2 D_x D_y + \gamma_3 D_y^2) f.f = 0. \tag{1.9}$$

It's obvious that if $f = f(x, y, t)$ in equations (1.4)–(1.7)are solutions of equation (1.5), then $u = 2(\ln f(x, y, t))_x$ will solve equation (1.2).

2. Complex one-, two, and three-soliton solutions

In this section, we construct special complex one-, two-, and three-soliton solutions to the introduced equation. To find complex soliton solutions, we require

$$w_m = -(k_m^2\alpha + k_m^2 l_m\beta + \gamma_1 + l_m\gamma_2 + l_m^2\gamma_3), \quad m = 1, 2, 3 \tag{2.1}$$

To determine special complex one-, two-, three-soliton solutions, we take

$$f = \mathrm{i} + e^{k_1(x+l_1y+w_1t)}, \tag{2.2}$$

$$f = \mathrm{i} + e^{\Omega_1} + e^{\Omega_2} + D_{12}e^{\Omega_1+\Omega_2}, \tag{2.3}$$

$$f = \mathrm{i} + e^{\Omega_1} + e^{\Omega_2} + e^{\Omega_3} + D_{12}e^{\Omega_1+\Omega_2} + D_{13}e^{\Omega_1+\Omega_3} + D_{23}e^{\Omega_2+\Omega_3} + D_{123}e^{\Omega_1+\Omega_2+\Omega_3}, \tag{2.4}$$

where

$$\Omega_m = k_m(x + l_m y + w_m t) + \alpha_m \, (m = 1, 2,\ldots,N) \tag{2.5}$$

$$D_{mn} = \frac{i(l_m - l_n)^2\gamma_3 - i(k_m - k_n)(3(k_m - k_n)\alpha + k_m(2l_m + l_n)\beta - k_2(l_m + 2l_n)\beta)}{(k_m + k_n)(3(k_m + k_n)\alpha + k_m(2l_m + l_n)\beta + k_n(l_m + 2l_n)\beta) - (l_m - l_n)^2\gamma_3}, \tag{2.6}$$

and

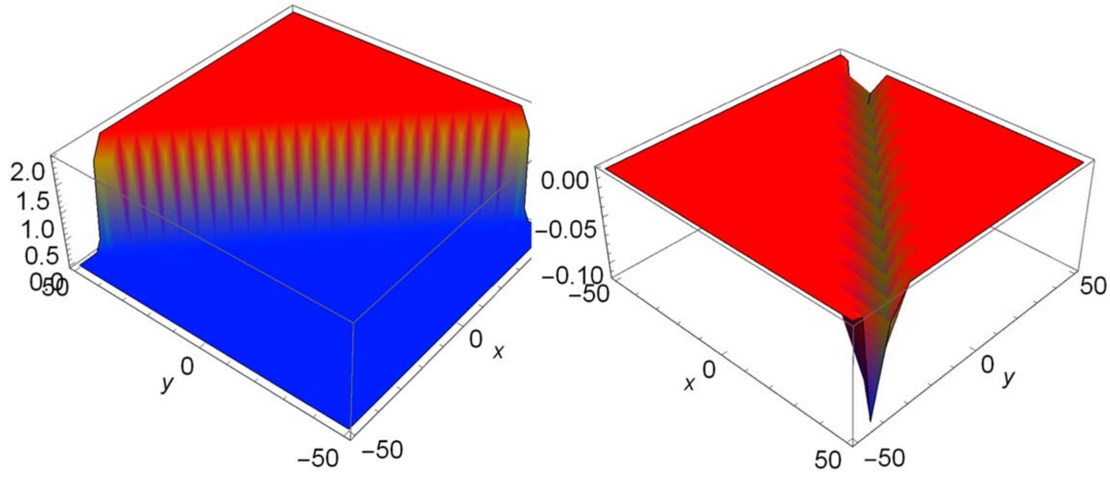
$$D_{123} = D_{12}D_{13}D_{23}. \tag{2.7}$$

Plugging equation (2.2) into equation (1.3), we get

$$u = \frac{2k_1 e^{k_1(x+l_1y-(k_1^2\alpha+k_1^2l_1\beta+\gamma_1+l_1\gamma_2+l_1^2\gamma_3)t+\alpha_0)}}{\mathrm{i} + e^{k_1(x+l_1y-(k_1^2\alpha+k_1^2l_1\beta+\gamma_1+l_1\gamma_2+l_1^2\gamma_3)t+\alpha_0)}}. \tag{2.8}$$

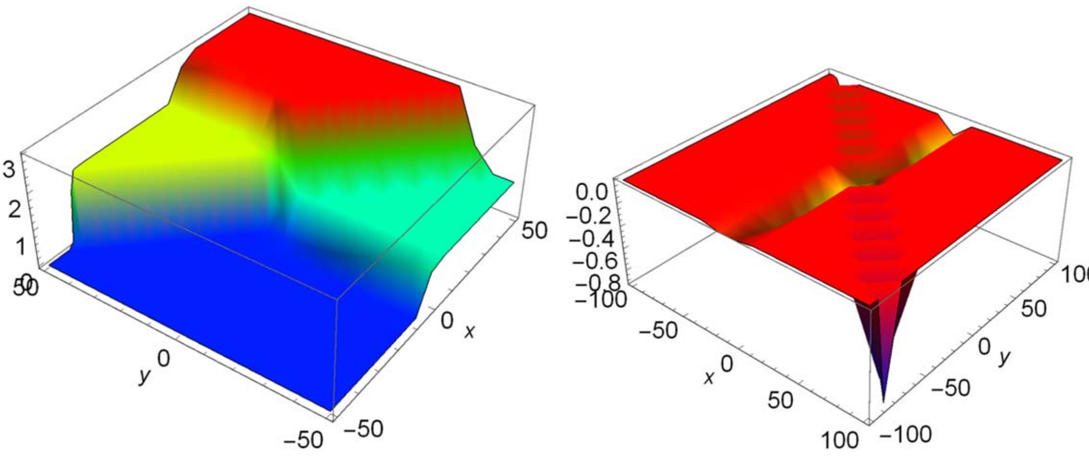
This is a complex 1-soliton solution as shown in figure 1.

By inserting equation (2.3) into equation (1.3), we have a complex two-soliton wave solution to the considered equation (see figure 2)



(a) Real solution (b) Imaginary solution

Figure 1. 3D surface of 1-soliton solution plotted under equation (2.6) when $t = 2, l_1 = 1, k_1 = 1, \alpha_1 = 0, \alpha = 1, \beta = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.



(a) Real solution (b) Imaginary solution

Figure 2. 3D surface of 2-soliton solution plotted under equation (2.7) when $t = 2, l_1 = -0.1, l_2 = 1, k_1 = 0.5, k_2 = 1, \alpha_1 = 0, \alpha_2 = 0, \alpha = 2, \beta = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.

$$u = \frac{2(k_1 e^{\Omega_1} + k_2 e^{\Omega_2} + D_{12}(k_1 + k_2)e^{\Omega_1+\Omega_2})}{i + e^{\Omega_1} + e^{\Omega_2} + D_{12}e^{\Omega_1+\Omega_2}}. \tag{2.9}$$

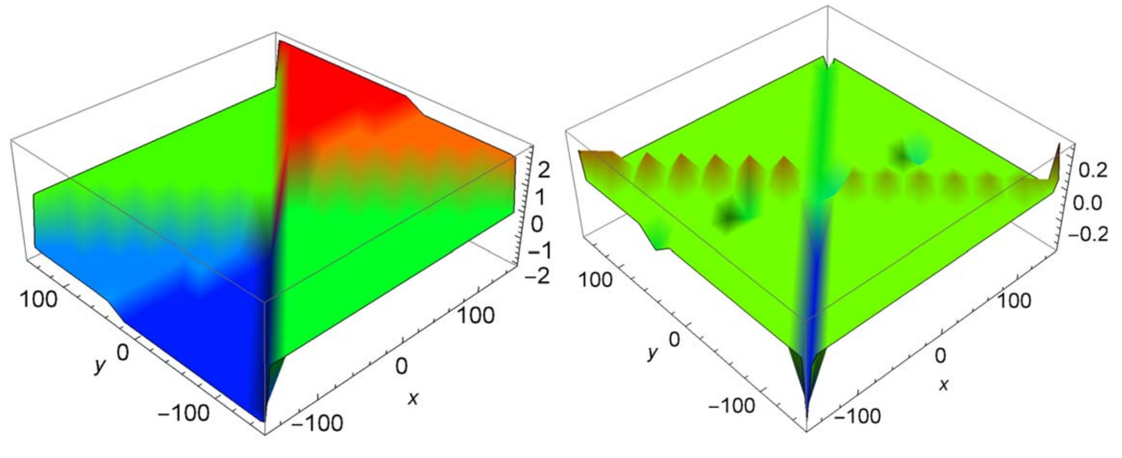
Substituting equation (2.4) into equation (1.3), we get

$$u = \frac{2\begin{pmatrix} e^{\theta_1}k_1 + e^{\theta_2}k_2 + D_{12}(k_1 + k_2)e^{\theta_1+\theta_2} + e^{\theta_3}k_3 + D_{13}(k_1 + k_3)e^{\theta_1+\theta_2} \\ +D_{23}(k_2 + k_3)e^{\theta_2+\theta_3} + D_{12}D_{13}D_{23}(k_1 + k_2 + k_3)e^{\theta_1+\theta_2+\theta_3} \end{pmatrix}}{i + e^{\theta_1} + e^{\theta_2} + D_{12}e^{\theta_1+\theta_2} + e^{\theta_3} + D_{13}e^{\theta_1+\theta_3} + D_{23}e^{\theta_2+\theta_3} + D_{12}D_{13}D_{23}e^{\theta_1+\theta_2+\theta_3}}. \tag{2.10}$$

This is a complex 3-soliton solution as seen in figure 3.

3. M-lump solutions

Lump solutions are analytical rational function solutions located in all directions in space. In this portion of the paper, we use a long-wave limit method to construct a rational solution to a gBK equation. consider equation (1.5), where



(a) Real solution (b) Imaginary solution

Figure 3. 3D surface of 3-soliton solution plotted under equation (2.8) when $t = 2, l_1 = 1, l_2 = -1, l_3 = 4, k_1 = -1, k_2 = 1, k_3 = 0.3, \alpha_1 = 0, \alpha_2 = -2, \alpha_3 = 5, \alpha = -1, \beta = -1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.

$$e^{A_{mn}} = \frac{(k_m - k_n)(3(k_m - k_n)\alpha + k_m(2l_m + l_n)\beta - k_n(l_m + 2l_n)\beta) - (l_m - l_n)^2\gamma_3}{(k_m + k_n)(3(k_m + k_n)\alpha + k_m(2l_m + l_n)\beta + k_n(l_m + 2l_n)\beta) - (l_m - l_n)^2\gamma_3}. \tag{3.1}$$

Here Ω_1, Ω_2 are given by equation (2.9). Using the long-wave limit method by taking the limit, $k_m \to 0$, $\frac{k_1}{k_2} = O(1)$, and $e^{\alpha_m} = -1 (m = 1, 2)$ we get

$$f_2 = \Phi_1\Phi_2 + B_{12}, \tag{3.2}$$

where

$$\Phi_i = x + l_i y + w_i t, \tag{3.3}$$

$$w_i = -(\gamma_1 + l_m\gamma_2 + {l_m}^2\gamma_3), \tag{3.4}$$

$$B_{ij} = \frac{6(2\alpha + (l_i + l_j)\beta)}{(l_i - l_j)^2\gamma_3}, \quad (i < j) \tag{3.5}$$

$$l_{\frac{N}{2}+i} = l_i^*, \left(i = 1, 2, ..., \frac{N}{2}\right). \tag{3.6}$$

Plugging equation (3.1) with equations (3.3)–(3.7) into equation (3.2) then into equation (1.3), we can construct a one-M-lump solution (see figure 4. Here, $l_1 = a + ib, l_2 = c + id, l_3 = a + ib$ and $l_4 = c - id$.

$$\begin{aligned} u &= 2\frac{\partial}{\partial x}\log\left((x' + ay')^2 + b^2y'^2 - \frac{3(\alpha + a\beta)}{b^2\gamma_3}\right) \\ &= \frac{4(x' + ay')}{(x' + ay')^2 + b^2y'^2 - \frac{3(\alpha + a\beta)}{b^2\gamma_3}}, \\ &= \frac{4(x - t\gamma_1 + a(y - t\gamma_2) - a^2t\gamma_3 + b^2t\gamma_3)}{(x - t\gamma_1 - (a - ib)(t(\gamma_2 + a\gamma_3 - ib\gamma_3)) - y)(x - t\gamma_1 - (a + ib)(t(\gamma_2 + a\gamma_3 + ib\gamma_3)) - y) - \frac{3(\alpha + a\beta)}{b^2\gamma_3}}, \end{aligned} \tag{3.7}$$

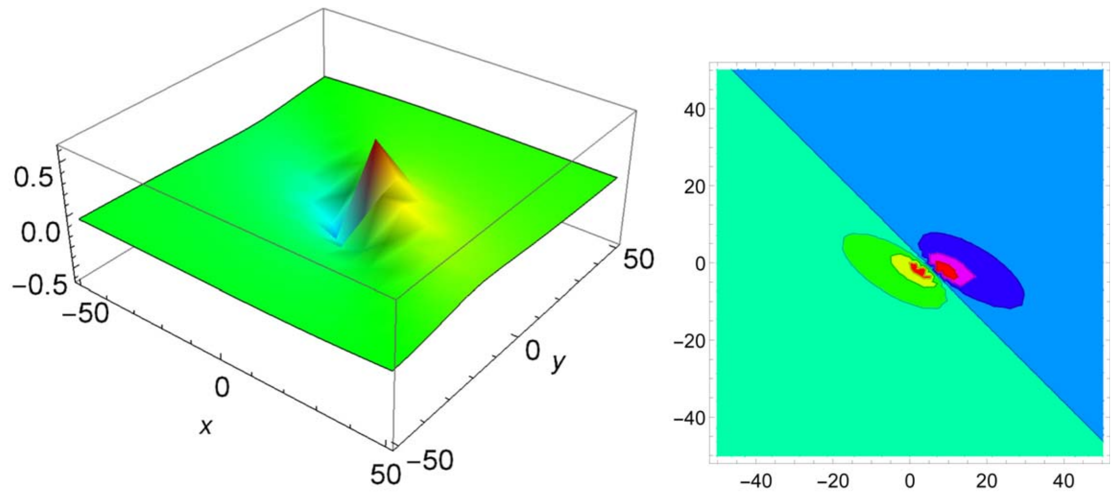
where

$$\begin{aligned} y' &= y - 2a\gamma_3 t - \gamma_2 t, \\ x' &= x + b^2\gamma_3 t - \gamma_1 t. \end{aligned} \tag{3.8}$$

The rational solution (3.8) is a permanent lump solution that decays as $O\left(\frac{1}{x^2}, \frac{1}{y^2}\right)$ for $|x|, |y| \to \infty$ and moves with the velocity

$$\begin{aligned} v_x &= 2a\gamma_3 - \gamma_2, \\ v_y &= \gamma_1 - b^2\gamma_3. \end{aligned} \tag{3.9}$$

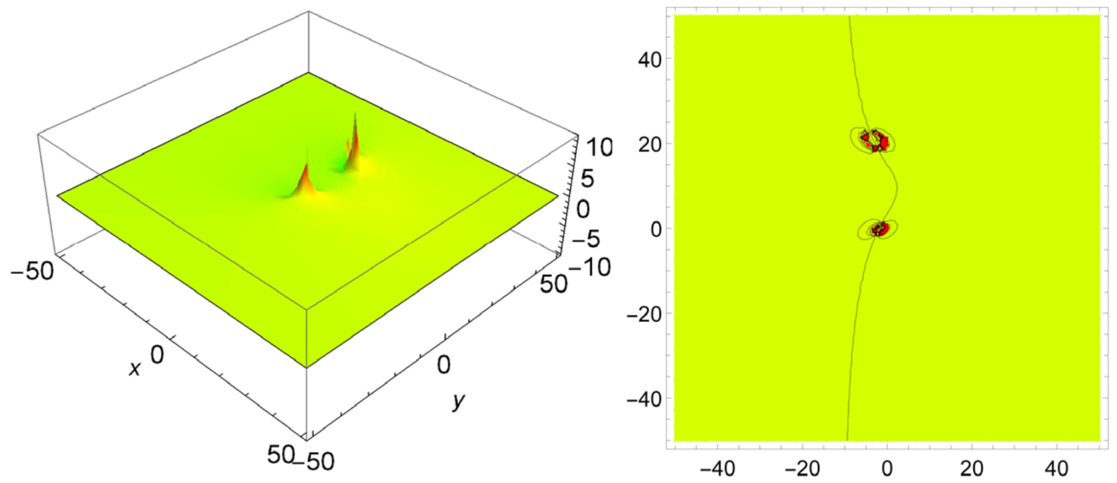
To study and reveal a 2-M-lump solution to a gBK equation, consider equation (1.7), $e^{\alpha_m} = -1 (m = 1, 2, 3, 4)$ and taking a limit $k_m \to 0$, we get



(a) One-lump solution

(b) Corresponding contour plot of one-lump solution

Figure 4. 3D surface and contour plot of equation (3.8) plotted when $t = 2, a = 1, b = 1, l_1 = 1, l_2 = 3, \alpha = 1, \beta = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = -1$.



(a) Two-lump solution

(b) Corresponding contour plot of two-lump solution

Figure 5. 3D surface and contour plot are drawn when $t = 10, a = 1/2, b = 1, c = -1/2, d = 1, l_1 = 0.1, l_2 = 1, l_3 = 2, \alpha = 1, \beta = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.

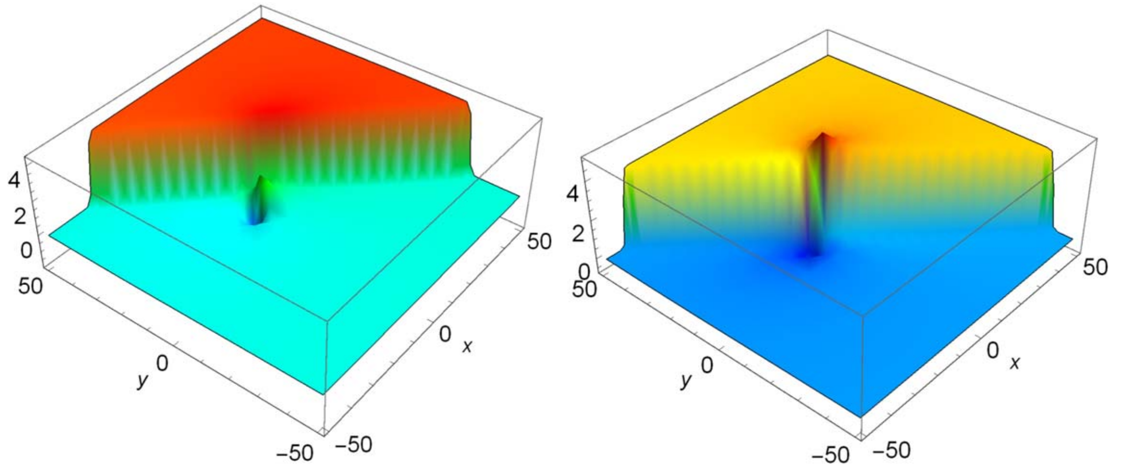
$$f_4 = \Phi_1\Phi_2\Phi_3\Phi_4 + B_{12}\Phi_3\Phi_4 + B_{13}\Phi_2\Phi_4 + B_{14}\Phi_2\Phi_3 + B_{23}\Phi_1\Phi_4 + B_{24}\Phi_1\Phi_3 + B_{34}\Phi_1\Phi_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}. \tag{3.10}$$

Here $\Phi_1, \Phi_2, \Phi_3, \Phi_4, w_i, B_{ij}(i < j)$, and $l_{\frac{N}{2}+i}$ are given by equations (3.4)–(3.7), respectively. Plugging equation (3.10) into equation (1.3), as a result, we can obtain a double-M-lump solution to the suggested equation as shown in figure 5.

4. Physical interactions between M-lump solution and soliton wave

In this section, we study the interaction physical phenomena between the 1-M-lump solution and the one-soliton solution. For this purpose, we consider equation (1.6), and take the limit $k_m \to 0, (m = 1, 2)$ and $\frac{k_1}{k_2} = O(1)$. As a result, f_3 could be rewritten as follows

$$f_3 = \Phi_1\Phi_2 + B_{12} + \Lambda_1 e^{\Omega_3}, \tag{4.1}$$



(a) Interaction phenomena via use Eq. (4.3)

(b) Interaction phenomena via use Eq. (4.4)

Figure 6. 3D surface of interaction between a one-lump solution and soliton wave solution drawn when $t = 5, a = 1/2, b = 1, l_3 = 1, k_3 = 2, \alpha_1 = 0, \alpha_2 = 1, \beta = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.

where

$$\Lambda_1 = \Phi_1\Phi_2 + B_{12} + C_{23}\Phi_1 + C_{13}\Phi_2 + C_{13}C_{23}. \tag{4.2}$$

Here Ω_3 is defined in equation (2.5), $\Phi_i (i = 1, 2)$ are defined in equation (3.4), and B_{12} is given in equation (3.6). The constants C_{13}, C_{23} are stated as follows

$$\begin{aligned}\alpha &= -\frac{\beta}{2}(l_1 + l_2) + \frac{(l_1 - l_3)(l_3 - l_2)\gamma_3}{k_3{}^2},\\ C_{13} &= -\frac{12(l_2 - l_3)}{k_3(l_1 + 3l_2 - 4l_3)},\ C_{23} = -\frac{12(l_1 - l_3)}{k_3(3l_1 + l_2 - 4l_3)},\end{aligned} \tag{4.3}$$

or

$$\begin{aligned}\alpha &= \frac{(l_1 - l_3)(l_3 - l_2)\gamma_3}{k_3{}^2} - l_3\beta,\ C_{13} = -\frac{6(k_3{}^2\beta + 2(l_3 - l_2)\gamma_3)}{k_3{}^3\beta - k_3(l_1 + 3l_2 - 4l_3)\gamma_3},\\ C_{23} &= -\frac{6(k_3{}^2\beta + 2(l_3 - l_1)\gamma_3)}{k_3{}^3\beta - k_3(3l_1 + l_2 - 4l_3)\gamma_3}.\end{aligned} \tag{4.4}$$

Substituting equations (4.1), (4.2)and (4.3)into equation (1.3), we get an equation that presents a collision between the 1-M-lump solution and the 1-soliton solution as seen in figure 6.

We can also choose

$$\begin{aligned}\beta &= \frac{(l_1 - l_3)(l_3 - l_2)\gamma_3 - k_3{}^2\alpha}{k_3{}^2 l_3},\\ C_{13} &= -\frac{6(k_3{}^2\alpha + (l_2 - l_3)(l_1 + l_3)\gamma_3)}{k_3(k_3{}^2\alpha + l_1 l_2\gamma_3 + (2l_2 - 3l_3)l_3\gamma_3)},\\ C_{23} &= -\frac{6(k_3{}^2\alpha + (l_1 - l_3)(l_2 + l_3)\gamma_3)}{k_3(k_3{}^2\alpha - 3l_3{}^2\gamma_3 + l_1(l_2 + 2l_3)\gamma_3)},\end{aligned} \tag{4.5}$$

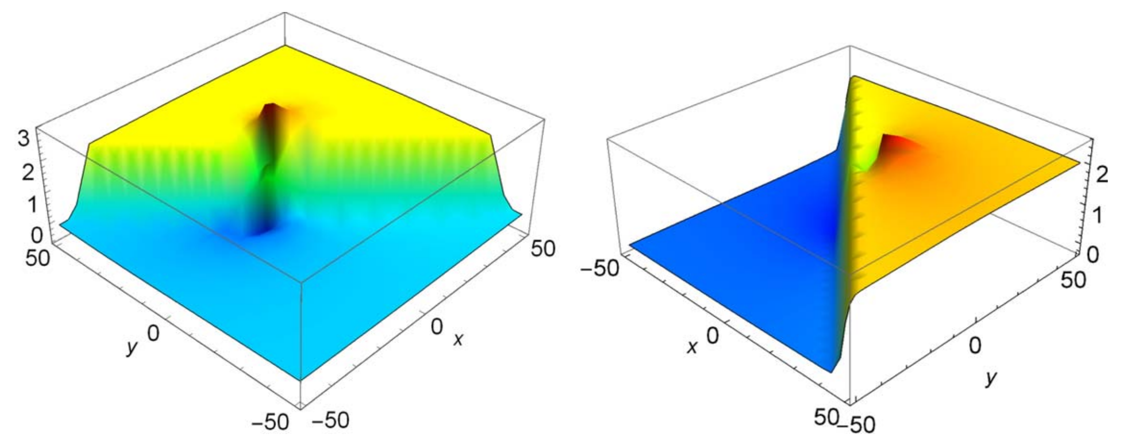
or

$$\beta = -\frac{2(k_3{}^2\alpha + (l_1 - l_3)(l_2 - l_3)\gamma_3)}{k_3{}^2(l_1 + l_2)},\ C_{13} = -\frac{12(l_2 - l_3)}{k_3(l_1 + 3l_2 - 4l_3)},\ C_{23} = -\frac{12(l_1 - l_3)}{k_3(3l_1 + l_2 - 4l_3)}. \tag{4.6}$$

Putting equations (4.1), (4.2)and (4.5)into equation (1.3), we can construct other solutions that describe the interaction between the one-lump solution and the 1-soliton wave solution as shown in figure 7.

5. Quadratic solutions

In this portion of the paper, we investigate single-lump solution via a quadratic function. In [38–41], quadratic function solutions and symbolic computation have been used to construct different kinds of exact solutions to



(a) Interaction phenomena via use Eq. (4.5)

(b) Interaction phenomena via use Eq. (4.6)

Figure 7. 3D surface of interaction between a one-lump solution and soliton wave solution drawn when $t = 5, a = 1/2, b = 1, l_3 = 1, k_3 = 1, \alpha_1 = 0, \alpha_2 = 1, \alpha = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$.

many classes of PDEs. Now, we define the solution of the equation (1.4) as follows

$$f(x, y, t) = (a_1x + a_2y + a_3t + a_4)^2 + (b_1x + b_2y + b_3t + b_4)^2 + c_1, \tag{5.1}$$

where a_i, b_i are constants to be determined later. Plugging equation (4.1) into equation (1.4), as a result, we gain a polynomial. Setting the coefficients of the polynomial with the same powers of the independent variables to zero, one can obtain the following cases of solutions:

Case 1: When we have

$$\begin{aligned} a_3 &= -\frac{a_1^3\gamma_1 + a_1b_1^2\gamma_1 + a_1^2a_2\gamma_2 + a_1(a_2^2 - b_2^2)\gamma_3 + a_2b_1(b_1\gamma_2 + 2b_2\gamma_3)}{a_1^2 + b_1^2},\\ b_3 &= -\frac{a_1^2(b_1\gamma_1 + b_2\gamma_2) + 2a_1a_2b_2\gamma_3 + b_1(b_1^2\gamma_1 + b_1b_2\gamma_2 + (b_2^2 - a_2^2)\gamma 3)}{a_1^2 + b_1^2},\\ c_1 &= -\frac{3(a_1^2 + b_1^2)^2(a_1^2\alpha + a_1a_2\beta + b_1(b_1\alpha + b_2\beta))}{(a_2b_1 - a_1b_2)^2\gamma_3} \end{aligned} \tag{5.2}$$

and substitute equation (5.2) into equation (5.1), then into equation (1.3), we get

$$u = \frac{2(2a_1(a_1x + a_2y + a_3t + a_4) + 2b_1(b_1x + b_2y + b_3t + b_4))}{c_1 + (a_1x + a_2y + a_3t + a_4)^2 + (b_1x + b_2y + b_3t + b_4)^2}. \tag{5.3}$$

This is a lump solution as present in figure 8 and reported in [1] .

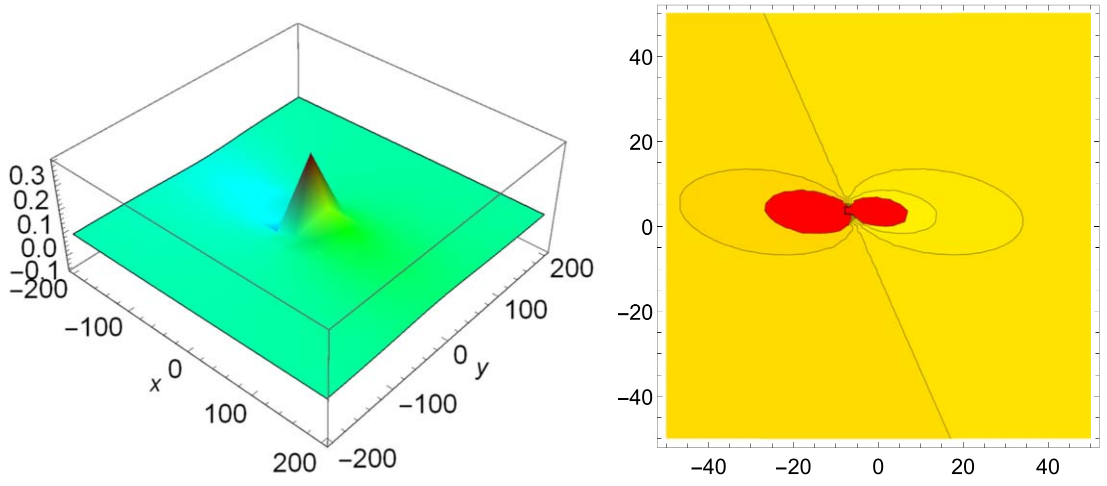
Case 2: When we have

$$\begin{aligned} &a_1 = b_1,\ a_3 = \frac{(b_2^2 - a_2^2 - 2a_2b_2)\gamma 3}{2b_1} - b_1\gamma_1 - a_2\gamma_2,\\ &b_3 = \frac{(a_2^2 - 2a_2b_2 - b_2^2)\gamma_3}{2b_1} - b_1\gamma_1 - b_2\gamma_2,\ c_1 = -\frac{12b_1^3(2b_1\alpha + (a_2 + b_2)\beta)}{(a_2 - b_2)^2\gamma_3} \end{aligned} \tag{5.4}$$

and plug them into equation (5.1), then into equation (1.3), one can obtain

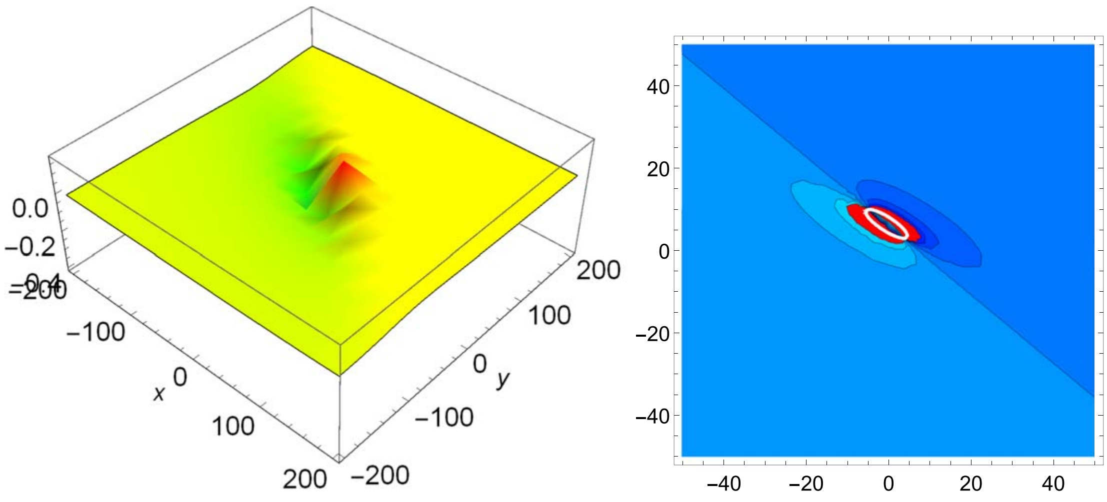
$$u = \frac{4b_1(2b_1x + (a_2 + b_2)y + (a_3 + b_3)t + a_4 + b_4)}{(a_4 + a_3t + b_1x + a_2y)^2 + (b_4 + b_3t + b_1x + b_2y)^2 - \frac{12b_1^3(2b_1\alpha + (a_2 + b_2)\beta)}{(a_2 - b_2)^2\gamma_3}}. \tag{5.5}$$

equation (5.5) is a lump solution as shown in figure 9.



(a) Lump solution (b) Corresponding contour plot of lump solution

Figure 8. 3D surface and contour plot of equation (5.3) plotted when $t = 2, a_1 = 0.01, a_2 = 1, a_3 = 0.1, a_4 = 0.5, b_1 = 0.5, b_2 = 0.2, b_4 = 0.2, c_1 = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1, \alpha = 0.5, \beta = 1$.



(a) Lump solution (b) Corresponding contour plot of lump solution

Figure 9. 3D surface and contour plot of equation (5.5) plotted when $t = 2, a_1 = 0.01, a_2 = 1, a_3 = 0.1, a_4 = 0.5, b_1 = 0.5, b_2 = 0.2, b_4 = 0.2, c_1 = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1, \alpha = 0.5, \beta = 1$.

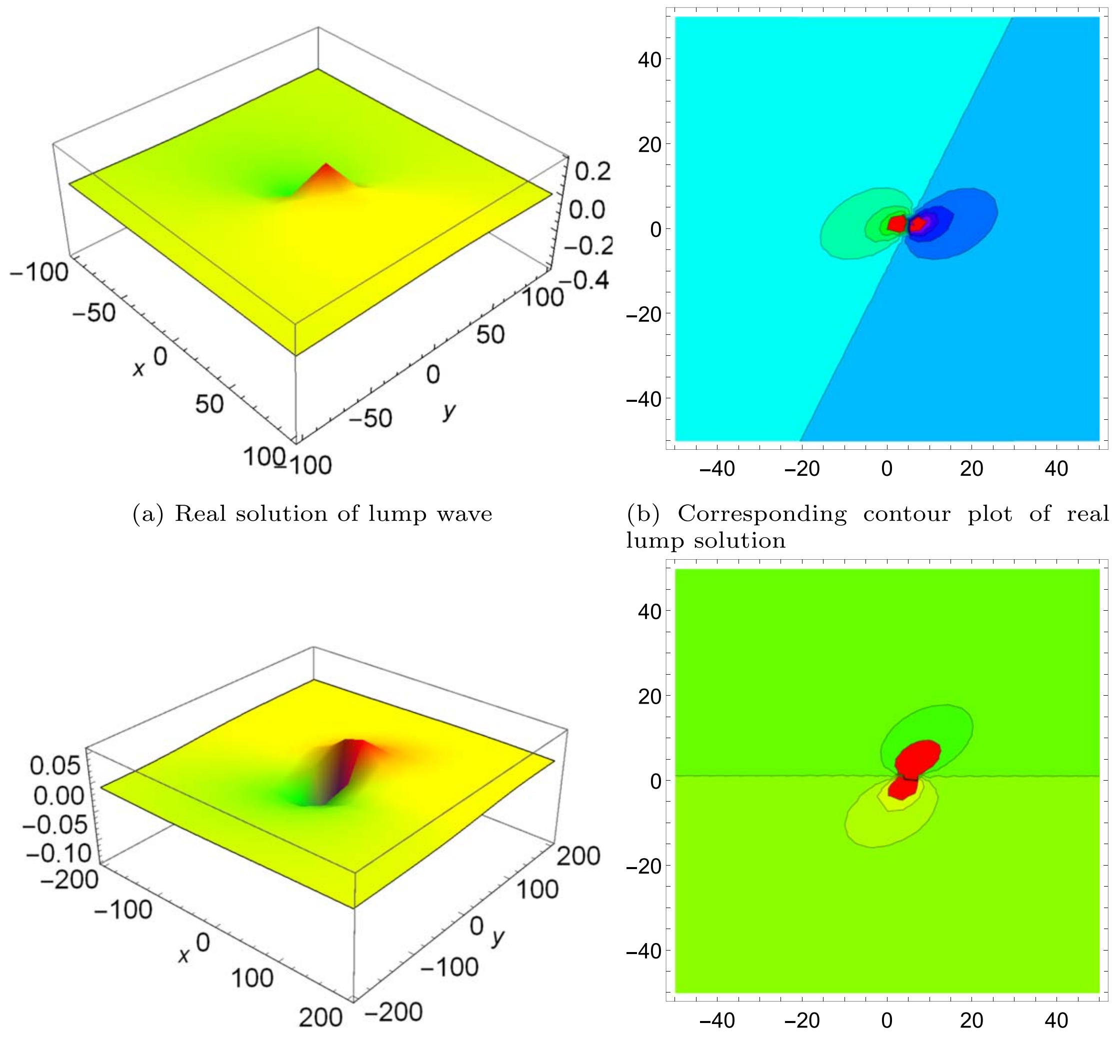
Case 3: In case

$$a_2 = -\frac{a_1\gamma_2 + i\sqrt{a_1(4a_3\gamma_3 + 4a_1\gamma_1\gamma_3 - a_1{\gamma_2}^2)}}{2\gamma_3},\ b_1 = \mathrm{i}a_1,\ b_3 = \mathrm{i}a_3,$$
$$b_2 = \frac{\sqrt{a_1(4a_3\gamma_3 + 4a_1\gamma_1\gamma_3 - a_1{\gamma_2}^2)} - \mathrm{i}a_1\gamma_2}{2\gamma_3}, \tag{5.6}$$

and using them into equation (5.1), then into equation (1.3), as a result equation (1.3) has the solution

$$u = \frac{4a_1(a_4 + \mathrm{i}b_4)\gamma_3}{({a_4}^2 + {b_4}^2 + c_1 + (2a_3a_4 + 2\mathrm{i}a_3b_4)t)\gamma_3 - a_1(a_4 + \mathrm{i}b_4)(\gamma_2 y - 2\gamma_3 x) + (b_4 - \mathrm{i}a_4)\sqrt{\phi}\, y}, \tag{5.7}$$

where $\phi = a_1(4a_3\gamma_3 + 4a_1\gamma_1\gamma_3 - a_1{\gamma_2}^2) > 0$. The above equation is a complex single-soliton solution to a gBK equation as seen in figure 10.



(a) Real solution of lump wave
(b) Corresponding contour plot of real lump solution
(c) Imaginary solution of lump wave
(d) Corresponding contour plot of imaginary lump solution

Figure 10. 3D surface and contour plot of equation (5.7) plotted when $t = 2, a_1 = -0.3, a_2 = 1, a_3 = 0.1, a_4 = 0.5, b_1 = 0.5, b_2 = 0.2, b_4 = 0.2, c_1 = 1, \gamma_1 = 2, \gamma_2 = 1, \gamma_3 = 1, \alpha = 1, \beta = 1$.

6. Interactions between single-lump wave and solitary wave

To study the lump waves and their interactions with the solitary waves, we let the solution of equation (1.2), according to equation (1.3), have the form

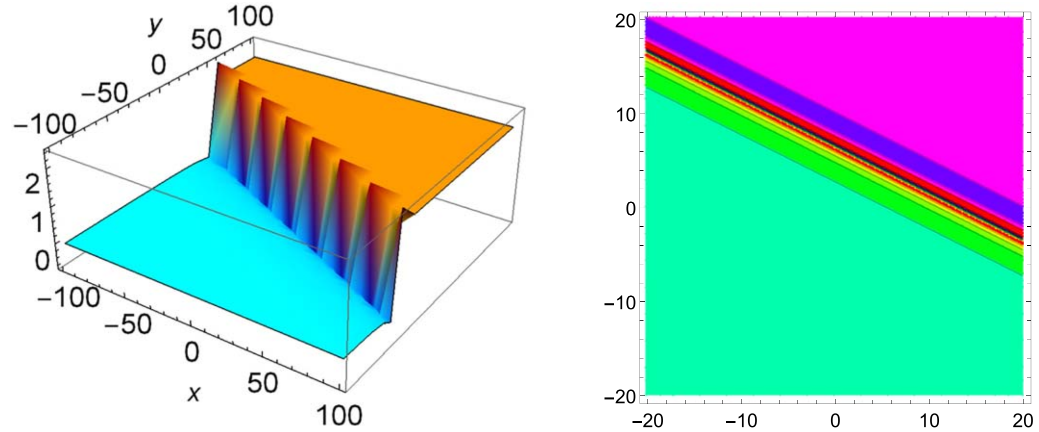
$$f(x, y, t) = e^{c_1x+c_2y+c_3t} + (a_1x + a_2y + a_3t + a_4)^2 + (b_1x + b_2y + b_3t + b_4)^2 + d_1. \tag{6.1}$$

Plugging equation (6.1) with equation (1.3) into equation (1.2), and making all the coefficients of dissimilar powers of t, x, y, exp and their product equal to zero, we get a system of polynomial equations. By solving the obtained system, we obtain the following cases of solutions:

Case 1: When

$$a_2 = \frac{a_1b_2}{b_1},\ a_3 = -\frac{a_1(b_1{}^2\gamma_1 + b_1b_2\gamma_2 + b_2{}^2\gamma_3)}{b_1{}^2},\ b_3 = -\frac{b_1{}^2\gamma 1 + b_1b_2\gamma_2 + b_2{}^2\gamma_3}{b_1},$$
$$c_2 = \frac{b_2c_1}{b1},\ c_3 = -\frac{c_1(b_1{}^2\gamma_1 + b_1b_2\gamma_2 + b_2{}^2\gamma_3)}{b_1{}^2},\ \alpha = -\frac{b_2\beta}{b_1} \tag{6.2}$$

and we use this case with equation (1.3), a collision physical phenomenon between the 1-soliton wave and the 1-lump solution (see figure 11) will equivalently become



(a) Interaction between single-lump wave and solitary waves (b) Corresponding contour plot of interaction between single-lump wave and solitary waves

Figure 11. 3D surface and contour plot of equation (6.3) plotted when $t = 2, a_1 = 2, a_4 = 1, b_1 = 1, b_2 = 2, b_4 = 2, c_1 = 1, d_1 = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1, \beta = 2$.

$$u = \frac{2\left(c_1 e^{c_1 x + \frac{b_2 c_1}{b_1} y + c_3 t} + 2b_1(b_4 + b_1(x - \gamma_1 t) + b_2(y - \gamma_2 t)) - 2{b_2}^2\gamma_3 t + 2a_1\left(a_4 + a_1 x + \frac{a_1 b_2 y}{b_1} - h(t)\right)\right)}{d_1 + e^{c_1 x + \frac{b_2 c_1}{b_1} y + c_3 t} + \left(a_4 + a_1 x + \frac{a_1 b_2 y}{b_1} - h(t)\right)^2 + (b_4 + b_1 x + b_2 y - h(t))^2}, \tag{6.3}$$

where $h(t) = \frac{a_1({b_1}^2\gamma_1 + b_1 b_2 \gamma_2 + {b_2}^2\gamma_3)t}{{b_1}^2}$.

Case 2: When

$$a_2 = \frac{a_1 b_2}{b_1},\ a_3 = -\frac{a_1({b_1}^2\gamma_1 + b_1 b_2\gamma_2 + {b_2}^2\gamma_3)}{{b_1}^2},$$
$$b_3 = -\frac{{b_1}^2\gamma_1 + b_1 b_2\gamma_2 + {b_2}^2\gamma_3}{b_1},\ b_4 = \frac{a_4 b_1}{a_1},\ c_2 = \frac{b_2 c_1}{b_1},$$
$$c_3 = -\frac{c_1({b_1}^2\gamma_1 + b_1 b_2\gamma_2 + {b_2}^2\gamma_3)}{{b_1}^2},\ \beta = -\frac{b_1\alpha}{b_2} \tag{6.4}$$

and input the case of the solution with equation (6.1) into equation (1.3), we get

$$u = \frac{2\left(c_{1e}^{c_1 x + \frac{b_2 c_1}{b_1} y + c_3 t} + 2a_1\left(a_4 + a_1 x + \frac{a_1 b_2 y}{b_1} - f(t)\right) + 2b_1\left(\frac{a_4 b_1}{a_1} + b_1 x + b_2 y - f(t)\right)\right)}{d_1 + e^{c_1 x + \frac{b_2 c_1}{b_1} y + c_3 t} + \left({}_4 + a_1 x + \frac{a_1 b_2 y}{b_1} - f(t)\right)^2 + \left(\frac{a_4 b_1}{a_1} + b_1 x + b_2 y - f(t)\right)^2}. \tag{6.5}$$

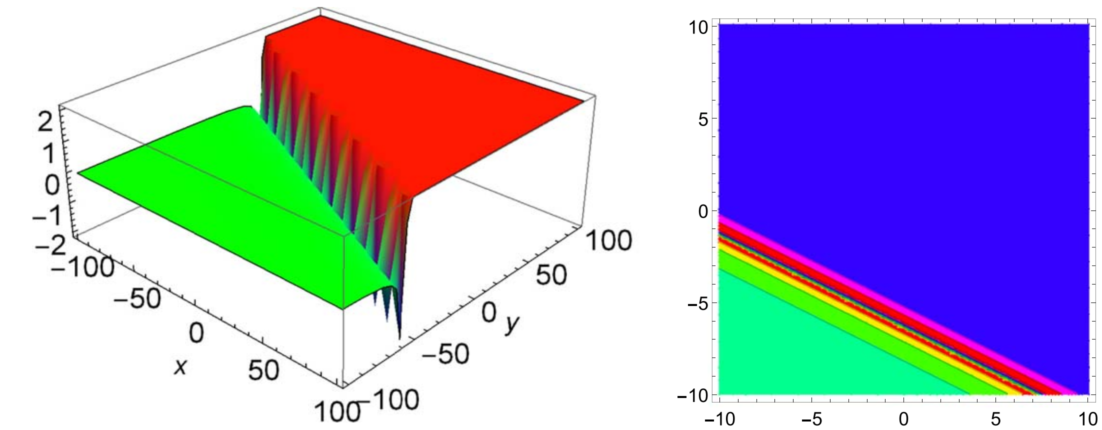
equation (6.5) describes an interaction between the 1-lump solution and the 1-soliton wave as shown in figure 12.

7. Periodic-lump wave solutions

In this portion, we attempt to reveal periodic lump solutions to a gBK equation. Let

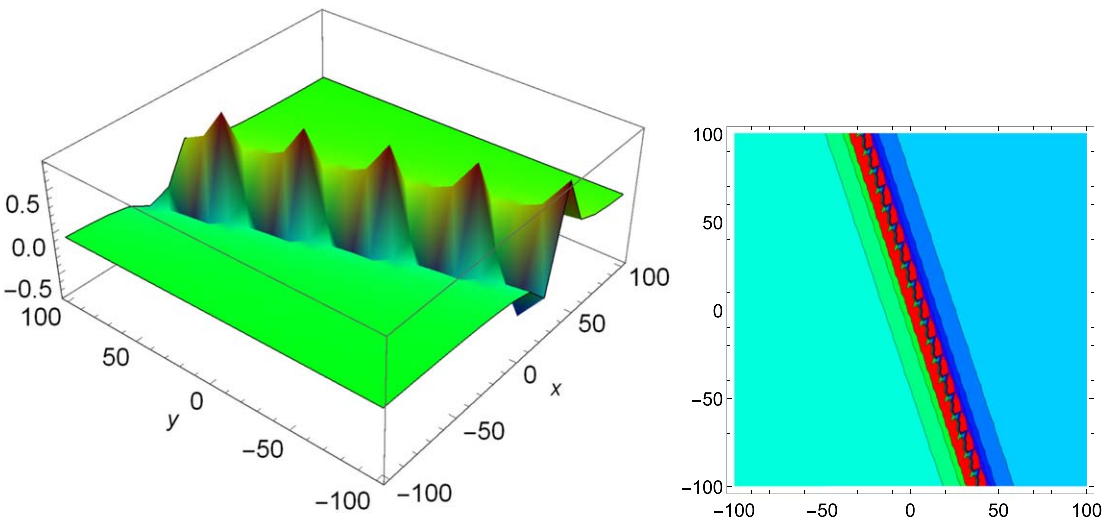
$$f(x, y, t) = d_1 + (a_1 x + a_2 y + a_3 t + a_4)^2 + (b_4 + b_3 t + b_1 x + b_2 y)^2$$
$$+ \sin(c_1 x + c_2 y + c_3 t), \tag{7.1}$$

and insert the trial assumption value of the $f(x, y, t)$ with equation (1.3) into equation (1.2), and make all the coefficients of dissimilar powers of t, x, y, sin,cos and their product equal to zero, we get a system of polynomial equations. Solving the obtained equations, we can evaluate the unknown value of parameters $a_i, b_i, c_i, d_1; (i = 1, 2, 3, 4)$ as follows



(a) Interaction between single-lump wave and solitary waves (b) Corresponding contour plot of interaction between single-lump wave and solitary waves

Figure 12. 3D surface and contour plot of equation (6.3) plotted when $t = 2, a_1 = 2, a_4 = 1, b_1 = 1, b_2 = 2, b_4 = 2, c_1 = 1, d_1 = 1, \gamma_1 = 2, \gamma_2 = 1, \gamma_3 = -1, \beta = 2$.



(a) Periodic-lump wave solution (b) Corresponding contour plot of periodic-lump wave solution

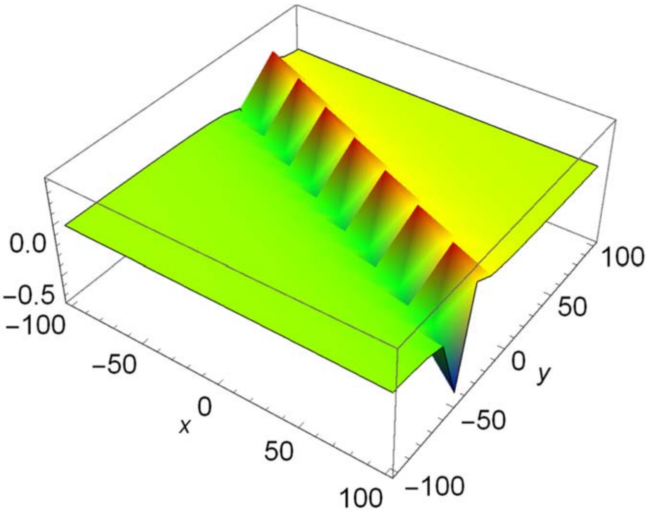
Figure 13. 3D surface and contour plot of equation (7.3) plotted when $t = 2, a_1 = 3, a_2 = 1, a_4 = 1, b_1 = 1, b_2 = 2, b_4 = 4, c_1 = 1, d_1 = 1, \gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 1$.

$$a_3 = -\frac{a_1^2\gamma_1 + a_1 a_2\gamma_2 + a_2^2\gamma_3}{a_1},$$
$$b_2 = \frac{a_2 b_1}{a_1},\ b_3 = -\frac{b_1(a_1^2\gamma_1 + a_1 a_2\gamma_2 + a_2^2\gamma_3)}{a_1^2},$$
$$c_2 = \frac{a_2 c_1}{a_1},\ c_3 = -\frac{c_1(a_1^2\gamma_1 + a_1 a_2\gamma_2 + a_2^2\gamma_3)}{a_1^2},\ \alpha = -\frac{a_2\beta}{a_1}. \tag{7.2}$$

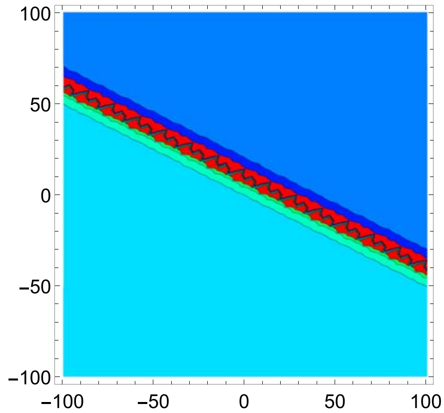
Plugging equation (7.2) with equation (7.1) into equation (1.3), we get

$$u = \frac{2\left(2a_1(a_4 + a_1 x + a_2 y - g(t)) + 2b_1\left(b_4 + b_1 x + \frac{a_2 b_1 y}{a_1} - b_1 g(t)\right) + c_1\cos\left(c_1\left(x + \frac{a_2 y}{a_1} - g(t)\right)\right)\right)}{d_1 + (a_4 + a_1 x + a_2 y - g(t))^2 + \left(b_4 + b_1 x + \frac{a_2 b_1 y}{a_1} - b_1 g(t)\right)^2 + \sin\left(c_1\left(x + \frac{a_2 y}{a_1} - g(t)\right)\right)}, \tag{7.3}$$

where $g(t) = \frac{(a_1^2\gamma_1 + a_1 a_2\gamma_2 + a_2^2\gamma_3)}{a_1^2}t$. Equation (7.3) is a periodic wave solution as seen in figure 13.



(a) Periodic-lump wave solution



(b) Corresponding contour plot of periodic-lump wave solution

Figure 14. 3D surface and contour plot of equation (7.6) plotted when $t = 2, a_1 = 3, a_2 = 1, a_4 = 1, b_1 = 1, b_2 = 2, b_4 = 4, c_1 = 1, d_1 = 1, \gamma_1 = 2, \gamma_2 = 2, \gamma_3 = 1$.

If the values of $a_i,\ b_i,\ c_i,\ d_1;\ (i = 1,\ 2,\ 3,\ 4)$ are as follows

$$a_2 = \frac{a_1 b_2}{b_1},\ a_3 = -\frac{a_1(b_1^2\gamma_1 + b_1 b_2\gamma_2 + b_2^2\gamma_3)}{b_1^2},$$
$$b_3 = -\frac{b_1^2\gamma_1 + b_1 b_2\gamma_2 + b_2^2\gamma_3}{b_1},$$
$$c_2 = \frac{b_2 c_1}{b_1},\ c_3 = -\frac{c_1(b_1^2\gamma_1 + b_1 b_2\gamma_2 + b_2^2\gamma 3)}{b_1^2},\ \beta = -\frac{b_1\alpha}{b_2} \tag{7.4}$$

and use the case of the solution with equation (7.3) into equation (1.3), we have

$$u = \frac{2\left(2a_1\left(a_1 x + \frac{a_1 b_2 y}{b_1} - a_1 h(t) + a_4\right) + 2b_1\left(b_1 x + b_2 y - \frac{b_1}{a_1}h(t) + b_4\right) + c_1\cos\left(c_1\left(x + \frac{b_2 y}{b_1} - \frac{1}{a_1}h(t)\right)\right)\right)}{d_1 + \left(a_1 x + \frac{a_1 b_2 y}{b_1} - a_1 h(t) + a_4\right)^2 + \left(b_1 x + b_2 y - \frac{b_1}{a_1}h(t) + b_4\right)^2 + \sin\left(c_1\left(x + \frac{b_2 y}{b_1} - \frac{1}{a_1}h(t)\right)\right)} \tag{7.5}$$

This is a periodic wave solution as shown in figure 14.

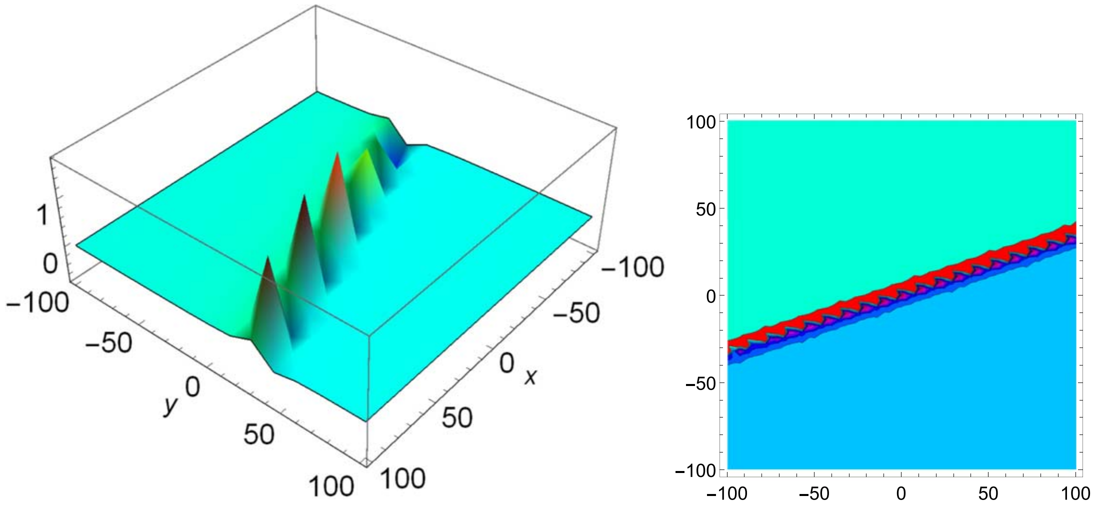
When the values of $a_i,\ b_i,\ c_i,\ d_1;\ (i = 1,\ 2,\ 3,\ 4)$ are as follows

$$a_2 = -\frac{a_1(b_1\gamma_2 + \sqrt{b_1^2(\gamma_2^2 - 4\gamma_1\gamma_3)})}{2b_1\gamma_3},\ a_3 = 0,$$
$$b_2 = -\frac{b_1\gamma_2 + \sqrt{b_1^2(\gamma_2^2 - 4\gamma_1\gamma_3)}}{2\gamma_3},\ b_3 = 0,$$
$$c_2 = -\frac{c_1(b_1\gamma_2 + \sqrt{b_1^2(\gamma_2^2 - 4\gamma_1\gamma_3)})}{2b_1\gamma_3},\ c_3 = 0,$$
$$\beta = \frac{b_1\gamma_2\alpha - \alpha\sqrt{b_1^2(\gamma_2^2 - 4\gamma_1\gamma_3)}}{2b_1\gamma_1}, \tag{7.6}$$

and insert the case of the solution with equation (7.1) into equation (1.3), we get

$$u = \frac{2\left(2b_1(b_4 + b_1 x - k(y)) + 2a_1\left(a_4 + a_1 x - \frac{a_1}{b_1}k(y)\right) + c_1\cos\left(c_1 x - \frac{c_1}{b_1}k(y)\right)\right)}{d_1 + (b_4 + b_1 x - k(y))^2 + \left(a_4 + a_1 x - \frac{a_1}{b_1}k(y)\right)^2 + \sin\left(c_1 x - \frac{c_1}{b_1}k(y)\right)}, \tag{7.7}$$

where $k(y) = \frac{b_1\gamma_2 + \sqrt{b_1^2(\gamma_2^2 - 4\gamma_1\gamma_3)}}{2\gamma_3}y$. Equation (7.7) represents a periodic wave solution as shown in figure 15.



(a) Periodic -lump wave solution

(b) Corresponding contour plot of periodic-lump wave solution

Figure 15. 3D surface and contour plot of equation (7.7) plotted when $t = 2, a_1 = 2, a_2 = 1, a_4 = 1, b_1 = 1, b_2 = 2, b_4 = 4, c_1 = 1, d_1 = 1, \gamma_1 = 0.1, \gamma_2 = 3, \gamma_3 = 1$.

8. Conclusions

The dynamical features of a generalized Bogoyavlensky-Konopelchenko equation in (2+1)-dimensions were explored, based on its Hirota bilinear form. N-soliton waves, M-lump solutions, lump waves, periodic lump waves, mixed kink-lump solitons, and interactions between lump solutions and 1-soliton waves were particularly presented. The long-wave limiting process was performed to construct M-lump solutions and symbolic computations were utilized to generate positive quadratic solutions and their interaction solutions with kink waves and solitary waves. Abundant 3d-plots and contour plots at different times were made and analyzed to explore dynamical and physical phenomena of the presented solutions.

Conflict of interest

The authors declare that they have no conflict of interest.

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