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The Riemann-Hilbert Approach to Initial-Boundary Value Problems for Integrable Coherently Coupled Nonlinear Schrödinger Systems on the Half-Line

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Abstract. An integrable coherently coupled nonlinear Schrödinger system describing the propagation of polarised optical waves in an isotropic medium with a generalized 4×4 matrix Ablowitz-Kaup-Newell-Segur-type Lax pair is studied. The corresponding initial-boundary value problem is reduced to a matrix Riemann-Hilbert problem in the complex plane. Moreover, it is shown that the associated spectral functions depend on each other and satisfy a global relationship.

AMS subject classifications: 35G31, 35Q51, 35Q15

Key words: Riemann-Hilbert problem, coherently coupled nonlinear Schrödinger system, initial-boundary value problem, unified transform method.

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$iq_t \pm q_{xx} + |q|^2 q = 0,$$

arises in plasma physics, solid-state physics, nonlinear optics and water waves. It describes the propagation of optical solitons in mono-mode fibers for scalar fields and the dependance of such solitons on the group velocity dispersion (GVD) and the self-phase modulation (SPM) [11]. Since nonlinear phase change comes from the cross-phase modulation

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(XPM) in birefringent or multi-mode fibers, the interaction of several field components at different frequencies or polarisations has to be taken into account. The dynamic features of such solitons are usually governed by coupled nonlinear Schrödinger (CNLS) systems [11]. Multicomponent solitons (MSs) are intriguing nonlinear objects where soliton is split into a number of components. These solitons are called the vector or the multicolour solitons.

Let us note that in optical fibers there are two types of vector solitons — viz. coherently and incoherently coupled vector solitons [11]. For incoherently coupled vector solitons the coupling is phase insensitive and special incoherently CNLS system —- the Manakov system, has the following form

$$\begin{aligned} &iu_{\zeta} \pm \frac{1}{2}u_{\tau\tau} + (|u|^2 + |v|^2)u = 0, \\ &iv_{\zeta} \pm \frac{1}{2}v_{\tau\tau} + (|u|^2 + |v|^2)v = 0, \end{aligned} \tag{1.1}$$

where ζ and τ, respectively, refer to the normalised spatial and temporal coordinates, the sign + or − corresponds to the anomalous dispersion (bright soliton) or normal dispersion (dark soliton) regime. Besides, $|u|^2u$ and $|v|^2v$ denote SPM effects, whereas the XPM effects $|u|^2v$ and $|v|^2u$ serve as incoherent coupling terms [15].

The initial-boundary value (IBV) problems for the system (1.1) on the half-line have been recently studied by using the Fokas method [10, 36]. This method can be also employed to consider the IBV problems for linear and nonlinear integrable evolution PDEs with 2×2 Lax pairs [6–8, 20, 21, 35, 42]. Similar to IST on a line, the Fokas approach allows to express the solutions of IBV problems via solutions of Riemann-Hilbert (RH) problems. Lenells [22] extended the Fokas approach to IBV problems for integrable nonlinear evolution equations with 3×3 Lax pairs. This stimulated the study of IBV problems with the Lax pairs of higher-order such as, the Degasperis-Procesi equation [23], the Ostrovsky-Vakhnenko equation [24], the Sasa-Satsuma equation [37], the three wave equation [38], the spin-1 Gross-Pitaevskii equation [40] and others [19,25,30]. Integrable equations with 2×2 or 3×3 Lax pairs have been also studied [12–14,42]. In particular, Deift and Zhou [5] investigated the asymptotic of the solutions by applying the steepest descent method to a RH problem.

There are also vector solitons associated with coherent CNLS systems. They can be used as the carriers of the switched information in optical fields [11]. The coupling effects depend on relative phases of the interacting fields, and coherent interactions usually occur when the nonlinear medium is weakly anisotropic or low birefringent [11]. Park and Shin [28] proposed new integrable CNLS equations — viz.

$$\begin{aligned} &iu_t + u_{xx} + 2(|u|^2 + 2|v|^2)u - 2u^*v^2 = 0, \\ &iv_t + v_{xx} + 2(2|u|^2 + |v|^2)v - 2v^*u^2 = 0, \end{aligned} \tag{1.2}$$

where u and v denote slowly varying envelopes of two interacting optical modes, x and t are, respectively, the normalised distance and time, and $*$ means the complex conjugation. Zhang *et al.* [41] used the Ablowitz-Kaup-Newell-Segur (AKNS) technology [1] to establish other new integrable CNLS equations

$$\begin{aligned} &iu_t + u_{xx} + 2(|u|^2 - 2|v|^2)u - 2u^*v^2 = 0, \\ &iv_t + v_{xx} + 2(2|u|^2 - |v|^2)v + 2v^*u^2 = 0. \end{aligned} \tag{1.3}$$

The systems (1.2) and (1.3) are called the coherently coupled NLS (CCNLS) systems. The different coefficients of the SPM and XPM are the incoherent coupling parameters. The terms $-2u^*v^2$ and $2v^*u^2$ in the system (1.3) describe the coherent coupling and govern the energy exchange between two axes of the fiber [11,41]. Let us note that the systems (1.2) and (1.3) can be simultaneously derived by using the AKNS technology — cf. Section 2.

The integrability of the CCNLS systems (1.2) and (1.3) have been studied from various points of view, including works focused on Lax pairs and Painlevé property [28,33], conservation laws [39,41], solitons obtained by the bilinear method [16–18,27–29,33], the Darboux transformation (DT) [9], classical Jacobi elliptic functions [3] and other methods [2]. The addition, propagation and collision of solitons are analysed [18,33], the existence of multi-speed solitary wave solutions for CCNLS systems (1.2), (1.3) is established in [34] and the dynamics of non-linear waves is considered in [26]. However, to the best of our knowledge, the IBV problems for the CCNLS systems (1.2), (1.3) have not been analysed. In this work we want to look at the IBV problems for (1.3) on the half-line via a unified approach. Note that the IBV problems for the system (1.2) can be also constructed.

Let $\Omega := \{0 < x < \infty, 0 < t < T\}$ denote the half-line domain — cf. Fig. 1. We consider the following IBV problems for (1.3):

$$\begin{aligned} &\text{Initial values:} && u_0(x) = u(x,0), \quad v_0(x) = v(x,0), \\ &\text{Dirichlet boundary values:} && p_0(t) = u(0,t), \quad q_0(t) = v(0,t), \\ &\text{Neumann boundary values:} && p_1(t) = u_x(0,t), \quad q_1(t) = v_x(0,t). \end{aligned} \tag{1.4}$$

Here we assume that $u_0(x)$ and $v_0(x)$ belong to the Schwartz space.

The outline of this paper is as follows. In Section 2, we recall the Lax pairs for systems (1.2) and (1.3). In Section 3, two sets of eigenfunctions $\{\mu_j\}_1^3$ and $\{M_n\}_1^4$ of the Lax pair

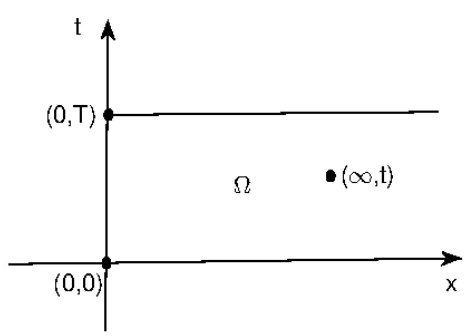

Figure 1: The region Ω in the (x,t) plan.

are used to determine spectral functions satisfying the so-called global relationship. In Section 4, we show that $\{u(x,t), v(x,t)\}$ can be expressed in terms of the unique solution of a 4×4 matrix RH problem. Our conclusions are in Section 5.

2. The Lax Pairs for Systems (1.2) and (1.3)

In order to obtain systems (1.2) and (1.3), we consider the linear eigenvalue problem

$$\psi_x = F\psi = (-i\lambda\Lambda + F_0)\psi,$$
$$\psi_t = G\psi = (-2i\lambda^2\Lambda + 2\lambda F_0 + iG_0)\psi,$$

where

$$\begin{aligned} \Lambda &= \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & R \\ -R^\dagger & 0 \end{pmatrix}, \\ G_0 &= \begin{pmatrix} RR^\dagger & R_x \\ R_x^\dagger & -R^\dagger R \end{pmatrix}, \quad R = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \end{aligned} \tag{2.1}$$

and $I_{2\times 2}$ is the 2×2 unit matrix [41]. The compatibility condition $F_x - G_t + FG - GF = 0$ leads to the following matrix NLS equation — cf. Refs. [4, 31, 32, 43]:

$$iR_t + R_{xx} + 2RR^\dagger R = 0. \tag{2.2}$$

The system (1.2) has been obtained in [2, 28] by substitution of the term R of (2.1) into the Eq. (2.2). If "$\dagger$" denotes the complex conjugation, then (2.2) yields (1.3).

3. Spectral Analysis

The Lax pair for the coherently coupled nonlinear Schrödinger (CCNLS) system (1.3) can be written as

$$\begin{aligned} &\psi_x + i\lambda\Lambda\psi = P(x,t)\psi, \\ &\psi_t + 2i\lambda^2\Lambda\psi = Q(x,t,\lambda)\psi, \end{aligned} \tag{3.1}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, Λ is 4×4 matrix, $\psi = \psi(x,t,\lambda)$ is 4×4 or 4×1 matrix-valued spectral function, F_0, G_0 are defined by (2.1) and

$$P(x,t) = F_0(x,t), \quad Q(x,t,\lambda) = 2\lambda F_0 + iG_0.$$

3.1. The closed one-form

Introducing a new eigenfunction $\mu(x,t,\lambda)$ by

$$\psi(x,t,\lambda) = \mu(x,t,\lambda)e^{-(i\lambda\Lambda x + 2i\lambda^2\Lambda t)},$$

we rewrite the Lax pair Eq. (3.1) as

$$\begin{aligned} &\mu_x + i\lambda[\Lambda,\mu] = P(x,t)\mu, \\ &\mu_t + 2i\lambda^2[\Lambda,\mu] = Q(x,t,\lambda)\mu, \end{aligned} \tag{3.2}$$

and, consequently,

$$d(e^{i\lambda\hat{\Lambda}x+2i\lambda^2\hat{\Lambda}t}\mu(x,t,\lambda))=W(x,t,\lambda),$$

where

$$W(x,t,\lambda)=e^{(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}(P(x,t)dx+Q(x,t,\lambda)dt)\mu. \tag{3.3}$$

By $\hat{\sigma}_4$ we denote the matrix operator acting on the space of 4×4 matrices X so that

$$\hat{\sigma}_4 X=[\sigma_4,X] \quad \text{and} \quad e^{x\hat{\sigma}_4}X=e^{x\sigma_4}Xe^{-x\sigma_4}.$$

3.2. Eigenfunctions μ_j

Assume that $u(x,t)$ and $v(x,t)$ are sufficiently smooth functions in the domain $\Omega=\{0<x<\infty,0<t<T\}$, which decay sufficiently fast as x tends to ∞, and let $\{\gamma_j\}_1^3$ be smooth curves connecting the points (x_j,t_j) and (x,t), where $(x_1,t_1)=(0,T)$, $(x_2,t_2)=(0,0)$, $(x_3,t_3)=(\infty,t)$ — cf. Fig. 2.

We consider 4×4 matrix functions $\{\mu_j(x,t,\lambda)\}_1^3$ defined by

$$\mu_j(x,t,\lambda)=I+\int_{\gamma_j}e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}W_j(\xi,\tau,\lambda), \quad j=1,2,3, \tag{3.4}$$

where I is the 4×4 identity matrix and W_j are of the form (3.3) with μ replaced by μ_j.

If $(\xi,\tau)\in\gamma_j$, $j=1,2,3$, then the following inequalities hold:

$$\begin{aligned}\gamma_1:\ & x-\xi\geq 0, \quad t-\tau\leq 0,\\ \gamma_2:\ & x-\xi\geq 0, \quad t-\tau\geq 0,\\ \gamma_3:\ & x-\xi\leq 0, \quad t-\tau=0.\end{aligned} \tag{3.5}$$

Since the one-form functions W_j are closed, then μ_j is independent of the path of integration. Therefore, integrating over the lines parallel to the axes x and t, we rewrite

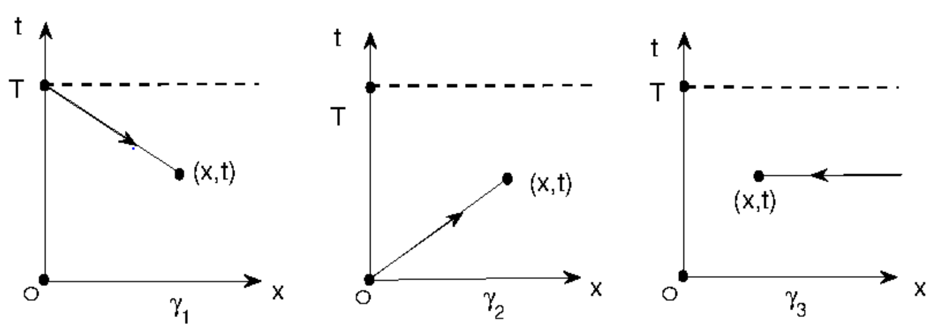

Figure 2: The three contours $\gamma_1,\gamma_2,\gamma_3$ in the (x,t)-domain.

the Eqs. (3.4) as

$$\mu_j(x,t,\lambda)=I+\int_{x_j}^{x}e^{-i\lambda(x-\xi)\hat{\Lambda}}(P\mu_j)(\xi,t,\lambda)d\xi$$
$$+e^{-i\lambda(x-x_j)\hat{\Lambda}}\int_{t_j}^{t}e^{-2i\lambda^2(t-\tau)\hat{\Lambda}}(Q\mu_j)(x_j,\tau,\lambda)d\tau,\quad j=1,2,3. \tag{3.6}$$

Let $[\mu_j]_k$ denote the k-th column vector of μ_j. The Eq. (3.5) implies that the first, second, third and fourth columns of the matrix equations (3.4) contain the following exponential terms

$$\begin{aligned}
&[\mu_j]_1: \ e^{2i\lambda(x-\xi)+4i\lambda^2(t-\tau)} \quad \text{and} \quad e^{2i\lambda(x-\xi)+4i\lambda^2(t-\tau)},\\
&[\mu_j]_2: \ e^{2i\lambda(x-\xi)+4i\lambda^2(t-\tau)} \quad \text{and} \quad e^{2i\lambda(x-\xi)+4i\lambda^2(t-\tau)},\\
&[\mu_j]_3: \ e^{-2i\lambda(x-\xi)-4i\lambda^2(t-\tau)} \quad \text{and} \quad e^{-2i\lambda(x-\xi)-4i\lambda^2(t-\tau)},\\
&[\mu_j]_4: \ e^{-2i\lambda(x-\xi)-4i\lambda^2(t-\tau)} \quad \text{and} \quad e^{-2i\lambda(x-\xi)-4i\lambda^2(t-\tau)}.
\end{aligned}$$

In order to describe the domains where the eigenfunctions $\{\mu_j(x,t,\lambda)\}_1^3$ are bounded, we introduce the contours $\mathrm{Im}\lambda=0$ and $\mathrm{Im}\lambda^2=0$ — cf. Fig. 3, which split the complex λ-plane into four regions

$$\begin{aligned}
&D_1=\{\lambda\in\mathbb{C}|\arg\lambda\in(0,\pi/2)\}, \quad D_2=\{\lambda\in\mathbb{C}|\arg\lambda\in(\pi/2,\pi)\},\\
&D_3=\{\lambda\in\mathbb{C}|\arg\lambda\in(\pi,3\pi/2)\}, \quad D_4=\{\lambda\in\mathbb{C}|\arg\lambda\in(3\pi/2,2\pi)\}.
\end{aligned}$$

It is easily seen that the eigenfunctions $\mu_j(x,t,\lambda)\}_1^3$ are analytic and bounded in the following domains:

$$\begin{aligned}
&\mu_1 \quad \text{in} \quad (D_1,D_1,D_3,D_3),\\
&\mu_2 \quad \text{in} \quad (D_2,D_2,D_4,D_4),\\
&\mu_3 \quad \text{in} \quad (C_-,C_-,C_+,C_+),
\end{aligned}$$

where $C_+=D_1\cup D_2$ and $C_-=D_3\cup D_4$ are the upper and lower half-planes, respectively.

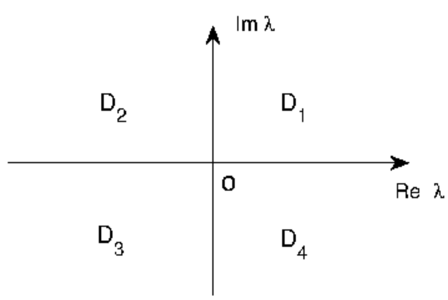

Figure 3: The sets $D_n, n=1,2,3,4$, which decompose the complex λ—plane.

The subsets D_n, $n=1,2,3,4$ have the following properties:

$$\begin{aligned}
D_1 &= \{\lambda\in\mathbb{C}|\mathrm{Re}a_1=\mathrm{Re}a_2>\mathrm{Re}a_3=\mathrm{Re}a_4, \quad \mathrm{Re}b_1=\mathrm{Re}b_2>\mathrm{Re}b_3=\mathrm{Re}b_4\},\\
D_2 &= \{\lambda\in\mathbb{C}|\mathrm{Re}a_1=\mathrm{Re}a_2>\mathrm{Re}a_3=\mathrm{Re}a_4, \quad \mathrm{Re}b_1=\mathrm{Re}b_2<\mathrm{Re}b_3=\mathrm{Re}b_4\},\\
D_3 &= \{\lambda\in\mathbb{C}|\mathrm{Re}a_1=\mathrm{Re}a_2<\mathrm{Re}a_3=\mathrm{Re}a_4, \quad \mathrm{Re}b_1=\mathrm{Re}b_2>\mathrm{Re}b_3=\mathrm{Re}b_4\},\\
D_4 &= \{\lambda\in\mathbb{C}|\mathrm{Re}a_1=\mathrm{Re}a_2<\mathrm{Re}a_3=\mathrm{Re}a_4, \quad \mathrm{Re}b_1=\mathrm{Re}b_2<\mathrm{Re}b_3=\mathrm{Re}b_4\},
\end{aligned}$$

where $a_i(\lambda)$ and $b_i(\lambda)$ are the diagonal elements of the matrices $-i\lambda\Lambda$ and $-2i\lambda^2\Lambda$, respectively.

We note that $\mu_1(x,t,\lambda)$ and $\mu_2(x,t,\lambda)$ are entire functions of λ and in the regions of the boundedness, we have

$$\mu_j(x,t,\lambda)=I+\mathcal{O}(1/\lambda), \quad \lambda\to\infty, \quad j=1,2,3.$$

In fact, $\mu_1(0,t,\lambda)$ is bounded in the region $(D_1\cup D_3, D_1\cup D_3, D_2\cup D_4, D_2\cup D_4)$ and $\mu_2(0,t,\lambda)$ in the region $(D_2\cup D_4, D_2\cup D_4, D_1\cup D_3, D_1\cup D_3)$.

3.3. Matrix-valued functions M_n

For each $n=1,2,3,4$, the solution $M_n(x,t,\lambda)$ of the Eq. (3.2) satisfies the integral equation

$$(M_n(x,t,\lambda))_{ij}=\delta_{ij}+\int_{\gamma_{ij}^n}(e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}W_n(\xi,\tau,\lambda))_{ij}, \quad i,j=1,2,3,4, \tag{3.7}$$

where $W_n(x,t,\lambda)$ is obtained from (3.3) if μ is replaced by M_n and the contours $\gamma_{ij}^n(\lambda)$, $\lambda\in D_n$, $n,i,j=1,2,3,4$ are defined by

$$\gamma_{ij}^n=\begin{cases}\gamma_1, & \text{if} \quad \mathrm{Re}\,a_i(\lambda)<\mathrm{Re}\,a_j(\lambda), \quad \mathrm{Re}\,b_i(\lambda)\ge\mathrm{Re}\,b_j(\lambda),\\ \gamma_2, & \text{if} \quad \mathrm{Re}\,a_i(\lambda)<\mathrm{Re}\,a_j(\lambda), \quad \mathrm{Re}\,b_i(\lambda)<\mathrm{Re}\,b_j(\lambda),\\ \gamma_3, & \text{if} \quad \mathrm{Re}\,a_i(\lambda)\ge\mathrm{Re}\,a_j(\lambda).\end{cases}$$

Thus

$$\gamma^1=\begin{pmatrix}\gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_2&\gamma_2&\gamma_3&\gamma_3\\ \gamma_2&\gamma_2&\gamma_3&\gamma_3\end{pmatrix}, \quad \gamma^2=\begin{pmatrix}\gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_1&\gamma_1&\gamma_3&\gamma_3\\ \gamma_1&\gamma_1&\gamma_3&\gamma_3\end{pmatrix},$$

$$\gamma^3=\begin{pmatrix}\gamma_3&\gamma_3&\gamma_1&\gamma_1\\ \gamma_3&\gamma_3&\gamma_1&\gamma_1\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\end{pmatrix}, \quad \gamma^4=\begin{pmatrix}\gamma_3&\gamma_3&\gamma_2&\gamma_2\\ \gamma_3&\gamma_3&\gamma_2&\gamma_2\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\\ \gamma_3&\gamma_3&\gamma_3&\gamma_3\end{pmatrix}.$$

The following proposition shows that M_n can be represented as a Riemann-Hilbert problem.

Proposition 3.1. *For each $n=1,2,3,4$ and $\lambda \in D_n$, the function $M_n(x,t,\lambda)$ is well defined by the Eq.* (3.7)*. If (x,t) is fixed, then M_n is analytic and bounded in the domain D_n except for a possible discrete set of singularities $\{\lambda_j\}$, where the corresponding Fredholm determinant vanishes. Moreover, M_n admits continuous and bounded extension to the axis* $\operatorname{Re}\lambda$ *and*

$$M_n(x,t,\lambda) = I + \mathcal{O}(1/\lambda), \quad \lambda \to \infty.$$

Proof. The analyticity and boundedness of M_n have been established in Ref. [22]. Substituting the expansion

$$M = M_0 + \frac{M^{(1)}}{\lambda} + \frac{M^{(2)}}{\lambda^2} + \cdots, \quad \lambda \to \infty, \tag{3.8}$$

into the Lax pair (3.2) and equating the coefficients at λ^j leads to the relation (3.8). □

3.4. The jump matrices

New spectral functions $S_n(\lambda)$, $n=1,2,3,4$ are defined by

$$S_n(\lambda) = M_n(0,0,\lambda), \quad \lambda \in D_n, \quad n=1,2,3,4.$$

Let $M(x,t,\lambda)$ be a sectionally analytic continuous function on the Riemann λ-sphere coinciding with $M_n(x,t,\lambda)$ for $\lambda \in D_n$. Then $M(x,t,\lambda)$ satisfies the jump conditions

$$M_n(x,t,\lambda) = M_m(x,t,\lambda) J_{m,n}(x,t,\lambda), \quad \lambda \in \bar{D}_n \cap \bar{D}_m, \quad n,m = 1,2,3,4,\ n \neq m,$$

where

$$J_{m,n}(x,t,\lambda) = e^{-(i\lambda x + 2i\lambda^2 t)\hat{\Lambda}}[S_m^{-1}(\lambda) S_n(\lambda)].$$

3.5. Adjugated eigenfunctions

Let us now show that the minors of the matrices $\mu_j(x,t,\lambda)$, $j=1,2,3$ are also bounded and analytic. We recall that if $m_{ij}(B)$ denotes the ij-minor of B, then the cofactor matrix B^A of a 4×4 matrix B is

$$B^A = \begin{pmatrix} m_{11}(B) & -m_{12}(B) & m_{13}(B) & -m_{14}(B) \\ -m_{21}(B) & m_{22}(B) & -m_{23}(B) & m_{24}(B) \\ m_{31}(B) & -m_{32}(B) & m_{33}(B) & -m_{34}(B) \\ -m_{41}(B) & m_{42}(B) & -m_{43}(B) & m_{44}(B) \end{pmatrix},$$

and $(B^A)^T B = \operatorname{adj}(B) B = \det B$, where T denotes the transposition operation.

It follows from the Eq. (3.2) that the matrix-valued functions μ^A satisfies the Lax pair

$$\begin{aligned} &\mu_x^A - i\lambda[\Lambda, \mu^A] = -P(x,t)^T \mu^A, \\ &\mu_t^A - 2i\lambda^2[\Lambda, \mu^A] = -Q(x,t,\lambda)^T \mu^A. \end{aligned}$$

Then the eigenfunctions $\mu_j^A(x,t,\lambda),\ j=1,2,3$ can be written as

$$\mu_j^A(x,t,\lambda)=I-\int_{x_j}^{x} e^{i\lambda(x-\xi)\hat{\Lambda}}(P\mu_j^A)(\xi,t,\lambda)d\xi \\ -e^{i\lambda(x-x_j)\hat{\Lambda}}\int_{t_j}^{t} e^{2i\lambda^2(t-\tau)\hat{\Lambda}}(Q\mu_j^A)(x_j,\tau,\lambda)d\tau.$$

Hence, the adjugated eigenfunctions μ_j^A are analytic and bounded in the following domains:

$$\begin{aligned} &\mu_1^A \quad \text{in} \quad (D_4,D_4,D_2,D_2),\\ &\mu_2^A \quad \text{in} \quad (D_3,D_3,D_1,D_1),\\ &\mu_3^A \quad \text{in} \quad (C_+,C_+,C_-,C_-). \end{aligned}$$

In fact, $\mu_1^A(0,t,\lambda)$ is bounded in the region $(D_2\cup D_4,D_2\cup D_4,D_1\cup D_3,D_1\cup D_3)$ and $\mu_2^A(0,t,\lambda)$ in the region $(D_1\cup D_3,D_1\cup D_3,D_2\cup D_4,D_2\cup D_4)$.

3.6. The symmetry of eigenfunctions

Let us write the 4×4 matrix $X=(X_{ij})_{4\times 4}$ as

$$X=\begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix}, \tag{3.9}$$

where

$$\tilde{X}_{11}=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},\quad \tilde{X}_{12}=\begin{pmatrix} X_{13} & X_{14} \\ X_{23} & X_{24} \end{pmatrix},\\ \tilde{X}_{21}=\begin{pmatrix} X_{31} & X_{32} \\ X_{41} & X_{42} \end{pmatrix},\quad \tilde{X}_{22}=\begin{pmatrix} X_{33} & X_{34} \\ X_{43} & X_{44} \end{pmatrix}.$$

Set $Z_\pm=\operatorname{diag}(\pm1,\pm1,\mp1,\mp1)$. Noting that

$$F(x,t,\lambda)=-i\lambda\Lambda+P(x,t),\quad G(x,t,\lambda)=2i\lambda^2\Lambda+Q(x,t,\lambda),\\ Z_\pm\overline{(F(x,t,\bar{\lambda}))}Z_\pm=-F(x,t,\lambda)^T,\quad Z_\pm\overline{(G(x,t,\bar{\lambda}))}Z_\pm=-G(x,t,\lambda)^T,$$

and the functions $F(x,t,\lambda)$ and $G(x,t,\lambda)$ are symmetric, we obtain the symmetry of the eigenfunctions $\mu(x,t,\lambda)$ — viz.

$$(\tilde{\mu}(x,t,\lambda))_{11}=Z^{-1}\overline{(\tilde{\mu}(x,t,\bar{\lambda}))_{22}}Z^{-1},\quad (\tilde{\mu}(x,t,\lambda))_{12}=\overline{(\tilde{\mu}(x,t,\bar{\lambda}))_{21}}^{T},$$

with $Z^{-1}:=\operatorname{diag}(1,-1)$.

According to Ref. [8], the eigenfunctions $\psi(x,t,\lambda)$ and $\mu(x,t,\lambda)$ of the respective Lax pairs (3.1) and (3.2) satisfy the same symmetric relations

$$\psi^{-1}(x,t,\lambda)=Z_{\pm}\overline{(\psi(x,t,\bar{\lambda}))}^{T}Z_{\pm},\quad \mu^{-1}(x,t,\lambda)=Z_{\pm}\overline{(\mu(x,t,\bar{\lambda}))}Z_{\pm}.$$

Moreover, in the domain where $\mu(x,t,\lambda)$ is bounded, it satisfies the relation

$$\mu(x,t,\lambda)=I+\mathcal{O}(1/\lambda),\quad \lambda\to\infty,$$

so that $\det[\mu(x,t,\lambda)]=1$ since $tr(F(x,t,\lambda))=tr(G(x,t,\lambda)=0$.

3.7. Spectral functions and the jump matrix computation

Let us define 4×4 matrix spectral functions $s(\lambda), S(\lambda)$ and $S'(\lambda)$ by

$$\begin{aligned}\mu_3(x,t,\lambda)&=\mu_2(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}s(\lambda),\\ \mu_1(x,t,\lambda)&=\mu_2(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S(\lambda),\\ \mu_3(x,t,\lambda)&=\mu_1(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S'(\lambda).\end{aligned} \tag{3.10}$$

Taking into account the equation $\mu_2(0,0,\lambda)=I$, we can write

$$\begin{aligned}&s(\lambda)=\mu_3(0,0,\lambda),\quad S(\lambda)=\mu_1(0,0,\lambda)=e^{2i\lambda^2T\hat{\Lambda}}\mu_2^{-1}(0,T,\lambda),\\ &S'(\lambda)=\mu_1^{-1}(0,0,\lambda)\mu_3(0,0,\lambda)=S^{-1}(\lambda)s(\lambda)=e^{2i\lambda^2T\hat{\Lambda}}\mu_3^{-1}(0,T,\lambda),\end{aligned} \tag{3.11}$$

and display the connections of the functions μ_j with each other in Fig. 4. Note that the functions $s(\lambda), S(\lambda)$ and $S'(\lambda)$ depend on each other. Therefore, here we consider only two of them — e.g. $s(\lambda)$ and $S(\lambda)$. It follows from (3.6), (3.11) that

$$\begin{aligned}&s(\lambda)=I-\int_0^{\infty}e^{i\lambda\xi\hat{\Lambda}}(P\mu_3)(\xi,0,\lambda)d\xi,\\ &S(\lambda)=I-\int_0^{T}e^{2i\lambda^2\tau\hat{\Lambda}}(Q\mu_1)(0,\tau,\lambda)d\tau=\left[I+\int_0^{T}e^{2i\lambda^2\tau\hat{\Lambda}}(Q\mu_2)(0,\tau,\lambda)d\tau\right]^{-1},\end{aligned} \tag{3.12}$$

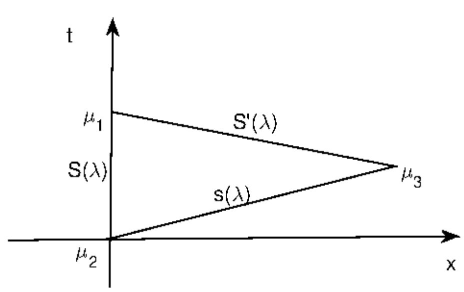

Figure 4: The relations among the dependent eigenfunctions $\mu_j(x,t,k), j=1,2,3$.

where $\mu_j(0,t,\lambda)$, $j=1,2$ and $\mu_3(x,t,\lambda)$, $0<x<\infty$, $0<t<T$ satisfy the Volterra integral equations

$$\begin{aligned}
\mu_1(0,t,\lambda) &= I - \int_t^T e^{-2i\lambda^2(t-\tau)\hat{\Lambda}}(Q\mu_1)(0,\tau,\lambda)d\tau, \\
&\qquad \lambda \in (D_1\cup D_3, D_1\cup D_3, D_2\cup D_4, D_2\cup D_4), \\
\mu_2(0,t,\lambda) &= I + \int_0^T e^{-2i\lambda^2(t-\tau)\hat{\Lambda}}(Q\mu_2)(0,\tau,\lambda)d\tau, \\
&\qquad \lambda \in (D_2\cup D_4, D_2\cup D_4, D_1\cup D_3, D_1\cup D_3), \\
\mu_3(x,0,\lambda) &= I - \int_x^\infty e^{-i\lambda(x-\xi)\hat{\Lambda}}(P\mu_3)(\xi,0,\lambda)d\xi, \quad \lambda \in (C_-, C_-, C_+, C_+).
\end{aligned} \tag{3.13}$$

Thus, the Eqs. (3.12), (3.13) show that $s(\lambda)$ and $S(\lambda)$ are determined by $U(x,0,\lambda)$ and $V(0,t,\mu)$ — i.e. by the initial $u_0(x), v_0(x)$ and boundary $p_0(t)$, $q_0(t)$, $p_1(t)$, $q_1(t)$ data. Indeed, the eigenfunctions $\mu_3(x,0,\lambda)$ and $\mu_j(0,t,\lambda)$, $j=1,2$ satisfy the x and t-parts of the Lax pair (3.2) at $t=0$ and $x=0$, respectively. Thus, for the x-part we have

$$\begin{aligned}
&\mu_x(x,0,\lambda) + i\lambda[\Lambda, \mu(x,0,\lambda)] = P(x,t=0)\mu(x,0,\lambda), \\
&\lim_{x\to\infty} \mu(x,0,\lambda) = I, \quad 0<x<\infty,
\end{aligned}$$

and for the t part

$$\begin{aligned}
&\mu_t(0,t,\lambda) + 2i\lambda^2[\Lambda, \mu(0,t,\lambda)] = Q(x=0,t)\mu(0,t,\lambda), \quad 0<t<T, \\
&\lim_{t\to 0} \mu(0,t,\lambda) = \mu(0,0,\lambda) = I, \quad \lim_{t\to T} \mu(0,t,\lambda) = \mu(0,T,\lambda) = I.
\end{aligned}$$

Besides, using the properties of $\{\mu_j\}_1^3$ and $\{\mu_j^A\}_1^3$, we note that $s(\lambda)$, $S(\lambda)$, $s^A(\lambda)$ and $S^A(\lambda)$ are bounded in the following regions:

$$\begin{aligned}
s(\lambda) &\quad \text{in} \quad (C_-, C_-, C_+, C_+), \\
S(\lambda) &\quad \text{in} \quad (D_1\cup D_3, D_1\cup D_3, D_2\cup D_4, D_2\cup D_4), \\
s^A(\lambda) &\quad \text{in} \quad (C_+, C_+, C_-, C_-), \\
S^A(\lambda) &\quad \text{in} \quad (D_2\cup D_4, D_2\cup D_4, D_1\cup D_3, D_1\cup D_3).
\end{aligned}$$

Proposition 3.2. *The matrix-valued functions $S_n(x,t,\lambda)$, $n=1,2,3,4$ defined by*

$$M_n(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S_n(\lambda), \quad \lambda \in D_n, \tag{3.14}$$

can be expressed with $s(\lambda)$ and $S(\lambda)$ elements as follows

$$S_1(\lambda)=\begin{pmatrix} \frac{m_{22}(s)}{n_{33,44}(s)} & \frac{m_{21}(s)}{n_{33,44}(s)} & s_{13} & s_{14} \\ \frac{m_{12}(s)}{n_{33,44}(s)} & \frac{m_{11}(s)}{n_{33,44}(s)} & s_{23} & s_{24} \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix}, \quad S_2(\lambda)=\begin{pmatrix} S_{11}^{(2)} & S_{12}^{(2)} & s_{13} & s_{14} \\ S_{21}^{(2)} & S_{22}^{(2)} & s_{23} & s_{24} \\ S_{31}^{(2)} & S_{32}^{(2)} & s_{33} & s_{34} \\ S_{41}^{(2)} & S_{42}^{(2)} & s_{43} & s_{44} \end{pmatrix},$$
$$S_4(\lambda)=\begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & \frac{m_{44}(s)}{n_{11,22}(s)} & \frac{m_{43}(s)}{n_{11,22}(s)} \\ s_{41} & s_{42} & \frac{m_{34}(s)}{n_{11,22}(s)} & \frac{m_{33}(s)}{n_{11,22}(s)} \end{pmatrix}, \quad S_3(\lambda)=\begin{pmatrix} s_{11} & s_{12} & S_{13}^{(3)} & S_{14}^{(3)} \\ s_{21} & s_{22} & S_{23}^{(3)} & S_{24}^{(3)} \\ s_{31} & s_{32} & S_{33}^{(3)} & S_{34}^{(3)} \\ s_{41} & s_{42} & S_{43}^{(3)} & S_{44}^{(3)} \end{pmatrix}, \tag{3.15}$$

with $n_{i1j1,i2j2}(X)$ denoting the determinant of the sub-matrix

$$X=\begin{pmatrix} X_{i1j1} & X_{i1j2} \\ X_{i2j1} & X_{i2j2} \end{pmatrix},$$

while

$$S_{1j}^{(2)}=\frac{n_{1j,2(3-j)}(S)m_{2(3-j)}(s)+n_{1j,3(3-j)}(S)m_{3(3-j)}(s)+n_{1j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)},$$
$$S_{2j}^{(2)}=\frac{n_{2j,1(3-j)}(S)m_{1(3-j)}(s)+n_{2j,3(3-j)}(S)m_{3(3-j)}(s)+n_{2j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)},$$
$$S_{3j}^{(2)}=\frac{n_{3j,1(3-j)}(S)m_{1(3-j)}(s)+n_{3j,2(3-j)}(S)m_{3(3-j)}(s)+n_{3j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)},$$
$$S_{4j}^{(2)}=\frac{n_{4j,1(3-j)}(S)m_{1(3-j)}(s)+n_{4j,2(3-j)}(S)m_{3(3-j)}(s)+n_{4j,3(3-j)}(S)m_{3(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)},$$

if $j=1,2$,

$$S_{1j}^{(3)}=\frac{n_{1j,2(7-j)}(S)m_{2(7-j)}(s)+n_{1j,3(7-j)}(S)m_{3(7-j)}(s)+n_{1j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)},$$
$$S_{2j}^{(3)}=\frac{n_{2j,1(7-j)}(S)m_{1(7-j)}(s)+n_{2j,3(7-j)}(S)m_{3(7-j)}(s)+n_{2j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)},$$
$$S_{3j}^{(3)}=\frac{n_{3j,1(7-j)}(S)m_{1(7-j)}(s)+n_{3j,2(7-j)}(S)m_{2(7-j)}(s)+n_{3j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)},$$
$$S_{4j}^{(3)}=\frac{n_{4j,1(7-j)}(S)m_{1(7-j)}(s)+n_{4j,2(7-j)}(S)m_{2(7-j)}(s)+n_{4j,3(7-j)}(S)m_{3(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)},$$

if $j=3,4$*, and*

$$\Delta([S]_1[S]_2[s]_3[s]_4) := \det(n([S]_1,[S]_2,[s]_3,[s]_4)),$$
$$\Delta([s]_1[s]_2[S]_3[S]_4) := \det(n([s]_1,[s]_2,[S]_3,[S]_4)),$$

with $[\cdot]_j$ *denoting the* j*-th column of the corresponding matrix* s *or* S.

3.8. Residue conditions

We recall that $\mu_2(x,t,\lambda)$ is an entire function, and according to (3.14), the function $M(x,t,\lambda)$ has singularities at the same points as S_n, $n=1,2,3,4$. Let $\{\lambda_j\}_1^L$ denote possible zeros of $M(x,t,\lambda)$.

Assumption 3.1. Assume that the points $\{\lambda_j\}_1^L$ satisfy the following conditions:

- The function $n_{33,44}(s)(\lambda)$ has at most n_1 simple zeros $\{\lambda_j\}_1^{n_1}$ in D_1.
- The function $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$ has at most $n_2-n_1\geq 0$ simple zeros $\{\lambda_j\}_{n_1+1}^{n_2}$ in D_2.
- The function $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$ has at most $n_3-n_2\geq 0$ simple zeros $\{\lambda_j\}_{n_2+1}^{n_3}$ in D_3.
- The function $n_{11,22}(s)(\lambda)$ has at most $N-n_3\geq 0$ simple zeros $\{\lambda_j\}_{n_3+1}^{L}$ in D_4.
- All these zeros are different and neither of the functions $n_{11,22}(s)(\lambda)$, $n_{33,44}(s)(\lambda)$, $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$ and $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$ have zeros on the boundaries of D_n, $n=1,2,3,4$.

Let $X=(X_{ij})_{4\times 4}$ be a 4×4 matrix. We define the matrix $e^{\theta\hat{\sigma}_4}X$ by

$$e^{\theta\hat{\sigma}_4}X := e^{\theta\sigma_4}Xe^{-\theta\sigma_4} = \begin{pmatrix} X_{11} & X_{12} & X_{13}e^{2\theta} & X_{14}e^{2\theta} \\ X_{21} & X_{22} & X_{23}e^{2\theta} & X_{24}e^{2\theta} \\ X_{31}e^{-2\theta} & X_{32}e^{-2\theta} & X_{33} & X_{34} \\ X_{41}e^{-2\theta} & X_{42}e^{-2\theta} & X_{43} & X_{44} \end{pmatrix}.$$

Proposition 3.3. *Let* $\{M_n(x,t,\lambda)\}_1^4$ *be the eigenfunctions* (3.7)*. If the points* $\{\lambda_j\}_1^L$ *satisfies Assumption* 3.1*, then*

$$\begin{aligned} \mathrm{Res}_{\lambda=\lambda_j}[M_1(x,t,\lambda)]_k =& \frac{m_{2(3-k)}(s)(\lambda_j)s_{24}(\lambda_j)-m_{1(3-k)}(s)(\lambda_j)s_{14}(\lambda_j)}{\dot{n}_{33,44}(s)(\lambda_j)n_{13,24}(s)(\lambda_j)}[M_1]_3e^{\theta_{31}(\lambda_j)} \\ &+\frac{m_{1(3-k)}(s)(\lambda_j)s_{13}(\lambda_j)-m_{2(3-k)}(s)(\lambda_j)s_{23}(\lambda_j)}{\dot{n}_{33,44}(s)(\lambda_j)n_{13,24}(s)(\lambda_j)}[M_1]_4e^{\theta_{31}(\lambda_j)}, \\ 1\leq j\leq n_1,\quad \lambda_j\in D_1,\quad k=1,2. \end{aligned} \qquad (3.16)$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_2(x,t,\lambda)]_k =& \frac{S^{(2)}_{1k}(\lambda_j)s_{24}(\lambda_j)-S^{(2)}_{2k}(\lambda_j)s_{14}(\lambda_j)}{\Delta([S]_1[S]_2[\dot{s}]_3[s]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_2]_3e^{\theta_{31}(\lambda_j)}\\ &+\frac{S^{(2)}_{2k}(\lambda_j)s_{13}(\lambda_j)-S^{(2)}_{1k}(\lambda_j)s_{23}(\lambda_j)}{\Delta([S]_1[S]_2[\dot{s}]_3[s]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_2]_4e^{\theta_{31}(\lambda_j)},\\ n_1+1\le j\le n_2,\quad &\lambda_j\in D_2,\quad k=1,2. \end{aligned} \tag{3.17}$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_3(x,t,\lambda)]_k =& \frac{S^{(3)}_{1k}(\lambda_j)s_{22}(\lambda_j)-S^{(3)}_{2k}(\lambda_j)s_{12}(\lambda_j)}{\Delta([s]_1[s]_2[\dot{S}]_3[S]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_3]_1e^{\theta_{13}(\lambda_j)}\\ &+\frac{S^{(3)}_{2k}(\lambda_j)s_{11}(\lambda_j)-S^{(3)}_{1k}(\lambda_j)s_{21}(\lambda_j)}{\Delta([s]_1[s]_2[\dot{S}]_3[S]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_3]_2e^{\theta_{13}(\lambda_j)},\\ n_2+1\le j\le n_3,\quad &\lambda_j\in D_3,\quad k=3,4. \end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_4(x,t,\lambda)]_k =& \frac{m_{4(7-k)}(s)(\lambda_j)s_{42}(\lambda_j)-m_{3(7-k)}(s)(\lambda_j)s_{32}(\lambda_j)}{n_{11,22}(\dot{s})(\lambda_j)n_{31,42}(s)(\lambda_j)}[M_4]_1e^{\theta_{13}(\lambda_j)}\\ &+\frac{m_{3(7-k)}(s)(\lambda_j)s_{31}(\lambda_j)-m_{4(7-k)}(s)(\lambda_j)s_{41}(\lambda_j)}{n_{11,22}(\dot{s})(\lambda_j)n_{31,42}(s)(\lambda_j)}[M_4]_2e^{\theta_{13}(\lambda_j)},\\ n_3+1\le j\le L,\quad &\lambda_j\in D_4,\quad k=3,4, \end{aligned} \tag{3.19}$$

where $\dot{f}=df/d\lambda$ and θ_{ij} have the form

$$\theta_{ij}(x,t,\lambda)=(l_i-l_j)x-(z_i-z_j)t,\quad i,j=1,2,3,4,$$

so that

$$\begin{aligned}&\theta_{12}=\theta_{21}=\theta_{34}=\theta_{43}=0,\\ &\theta_{13}=\theta_{14}=\theta_{23}=\theta_{24}=-2i\lambda x-4i\lambda^2t,\\ &\theta_{31}=\theta_{41}=\theta_{32}=\theta_{42}=2i\lambda x+4i\lambda^2t.\end{aligned}$$

Proof. Let us start with the relation (3.19). Writing the expression (3.14) for $n=4$, we obtain that

$$M_4(x,t,\lambda)=\mu_2(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2t)\hat{\Lambda}}S_4,$$

where S_4 is defined in (3.15). The matrix $M_4(x,t,\lambda)$ has the following columns

$$[M_4]_1=[\mu_2]_1s_{11}+[\mu_2]_2s_{21}+[\mu_2]_3s_{31}e^{\theta_{31}}+[\mu_2]_4s_{41}e^{\theta_{31}}, \tag{3.20}$$

$$[M_4]_2=[\mu_2]_1s_{12}+[\mu_2]_2s_{22}+[\mu_2]_3s_{32}e^{\theta_{31}}+[\mu_2]_4s_{42}e^{\theta_{31}}, \tag{3.21}$$

$$[M_4]_3=[\mu_2]_3\frac{m_{44}(s)}{n_{11,22}(s)}+[\mu_2]_4\frac{m_{34}(s)}{n_{11,22}(s)}, \tag{3.22}$$

$$[M_4]_4=[\mu_2]_3\frac{m_{43}(s)}{n_{11,22}(s)}+[\mu_2]_4\frac{m_{33}(s)}{n_{11,22}(s)}. \tag{3.23}$$

If $\lambda_j \in D_4$ is a simple zero of $n_{11,22}(s)(\lambda)$, then solving the Eqs. (3.20), (3.21) for $[\mu_2]_3$, $[\mu_2]_4$ and substituting the solutions in (3.22) and (3.23), we obtain that

$$\begin{aligned}[M_4]_3 =& \frac{m_{44}(s)s_{42}-m_{34}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_4]_1e^{\theta_{13}} + \frac{m_{34}(s)s_{31}-m_{44}(s)s_{41}}{n_{11,22}(s)n_{13,24}(s)}[M_4]_2e^{\theta_{13}} \\ &+ \frac{m_{24}(s)[\mu_2]_1+m_{14}(s)[\mu_2]_2}{n_{31,42}(s)}e^{\theta_{13}}, \\ [M_4]_4 =& \frac{m_{43}(s)s_{42}-m_{33}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_4]_1e^{\theta_{13}} + \frac{m_{33}(s)s_{31}-m_{43}(s)s_{41}}{n_{11,22}(s)n_{31,42}(s)}[M_4]_2e^{\theta_{13}} \\ &+ \frac{m_{23}(s)[\mu_2]_1+m_{13}(s)[\mu_2]_2}{n_{31,42}(s)}e^{\theta_{13}}.\end{aligned}$$

Calculating the residues of these functions at λ_j, we arrive at the formula (3.19). The representations (3.16)-(3.18) can be derived analogously. □

3.9. The global relationship

The spectral functions $S(\lambda)$ and $s(\lambda)$ are not independent. Thus the Eqs. (3.10), (3.11) yield

$$\mu_3(x,t,\lambda) = \mu_1(x,t,\lambda)e^{-(i\lambda x+2i\lambda^2 t)\hat{\Lambda}}S^{-1}(\lambda)s(\lambda),$$

since $\mu_1(0,T,\lambda) = I$. Therefore, we obtain the global relation

$$S^{-1}(\lambda)s(\lambda) = e^{2i\lambda^2 T\hat{\Lambda}}c(T,\lambda) = e^{2i\lambda^2 T\hat{\Lambda}}\mu_3(0,T,\lambda),$$

where

$$c(T,\lambda) = \mu_3(0,T,\lambda) = I - \int_0^\infty e^{i\lambda\xi\hat{\Lambda}}(U\mu_3)(\xi,T,\lambda)d\xi, \quad \lambda \in (C_-,C_-,C_+,C_+).$$

4. Riemann-Hilbert Problem

In Section 3, we defined a sectionally analytic function $M(x,t,\lambda)$ that satisfies a RH problem formulated in terms of initial and boundary values. The solution of system (1.3) can be determined from the solutions of this RH problem.

Theorem 4.1. *If $\{u(x,t),v(x,t)\}$ is a sufficiently smooth and fast decaying (as x tens to ∞) solution of the system* (1.3) *in the domain Ω, then it can be reconstructed from the initial values $\{u_0(x),v_0(x)\}$ and the boundary values $\{p_0(t),q_0(t),p_1(t),q_1(t)\}$ of* (1.4). *More precisely, using the initial and boundary data, we establish the spectral functions $s(\lambda)$ and $S(\lambda)$ of* (3.10) *and define the jump matrix $J_{m,n}(x,t,\lambda)$. If the zeros $\{\lambda_j\}_1^L$ of the functions $n_{33,44}(s)(\lambda)$, $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$, $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$ and $n_{11,22}(s)(\lambda)$ satisfy Assumption* 3.1, *then*

$$u(x,t) = -2i\lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{24},$$
$$v(x,t) = -2i\lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{14},$$

where

- $M(x,t,\lambda)$ *is a sectionally meromorphic on the Riemann* λ*-sphere and has jumps on the contours* $\bar{D}_n \cap \bar{D}_m$, $n,m = 1,2,3,4$ — *cf. Fig.* 3.
- *On contours* $\bar{D}_n \cap \bar{D}_m$, $n,m = 1,2,3,4$ *the function* $M(x,t,\lambda)$ *satisfies the jump condition*

$$M_n(x,t,\lambda) = M_m(x,t,\lambda) J_{m,n}(x,t,\lambda), \quad \lambda \in \bar{D}_n \cap \bar{D}_m, \quad n \neq m.$$

- $M(x,t,\lambda) = I + \mathcal{O}(1/\lambda), \quad \lambda \to \infty.$
- *The function* $M(x,t,\lambda)$ *satisfies the residue condition of Proposition* 3.3.

The proof of this theorem is similar to the corresponding proof in Ref. [37].

5. Conclusions

We considered IBV problems for a coherently coupled NLS system on the half-line. Applying the Fokas unified transform method for nonlinear evolution equations in the form of Lax isospectral deformations and continuous spectra of the corresponding Lax operators, we reduce the initial problem to a matrix Riemann-Hilbert problem in the complex plane. Other integrable equations with 4×4 matrix Lax pairs can be discussed elsewhere.

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