Applied Mathematics and Computation 332 (2018) 148–159

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Riemann–Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg-de Vries equation on the half-line

Bei-Bei Hu [a,b,*], Tie-Cheng Xia [a,*], Wen-Xiu Ma [c,d,e]

[a] Department of Mathematics, Shanghai University, Shanghai 200444, China
[b] School of Mathematics and Finance, Chuzhou University, Anhui 239000, China
[c] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[d] Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa
[e] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China

ARTICLE INFO

Keywords:
Riemann–Hilbert problem
Two-component mKdV equation
Initial-boundary value problem
Unified transform method

ABSTRACT

In this work, we investigate the two-component modified Korteweg-de Vries (mKdV) equation, which is a complete integrable system, and accepts a generalization of 4×4 matrix Ablowitz–Kaup–Newell-Segur (AKNS)-type Lax pair. By using of the unified transform approach, the initial-boundary value (IBV) problem of the two-component mKdV equation associated with a 4×4 matrix Lax pair on the half-line will be analyzed. Supposing that the solution $\{u_1(x,t), u_2(x,t)\}$ of the two-component mKdV equation exists, we will show that it can be expressed in terms of the unique solution of a 4×4 matrix Riemann–Hilbert problem formulated in the complex λ-plane. Moreover, we will prove that some spectral functions $s(\lambda)$ and $S(\lambda)$ are not independent of each other but meet the global relationship.

1. Introduction

It is well known that the modified Korteweg-de Vries (mKdV) equation is one of the most important integrable systems in mathematics and physics which reads

$$u_t + u_{xxx} + 6\epsilon u^2 u_x = 0, \quad \epsilon = \pm 1, \tag{1.1}$$

and which is a natural extension of the KdV equation, whose can be describe acoustic wave in a certain anharmonic lattice and the Alfven wave in a cold collision-free plasma. The mKdV equation have been studied extensively because of their physical significance [1–5].

Currently, much attention is paid to the multi-component equations [3,6–8] for their important physical applications in many fields, such as fluid mechanics, nonlinear optics to Bose–Einstein condensates, and field theories [6–8]. In addition, multi-component equations possess abundant solution structures and appealing soliton collision phenomena [9–11]. For instance, when the polarization effect is included in the problem of pulse propagation along the optical fiber, the scalar nonlinear Schrödinger (NLS) [12] equation which governs the propagation of light pulses with fixed polarization should be replaced by a two-component generalization of the NLS equation.

* Corresponding authors at: Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail addresses: hu_chzu@shu.edu.cn (B.-B. Hu), xiatc@shu.edu.cn (T.-C. Xia).

https://doi.org/10.1016/j.amc.2018.03.049

Like the NLS equation, generally speaking, there exist two categories of the mKdV equation. One example is a vector version of the mKdV equation proposed by Yajima and Oikawa [2], the vector mKdV (vmKdV) equation has been studied extensively on the integrability associated with explicit form of the Lax pair [13], the initial value problem [9], infinitely many conservation laws [14], soliton solutions given by the Darboux transformation (DT) [14] and by the bilinear approach [7] and by the inverse scattering transform (IST) [9], etc. Besides, the algebro-geometric solutions of the vmKdV equation associated with a 3×3 matrix spectral problem on the basis of the theory of algebraic curves [15].

An other example is a matrix version of the mKdV equation studied by Athorne and Fordy [3]. In 1997, Iwao and Hirota [7] discussed a simple coupled version of the mKdV equation

$$u_{j,t} + 6\left(\sum_{k,l=1}^{N} c_{kl} u_k u_l\right) u_{j,x} + u_{j,xxx} = 0, \quad j = 1, 2, \ldots, N, \tag{1.2}$$

it is not difficulty to find that system (1.2) can be transformed into the following normalized multi-component mKdV equations [8]

$$u_{j,t} + 6\left(\sum_{k=1}^{N} \epsilon_k u_k^2\right) u_{j,x} + u_{j,xxx} = 0, \quad \epsilon_k = \pm 1, \quad j = 1, 2, \ldots, N. \tag{1.3}$$

Recently, the initial-boundary value (IBV) problems for the mKdV equation on the half-line are studied via the unified transform approach [5], the unified transform approach can be used to study the IBV problems for both linear and nonlinear integrable evolution PDEs with 2×2 Lax pairs [16–22]. Just like the IST on the line, the unified transform method gets an expression for the solution of an IBV problem in terms of the solution of a Riemann–Hilbert (RH) problem. In 2012, Lenells extended the unified transform method to study the IBV problems for integrable nonlinear equations with 3×3 Lax pairs [23]. After that, many researchers have devoted their attention to studying IBV problems for integrable evolution equations with higher-order Lax pairs, the IBV problem for the many integrable system with 3×3 or 4×4 Lax pairs are studied, such as, the Degasperis–Procesi equation [24,25], the Ostrovsky–Vakhnenko equation [26], the Sasa–Satsuma equation [27], the three wave equation [28], the spin-1 Gross–Pitaevskii equations [29] and others [30–33]. These authors have also done some work on integrable equations with 2×2 or higher-order Lax pairs [21,34–36]. In particular, an effective way analyzing the asymptotic behaviour of the solution is based on this RH problem and by using the nonlinear version of the steepest descent method introduced by Deift and Zhou [37].

Most recently, Geng et al. [13,38] discussed the IBV problem of the coupled mKdV (vector version) equation on the half-line. To the best of our knowledge, the IBV problem of the two-component mKdV equation (simple matrix version) has not been studied. In this work, by using the unified transform method, we investigate the self-focusing type case of two-component mKdV equation which reads

$$\begin{cases} u_{1,t} + 6(u_1^2 + u_2^2) u_{1,x} + u_{1,xxx} = 0, \\ u_{2,t} + 6(u_1^2 + u_2^2) u_{2,x} + u_{2,xxx} = 0. \end{cases} \tag{1.4}$$

That is to say, we consider the following initial-boundary value problem of the two-component mKdV equation. The IBV problems of system (1.4) on the interval will be presented in another paper.

Throughout this paper, we consider the half-line domain Ω and the IBV problems for system (1.4) as follows

$$\begin{aligned} &\text{Half-line domain (see Figure 1)} : \Omega = \{0 < x < \infty, 0 < t < T\}; \\ &\text{Initial values} : u_0(x) = u_1(x, t = 0),\ v_0(x) = u_2(x, t = 0); \\ &\text{Dirichlet boundary values} : p_0(t) = u_1(x = 0, t),\ q_0(t) = u_2(x = 0, t); \\ &\text{Neumann boundary values} : p_1(t) = u_{1,x}(x = 0, t),\ q_1(t) = u_{2,x}(x = 0, t). \end{aligned} \tag{1.5}$$

Here $u_0(x)$ and $v_0(x)$ belong to the Schwartz space.

The outline of the present paper is as follows. In Section 2, two sets of eigenfunctions $\{\mu_j\}_1^3$ and $\{M_n\}_1^4$ of the Lax pair for spectral analysis are given, and further determine some spectral functions which meet a so-called global relationship. In Section 3, we prove that $\{u_1(x, t), u_2(x, t)\}$ can be described in terms of the unique solution of a 4×4 matrix Riemann–Hilbert problem formulated in the complex λ-plane. The last section is devoted to giving some conclusions.

2. The spectral analysis

The two-component mKdV Eq. (1.4) admits the 4×4 Lax pair

$$\begin{cases} \psi_x = F\psi = (-i\lambda\Lambda + F_0)\psi, \\ \psi_t = G\psi = (-4i\lambda^3\Lambda + 4\lambda^2 F_0 + 2i\lambda G_1 + G_0)\psi, \end{cases} \tag{2.1}$$

here $\psi=\psi(x,t,\lambda)$ is a 4×4 matrix-valued or a 4×1 column vector-valued spectral function, and 4×4 matrix $\Lambda=diag\{1,1,-1,-1\}$ and the 4×4 matrix-valued functions F_0, G_0 and G_1 are defined by

$$F_0(x,t)=\begin{pmatrix}0&0&u_1&u_2\\0&0&-u_2&u_1\\-\bar{u}_1&\bar{u}_2&0&0\\-\bar{u}_2&-\bar{u}_1&0&0\end{pmatrix},\quad G_0(x,t)=\begin{pmatrix}G_{11}^{(0)}&G_{12}^{(0)}&G_{13}^{(0)}&G_{14}^{(0)}\\-G_{12}^{(0)}&G_{11}^{(0)}&-G_{14}^{(0)}&G_{13}^{(0)}\\G_{31}^{(0)}&G_{32}^{(0)}&-G_{11}^{(0)}&-G_{12}^{(0)}\\-G_{32}^{(0)}&G_{31}^{(0)}&G_{12}^{(0)}&-G_{11}^{(0)}\end{pmatrix},$$
$$G_1(x,t)=\begin{pmatrix}|u_1|^2+|u_2|^2&-u_1\bar{u}_2+\bar{u}_1u_2&u_{1,x}&u_{2,x}\\u_1\bar{u}_2-\bar{u}_1u_2&|u_1|^2+|u_2|^2&-u_{2,x}&u_{1,x}\\\bar{u}_{1,x}&-\bar{u}_{2,x}&-|u_1|^2-|u_2|^2&u_1\bar{u}_2-\bar{u}_1u_2\\\bar{u}_{2,x}&\bar{u}_{1,x}&-u_1\bar{u}_2+\bar{u}_1u_2&-|u_1|^2-|u_2|^2\end{pmatrix}. \tag{2.2}$$

with

$$G_{11}^{(0)}=-\bar{u}_1u_{1,x}+u_1\bar{u}_{1,x}-\bar{u}_2u_{2,x}+u_2\bar{u}_{2,x},\quad G_{12}^{(0)}=u_{1,x}\bar{u}_2-u_1\bar{u}_{2,x}+\bar{u}_{1,x}u_2-\bar{u}_1u_{2,x},$$
$$G_{13}^{(0)}=-u_{1,xx}-2\bar{u}_1(u_1^2-u_2^2)-4u_1|u_2|^2,\quad G_{14}^{(0)}=-u_{2,xx}+2\bar{u}_1(u_1^2-u_2^2)-4u_2|u_1|^2,$$
$$G_{31}^{(0)}=u_{1,xx}+2u_1(|\bar{u}_1|^2-|\bar{u}_2|^2)+4\bar{u}_1|u_2|^2,\quad G_{32}^{(0)}=-u_{2,xx}+2u_2(|\bar{u}_1|^2-|\bar{u}_2|^2)+4\bar{u}_2|u_1|^2.$$

Direct computations display that the zero-curvature equation $U_x-V_t+UV-VU=0$ exactly gives system (1.4).

2.1. The closed one-form

We find that Lax pair Eq. (2.1) can be written as

$$\begin{cases}\psi_x+i\lambda\Lambda\psi=P(x,t)\psi,\\ \psi_t+4i\lambda^3\Lambda\psi=Q(x,t,\lambda)\psi,\end{cases} \tag{2.3}$$

where

$$P(x,t)=F_0(x,t),\quad Q(x,t,\lambda)=4\lambda^2F_0+2i\lambda G_1+G_0. \tag{2.4}$$

Introduce a new eigenfunction $\mu(x,t,\lambda)$ by the transform

$$\psi(x,t,\lambda)=\mu(x,t,\lambda)e^{-(i\lambda\Lambda x+4i\lambda^3\Lambda t)}, \tag{2.5}$$

and then the Lax pair Eq. (2.3) becomes

$$\begin{cases}\mu_x+i\lambda[\Lambda,\mu]=P(x,t)\mu,\\ \mu_t+4i\lambda^3[\Lambda,\mu]=Q(x,t,\lambda)\mu,\end{cases} \tag{2.6}$$

and Eq. (2.6) gets to a full derivative form

$$d(e^{i\lambda\hat{\Lambda}x+4i\lambda^3\hat{\Lambda}t}\mu(x,t,\lambda))=W(x,t,\lambda), \tag{2.7}$$

where the closed one-form $W(x,t,\lambda)$ is defined by

$$W(x,t,\lambda)=e^{(i\lambda x+4i\lambda^3t)\hat{\Lambda}}(P(x,t)dx+Q(x,t,\lambda)dt)\mu, \tag{2.8}$$

and $\hat{\sigma}_4$ represents a matrix operator acting on 4×4 matrix X by $\hat{\sigma}_4X=[\sigma_4,X]$ and by $e^{x\hat{\sigma}_4}X=e^{x\sigma_4}Xe^{-x\sigma_4}$ (see Lemma 2.4).

2.2. The basic eigenfunction μ_j's

Suppose that $u_1(x,t)$ and $u_2(x,t)$ are sufficiently smooth functions in the half-line region $\Omega=\{0<x<\infty,0<t<T\}$, and decay sufficiently when $x\to\infty$. $\{\mu_j(x,t,\lambda)\}_1^3$ are the 4×4 matrix valued functions, based on the Volterra integral equation, we can define the three eigenfunctions $\{\mu_j(x,t,\lambda)\}_1^3$ of Eq. (2.6) by

$$\mu_j(x,t,\lambda)=\mathbb{I}+\int_{\gamma_j}e^{-(i\lambda x+4i\lambda^3t)\hat{\Lambda}}W_j(\xi,\tau,\lambda),\quad j=1,2,3, \tag{2.9}$$

where $\mathbb{I}=diag\{1,1,1,1\}$ is a 4×4 identity matrix, W_j is determined Eq. (2.8), where μ_j replaces μ, and the contours $\{\gamma_j\}_1^3$ are smooth curves from (x_j,t_j) to (x,t), and $(x_1,t_1)=(0,T)$, $(x_2,t_2)=(0,0)$, $(x_3,t_3)=(\infty,t)$ (see Fig. 2).

Thus, we have the following inequalities are hold true for the point (ξ,τ) on the each contour

$$\begin{aligned}\gamma_1:&\quad x-\xi\geq0,\quad t-\tau\leq0;\\ \gamma_2:&\quad x-\xi\geq0,\quad t-\tau\geq0;\\ \gamma_3:&\quad x-\xi\leq0,\quad t-\tau=0.\end{aligned} \tag{2.10}$$

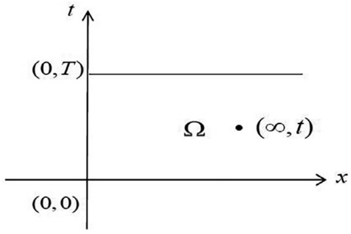

Fig. 1. The region Ω in the (*x*, *t*) plan.

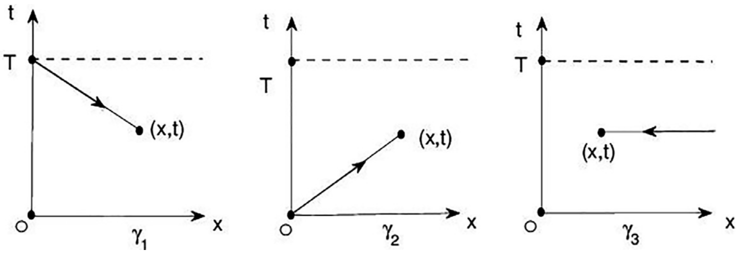

Fig. 2. The three contours $\gamma_1, \gamma_2, \gamma_3$ in the (*x*, *t*)-domaint.

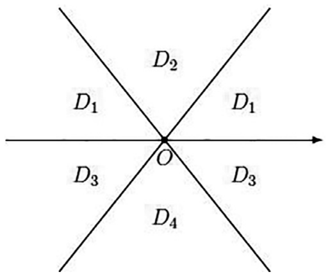

Fig. 3. The sets $D_n, n = 1, 2,$ which decompose the complex $\lambda-$plane.

As the one-form W_j is closed, thus μ_j are independent of the path of integration. Choose the paths of integration to be parallel to the x and t axes, then Eq. (2.9) is change into the follow form

$$\mu_j(x,t,\lambda)=\mathbb{I}+\int_{x_j}^{x} e^{-i\lambda(x-\xi)\hat{\Lambda}}(P\mu_j)(\xi,t,\lambda)d\xi + e^{-i\lambda(x-x_j)\hat{\Lambda}}\int_{t_j}^{t} e^{-4i\lambda^3(t-\tau)\hat{\Lambda}}(Q\mu_j)(x_j,\tau,\lambda)d\tau, \quad (j=1,2,3), \tag{2.11}$$

Let $[\mu_j]_k$ denote the kth column vector of μ_j, Eq. (2.10) implies that the first, second, third and fourth columns of the matrices Eq. (2.9) contain the following exponential term

$$\begin{aligned}
&[\mu_j]_1 : e^{2i\lambda(x-\xi)+8i\lambda^3(t-\tau)}, \quad e^{2i\lambda(x-\xi)+8i\lambda^3(t-\tau)},\\
&[\mu_j]_2 : e^{2i\lambda(x-\xi)+8i\lambda^3(t-\tau)}, \quad e^{2i\lambda(x-\xi)+8i\lambda^3(t-\tau)},\\
&[\mu_j]_3 : e^{-2i\lambda(x-\xi)-8i\lambda^3(t-\tau)}, \quad e^{-2i\lambda(x-\xi)-8i\lambda^3(t-\tau)},\\
&[\mu_j]_4 : e^{-2i\lambda(x-\xi)-8i\lambda^3(t-\tau)}, \quad e^{-2i\lambda(x-\xi)-8i\lambda^3(t-\tau)}.
\end{aligned} \tag{2.12}$$

Thus, we can show that the eigenfunctions $\{\mu_j(x,t,\lambda)\}_1^3$ are bounded and analytic for $\lambda \in \mathbb{C}$ such that λ belongs to

$$\begin{aligned}
&\mu_1 \text{ is bounded and analytic for } \lambda \in (D_2, D_2, D_4, D_4),\\
&\mu_2 \text{ is bounded and analytic for } \lambda \in (D_1, D_1, D_3, D_3),\\
&\mu_3 \text{ is bounded and analytic for } \lambda \in (C_-, C_-, C_+, C_+),
\end{aligned} \tag{2.13}$$

where $C_- = D_3 \cup D_4$ and $C_+ = D_1 \cup D_2$ denote the lower half-plane and the upper half-plane, respectively, and $\{D_n\}_1^4$ denote four open, pairwisely disjoint sub-sets of the Riemann λ-plane are shown in Fig. 3.

And these sets $\{D_n\}_1^4$ have the following properties:

$$\begin{aligned}
D_1 &= \{\lambda \in \mathbb{C} | \mathrm{Re}l_1 = \mathrm{Re}l_2 > \mathrm{Re}l_3 = \mathrm{Re}l_4, \quad \mathrm{Re}z_1 = \mathrm{Re}z_2 > \mathrm{Re}z_3 = \mathrm{Re}z_4\},\\
D_2 &= \{\lambda \in \mathbb{C} | \mathrm{Re}l_1 = \mathrm{Re}l_2 > \mathrm{Re}l_3 = \mathrm{Re}l_4, \quad \mathrm{Re}z_1 = \mathrm{Re}z_2 < \mathrm{Re}z_3 = \mathrm{Re}z_4\},
\end{aligned}$$

$$D_3 = \{\lambda \in \mathbb{C} | \mathrm{Re} l_1 = \mathrm{Re} l_2 < \mathrm{Re} l_3 = \mathrm{Re} l_4, \quad \mathrm{Re} z_1 = \mathrm{Re} z_2 < \mathrm{Re} z_3 = \mathrm{Re} z_4\},$$
$$D_4 = \{\lambda \in \mathbb{C} | \mathrm{Re} l_1 = \mathrm{Re} l_2 < \mathrm{Re} l_3 = \mathrm{Re} l_4, \quad \mathrm{Re} z_1 = \mathrm{Re} z_2 > \mathrm{Re} z_3 = \mathrm{Re} z_4\}, \tag{2.14}$$

where $l_i(\lambda)$ and $z_i(\lambda)$ are the diagonal elements of the 4×4 matrix $-i\lambda\Lambda$ and $-4i\lambda^3\Lambda$, respectively.

Moreover, since $\mu_1(x, t, \lambda)$ and $\mu_2(x, t, \lambda)$ are entire functions of λ, and in their corresponding regions of boundedness, we have

$$\mu_j(x,t,\lambda) = \mathbb{I} + O\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty, \quad j = 1,2,3. \tag{2.15}$$

In fact, $\mu_1(0, t, \lambda)$ and $\mu_2(0, t, \lambda)$ can be enlarged the domain of boundedness ($D_2\cup D_3$, $D_2\cup D_3$, $D_1\cup D_4$, $D_1\cup D_4$) and ($D_1\cup D_4$, $D_1\cup D_4$, $D_2\cup D_3$, $D_2\cup D_3$) for $x = 0$, respectively.

2.3. The symmetry of eigenfunctions

For the convenience of the analysis, we write a 4×4 matrix $X = (X_{ij})_{4\times 4}$ as follows

$$X = \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix}, \quad \tilde{X}_{11} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad \tilde{X}_{12} = \begin{pmatrix} X_{13} & X_{14} \\ X_{23} & X_{24} \end{pmatrix}, \quad \tilde{X}_{21} = \begin{pmatrix} X_{31} & X_{32} \\ X_{41} & X_{42} \end{pmatrix}, \quad \tilde{X}_{22} = \begin{pmatrix} X_{33} & X_{34} \\ X_{43} & X_{44} \end{pmatrix}. \tag{2.16}$$

As $F(x,t,\lambda) = -i\lambda\Lambda + P(x,t)$, $G(x,t,\lambda) = 4i\lambda^3\Lambda + Q(x,t,\lambda)$, and by the symmetry properties of $F(x, t, \lambda)$ and $G(x, t, \lambda)$, it is not difficult to find that the eigenfunction $\mu(x, t, \lambda)$ have the symmetries

$$(\tilde{\mu}(x,t,\lambda))_{11} = Y^{-1}\overline{(\tilde{\mu}(x,t,\bar{\lambda}))_{22}}Y^{-1}, \quad (\tilde{\mu}(x,t,\lambda))_{12} = \overline{(\tilde{\mu}(x,t,\bar{\lambda}))_{21}}^{T}, \tag{2.17}$$

where $Y^{-1} = diag(1, -1)$.

Notice that

$$Y_\pm \overline{(F(x,t,\bar{\lambda}))} Y_\pm = -F(x,t,\lambda)^T, \quad Y_\pm \overline{(G(x,t,\bar{\lambda}))} Y_\pm = -G(x,t,\lambda)^T, \tag{2.18}$$

where $P_\pm = diag(\pm 1, \pm 1, \mp 1, \mp 1)$.

Indeed, we note that the eigenfunctions $\psi(x, t, \lambda)$ defined by the (2.3) and $\mu(x, t, \lambda)$ defined by the (2.6) have the same symmetric relation

$$\psi^{-1}(x,t,\lambda) = Y_\pm \overline{(\psi(x,t,\bar{\lambda}))}^{T} Y_\pm, \quad \mu^{-1}(x,t,\lambda) = Y_\pm \overline{(\mu(x,t,\bar{\lambda}))} Y_\pm. \tag{2.19}$$

Moreover, as $tr(F(x,t,\lambda)) = tr(G(x,t,\lambda) = 0$ and $\mu(x, t, \lambda)$ is bounded in their corresponding regions, we find

$$\mu(x,t,\lambda) = \mathbb{I} + O\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty, \tag{2.20}$$

and $\det[\mu(x,t,\lambda)] = 1$.

2.4. The minors of eigenfunctions

We also need to consider the bounded and analytical properties of the minors of the matrices $\{\mu_j(x,t,\lambda)\}_1^3$. Recall that the cofactor matrix Z^A (or the transpose of the adjugate) of a 4×4 matrix Z is defined by

$$Z^A = \mathrm{adj}(Z)^T = \begin{pmatrix} m_{11}(Z) & -m_{12}(Z) & m_{13}(Z) & -m_{14}(Z) \\ -m_{21}(Z) & m_{22}(Z) & -m_{23}(Z) & m_{24}(Z) \\ m_{31}(Z) & -m_{32}(Z) & m_{33}(Z) & -m_{34}(Z) \\ -m_{41}(Z) & m_{42}(Z) & -m_{43}(Z) & m_{44}(Z) \end{pmatrix}, \tag{2.21}$$

here $m_{ij}(Z)$ denote the (ij)th minor of a 4×4 matrix Z and $(Z^A)^T Z = \mathrm{adj}(Z)Z = \det Z$.

It follows from Eq. (2.6) that the matrix-valued functions μ^A's satisfies the Lax pair

$$\begin{cases} \mu_x^A - i\lambda[\Lambda, \mu^A] = -P(x,t)^T \mu^A, \\ \mu_t^A - 4i\lambda^3[\Lambda, \mu^A] = -Q(x,t,\lambda)^T \mu^A, \end{cases} \tag{2.22}$$

where the superscript T denotes a matrix transpose. Then the eigenfunctions $\{\mu_j^A(x,t,\lambda)\}_1^3$ are solutions that can be expressed as

$$\mu_j^A(x,t,\lambda) = \mathbb{I} - \int_{x_j}^{x} e^{i\lambda(x-\xi)\hat{\Lambda}} (P\mu_j^A)(\xi,t,\lambda)d\xi - e^{i\lambda(x-x_j)\hat{\Lambda}} \int_{t_j}^{t} e^{4i\lambda^3(t-\tau)\hat{\Lambda}} (Q\mu_j^A)(x_j,\tau,\lambda)d\tau, \tag{2.23}$$

by using the Volterra integral equations, respectively. Thus, we can obtain the eigenfunctions $\{\mu_j^A(x,t,\lambda)\}_1^3$ which satisfies the following analytic properties

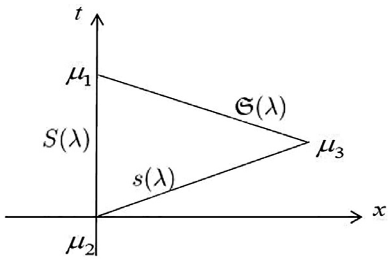

Fig. 4. The relations among the dependent eigenfunctions $\mu_j(x, t, k), j = 1, 2, 3$.

$$\begin{aligned}&\mu_1^A \text{ is bounded and analytic for } \lambda \in (D_4, D_4, D_2, D_2),\\&\mu_2^A \text{ is bounded and analytic for } \lambda \in (D_3, D_3, D_1, D_1),\\&\mu_3^A \text{ is bounded and analytic for } \lambda \in (C_+, C_+, C_-, C_-).\end{aligned} \tag{2.24}$$

In fact, $\mu_1^A(0, t, \lambda)$ and $\mu_2^A(0, t, \lambda)$ can be enlarged the domain of boundedness ($D_1 \cup D_4$, $D_1 \cup D_4$, $D_2 \cup D_3$, $D_2 \cup D_3$) and ($D_2 \cup D_3$, $D_2 \cup D_3$, $D_1 \cup D_4$, $D_1 \cup D_4$) for $x = 0$, respectively.

2.5. The spectral functions and the global relation

We also need to define the 4×4 matrix value spectral function $s(\lambda)$, $S(\lambda)$ and $\mathfrak{S}(\lambda)$ as follows

$$\begin{aligned}&\mu_3(x, t, \lambda) = \mu_2(x, t, \lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}s(\lambda),\\&\mu_1(x, t, \lambda) = \mu_2(x, t, \lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}S(\lambda),\\&\mu_3(x, t, \lambda) = \mu_1(x, t, \lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}\mathfrak{S}(\lambda).\end{aligned} \tag{2.25}$$

As $\mu_2(0, 0, \lambda) = \mathbb{I}$, we obtain

$$\begin{aligned}&s(\lambda) = \mu_3(0, 0, \lambda), \quad S(\lambda) = \mu_1(0, 0, \lambda) = e^{4i\lambda^3 T\hat{\Lambda}}\mu_2^{-1}(0, T, \lambda),\\&\mathfrak{S}(\lambda) = \mu_1^{-1}(0, 0, \lambda)\mu_3(0, 0, \lambda) = S^{-1}(\lambda)s(\lambda) = e^{4i\lambda^3 T\hat{\Lambda}}\mu_3^{-1}(0, T, \lambda).\end{aligned} \tag{2.26}$$

These relationship among μ_j are shown in Fig. 4. Thus three matrix value spectral functions $s(\lambda)$, $s(\lambda)$ and $\mathfrak{S}(\lambda)$ are dependent such that we only consider two of them, that is to say, we only consider spectral functions $s(\lambda)$ and $S(\lambda)$ in this paper.

According to the definition of eigenfunction μ_j in (2.11), Eq. (2.26) means that

$$\begin{aligned}&s(\lambda) = \mathbb{I} - \int_0^\infty e^{i\lambda\xi\hat{\Lambda}}(P\mu_3)(\xi, 0, \lambda)d\xi,\\&S(\lambda) = \mathbb{I} - \int_0^T e^{4i\lambda^3\tau\hat{\Lambda}}(Q\mu_1)(0, \tau, \lambda)d\tau = \left[\mathbb{I} + \int_0^T e^{4i\lambda^3\tau\hat{\Lambda}}(Q\mu_2)(0, \tau, \lambda)d\tau\right]^{-1},\end{aligned} \tag{2.27}$$

where $\mu_1(0, t, \lambda)$, $\mu_2(0, t, \lambda)$ and $\mu_3(x, t, \lambda)$, $0 < x < \infty$, $0 < t < T$ are given by the Volterra integral equations

$$\begin{aligned}&\mu_1(0, t, \lambda) = \mathbb{I} - \int_t^T e^{-4i\lambda^3(t-\tau)\hat{\Lambda}}(Q\mu_1)(0, \tau, \lambda)d\tau, \quad \lambda \in (D_2 \cup D_3, D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4),\\&\mu_2(0, t, \lambda) = \mathbb{I} + \int_0^T e^{-4i\lambda^3(t-\tau)\hat{\Lambda}}(Q\mu_2)(0, \tau, \lambda)d\tau, \quad \lambda \in (D_1 \cup D_4, D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3),\\&\mu_3(x, 0, \lambda) = \mathbb{I} - \int_x^\infty e^{-i\lambda(x-\xi)\hat{\Lambda}}(P\mu_3)(\xi, 0, \lambda)d\xi, \quad \lambda \in (C_-, C_-, C_+, C_+).\end{aligned} \tag{2.28}$$

Thus, it follows from Eqs. (2.27) and (2.28), we find that the spectral functions $s(\lambda)$ and $S(\lambda)$ are determined by $P(x, 0, \lambda)$ and $Q(0, t, \lambda)$, that is to say, the spectral functions $s(\lambda)$ and $S(\lambda)$ determined by the initial data $u_0(x), v_0(x)$ and the boundary data $p_0(t)$, $q_0(t)$, $p_1(t)$, $q_1(t)$.

Indeed, the eigenfunction $\mu_3(x, 0, \lambda)$ satisfy the x-part of the Lax pair (2.6) at $t = 0$. Thus, we have

$$x - part: \begin{cases}\mu_x(x, 0, \lambda) + i\lambda[\Lambda, \mu(x, 0, \lambda)] = P(x, t = 0)\mu(x, 0, \lambda),\\ \lim_{x\to\infty}\mu(x, 0, \lambda) = \mathbb{I}, \quad 0 < x < \infty,\end{cases} \tag{2.29}$$

and the eigenfunctions $\mu_j(0, t, \lambda)$, $\mu_2(0, t, \lambda)$ satisfies the t-part of the Lax pair (2.6) at $x = 0$. Thus, we have

$$t - part: \begin{cases}\mu_t(0, t, \lambda) + 4i\lambda^3[\Lambda, \mu(0, t, \lambda)] = Q(x = 0, t)\mu(0, t, \lambda), \quad 0 < t < T,\\ \lim_{t\to 0}\mu(0, t, \lambda) = \mu(0, 0, \lambda) = \mathbb{I}, \quad and \quad \lim_{t\to T}\mu(0, t, \lambda) = \mu(0, T, \lambda) = \mathbb{I}.\end{cases} \tag{2.30}$$

Moreover, by the properties of $\{\mu_j\}_1^3$ and $\{\mu_j^A\}_1^3$, we can obtain that $s(\lambda)$, $S(\lambda)$, $s^A(\lambda)$ and $S^A(\lambda)$ have the following bounded properties

$$\begin{aligned} &s(\lambda) \text{ is bounded for } \lambda \in (C_-, C_-, C_+, C_+),\\ &S(\lambda) \text{ is bounded for } \lambda \in (D_2 \cup D_3, D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4),\\ &s^A(\lambda) \text{ is bounded for } \lambda \in (C_+, C_+, C_-, C_-),\\ &S^A(\lambda) \text{ is bounded for } \lambda \in (D_1 \cup D_4, D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3). \end{aligned} \tag{2.31}$$

In fact, the spectral functions $s(\lambda)$ and $S(\lambda)$ are not independent of each other but satisfy an important relation. Indeed, it follows from Eq. (2.26), we obtain

$$\mu_3(x,t,\lambda) = \mu_1(x,t,\lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}} S^{-1}(\lambda)s(\lambda), \tag{2.32}$$

since $\mu_1(0,t,\lambda) = \mathbb{I}$, when $(x,t) = (0,T)$. We can evaluate the following relationship which is the global relation as follows

$$S^{-1}(\lambda)s(\lambda) = e^{4i\lambda^3 T\hat{\Lambda}} c(T,\lambda), \quad \lambda \in (C_-, C_-, C_+, C_+), \tag{2.33}$$

here

$$c(T,\lambda) = \mu_3(0,T,\lambda) = \mathbb{I} - \int_0^\infty e^{i\lambda\xi\hat{\Lambda}} (P\mu_3)(\xi,T,\lambda)d\xi. \tag{2.34}$$

2.6. The definition of matrix-valued functions $\mathrm{M_n}$*'s*

For each $n = 1,\dots,4$, the solution $M_n(x, t, \lambda)$ of Eq. (2.6) is defined by the following integral equation

$$(M_n(x,t,\lambda))_{ij} = \delta_{ij} + \int_{\gamma_{ij}^n} (e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}} W_n(\xi,\tau,\lambda))_{ij}, \quad i,j = 1,\dots,4, \tag{2.35}$$

where $W_n(x, t, \lambda)$ is given by Eq. (2.8), where M_n replaces μ, and the contours $\gamma_{ij}^n (n,i,j = 1,\dots,4)$ are defined as follows

$$\gamma_{ij}^n = \begin{cases} \gamma_1, & if \quad Rel_i(\lambda) < Rel_j(\lambda) \quad and \quad Rez_i(\lambda) \geq Rez_j(\lambda), \\ \gamma_2, & if \quad Rel_i(\lambda) < Rel_j(\lambda) \quad and \quad Rez_i(\lambda) < Rez_j(\lambda), for\, \lambda \in D_n \\ \gamma_3, & if \quad Rel_i(\lambda) \geq Rel_j(\lambda). \end{cases}$$

According to the definition of γ^n in (2.36), we obtain

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_1 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_1 & \gamma_3 & \gamma_3 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}. \tag{2.37}$$

Next, the following proposition guarantees that the previous definition of M_n can be represented as a RH problem.

Proposition 2.1. *For each $n = 1,\dots,4$ and $\lambda \in D_n$, the function $M_n(x, t, \lambda)$ is defined well by Eq. (2.35). For any identified point (x, t), M_n is bounded and analytical as a function of $\lambda \in D_n$ away from a possible discrete set of singularities $\{\lambda_j\}$ at which the Fredholm determinant vanishes. Moreover, M_n admits a bounded and continuous extension to* Reλ*-axis and*

$$M_n(x,t,\lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right). \tag{2.38}$$

Proof. The associated bounded and analytical properties have been established in Appendix B in [23]. Substituting the following expansion

$$M = M_0 + \frac{M^{(1)}}{\lambda} + \frac{M^{(2)}}{\lambda^2} + \cdots \quad \lambda \to \infty, \tag{2.39}$$

into the Lax pair Eq. (2.6) and comparing the coefficients of the same order of λ, we can obtain Eq. (2.39). □

2.7. The jump matrix and computations

There are four new spectral functions $\{S_n(\lambda)\}_1^4$ can be defined by

$$S_n(\lambda) = M_n(0,0,\lambda), \quad \lambda \in D_n, \quad n = 1,\dots,4. \tag{2.40}$$

Let $M(x, t, \lambda)$ be a sectionally analytical continuous function in the Riemann λ-sphere, which equals $M_n(x, t, \lambda)$ for $\lambda \in D_n$. Then $M(x, t, \lambda)$ satisfies the following jump conditions

$$M_n(x,t,\lambda) = M_m(x,t,\lambda)J_{m,n}(x,t,\lambda), \quad \lambda \in \bar{D}_n \cap \bar{D}_m, \quad n,m = 1,\dots,4, \quad n \neq m, \tag{2.41}$$

where $J_{m,\,n}(x, t, \lambda)$ are the jump matrices which defined by

$$J_{m,n}(x,t,\lambda) = e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}[S_m^{-1}(\lambda)S_n(\lambda)]. \tag{2.42}$$

Proposition 2.2. *The matrix-valued functions $\{S_n(x,t,\lambda)\}_1^4$ defined by*

$$M_n(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}S_n(\lambda), \quad \lambda \in D_n, \tag{2.43}$$

can be expressed in terms of the entries $s(\lambda)$ and $S(\lambda)$ elements as follows

$$S_1(\lambda) = \begin{pmatrix} \frac{m_{22}(s)}{n_{33,44}(s)} & \frac{m_{21}(s)}{n_{33,44}(s)} & s_{13} & s_{14} \\ \frac{m_{12}(s)}{n_{33,44}(s)} & \frac{m_{11}(s)}{n_{33,44}(s)} & s_{23} & s_{24} \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix}, \quad S_2(\lambda) = \begin{pmatrix} S_{11}^{(2)} & S_{12}^{(2)} & s_{13} & s_{14} \\ S_{21}^{(2)} & S_{22}^{(2)} & s_{23} & s_{24} \\ S_{31}^{(2)} & S_{32}^{(2)} & s_{33} & s_{34} \\ S_{41}^{(2)} & S_{42}^{(2)} & s_{43} & s_{44} \end{pmatrix},$$

$$S_3(\lambda) = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & \frac{m_{44}(s)}{n_{11,22}(s)} & \frac{m_{43}(s)}{n_{11,22}(s)} \\ s_{41} & s_{42} & \frac{m_{34}(s)}{n_{11,22}(s)} & \frac{m_{33}(s)}{n_{11,22}(s)} \end{pmatrix}, \quad S_4(\lambda) = \begin{pmatrix} s_{11} & s_{12} & S_{13}^{(4)} & S_{14}^{(4)} \\ s_{21} & s_{22} & S_{23}^{(4)} & S_{24}^{(4)} \\ s_{31} & s_{32} & S_{33}^{(4)} & S_{34}^{(4)} \\ s_{41} & s_{42} & S_{43}^{(4)} & S_{44}^{(4)} \end{pmatrix}, \tag{2.44}$$

where $n_{i_1 j_1, i_2 j_2}(Z)$ denotes the determinant of the sub-matrix generated by taking the cross elements of $i_{1,\,2}$ th rows and $j_{1,\,2}$ th columns of the 4×4 matrix Z, that is to say

$$n_{i_1 j_1, i_2 j_2}(Z) = \begin{vmatrix} Z_{i_1 j_1} & Z_{i_1 j_2} \\ Z_{i_2 j_1} & Z_{i_2 j_2} \end{vmatrix},$$

and

$$\begin{cases} S_{1j}^{(2)} = \frac{n_{1j,2(3-j)}(S)m_{2(3-j)}(s) + n_{1j,3(3-j)}(S)m_{3(3-j)}(s) + n_{1j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)}, \\ S_{2j}^{(2)} = \frac{n_{2j,1(3-j)}(S)m_{1(3-j)}(s) + n_{2j,3(3-j)}(S)m_{3(3-j)}(s) + n_{2j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)}, \\ S_{3j}^{(2)} = \frac{n_{3j,1(3-j)}(S)m_{1(3-j)}(s) + n_{3j,2(3-j)}(S)m_{3(3-j)}(s) + n_{3j,4(3-j)}(S)m_{4(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)}, \\ S_{4j}^{(2)} = \frac{n_{4j,1(3-j)}(S)m_{1(3-j)}(s) + n_{4j,2(3-j)}(S)m_{3(3-j)}(s) + n_{4j,3(3-j)}(S)m_{3(3-j)}(s)}{\Delta([S]_1[S]_2[s]_3[s]_4)}, \end{cases} \quad j = 1, 2, \tag{2.45}$$

$$\begin{cases} S_{1j}^{(4)} = \frac{n_{1j,2(7-j)}(S)m_{2(7-j)}(s) + n_{1j,3(7-j)}(S)m_{3(7-j)}(s) + n_{1j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)}, \\ S_{2j}^{(4)} = \frac{n_{2j,1(7-j)}(S)m_{1(7-j)}(s) + n_{2j,3(7-j)}(S)m_{3(7-j)}(s) + n_{2j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)}, \\ S_{3j}^{(4)} = \frac{n_{3j,1(7-j)}(S)m_{1(7-j)}(s) + n_{3j,2(7-j)}(S)m_{2(7-j)}(s) + n_{3j,4(7-j)}(S)m_{4(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)}, \\ S_{4j}^{(4)} = \frac{n_{4j,1(7-j)}(S)m_{1(7-j)}(s) + n_{4j,2(7-j)}(S)m_{2(7-j)}(s) + n_{4j,3(7-j)}(S)m_{3(7-j)}(s)}{\Delta([s]_1[s]_2[S]_3[S]_4)}, \end{cases} \quad j = 3, 4, \tag{2.46}$$

where $\Delta([S]_1[S]_2[s]_3[s]_4) = \det(n([S]_1,[S]_2,[s]_3,[s]_4))$ denotes the determinant of the matrix by choosing the first and second columns of $S(\lambda)$ and the third and fourth columns of $s(\lambda)$, and $\Delta([s]_1[s]_2[S]_3[S]_4) = \det(n([s]_1,[s]_2,[S]_3,[S]_4))$ denotes the determinant of the matrix by choosing the first and second columns of $s(\lambda)$ and the third and fourth columns of $S(\lambda)$, respectively.

Proof. We can use a similar method to the one in Ref. [23] to prove this Theorem. □

2.8. The residue conditions

Since $\mu_2(x, t, \lambda)$ is an entire function, and from Eq. (2.43) we know that $M(x, t, \lambda)$ only produces singularities in $S_n(\lambda)$ where there are singular points, from the exact expression Eq. (2.44), we know that $M(x, t, \lambda)$ may be singular as follows

- $[M_1]_1$ and $[M_1]_2$ could have poles in D_1 at the zeros of $n_{33,\,44}(s)(\lambda)$,
- $[M_2]_1$ and $[M_2]_2$ could have poles in D_2 at the zeros of $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$,
- $[M_3]_3$ and $[M_3]_4$ could have poles in D_3 at the zeros of $n_{11,\,22}(s)(\lambda)$,
- $[M_4]_3$ and $[M_4]_4$ could have poles in D_4 at the zeros of $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$.

We use $\{\lambda_j\}_1^N$ to denote the possible zero point above, and assume that these zeros satisfy the following assumptions

Assumption 2.3. Suppose that

- $n_{33,\,44}(s)(\lambda)$ admits $n_1 \geq 0$ possible simple zeros in D_1 denoted by $\{\lambda_j\}_1^{n_1}$,
- $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$ admits $n_2 - n_1 \geq 0$ possible simple zeros in D_2 denoted by $\{\lambda_j\}_{n_1+1}^{n_2}$,
- $n_{11,\,22}(s)(\lambda)$ admits $n_3 - n_2 \geq 0$ possible simple zeros in D_3 denoted by $\{\lambda_j\}_{n_2+1}^{n_3}$,
- $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$ admits $N - n_3 \geq 0$ possible simple zeros in D_4 denoted by $\{\lambda_j\}_{n_3+1}^{N}$.

And these zeros are each different. Moreover assume that there is no zero on the boundary of D_n's$(n = 1, 2, 3, 4)$.

Lemma 2.4. *For a 4×4 matrix $X = (X_{ij})_{4\times 4}$, $e^{\theta\hat{\sigma}_4}X$ is given by*

$$e^{\theta\hat{\sigma}_4}X = e^{\theta\sigma_4}Xe^{-\theta\sigma_4} = \begin{pmatrix} X_{11} & X_{12} & X_{13}e^{2\theta} & X_{14}e^{2\theta} \\ X_{21} & X_{22} & X_{23}e^{2\theta} & X_{24}e^{2\theta} \\ X_{31}e^{-2\theta} & X_{32}e^{-2\theta} & X_{33} & X_{34} \\ X_{41}e^{-2\theta} & X_{42}e^{-2\theta} & X_{43} & X_{44} \end{pmatrix}.$$

We can deduce the residue conditions at these zeros in the following expressions:

Proposition 2.5. *Let $\{M_n(x, t, \lambda)\}_1^4$ be the eigenfunctions are defined by Eq. (2.35) and assume that the set $\{\lambda_j\}_1^N$ of singularities are as the above Assumption 2.3. Then the following residue conditions hold true:*

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_1(x,t,\lambda)]_k = {} & \frac{m_{2(3-k)}(s)(\lambda_j)s_{24}(\lambda_j) - m_{1(3-k)}(s)(\lambda_j)s_{14}(\lambda_j)}{\dot{n}_{33,44}(s)(\lambda_j)n_{13,24}(s)(\lambda_j)}[M_1(x,t,\lambda_j)]_3 e^{\theta_{31}} \\ & + \frac{m_{1(3-k)}(s)(\lambda_j)s_{13}(\lambda_j) - m_{2(3-k)}(s)(\lambda_j)s_{23}(\lambda_j)}{\dot{n}_{33,44}(s)(\lambda_j)n_{13,24}(s)(\lambda_j)}[M_1(x,t,\lambda_j)]_4 e^{\theta_{31}}, \\ & 1 \leq j \leq n_1; \quad \lambda_j \in D_1, \quad k = 1, 2. \end{aligned} \tag{2.47}$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_2(x,t,\lambda)]_k = {} & \frac{S_{1k}^{(2)}(\lambda_j)s_{24}(\lambda_j) - S_{2k}^{(2)}(\lambda_j)s_{14}(\lambda_j)}{\Delta([S]_1[S]_2[\dot{s}]_3[s]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_2(x,t,\lambda_j)]_3 e^{\theta_{31}} \\ & + \frac{S_{2k}^{(2)}(\lambda_j)s_{13}(\lambda_j) - S_{1k}^{(2)}(\lambda_j)s_{23}(\lambda_j)}{\Delta([S]_1[S]_2[\dot{s}]_3[s]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_2(x,t,\lambda_j)]_4 e^{\theta_{31}}, \\ & n_1 + 1 \leq j \leq n_2; \quad \lambda_j \in D_2, \quad k = 1, 2. \end{aligned} \tag{2.48}$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_3(x,t,\lambda)]_k = {} & \frac{m_{4(7-k)}(s)(\lambda_j)s_{42}(\lambda_j) - m_{3(7-k)}(s)(\lambda_j)s_{32}(\lambda_j)}{\dot{n}_{11,22}(s)(\lambda_j)n_{31,42}(s)(\lambda_j)}[M_3(x,t,\lambda_j)]_1 e^{\theta_{13}} \\ & + \frac{m_{3(7-k)}(s)(\lambda_j)s_{31}(\lambda_j) - m_{4(7-k)}(s)(\lambda_j)s_{41}(\lambda_j)}{\dot{n}_{11,22}(s)(\lambda_j)n_{31,42}(s)(\lambda_j)}[M_3(x,t,\lambda_j)]_2 e^{\theta_{13}}, \\ & n_2 + 1 \leq j \leq n_3; \quad \lambda_j \in D_3, \quad k = 3, 4. \end{aligned} \tag{2.49}$$

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}[M_4(x,t,\lambda)]_k = {} & \frac{S_{1k}^{(4)}(\lambda_j)s_{22}(\lambda_j) - S_{2k}^{(4)}(\lambda_j)s_{12}(\lambda_j)}{\Delta([s]_1[s]_2[\dot{S}]_3[S]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_4(x,t,\lambda_j)]_1 e^{\theta_{13}} \\ & + \frac{S_{1k}^{(4)}(\lambda_j)s_{11}(\lambda_j) - S_{1k}^{(4)}(\lambda_j)s_{21}(\lambda_j)}{\Delta([s]_1[s]_2[\dot{S}]_3[S]_4)(\lambda_j)n_{11,22}(s)(\lambda_j)}[M_4(x,t,\lambda_j)]_2 e^{\theta_{13}}, \\ & n_3 + 1 \leq j \leq N; \quad \lambda_j \in D_4, \quad k = 3, 4. \end{aligned} \tag{2.50}$$

where $\dot{f} = \frac{df}{d\lambda}$ *and* θ_{ij} *defined by*

$$\theta_{ij}(x,t,\lambda) = (l_i - l_j)x - (z_i - z_j)t, \quad i,j = 1,2,3,4, \tag{2.51}$$

and thus, we have

$$\begin{aligned} \theta_{12} &= \theta_{21} = \theta_{34} = \theta_{43} = 0, \\ \theta_{13} &= \theta_{14} = \theta_{23} = \theta_{24} = -2i\lambda x - 8i\lambda^3 t, \\ \theta_{31} &= \theta_{41} = \theta_{32} = \theta_{42} = 2i\lambda x + 8i\lambda^3 t. \end{aligned}$$

Proof. We will only prove (2.49), the other conditions follow by similar arguments. It follows from (2.43) that

$$M_3(x,t,\lambda) = \mu_2(x,t,\lambda)e^{-(i\lambda x + 4i\lambda^3 t)\hat{\Lambda}}S_3, \tag{2.52}$$

In view of the expressions for S_3 given in (2.44), we find the four columns of Eq. (2.52) read

$$[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} + [\mu_2]_3 s_{31} e^{\theta_{31}} + [\mu_2]_4 s_{41} e^{\theta_{31}}, \tag{2.53}$$

$$[M_3]_2 = [\mu_2]_1 s_{12} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32} e^{\theta_{31}} + [\mu_2]_4 s_{42} e^{\theta_{31}}, \tag{2.54}$$

$$[M_3]_3 = [\mu_2]_3 \frac{m_{44}(s)}{n_{11,22}(s)} + [\mu_2]_4 \frac{m_{34}(s)}{n_{11,22}(s)}, \tag{2.55}$$

$$[M_3]_4 = [\mu_2]_3 \frac{m_{43}(s)}{n_{11,22}(s)} + [\mu_2]_4 \frac{m_{33}(s)}{n_{11,22}(s)}. \tag{2.56}$$

Suppose that $\lambda_j \in D_3$ is a simple zero of $n_{11,\,22}(s)(\lambda)$. Solving Eqs. (2.53) and (2.54) for $[\mu_2]_3$, $[\mu_2]_4$ and substituting the result into Eqs. (2.55) and (2.56) yields

$$[M_3]_3 = \frac{m_{44}(s)s_{42} - m_{34}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_3]_1 e^{\theta_{13}} + \frac{m_{34}(s)s_{31} - m_{44}(s)s_{41}}{n_{11,22}(s)n_{13,24}(s)}[M_3]_2 e^{\theta_{13}} + \frac{m_{24}(s)[\mu_2]_1 + m_{14}(s)[\mu_2]_2}{n_{31,42}(s)} e^{\theta_{13}}, \tag{2.57}$$

$$[M_3]_4 = \frac{m_{43}(s)s_{42} - m_{33}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_3]_1 e^{\theta_{13}} + \frac{m_{33}(s)s_{31} - m_{43}(s)s_{41}}{n_{11,22}(s)n_{31,42}(s)}[M_3]_2 e^{\theta_{13}} + \frac{m_{23}(s)[\mu_2]_1 + m_{13}(s)[\mu_2]_2}{n_{31,42}(s)} e^{\theta_{13}}, \tag{2.58}$$

By taking the residue of this equations at λ_j, it implies that condition Eqs. (2.57) and (2.58) in the case when $\lambda_j \in D_3$. □

3. The Riemann–Hilbert problem

For all values of x, t, the solution of system (1.4) can be recovered by solving this 4×4 matrix RH problem. So we can establish the following theorem.

Theorem 3.1. *Suppose that* $\{u_1(x,t), u_2(x,t)\}$ *is a solution of system (1.4) in the half-line domain* Ω, *and it is sufficient smoothness and decays when* $x \to \infty$. *Then the solution* $\{u_1(x,t), u_2(x,t)\}$ *of system (1.4) can be reconstructed from the initial values* $\{u_0(x), v_0(x)\}$ *and boundary values* $\{p_0(t), q_0(t), p_1(t), q_1(t)\}$ *defined as follows*

$$\begin{aligned} &\textit{Initial values}: u_0(x) = u_1(x, t=0),\ v_0(x) = u_2(x,t=0); \\ &\textit{Dirichlet boundary values}: p_0(t) = u_1(x=0,t),\ q_0(t) = u_2(x=0,t); \\ &\textit{Neumann boundary values}: p_1(t) = u_{1,x}(x=0,t),\ q_1(t) = u_{2,x}(x=0,t). \end{aligned} \tag{3.1}$$

Like Eq. (2.25), by using the initial and boundary data to define the spectral functions $s(\lambda)$ *and* $S(\lambda)$, *we can further get the jump matrix* $J_{m,\,n}(x,t,\lambda)$. *Assume that the zero points of the* $n_{33,\,44}(s)(\lambda)$, $\Delta([S]_1[S]_2[s]_3[s]_4)(\lambda)$, $n_{11,\,22}(s)(\lambda)$ *and* $\Delta([s]_1[s]_2[S]_3[S]_4)(\lambda)$ *are* $\{\lambda_j\}_1^N$ *just like in Assumption 2.3. We then have the following results*

$$\begin{aligned} u_1(x,t) &= -2i \lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{13}, \\ u_2(x,t) &= -2i \lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{14}, \end{aligned} \tag{3.2}$$

where $M(x,t,\lambda)$ *satisfies the following* 4×4 *matrix RH problem:*

- $M(x,t,\lambda)$ *is a sectionally meromorphic on the Riemann* λ*-sphere with jumps across the contours on* $\bar{D}_n \cap \bar{D}_m (n,m = 1,\ldots,4)$ *(see Fig. 3).*

- *$M(x, t, \lambda)$ satisfies the jump condition with jumps across the contours on $\bar{D}_n \cap \bar{D}_m (n, m = 1, \dots, 4)$*

$$M_n(x,t,\lambda) = M_m(x,t,\lambda) J_{m,n}(x,t,\lambda), \quad \lambda \in \bar{D}_n \cap \bar{D}_m, \quad n \neq m. \tag{3.3}$$

- $M(x,t,\lambda) = \mathbb{I} + O(\frac{1}{\lambda}), \ \lambda \to \infty.$
- *The residue condition of $M(x, t, \lambda)$ is showed in Proposition 2.5.*

Proof. The Eq. (3.2) can be deduced from the large λ asymptotics of the eigenfunctions. We can follow the similar one in Refs. [17,18,27] to show the rest proof of the Theorem. We omit the proof here due to a long and complicated computation. □

4. Conclusions

In this work, we have considered the IBV problems of the two-component mKdV equation on the half-line. Using the Fokas method for nonlinear evolution equations, which takes the form of Lax pair isospectral deformations and the corresponding continuous spectra Lax operators, under the assumption that the solution $\{u_1(x, t), u_2(x, t)\}$ exists, we have showed that the solution can be represented in terms of the solution of a 4×4 matrix RH problem formulated in the plane of the complex spectral parameter λ. The spectral functions $s(\lambda)$ and $S(\lambda)$ are not independent, but related by a compatibility condition – the so-called global relation. For other integrable equations with high-order matrix Lax pairs, can we construct their solutions of associated matrix RH problems formulated in the plane of the complex spectral parameter λ similarly? This question will be discussed in our future paper.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under the grant 12271008, 11601055 and 51771083, National Science Foundation under the grant DMS-1664561, Natural Science Foundation of Anhui Province of China under the grant 1408085QA06, and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University.

References

[1] M. Wadati, The modified Korteweg-de Vries equation, J. Phys. Soc. Jpn. 34 (1973) 1289–1296.
[2] N. Yajima, M. Oikawa, A class of exactly solvable nonlinear evolution equations, Prog. Theor. phys. 54 (1975) 1576–1577.
[3] C. Athorne, A.P. Fordy, Generalised KdV and MKdV equations associated with symmetric spaces, J. Phys. A Math. Gen. 20 (1987) 1377–1386.
[4] I.T. Habibullin, An initial-boundary value problem on the half-line for the MKdv equation, Funct. Anal. Appl. 34 (2000) 52–59.
[5] A.B. de Monvel, A.S. Fokas, D. Shepelsky, The MKdV equation on the half-line, J. Inst. Math. Jussieu 3 (2004) 139–164.
[6] R. Hirota, Molecule solution of coupled modified KdV equation, J. Phys. Soc. Jpn. 66 (1997) 2530–2532.
[7] M. Iwao, R. Hirota, Soliton solutions of a coupled modified KdV equations, J. Phys. Soc. Jpn. 66 (1997) 577–588.
[8] T. Tsuchida, M. Wadati, The coupled modified Korteweg-de Vries equations, J. Phys. Soc. Jpn. 67 (1998) 1175–1187.
[9] N. Sasa, J. Satsuma, New-type of soliton solutions for a higher-order nonlinear Schrödinger equation, J. Phys. Soc. Jpn. 60 (1991) 409–417.
[10] T. Kanna, M. Lakshmanan, P.T. Dinda, N. Akhmediev, Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations, Phys. Rev. E 73 (2006) 026604.
[11] H.Q. Zhang, B. Tian, T. Xu, H. Li, C. Zhang, H. Zhang, Lax pair and Darboux transformation for multi-component modified Korteweg-de Vries equations, J. Phys. A Math. Theor. 41 (2008) 355210.
[12] Q.H. Park, H.J. Shin, Darboux transformation and Crum's formula for multi-component integrable equations, Phys. D 157 (2001) 1–15.
[13] H. Liu, X.G. Geng, Initial-boundary problems for the vector modified Korteweg-dev Ries equation via Fokas unified transform method, J. Math. Anal. Appl. 440 (2016) 578–596.
[14] B. Xue, F. Li, G. Yang, Explicit solutions and conservation laws of the coupled modified Korteweg-dev Ries equation, Phys. Scr. 90 (2015) 085204.
[15] X.G. Geng, Y.Y. Zhai, H.H. Dai, Algebro-geometric solutions of the coupled modified Korteweg-dev Ries hierarchy, Adv. Math. 263 (2014) 123–153.
[16] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, Proc. R. Soc. Lond. A 453 (1997) 1411–1443.
[17] A.S. Fokas, Integrable nonlinear evolution equations on the half-line, Commun. Math. Phys. 230 (2002) 1–39.
[18] A.S. Fokas, A.R. Its, L.Y. Sung, The nonlinear Schrödinger equation on the half-line, Nonlinearity 18 (2005) 1771–1822.
[19] J. Lenells, A.S. Fokas, Boundary-value problems for the stationary axisymmetric einstein equations: a rotating disc, Nonlinearity 24 (2011) 177–206.
[20] J. Lenells, Boundary value problems for the stationary axisymmetric einstein equations: a disk rotating around a black hole, Commun. Math. Phys. 304 (2011) 585–635.
[21] N. Zhang, T.C. Xia, B.B. Hu, A Riemann–Hilbert approach to the complex Sharma–Tasso–Olver equation on the half line, Commun. Theor. Phys. 68 (5) (2017) 580–594.
[22] B.Q. Xia, A.S. Fokas, Initial-boundary value problems associated with the Ablowitz–Ladik system, Phys. D 364 (2018) 27–61.
[23] J. Lenells, Initial-boundary value problems for integrable evolution equations with 3 × 3 lax pairs, Phys. D 241 (2012) 857–875.
[24] J. Lenells, The Degasperis–Procesi equation on the half-line, Nonlinear Anal. 76 (2013) 122–139.
[25] A.B. de Monvel, D. Shepelsky, A Riemann–Hilbert approach for the Degasperis–Procesi equation, Nonlinearity 26 (2013) 2081–2107.
[26] A.B. de Monvel, D. Shepelsky, The Ostrovsky–Vakhnenko equation by a Riemann–0Hilbert approach, J. Phys. A Math. Theor. 48 (2015) 035204.
[27] J. Xu, E.G. Fan, The unified method for the Sasa–Satsuma equation on the half-line, Proc. R. Soc. A, Math. Phys. Eng. Sci. 469 (2013) 1–25.
[28] J. Xu, E.G. Fan, The three wave equation on the half-line, Phys. Lett. A 378 (2014) 26–33.
[29] Z.Y. Yan, An initial-boundary value problem for the integrable spin-1 gross-Pitaevskii equations with a 4 × 4 lax pair on the half-line, CHAOS 27 (2017) 053117.
[30] X.G. Geng, H. Liu, J.Y. Zhu, Initial-boundary value problems for the coupled nonlinear Schrödinger equation on the half-line, Stud. Appl. Math. 135 (2015) 310–346.
[31] J. Xu, Initial-boundary value problem for the two-component nonlinear Schrödinger equation on the half-line, J. Nonlinear Math. Phys. 23 (2) (2016) 167–189.
[32] A.B. de Monvel, D. Shepelsky, L. Zielinski, A Riemann–Hilbert approach for the Novikov equation, SIGMA 12 (2016) 095.
[33] S.F. Tian, Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method, J. Differ. Equ. 262 (1) (2017) 506–558.

[34] B.B. Hu, T.C. Xia, N. Zhang, J.B. Wang, Initial-boundary value problems for the coupled higher-order nonlinear Schrödinger equations on the half-line, Int. J. Nonlinear Sci. Numer. Simul. 19 (1) (2018) 83–92.
[35] B.B. Hu, T.C. Xia, The coupled Fokas–Lenells equations by a Riemann–Hilbert approach, 2017, arXiv:1711.03861.
[36] W.X. Ma, H.H. Dong, Modeling Riemann–Hilbert problems to get soliton solutions, Math. Model. Appl. 6 (3) (2017) 16–25.
[37] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems, Ann. Math. 137 (1993) 295–368.
[38] S.F. Tian, Initial-boundary value problems of the coupled modified Korteweg-de Vries equation on the half-line via the Fokas method, J. Phys. A Math. Theor. 50 (39) (2017) 395204.