ISSN 1560-3547, Regular and Chaotic Dynamics, 2020, Vol. 25, No. 4, pp. 383–391.

A New $(3+1)$-dimensional Hirota Bilinear Equation: Its Bäcklund Transformation and Rational-type Solutions

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Received May 07, 2020; revised June 11, 2020; accepted June 15, 2020

Abstract—The behavior of specific dispersive waves in a new $(3+1)$-dimensional Hirota bilinear (3D-HB) equation is studied. A Bäcklund transformation and a Hirota bilinear form of the model are first extracted from the truncated Painlevé expansion. Through a series of mathematical analyses, it is then revealed that the new 3D-HB equation possesses a series of rational-type solutions. The interaction of lump-type and 1-soliton solutions is studied and some interesting and useful results are presented.

MSC2010 numbers: 45K05, 83C15

DOI: 10.1134/S156035472004005X

Keywords: new (3+1)-dimensional Hirota bilinear equation, Bäcklund transformation, Hirota bilinear form, rational-type solutions

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1. INTRODUCTION

In the last few decades, nonlinear evolution (NLE) equations have been considered by a lot of researchers in the field of mathematical physics. It is known that exact solutions extracted from NLE equations provide comprehensive information about real-world phenomena. For this reason, searching for exact solutions of NLE equations plays a vital role in mathematical physics and is the major topic of many works. An attractive kind of exact solutions is referred to as rational-type solutions which include soliton, lump, lump-kink, breather-wave, and rogue-wave solutions. Because of the importance of these types of exact solutions, a wide range of scholars have devoted their studies to looking for rational-type solutions of nonlinear evolution equations. For example, Wazwaz and El-Tantawy in [1] exerted the simplified Hirota method to seek solitons of $(3+1)$-dimensional KP−Boussinesq and BKP−Boussinesq equations. In another work performed by Manukure et al. [2], lump solutions of a $(2+1)$-dimensional extended Kadomtsev−Petviashvili equation were obtained by making use of quadratic test functions. Lump-kink solutions to the KP equation were constructed in [3] by considering an ansatz which is a combination of positive quadratic and exponential functions. Lan [4] reported breather-wave and rogue-wave solutions of a generalized $(3+1)$-dimensional B-type Kadomtsev−Petviashvili equation with the variable coefficient using the homoclinic test technique. For more papers, see [5–36].

The fundamental purpose of the present article is to study a new 3D-HB equation in fluids in the following form:

$$u_{yt} + c_1(u_{xxxy} + 6u_xu_y + 3u_{xy}u + 3u_{xx}\int u_y dx) + c_2u_{yy} + c_3u_{zz} = 0, \tag{1.1}$$

or

$$u_{yt} + c_1(u_{xxxy} + 6u_xu_y + 3u_{xy}u + 3u_{xx}v) + c_2u_{yy} + c_3u_{zz} = 0, \quad u_y = v_x, \tag{1.2}$$

and to retrieve a bunch of rational-type solutions for it by exerting a number of effective techniques.

It is worthy of note that by considering $c_3 = 0$, the above new 3D-HB equation is reduced to the 2D-HB equation which has been considered by Hua et al. in [37]. Hua and his collaborators extracted interaction solutions of the 2D-HB equation by utilizing a series of ansatz techniques. Hosseini et al. [38] also found rational wave solutions of the 2D-HB equation with different structures by means of a number of useful approaches.

The structure of this paper is as follows: In Section 2, by using the truncated Painlevé expansion, the Bäcklund transformation and Hirota bilinear form of the new 3D-HB equation are formally derived. In Section 3, a series of test functions is applied to obtain rational-type solutions of the new 3D-HB equation. The last section presents the outcomes of the current article.

2. BÄCKLUND TRANSFORMATION AND HIROTA BILINEAR FORM OF THE MODEL

According to the truncated Painlevé expansion, the Bäcklund transformation of the system (1.2) can be written as

$$u = \frac{u_0}{f^2} + \frac{u_1}{f} + u_2, \quad v = \frac{v_0}{f^2} + \frac{v_1}{f} + v_2, \tag{2.1}$$

where f is a function of variables x, y, z, and t. The functions u_2 and v_2 are arbitrary solutions of the new 3D-HB equation and u_0, u_1, v_0, and v_1 are unknown functions including the derivatives of f.

Now, by inserting (2.1) into the system (1.2) and solving the resulting system obtained by equating the coefficients of f^{-6} and f^{-3} to zero, we obtain

$$u_0 = -2f_x^2, \quad v_0 = -2f_xf_y. \tag{2.2}$$

Similarly, by considering $u_2 = v_2 = 0$, the relations presented in (2.2), and equating the coefficients of f^{-5} and f^{-2} to zero, we arrive at a system of nonlinear PDEs whose solution gives

$$u_1 = 2f_{xx}, \quad v_1 = 2f_{xy}.$$

Substituting the above results into (2.1) finally results in the following Bäcklund transformation for the new 3D-HB equation:

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}. \tag{2.3}$$

Based on the Bäcklund transformation (2.3), Hirota bilinear form corresponding to the new 3D-HB equation can be written as

$$\begin{aligned} B_{3D-HB}(f) := (D_y D_t + c_1 D_x^3 D_y + c_2 D_y^2 + c_3 D_z^2)f.f = 2(ff_{yt} - f_y f_t + c_1(ff_{xxxy} - 3(f_{xxy} f_x \\ - f_{xy} f_{xx}) - f_y f_{xxx}) + c_2(ff_{yy} - f_y^2) + c_3(ff_{zz} - f_z^2)) = 0, \end{aligned} \tag{2.4}$$

where D_x, D_y, D_y, and D_z are Hirota's bilinear operators.

3. RATIONAL-TYPE SOLUTIONS OF THE NEW 3D-HB EQUATION

In the present section, rational-type solutions of the new 3D-HB equation with different structures are formally established by exerting a number of effective techniques.

3.1. Soliton Solutions

To derive soliton solutions of the governing model, we substitute $u = e^{k_i x + \tau_i y + \xi_i z + \varsigma_i t}$ into the linear terms of (1.1) and solve the resulting equation for ς_i. After that, the dispersion relation ς_i can be written as

$$\varsigma_i = -\frac{c_1 k_i^3 \tau_i + c_2 \tau_i^2 + c_3 \xi_i^2}{\tau_i},$$

and so

$$\theta_i = k_i x + \tau_i y + \xi_i z - \frac{c_1 k_i^3 \tau_i + c_2 \tau_i^2 + c_3 \xi_i^2}{\tau_i} t,$$

where $\theta_i, 1 \leqslant i \leqslant 3$ are phase variables.
Now, by considering the dependent variables

$$u = R(\ln f)_{xx}, \quad v = R(\ln f)_{xy},$$

and the exponential function

$$f = 1 + e^{k_1 x + \tau_1 y + \xi_1 z - \frac{c_1 k_1^3 \tau_1 + c_2 \tau_1^2 + c_3 \xi_1^2}{\tau_1} t},$$

we obtain $R = 2$. Accordingly, the following 1-soliton solution can be obtained:

$$u = 2\left(\ln\left(1 + e^{k_1 x + \tau_1 y + \xi_1 z - \frac{c_1 k_1^3 \tau_1 + c_2 \tau_1^2 + c_3 \xi_1^2}{\tau_1} t}\right)\right)_{xx}, \quad v = 2\left(\ln\left(1 + e^{k_1 x + \tau_1 y + \xi_1 z - \frac{c_1 k_1^3 \tau_1 + c_2 \tau_1^2 + c_3 \xi_1^2}{\tau_1} t}\right)\right)_{xy}.$$

Now, we formally take

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}$$

as the auxiliary function in order to retrieve the following 2-soliton solution:

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}.$$

It is noted that the phase variables $\theta_i, i = 1, 2$ are defined as above and the phase shift a_{12} is given as

$$a_{12} = \frac{-3c_1 k_1^2 k_2 \tau_1^2 \tau_2 + 3c_1 k_1^2 k_2 \tau_1 \tau_2^2 + 3c_1 k_1 k_2^2 \tau_1^2 \tau_2 - 3c_1 k_1 k_2^2 \tau_1 \tau_2^2 + c_3 \tau_1^2 \xi_2^2 - 2c_3 \tau_1 \tau_2 \xi_1 \xi_2 + c_3 \tau_2^2 \xi_1^2}{-3c_1 k_1^2 k_2 \tau_1^2 \tau_2 - 3c_1 k_1^2 k_2 \tau_1 \tau_2^2 - 3c_1 k_1 k_2^2 \tau_1^2 \tau_2 - 3c_1 k_1 k_2^2 \tau_1 \tau_2^2 + c_3 \tau_1^2 \xi_2^2 - 2c_3 \tau_1 \tau_2 \xi_1 \xi_2 + c_3 \tau_2^2 \xi_1^2}.$$

The special 3-soliton solution of the new 3D-HB equation, namely,

$$u = 2(\ln f)_{xx}, \; v = 2(\ln f)_{xy}, \; f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + a_{12} a_{13} a_{23} e^{\theta_1 + \theta_2 + \theta_3},$$

where

$$\theta_i = k_i x + \tau_i y + \xi_i z - \frac{c_1 k_i^3 \tau_i + c_2 \tau_i^2 + c_3 \xi_i^2}{\tau_i} t, \qquad 1 \leqslant i \leqslant 3,$$

$$a_{ij} = \frac{-3c_1 k_i^2 k_j \tau_i^2 \tau_j + 3c_1 k_i^2 k_j \tau_i \tau_j^2 + 3c_1 k_i k_j^2 \tau_i^2 \tau_j - 3c_1 k_i k_j^2 \tau_i \tau_j^2 + c_3 \tau_i^2 \xi_j^2 - 2c_3 \tau_i \tau_j \xi_i \xi_j + c_3 \tau_j^2 \xi_i^2}{-3c_1 k_i^2 k_j \tau_i^2 \tau_j - 3c_1 k_i^2 k_j \tau_i \tau_j^2 - 3c_1 k_i k_j^2 \tau_i^2 \tau_j - 3c_1 k_i k_j^2 \tau_i \tau_j^2 + c_3 \tau_i^2 \xi_j^2 - 2c_3 \tau_i \tau_j \xi_i \xi_j + c_3 \tau_j^2 \xi_i^2},$$
$$1 \leqslant i, j \leqslant 3,$$

can be achieved if and only if the following 3-soliton condition is satisfied [39, 40]:

$$\sum\nolimits_{\mu_1,\mu_2,\mu_3=\pm 1} P(\mu_1 V_1 + \mu_2 V_2 + \mu_3 V_3) P(\mu_1 V_1 - \mu_2 V_2) P(\mu_2 V_2 - \mu_3 V_3) P(\mu_1 V_1 - \mu_3 V_3)$$
$$= 2 \sum\nolimits_{(\mu_1,\mu_2,\mu_3)\in S} P(\mu_1 V_1 + \mu_2 V_2 + \mu_3 V_3) P(\mu_1 V_1 - \mu_2 V_2) P(\mu_2 V_2 - \mu_3 V_3) P(\mu_1 V_1 - \mu_3 V_3) = 0,$$

in which $P = yt + c_1 x^3 y + c_2 y^2 + c_3 z^2, V_i = (k_i, \tau_i, \xi_i, \varsigma_i)$, and $S = \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1)\}$. The 1-soliton, 2-soliton, and special 3-soliton solutions of the new 3D-HB equation are presented in Figs. 1–3, presenting the behavior of dispersive waves.

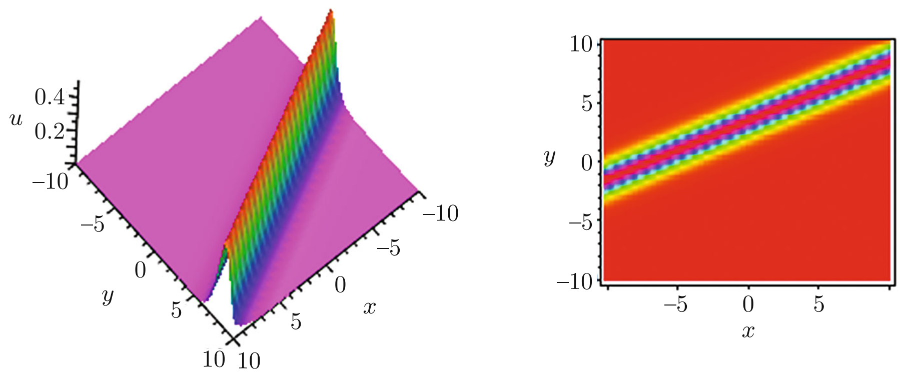


Fig. 1. 1-soliton solution on the $x - y$ plane for $k_1 = 1, \tau_1 = -2, \xi_1 = 2, c_1 = 1, c_2 = 1, c_3 = 2, z = 1$, and $t = 1$.

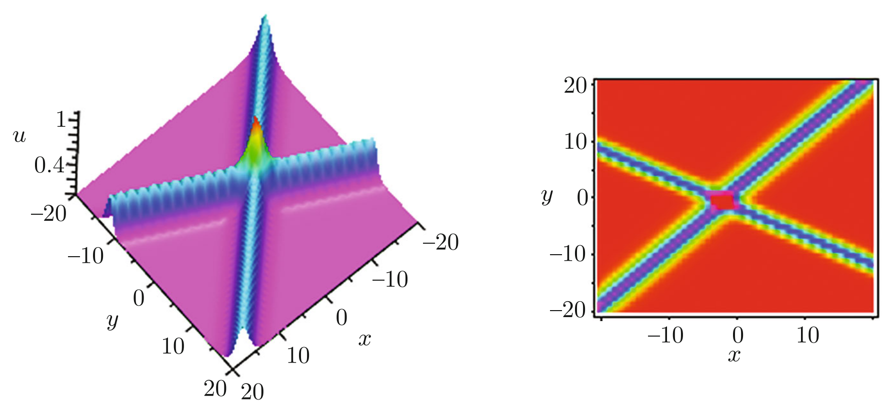


Fig. 2. 2-soliton solution on the $x - y$ plane for $k_1 = 1$, $\tau_1 = 2$, $\xi_1 = -1$, $k_2 = -1$, $\tau_2 = 1$, $\xi_2 = 2$, $c_1 = -1$, $c_2 = -2$, $c_3 = 1$, $z = 1$, and $t = 1$.

3.2. Lump-type Solutions

To retrieve lump-type solutions of the new 3D-HB equation, we exert a test function as follows:

$$f = g^2 + h^2 + a_{11}, \tag{3.1}$$

where

$$g = a_1 x + a_2 y + a_3 z + a_4 t + a_5, \quad h = a_6 x + a_7 y + a_8 z + a_9 t + a_{10},$$

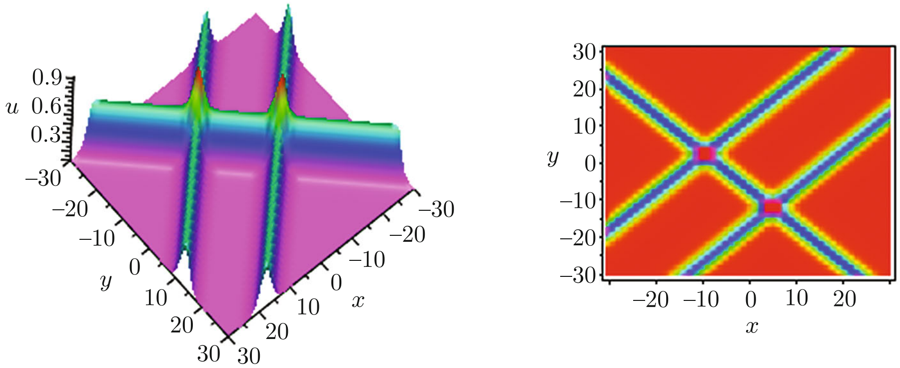


Fig. 3. Special 3-soliton on the $x - y$ plane for $k_1 = -1$, $\tau_1 = 1$, $\xi_1 = 2$, $k_2 = 1$, $\tau_2 = -1$, $\xi_2 = 1$, $k_3 = 1$, $\tau_3 = 1$, $\xi_3 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $z = 10$, and $t = 1$.

and $a_i, 1 \leqslant i \leqslant 11$ are real constants to be computed later. By setting (3.1) in (2.4) and adopting specific operations, we find the following results:

$$a_1 = -\frac{a_6 a_7}{a_2}, \quad a_4 = -\frac{a_2^2 c_2 + a_3^2 c_3}{a_2}, \quad a_8 = \frac{a_3 a_7}{a_2}, \quad a_9 = -\frac{a_7\left(a_2^2 c_2 + a_3^2 c_3\right)}{a_2^2}.$$

Now, a lump-type solution is obtained as

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}, \tag{3.2}$$

such that

$$f = \left(-\frac{a_6 a_7}{a_2}x + a_2 y + a_3 z - \frac{a_2^2 c_2 + a_3^2 c_3}{a_2}t + a_5\right)^2 + \left(a_6 x + a_7 y + \frac{a_3 a_7}{a_2}z - \frac{a_7(a_2^2 c_2 + a_3^2 c_3)}{a_2^2}t + a_{10}\right)^2 + a_{11}.$$

The lump-type solution of the new 3D-HB equation given by (3.2) is demonstrated in Fig. 4 for a special choice of free parameters.

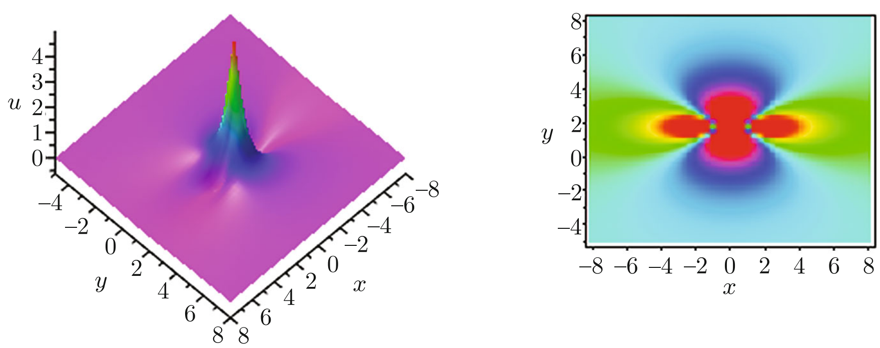


Fig. 4. Lump-type solution on the $x - y$ plane for $a_2 = 2, a_3 = -1, a_5 = -1, a_6 = 1, a_7 = 1, a_{10} = -0.5, a_{11} = 1, c_2 = 1, c_3 = -1, z = 1$, and $t = 1$.

3.3. Interaction Solutions

To derive interaction solutions of the new 3D-HB equation, we employ the following test function:

$$f = g^2 + h^2 + ke^{k_1 x + k_2 y + k_3 z + k_4 t} + a_{11}, \tag{3.3}$$

where

$$g = a_1 x + a_2 y + a_3 z + a_4 t + a_5, \quad h = a_6 x + a_7 y + a_8 z + a_9 t + a_{10},$$

and $a_i, 1 \leqslant i \leqslant 11$ and $k_j, 1 \leqslant j \leqslant 4$ are real constants to be calculated, and k is a positive real constant. By substituting (3.3) into (2.4) and applying specific operations, we find

$$a_1 = -\frac{a_6 a_7}{a_2}, \quad a_4 = -\frac{a_2^2 c_2 + a_3^2 c_3}{a_2}, \quad a_8 = \frac{a_3 a_7}{a_2}, \quad a_9 = -\frac{a_7\left(a_2^2 c_2 + a_3^2 c_3\right)}{a_2^2},$$
$$k_2 = 0, \quad k_3 = 0, \quad k_4 = -c_1 k_1^3.$$

Now we find the following interaction solution:

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}, \tag{3.4}$$

in which

$$f = \left(-\frac{a_6 a_7}{a_2}x + a_2 y + a_3 z - \frac{a_2^2 c_2 + a_3^2 c_3}{a_2}t + a_5\right)^2 + \left(a_6 x + a_7 y + \frac{a_3 a_7}{a_2}z - \frac{a_7(a_2^2 c_2 + a_3^2 c_3)}{a_2^2}t + a_{10}\right)^2$$
$$+ k e^{k_1 x - c_1 k_1^3 t} + a_{11}.$$

Three-dimensional and density plots of the interaction solution (3.4) are illustrated in Figs. 5 and 6 for different choices of t.

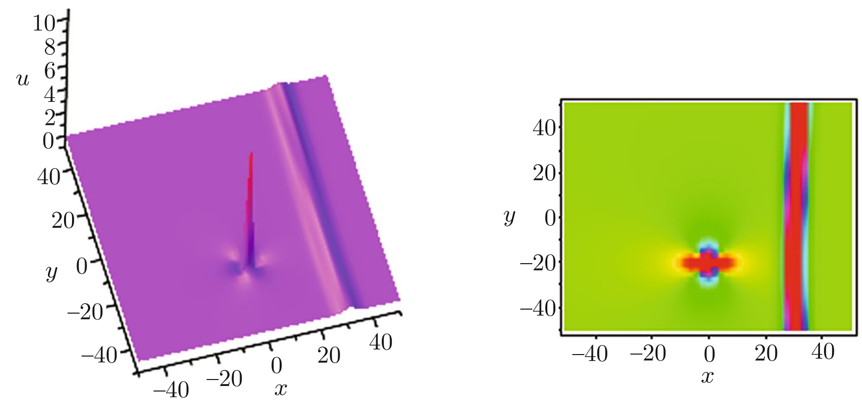


Fig. 5. Interaction solution on the $x - y$ plane for $a_2 = -5, a_3 = 1, a_5 = -1, a_6 = 5, a_7 = 0.1, a_{10} = 1, a_{11} = 1, k = 1, k_1 = 1, c_1 = -0.2, c_2 = 0.2, c_3 = 0.1, z = 1$, and $t = -100$.

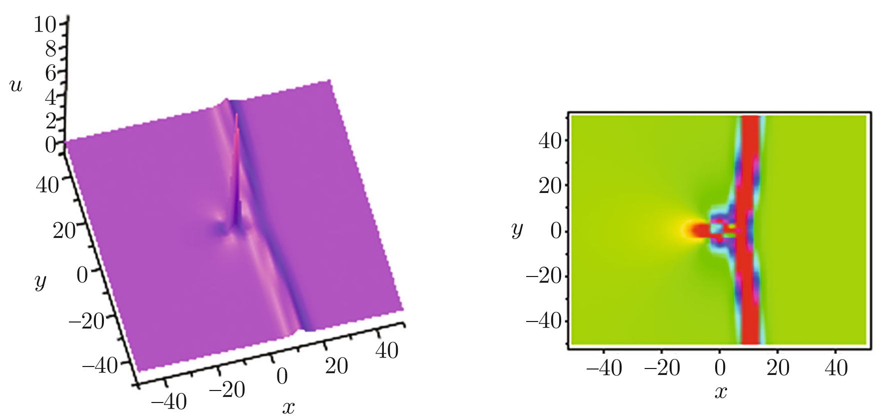


Fig. 6. Interaction solution on the $x - y$ plane for $a_2 = -5$, $a_3 = 1$, $a_5 = -1$, $a_6 = 5$, $a_7 = 0.1$, $a_{10} = 1$, $a_{11} = 1$, $k = 1$, $k_1 = 1$, $c_1 = -0.2$, $c_2 = 0.2$, $c_3 = 0.1$, $z = 1$, and $t = 1$.

It is worthy of note that when $-c_1 k_1^3 > 0$ and $t \to +\infty$, the lump-type solution vanishes and the 1-soliton solution stays.

3.4. Breather-wave and Rogue-wave Solutions

To obtain breather-wave solutions of the new 3D-HB equation, we use the following ansatz:

$$f = e^{-kg} + b_0 \cos(\tau h) + b_1 e^{kg}, \tag{3.5}$$

where

$$g = a_1 x + a_2 y + a_3 z + a_4 t + a_5, \quad h = a_6 x + a_7 y + a_8 z + a_9 t + a_{10}.$$

By setting (3.5) in (2.4) and applying specific operations, we derive

$$a_2 = 0, \ a_4 = -\frac{k^2 a_1^3 a_7 c_1 + 2a_3 a_8 c_3}{a_7}, \ a_6 = 0, \ a_9 = -\frac{\tau^2 a_7^2 c_2 + \tau^2 a_8^2 c_3 - k^2 a_3^2 c_3}{\tau^2 a_7}, \ b_1 = \frac{1}{4} b_0^2.$$

Now a breather-wave solution is found as

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy},$$

where

$$f = e^{-k\left(a_1 x + a_3 z - \frac{k^2 a_1^3 a_7 c_1 + 2a_3 a_8 c_3}{a_7} t + a_5\right)} + b_0 \cos\left(\tau\left(a_7 y + a_8 z - \frac{\tau^2 a_7^2 c_2 + \tau^2 a_8^2 c_3 - k^2 a_3^2 c_3}{\tau^2 a_7} t + a_{10}\right)\right) + \frac{1}{4} b_0^2 e^{k\left(a_1 x + a_3 z - \frac{k^2 a_1^3 a_7 c_1 + 2a_3 a_8 c_3}{a_7} t + a_5\right)}.$$

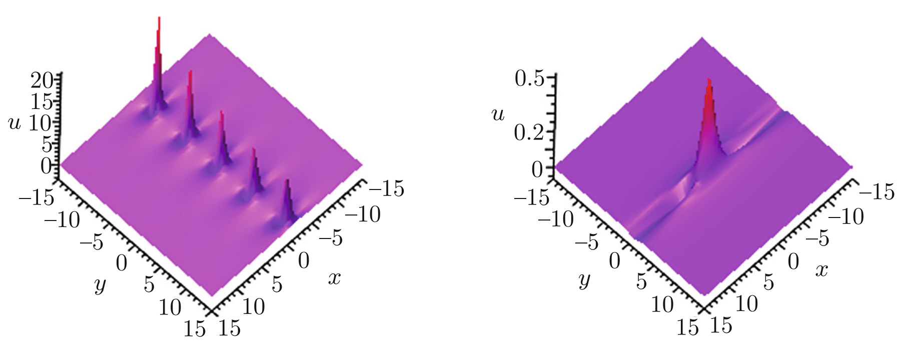
Now, by assuming $b_0 = -2, \tau = k$, and $k \to 0$, the following rogue-wave solution can be obtained:

$$u(x,y,z,t) = \frac{4a_1^2}{\theta^2 + \nu^2} - \frac{8a_1^2 \theta^2}{(\theta^2 + \nu^2)^2}, \quad v(x,y,z,t) = -\frac{8a_1 a_7 \theta \nu}{(\theta^2 + \nu^2)^2},$$

where

$$\theta = a_1 x + a_3 z - \frac{2a_3 a_8 c_3}{a_7} t + a_5, \quad \nu = a_7 y + a_8 z - \frac{a_7^2 c_2 + a_8^2 c_3 - a_3^2 c_3}{a_7} t + a_{10}.$$

The breather-wave and rogue-wave solutions are formally plotted in Fig. 7 for a special choice of free parameters.



(a) $a_1 = 0.1$, $a_3 = 1$, $a_5 = -0.1$, $a_7 = 1$, $a_8 = 1$, $a_{10} = -1$, $k = -5$, $\tau = 1$, $b_0 = -1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $z = 0.1$ and $t = 0.001$

(b) $a_1 = 0.1$, $a_3 = 1$, $a_5 = -0.1$, $a_7 = 1$, $a_8 = 1$, $a_{10} = -1$, $c_2 = 1$, $c_3 = 1$, $z = 0.1$ and $t = 0.001$

Fig. 7. (a) Breather-wave solution on the $x - y$ plane; (b) Rogue-wave solution on the $x - y$ plane.

4. CONCLUSION

In the present paper, a new $(3+1)$-dimensional Hirota bilinear equation has been developed and its rational-type solutions have been obtained successfully. In this respect,

- the truncated Painlevé expansion was utilized to derive the Bäcklund transformation and Hirota bilinear form of the model;
- soliton solutions were extracted by applying the simplified Hirota method and the 3-soliton condition;
- the lump-type solution was established by considering two positive quadratic functions as an ansatz;
- the interaction solution was retrieved by exerting an ansatz composed of two positive quadratic functions and an exponential function;
- the breather-wave solution and its corresponding rogue-wave solution were constructed by adopting the homoclinic test technique.

In the end, the interaction of lump-type and 1-soliton solutions was studied and some interesting and useful results were presented.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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