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Revealing abundant novel optical wave solutions of the two nonlinear models with dynamical behaviors: Bifurcation, chaotic, and stability analyses

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The Benney–Luke equation is a nonlinear partial differential equation that has abundant purposes in numerous scientific domains, including shallow water waves, coastal engineering, water wave theory, nonlinear optics, wave propagation in plasmas, seismic waves, nonlinear acoustics, and wave interaction studies. Here, we explore the novel optical abundant solutions for the Benney–Luke equation as well as the Burgers equation applying the new generalized (G'/G)-expansion technique. Additionally, bifurcation and chaotic studies of the Benney–Luke

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equation, along with stability investigation on the Burgers equation, are conducted to elucidate the mechanisms underlying the derived soliton solutions. The attained consequences are symbolized by trigonometric, hyperbolic, and rational functions, which have been used in practical life to explain the facts that have come out in everyday activities. Some of the results are revealed in 3D and contour form with the selection of appropriate parameters that show the singular kink, singular periodic, kink, singular bell, anti-bell, and anti-kink-shaped soliton. Unique features are exhibited with the help of our discovered solutions that have not been noticed in the existing outcomes. Additionally, the initiated expansion procedure is more practical, authentic, and highly effective in resolving these types of equations that appear in modern physics and applied mathematics.

Keywords: Benney–Luke equation; Burgers equation; new generalized (G'/G)-expansion method; exact solution; dynamical aspects; nonlinear PDEs.

1. Introduction

Most of the events around us in the real world are nonlinear — water wave propagation, gas dynamics, and traffic flow — are a few examples of many such complex phenomena in our everyday experience. Typically, nonlinear evolution equations (NLEEs) serve the purpose of modeling these facts, leading to different categories of wave solutions, commonly in the form of shock waves, solitary waves, and cnoidal waves.[1,2] Waves manifest in numerous contexts in daily life; common instances include shallow water settings, nonlinear acoustics, which occur both in fluids and solids, and various branches of applied mathematics and physics to describe nonlinear phenomena, gas dynamics, traffic flow, modeling of fluid mechanics, boundary layer activities, and shock wave development. The terms diffusion, reaction, dissipation, convection, and dispersion are closely associated with the aforementioned phenomena and can be examined exactly through NLEEs. The closed-form wave solutions of these NLEEs play an essential role in nonlinear physical science as they can provide a higher degree of physical information regarding complex phenomena. However, it is challenging to search for the closed-form solutions of NLEEs; many researchers have devoted themselves to the advancement of various potent and effective techniques to discover the precise solutions of NLEEs, such as the sine-Gordon expansion method,[3] Darboux transformation technique,[4] two variable $(G'/G, 1/G)$-expansion process,[5,6] bilinear neural network method,[7–10] bilinear residual network method,[11–13] generalized Kudryashov technique,[14] unified solver method,[15] improved simple equation scheme,[16] modification of mathematical method,[17] modified direct similarity reduction technique,[18] extended hyperbolic function method,[19] modified tanh–coth method,[20] multiple exp-function procedure,[21] the Lie symmetry analysis,[22] Lyapunov exponents,[23–25] Backlund transformations,[25] Sardar sub-equation method,[27] the $(\frac{G'}{G'+G+A})$-expansion technique,[28] and several others. To reveal the analytic solutions of NLEEs, Wang *et al.*[29] initiated the $(\frac{G'}{G})$-expansion scheme and also analyzed the KdV equation, the mKdV equation, the variant Boussinesq equations, and the Hirota–Satsuma equations. In Ref. 30, scholars introduced the improved $(\frac{G'}{G})$-expansion method to identify the wave solutions of the Zakharov–Kuznetsov–Benjamin–Bona–Mahony equation and the (2+1)-dimensional dispersive long wave equations. Most recently, Borhan *et al.*[31] applied the new generalized $(\frac{G'}{G})$-expansion technique to examine the analytic solutions of the (2+1)-dimensional first integro-differential KP hierarchy equation and the (2+1)-dimensional second integro-differential KP hierarchy equation. This paper will address the new extended version of the generalized (G'/G)-expansion method, which will be utilized for the Benney–Luke equation (BLE) and the Burgers equation (BE) to retrieve novel traveling wave solutions.

The BLE is an NLEE that emerges in the analysis of long-wave propagation in fluids, particularly in the context of shallow water waves. The researchers David J. Benney and J. C. Luke contributed to its development. This equation extends simpler models like the Korteweg–de Vries (KdV) equation by incorporating higher-order dispersion and nonlinear effects. This makes it suitable for modeling more complex wave phenomena where both effects play a significant role. The BLE model has diverse utilization in fluid dynamics, shallow water waves, coastal engineering, water wave theory, communication systems, signal transmission, nonlinear optics, wave propagation in plasmas, nonlinear acoustics, and wave interaction studies.

The kink shape and singular kink shape solutions of the BLE have been examined in Ref. 32. Khan *et al.*[33] achieved new solitary wave solutions including hyperbolic, trigonometric, and rational function solutions of the BLE. In Ref. 34, scholars introduced the closed-form wave solutions of the BLE having periodic solitons, dark peakon solitons, and bright solitons. Very recently, Vivas-Cortez *et al.*[35] exposed the lump soliton, periodic cross-kink waves, multi-waves, breather waves, Ma-breather, Kuznetsov–Ma-breather, periodic waves, and rogue waves solutions of the BLE. Dutch physicist Johannes Martinus Burgers introduced the BE, a fundamental nonlinear partial differential equation in fluid dynamics and applied mathematics. The BE, in its various forms, is a powerful model that encapsulates the interplay between nonlinearity and diffusion in many physical systems. The equation models multiple physical phenomena, particularly those involving nonlinear wave propagation and viscous effects. It is particularly well-known in fluid dynamics, turbulence modeling, shock waves, traffic flow modeling, gas dynamics, nonlinear wave phenomena, mathematical biology, population dynamics, financial mathematics, and nonlinear acoustics. In Ref. 36, scholars applied the power index method to the BE and launched some wave solutions. Qi and Zhu[37] studied the BE and then originated the parabolic kink, tine-shaped, curved-shaped kink, and shock solitons. Regular and singular coupled BEs have also been scrutinized in Ref. 38. Lately, the bright soliton, kink wave solution, and periodic wave solution have been identified by Hussain *et al.*[39]

Most recently, various analyses, such as bifurcation analysis, chaotic analysis, and stability analysis, have become more involved with diverse nonlinear physical models.[40–43] Bifurcation analysis, incorporating phase portraiture, operates as a method within dynamical systems theory to examine the variations in a system's behavior in response to alterations in parameters. The aforementioned framework provides a significant mathematical basis for examining physical events within classical mechanics. Essentially, when actions seem arbitrary or unexpected but are actually controlled by rules that everyone can predict, we say that they are engaging in chaotic behavior. Chaos theory is useful in many fields, such as ecology, weather prediction, and the financial markets. The relevance of a model to the real world can be ascertained through stability analysis of solutions to NLEEs. The capacity to accurately anticipate the behavior of systems throughout time makes stability analysis useful in many domains; these include biochemical pathways, control systems, electrical power systems, financial markets, ecological systems, and population dynamics.

From the prior dissertation, we perceive that no one investigated the mentioned equations through this expansion technique. One can obtain numerous unique and novel closed-form traveling wave solutions, which can explain the genuine nature of NLEEs by applying the new generalized (G'/G)-expansion method. From this enthusiasm, our recent research will address the new expansion process, which will be applied to the stated models to achieve abundant fresh closed-form traveling wave solutions. Besides, we inspect the bifurcation and chaotic analyses for the BLE and stability analysis for the BE. The application of the method will demonstrate the efficiency, effectiveness, and simplicity of the extended version of the generalized (G'/G)-expansion method.

The remaining portion of this paper is structured as follows. Interpretation of the applied method is discussed in Sec. 2. Section 3 is categorized for the materialization of the solutions. Moreover, several analyses are positioned in Sec. 4. Graphical representation and physical explanations are engaged in Sec. 5. Finally, the conclusion is placed in Sec. 6.

2. Interpretation of the New Generalized (G'/G)-Expansion Method

In this section, we outline the main steps of the new generalized (G'/G)-expansion method used for exploring the optical wave solutions of the BLE and the BE. The generic NLEE can be expressed in the succeeding manner:

$$L(u, u_t, u_x, u_y, u_z, u_{tt}, u_{tx}, u_{xx}, u_{xy}, u_{tz}, \ldots) = 0, \tag{1}$$

where $u = u(x, y, z, t)$ is an anonymous function of the independent variables x, y, z, and t, and L represents a polynomial in $u(x, y, z, t)$ and its several partial derivatives. In Eq. (1), the subscripts stand for the partial derivatives of the respective variables where the highest-order derivatives and nonlinear terms are associated. The composition of this method is as follows:

Step 1: We consider the fusion of geometric variables x, y, z, and time variable t can be expressed by the new variable η as follows:

$$u(x, y, z, t) = u(\eta), \quad \eta = x + y + z \pm ct, \tag{2}$$

where c stands for the speed of wave. The new variable of Eq. (2) allows us to transform Eq. (1) into an ordinary differential equation (ODE) in the following form:

$$R(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where R is the polynomial of function u and its derivatives, and the superscripts denote the derivatives with respect to η.

Step 2: On the basis of possibility, Eq. (3) can be integrated one or more times as required and generates a constant of integration; it may be zero for simplicity.

Step 3: We presume that the traveling wave solution of (3) can be constructed in the form of a polynomial and expressed as follows:

$$u(\eta) = \sum_{l=0}^{P} a_l (d+S)^l + \sum_{l=1}^{P} b_l (d+S)^{-l}, \tag{4}$$

whereas a_l and b_l are unable to be identical to zero simultaneously; however, either one of them may be zero. Here, a_l $(l = 0, 1, 2, 3, \ldots, P)$, $b_l(l = 0, 1, 2, 3, \ldots, P)$, and d represent arbitrary constants which are required to be fixed, and $S(\eta)$ is

$$S(\eta) = (G'/G), \tag{5}$$

where $G = G(\eta)$ justifies the succeeding auxiliary nonlinear ODE:

$$AGG'' - BGG' - EG^2 - C(G')^2 = 0, \tag{6}$$

where A, B, and C are real parameters, and the derivative with regard to η is indicated by prime.

Step 4: To originate the value of positive integer P, the homogeneous balance principle is applied in Eq. (3).

Step 5: Combining Eqs. (3)–(5), we acquire a polynomial in $(d + S)^P$ $(P = 0, 1, 2, \ldots)$ and $(d + S)^{-P}$ $(P = 0, 1, 2, \ldots)$. To achieve a system of algebraic equations for a_l, b_l, d, and c, we set each coefficient of the resulting polynomial to zero.

Step 6: We consider that the value of constants a_l $(l = 0, 1, 2, 3, \ldots, P)$, $b_l(l = 0, 1, 2, 3, \ldots, P)$, d, and c may be gained after solving the system of equations attained in the former step. As we know the general solution of Eq. (6), we substitute the values of a_l, b_l, d, and c in Eq. (4). We accomplish novel closed-form solutions of the NLEE placed in Eq. (1), which are more of a generic type.

Step 7: Applying the standard solution of Eq. (6), one can manage the subsequent solution of Eq. (5).

Set 1: Whereas $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) > 0$, we acquire the hyperbolic function solution:

$$S(\eta) = \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi} \frac{m_1 \sinh\left(\frac{\sqrt{Q}}{2A} \cdot \eta\right) + m_2 \cosh\left(\frac{\sqrt{Q}}{2A} \cdot \eta\right)}{m_1 \cosh\left(\frac{\sqrt{Q}}{2A} \cdot \eta\right) + m_2 \sinh\left(\frac{\sqrt{Q}}{2A} \cdot \eta\right)}. \tag{7}$$

Set 2: Whereas $B \neq 0, \phi = A - C$, and $Q = B^2 + 4E(A - C) < 0$, we obtain the trigonometric solution:

$$S(\eta) = \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi} \frac{-m_1 \sin\left(\frac{\sqrt{-Q}}{2A} \cdot \eta\right) + m_2 \cos\left(\frac{\sqrt{-Q}}{2A} \cdot \eta\right)}{m_1 \cos\left(\frac{\sqrt{-Q}}{2A} \cdot \eta\right) + m_2 \sin\left(\frac{\sqrt{-Q}}{2A} \cdot \eta\right)}. \tag{8}$$

Set 3: Whereas $B \neq 0$, $\phi = A - C$ and $Q = B^2 + 4E(A - C) = 0$, we attain the rational form of solution:

$$S(\eta) = \frac{B}{2\phi} + \frac{m_2}{m_1 + m_2\eta}. \tag{9}$$

Set 4: Whereas $B = 0$, $\phi = A - C$ and $P = \phi E > 0$, we achieve hyperbolic function solution:

$$S(\eta) = \frac{\sqrt{P}}{\phi} \frac{m_1 \sinh\left(\frac{\sqrt{P}}{A} \cdot \eta\right) + m_2 \cosh\left(\frac{\sqrt{P}}{A} \cdot \eta\right)}{m_1 \cosh\left(\frac{\sqrt{P}}{A} \cdot \eta\right) + m_2 \sinh\left(\frac{\sqrt{P}}{A} \cdot \eta\right)}. \tag{10}$$

Set 5: Whereas $B = 0, \phi = A - C$ and $P = \phi E < 0$, we find the trigonometric solution:

$$S(\eta) = \frac{\sqrt{-P}}{\phi} \frac{-m_1 \sin\left(\frac{\sqrt{-P}}{A} \cdot \eta\right) + m_2 \cos\left(\frac{\sqrt{-P}}{A} \cdot \eta\right)}{m_1 \cos\left(\frac{\sqrt{-P}}{A} \cdot \eta\right) + m_2 \sin\left(\frac{\sqrt{-P}}{A} \cdot \eta\right)}. \tag{11}$$

These are the common solutions of Eq. (5), which can be implemented to deduce the elucidations of the BLE and the BE. Even though it has proven to be a versatile and effective approach, there are some limitations associated with it. The method depends on the appropriate choice of the auxiliary function $G(x, t)$. The success of the method often relies on finding suitable forms for this function, which may not always be obvious. For higher-dimensional nonlinear partial differential equations, the application of the selected method becomes increasingly complicated. Solving higher-dimensional problems may require more modifications to the method.

3. Materialization of the Solutions

In this segment, the new expansion scheme is implemented to explore the novel closed-form wave solutions to the BLE and the BE. These will be elaborated in detail in the following subsections.

3.1. *Benney–Luke equation*

The dispersive, as well as weakly nonlinear long water waves with small amplitude and complex wave phenomena over long distances and timescales, can be explained by the BLE, and it is named after the researchers David J. Benney and J.C. Luke, who contributed to its development. Moreover, this equation delineates bi-directional water wave flow aptly and can be employed to explore the evaluation of the full water wave equation. At this instant, we choose the BLE of the ensuing form[33,34]:

$$u_{tt} - k^2 u_{xx} + a u_{xxxx} + b u_{xxtt} + c u_t u_{xx} + d u_x u_{xt} = 0, \tag{12}$$

where k, a, b, c, and d are parameters, and $a - b = \mathfrak{B} - \frac{1}{3}$, where a and b are positive. $\mathfrak{B}$ stands for the Bond number. To explore the traveling wave solution of Eq. (12), we introduce the transformation $u(x,t) = u(\eta)$, $\eta = x - vt$, where v is the wave speed to be determined later, and integrating it with respect to η once yields

$$(v^2 - k^2)u' + (a + bv^2)u''' - \frac{1}{2}(cv + dv)u'^2 + l = 0, \tag{13}$$

where l is a constant of integration which will be computed later. To attain the value of P, we use the homogeneous balance between linear and nonlinear term; we get hold of $P = 1$. Hence, we attain the solution of Eq. (13) in the subsequent form:

$$u(\eta) = a_0 + a_1(d + S) + b_1(d + S)^{-1}, \tag{14}$$

where a_0, a_1, b_1, and d are constants which are to be finalized afterward. To evaluate the constants a_0,a_1, b_1, v, l, and d, we have to apply step 6. After a fruitful accomplishment, we derive three families of solutions as follows:

Family 1:

$$\begin{aligned} &d = -\frac{B}{2\phi}, \quad l = 0, \quad a_0 = a_0, \quad a_1 = \pm\frac{24A\{(a + bk^2)(A^2 + C^2) - 2A(aC + bCk^2)\}}{\sqrt{(A^2k^2 - 4aB^2 - 16a\phi E)(A^2 + 4bB^2 + 16b\phi E)}(2c\phi - B)}, \\ &b_1 = \pm\frac{6A(-4k^2CEb - 4aCE + 4aAE + aB^2 + k^2B^2b + 4k^2bEA)}{\sqrt{(A^2k^2 - 4aB^2 - 16a\phi E)(A^2 + 4bB^2 + 16b\phi E)}(2c\phi - B)}, \quad v = \pm\sqrt{\frac{(A^2k^2 - 4aB^2 - 16a\phi E)}{A^2 + 4bB^2 + 16b\phi E}}. \end{aligned} \tag{15}$$

Family 2:

$$\begin{aligned} &d = d, \quad l = 0, \quad a_0 = a_0, \quad a_1 = \pm\frac{12\phi A(a + bk^2)}{\sqrt{(A^2k^2 - aB^2 - 4a\phi E)(A^2 + bB^2 + 4b\phi E)}(d + c)}, \quad b_1 = 0, \\ &v = \pm\sqrt{\frac{(A^2k^2 - 4aB^2 - 16a\phi E)}{A^2 + 4bB^2 + 16b\phi E}}. \end{aligned} \tag{16}$$

Family 3:

$$\begin{aligned} &d = d, \quad l = 0, \quad a_0 = a_0, \quad a_1 = 0, \quad b_1 = \pm\frac{12A(bk^2d^2\phi + ad^2\phi - Ea - k^2Eb + Bk^2bd + Bad)}{\sqrt{(A^2k^2 - aB^2 - 4a\phi E)(A^2 + bB^2 + 4b\phi E)}(d + c)}, \\ &v = \pm\sqrt{\frac{(A^2k^2 - 4aB^2 - 16a\phi E)}{A^2 + 4bB^2 + 16b\phi E}}, \end{aligned} \tag{17}$$

where $\phi = A - C$ and A, B, C, E are parameters.

In consideration of Family 1:
When $\phi = A - C$, $Q = B^2 + 4E(A - C) > 0$ and $B \neq 0$, we utilize Eq. (15) in Eq. (14) as well as Eqs. (7)–(11). We have the resulting traveling wave solutions used for $m_1 = 0$ when $m_2 \neq 0$ and for $m_2 = 0$ when $m_1 \neq 0$ gradually:

$$u_{11} = a_0 + n_1\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + n_2\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{12} = a_0 + n_1\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + n_2\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{13} = a_0 + n_1\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + n_2\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{14} = a_0 - n_1\left\{\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} - n_2\left\{\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{15} = a_0 + n_1\left(\frac{m_2}{m_1 + m_2\eta}\right) + n_2\left(\frac{m_2}{m_1 + m_2\eta}\right)^{-1},$$

$$u_{16} = a_0 + \frac{n_3\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right) + \frac{n_4\phi}{\sqrt{P}}\tanh\left(\frac{\sqrt{P}}{A}\eta\right),$$

$$u_{17} = a_0 + \frac{n_3\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right) + \frac{n_4\phi}{\sqrt{P}}\coth\left(\frac{\sqrt{P}}{A}\eta\right),$$

$$u_{18} = a_0 + \frac{n_3\sqrt{-P}}{\phi}\cot\left.\frac{\sqrt{-P}}{A}\eta\right) + \frac{n_4\phi}{\sqrt{-P}}\tan\left.\frac{\sqrt{-P}}{A}\eta\right),$$

$$u_{19} = a_0 - \frac{n_3\sqrt{-P}}{\phi}\tan\left.\frac{\sqrt{-P}}{A}\eta\right) - \frac{n_4\phi}{\sqrt{-P}}\cot\left.\frac{\sqrt{-P}}{A}\eta\right),$$

where

$$n_1 = \pm\frac{24A\{(a + bk^2)(A^2 + C^2) - 2A(aC + bCk^2)\}}{\sqrt{(A^2k^2 - 4aB^2 - 16a\phi E)(A^2 + 4bB^2 + 16b\phi E)}(2c\phi - B)},$$

$$n_2 = \pm\frac{6A(-4k^2CEb - 4aCE + 4aAE + aB^2 + k^2B^2b + 4k^2bEA)}{\sqrt{(A^2k^2 - 4aB^2 - 16a\phi E)(A^2 + 4bB^2 + 16b\phi E)}(2c\phi - B)},$$

$$n_3 = \pm\frac{12A\{(a + bk^2)(A^2 + C^2) - 2A(aC + bCk^2)\}}{c\phi\sqrt{(A^2k^2 - 16a\phi E)(A^2 + 16b\phi E)}},$$

$$n_4 = \pm\frac{3A(-4k^2CEb - 4aCE + 4aAE + 4k^2bEA)}{c\phi\sqrt{(A^2k^2 - 16a\phi E)(A^2 + 16b\phi E)}}.$$

If we apply the transformation $u(x, t) = u(\eta)$ with $\eta = x - vt$ in the above-established solutions, we would have the desired solutions of Eq. (12).

In consideration of Family 2:
When $\phi = A - C$, $Q = B^2 + 4E(A - C) > 0$ and $B \neq 0$, by employing Eq. (16) in Eq. (14) together with Eqs. (7)–(11), we find subsequent wave solutions used for

$m_1 = 0$ when $m_2 \neq 0$ and for $m_2 = 0$ when $m_1 \neq 0$ as follows:

$$u_{21} = a_0 + r_1\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{22} = a_0 + r_1\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{23} = a_0 + r_1\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{24} = a_0 + r_1\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{25} = a_0 + r_1\left(d + \frac{B}{2\phi} + \frac{m_2}{m_1 + m_2\eta}\right),$$

$$u_{26} = a_0 + r_2\left\{d + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{27} = a_0 + r_2\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{28} = a_0 + r_2\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left.\frac{\sqrt{-P}}{A}\eta\right)\right\},$$

$$u_{29} = a_0 + r_2\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left.\frac{\sqrt{-P}}{A}\eta\right)\right\},$$

where $r_1 = \pm\frac{12\phi A(a+bk^2)}{\sqrt{(A^2k^2-aB^2-4a\phi E)(A^2+bB^2+4b\phi E)(d+c)}}$, $r_2 = \pm\frac{12\phi A(a+bk^2)}{\sqrt{(A^2k^2-4a\phi E)(A^2+4b\phi E)(d+c)}}$.

If we apply the modification $u(x,t) = u(\eta)$ along with $\eta = x - vt$ in the above-recognized solutions, we obtain the desired solutions of Eq. (12).

In consideration of Family 3:
When $\phi = A - C$, $Q = B^2 + 4E(A - C) > 0$ and $B \neq 0$, introducing the outcomes in Eq. (17) in Eq. (14) alongside Eqs. (7)–(11), we obtain the resulting traveling wave solutions used for $m_1 = 0$ when $m_2 \neq 0$ and for $m_2 = 0$ when $m_1 \neq 0$ as follows:

$$u_{31} = a_0 + r_3\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{32} = a_0 + r_3\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{33} = a_0 + r_3\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{34} = a_0 + r_3\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{35} = a_0 + r_3\left(d + \frac{B}{2\phi} + \frac{m_2}{m_1 + m_2\eta}\right)^{-1},$$

$$u_{36} = a_0 + r_4\left\{d + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{37} = a_0 + r_4\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{38} = a_0 + r_4\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left.\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{39} = a_0 + r_4\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left.\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

where $r_3 = \pm\frac{12A(bk^2d^2\phi + ad^2\phi - Ea - k^2Eb + Bk^2bd + Bad)}{\sqrt{(A^2k^2 - aB^2 - 4a\phi E)(A^2 + bB^2 + 4b\phi E)(d+c)}}$, $r_4 = \pm\frac{12A(bk^2d^2\phi + ad^2\phi - Ea - k^2Eb)}{\sqrt{(A^2k^2 - 4a\phi E)(A^2 + 4b\phi E)(d+c)}}$.

If we use the alteration $u(x,t) = u(\eta)$ with $\eta = x - vt$ in the above-induced solutions, then we obtain the expected solutions of Eq. (12).

3.2. *Burgers equation*

The BE remains a fundamental model in nonlinear science, often studied in the context of chaos, turbulence, and wave phenomena. It is initially proposed as an elementary model of the turbulence as demonstrated by the notable Navier–Stokes equations. This equation is propitiously adapted in fluid dynamics, turbulence modeling, shock waves, traffic flow modeling, gas dynamics, mathematical biology, and financial mathematics. The equation can be written as follows[36,37]:

$$u_t = 2uu_x + u_{xx}. \tag{18}$$

By applying the conversion $u(x,t) = u(\eta)$, where $\eta = x - vt$, we reduce Eq. (18) to the following ODE:

$$vu' + 2uu' + u'' = 0. \tag{19}$$

Performing integration of Eq. (19) with regard to η for one time, we have

$$vu + u^2 + u' + l_1 = 0, \tag{20}$$

where l_1 represents the integral constant. By applying step 4 in Eq. (20), one may have $P = 1$. Hence, we achieve the outcome of Eq. (20) in the succeeding form:

$$u(\eta) = a_0 + a_1(d+S) + b_1(d+S)^{-1}, \tag{21}$$

where a_0, a_1, b_1 and d stay for constants which are to be established. Subsequently, an effective execution of step 6, we established three sets of findings as follows:

Family 4:

$$\begin{aligned} &d = -\frac{B}{2\phi}, \quad a_0 = a_0, \quad a_1 = \frac{\phi}{A}, \quad b_1 = \frac{4E\phi + B^2}{4A\phi}, \\ &v = 2a_0, \quad l_1 = \frac{a_0^2 A^2 - B^2 - 4E\phi}{A^2}. \end{aligned} \tag{22}$$

Family 5:

$$\begin{aligned} &d = d, \quad a_0 = a_0, \quad a_1 = \frac{\phi}{A}, \quad b_1 = 0, \\ &l_1 = \frac{d^2(A^2 + C^2) - 2Cd^2 A + AB(d + a_0) + a_0^2 A^2 - E\phi + 2a_0 A d\phi}{A^2}, \\ &v = -\frac{2d\phi + 2a_0 A + B}{A}. \end{aligned} \tag{23}$$

Family 6:

$$\begin{aligned} &d = d, \quad a_0 = a_0, \quad a_1 = 0, \quad b_1 = -\frac{Bd - E + d^2\phi}{A}, \quad v = \frac{B - 2a_0 A + 2d\phi}{A}, \\ &l_1 = \frac{d^2(A^2 + C^2) + Bd\phi - 2Cd^2 A + a_0 AB + a_0^2 A^2 - E\phi - 2a_0 A d\phi}{A^2}. \end{aligned} \tag{24}$$

In case of Family 4:
When $B \neq 0$, $\phi = A - C$ and $Q = B^2 + 4E(A - C) > 0$, relating Eq. (22) to Eq. (21) along with Eqs. (7)–(11) and by simplifying, we get the subsequent traveling wave solutions used for $m_1 = 0$ when $m_2 \neq 0$ and for $m_2 = 0$ when $m_1 \neq 0$ gradually:

$$u_{41} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{B^2 + 4\phi E}{4\phi A}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{42} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{B^2 + 4\phi E}{4\phi A}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{43} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + \frac{B^2 + 4\phi E}{4\phi A}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{44} = a_0 - \frac{\phi}{A}\left\{\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} - \frac{B^2 + 4\phi E}{4\phi A}\left\{\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{45} = a_0 + \frac{\phi}{A}\left(\frac{m_2}{m_1 + m_2\eta}\right) + \frac{B^2 + 4\phi E}{4\phi A}\left(\frac{m_2}{m_1 + m_2\eta}\right)^{-1},$$

$$u_{46} = a_0 + \frac{\sqrt{P}}{A}\coth\left(\frac{\sqrt{P}}{A}\eta\right) + \frac{A\phi}{E\sqrt{P}}\tanh\left(\frac{\sqrt{P}}{A}\eta\right),$$

$$u_{47} = a_0 + \frac{\sqrt{P}}{A} \tanh\left(\frac{\sqrt{P}}{A}\eta\right) + \frac{A\phi}{E\sqrt{P}} \coth\left(\frac{\sqrt{P}}{A}\eta\right),$$

$$u_{48} = a_0 + \frac{\sqrt{-P}}{A} \cot \left. \frac{\sqrt{-P}}{A}\eta\right) + \frac{A\phi}{E\sqrt{-P}} \tan \left. \frac{\sqrt{-P}}{A}\eta\right),$$

$$u_{49} = a_0 - \frac{\sqrt{-P}}{A} \tan \left. \frac{\sqrt{-P}}{A}\eta\right) - \frac{A\phi}{E\sqrt{-P}} \cot \left. \frac{\sqrt{-P}}{A}\eta\right).$$

To get the closed-form solutions of Eq. (18), we have to operate the transformation $u(x,t) = u(\eta)$ with $\eta = x - vt$ in the above-established solutions.

In case of Family 5:
When $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) > 0$, joining Eq. (23) into (21), with Eqs. (7)–(11), and by abridging, we have the subsequent closed-form solutions used for $m_1 = 0$ when $m_2 \neq 0$ as well as $m_2 = 0$ when $m_1 \neq 0$ progressively:

$$u_{51} = a_0 + \frac{\phi}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi} \coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{52} = a_0 + \frac{\phi}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi} \tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{53} = a_0 + \frac{\phi}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi} \cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{54} = a_0 + \frac{\phi}{A}\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi} \tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{55} = a_0 + \frac{\phi}{A}\left(d + \frac{B}{2\phi} + \frac{m_2}{m_1 + m_2\eta}\right),$$

$$u_{56} = a_0 + \frac{\phi}{A}\left\{d + \frac{\sqrt{P}}{\phi} \coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{57} = a_0 + \frac{\phi}{A}\left\{d + \frac{\sqrt{P}}{\phi} \tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{58} = a_0 + \frac{\phi}{A}\left\{d + \frac{\sqrt{-P}}{\phi} \cot \left. \frac{\sqrt{-P}}{A}\eta\right)\right\},$$

$$u_{59} = a_0 + \frac{\phi}{A}\left\{d - \frac{\sqrt{-P}}{\phi} \tan \left. \frac{\sqrt{-P}}{A}\eta\right)\right\}.$$

To generate the closed-form solutions of Eq. (18), we have to organize the transformation $u(x,t) = u(\eta)$ along with $\eta = x - vt$ in the above-established solutions.

In case of Family 6:
When $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) > 0$, setting the results of constants held in Eq. (24) into Eq. (21) together with Eqs. (7)–(11) and by minimizing, we obtain the traveling wave solutions used for $m_1 = 0$ when $m_2 \neq 0$ and for $m_2 = 0$ when $m_1 \neq 0$ gradually:

$$u_{61} = a_0 - \frac{(d^2\phi - E + Bd)}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{62} = a_0 - \frac{(d^2\phi - E + Bd)}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{63} = a_0 - \frac{(d^2\phi - E + Bd)}{A}\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{64} = a_0 - \frac{(d^2\phi - E + Bd)}{A}\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{65} = a_0 - \frac{(d^2\phi - E + Bd)}{A}\left(d + \frac{B}{2\phi} + \frac{m_2}{m_1 + m_2\eta}\right)^{-1},$$

$$u_{66} = a_0 - \frac{(d^2\phi - E)}{A}\left\{d + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{67} = a_0 - \frac{(d^2\phi - E)}{A}\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{68} = a_0 - \frac{(d^2\phi - E)}{A}\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left.\ \frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{69} = a_0 - \frac{(d^2\phi - E)}{A}\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left.\ \frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

To form the closed-form solutions of Eq. (18), we have to manage the conversion $u(x,t) = u(\eta)$ together with $\eta = x - vt$ in the above-established solutions.

4. Several Analyses

This section is organized with the diverse analyses of our considered models.

4.1. *Bifurcation analysis of BLE*

In several fields, bifurcation analysis has become an advantageous tool for comprehending how complex systems behave, which helps with optimization, control, and prediction. It assists researchers in comprehending how minor adjustments to parameters may result in substantial alterations in the dynamics of complicated systems. This subsection demonstrates the bifurcation analysis for the BLE. To construct the dynamical system by using the Galilean transformation, we assume $u' = U$ in Eq. (13) that one can write as

$$\begin{cases} \dfrac{dU}{d\eta} = W, \\ \dfrac{dW}{d\eta} = M_1 U^2 + M_2 U, \end{cases} \tag{25}$$

where $M_1 = \frac{1}{2}\frac{cv+dv}{a+bv^2}$ and $M_2 = \frac{k^2-v^2}{a+bv^2}$. In the theoretical framework of classical mechanics, the above dynamical system provides an efficient mathematical basis for evaluating physical mechanisms. The Hamiltonian function generated by the previously mentioned dynamical system is as follows:

$$h(U, W) = \frac{W^2}{2} - \frac{M_1 U^3}{3} - \frac{M_2 U^2}{2} = h_1,$$

where h_1 means the Hamiltonian constant. The equilibrium points $(0, 0)$ and $(-\frac{M_2}{M_1}, 0)$ of the dynamical system in Eq. (25) are examined after resolving the following dynamical system:

$$\begin{cases} W = 0, \\ M_1 U^2 + M_2 U = 0. \end{cases} \tag{26}$$

The dynamical system in Eq. (25) produces a Jacobian:

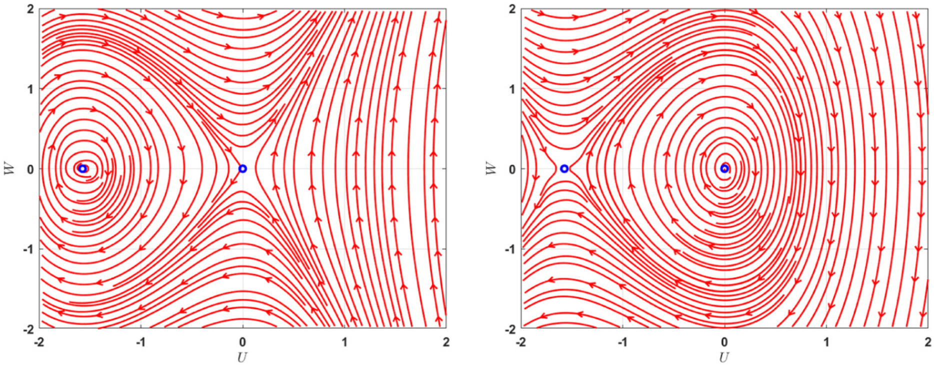
$$\mathfrak{D}(U, W) = \begin{vmatrix} 0 & 1 \\ 2M_1 U + M_2 & 0 \end{vmatrix} = -2M_1 U - M_2.$$

By following the theory of planar dynamical systems, we get that

(1) one saddle point is clarified by the equilibrium position (U, W), whereas $\mathfrak{D}(U, W) < 0$;
(2) one center point is demonstrated by the equilibrium position (U, W), whereas $\mathfrak{D}(U, W) > 0$; and
(3) one cuspid point is exemplified by the equilibrium position (U, W), whereas $\mathfrak{D}(U, W) = 0$.

A set of feasible results can be achieved by adjusting the appropriate parameters.
Condition 1: $M_1 > 0$ and $M_2 > 0$.
The equilibrium points $(0, 0)$ and $(-1.5725, 0)$ are identified as a saddle point and center point, as demonstrated in Fig. 1(a), for the value of the parameters $a = 1, b = 1, c = 1, d = 7, k = 2.7$, and $v = 1$, respectively.

Condition 2: $M_1 < 0$ and $M_2 < 0$.
By setting the value of the parameters $a = -1, b = -1, c = 1, d = 7, k = 2.7$, and $v = 1$, one can discover the center point and saddle point at the equilibrium position at $(0, 0)$ and $(-1.5725, 0)$ as displayed in Fig. 1(b), accordingly.



(a)

(b)

Fig. 1. The bifurcation analysis of BLE for different situation of M_1 and M_2 with proper value of the parameters.

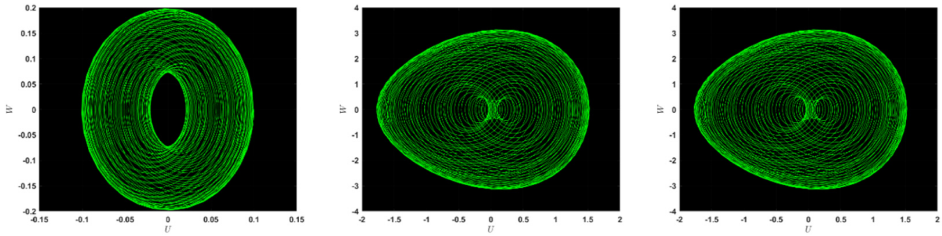
4.2. *Chaotic analysis of BLE*

The notion of chaos was developed through the study of predetermined dynamical systems, which produce seemingly inconsequential and arbitrary results due to their very sensitive behavior to beginning conditions. In this subsection, our goal is to figure out the BLE's chaotic nature. In order to assess whether chaotic activity could exist in the developing system (25), we comprised a perturbation term $\Upsilon_1 \cos(\Upsilon_2 t)$, with the amplitude Υ_1 and the frequency Υ_2,

$$\begin{cases} \dfrac{dU(t)}{dt} = W, \\ \dfrac{dW(t)}{dt} = M_1 U(t)^2 + M_2 U(t) + \Upsilon_1 \cos(\Upsilon_2 t). \end{cases} \tag{27}$$

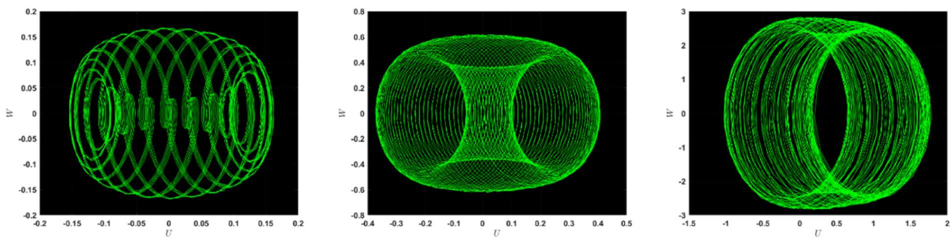
We analyze the impact of the newly added term on the dynamical system in Eq. (27), as shown in Figs. 2–4. Now, we select the parameter values $a = -2, b = 1, c = 3, d = -2, v = 1, k = \sqrt{6}, \Upsilon_1 = 0.1$, and $\Upsilon_2 = \frac{\pi}{2}$ for Fig. 2(a), $a = -2, b = 1, c = 3, d = -2, v = 1, k = \sqrt{6}, \Upsilon_1 = 1.55$, and $\Upsilon_2 = \frac{\pi}{2}$ for Fig. 2(b), $a = -2, b = 1, c = 3, d = -2, v = 1, k = \sqrt{6}, \Upsilon_1 = 2$, and $\Upsilon_2 = \frac{\pi}{2}$ for Fig. 2(c), and likewise for other figures. After analyzing the established figures we detect that ringlet dynamics is exhibited in Figs. 2(a) and 3(c); complex dynamics is exposed in Figs. 2(b), 2(c), and 3(b); periodic dynamics is displayed in Figs. 3(a) and 4(c); strange dynamics is demonstrated in Fig. 4(a); and scroll dynamics is presented in Fig. 4(b).

The illustrations reported here show how the perturbed term affects the system's global behavior and reveals its vulnerability. These observations assist in clarifying how even small modifications to parameters might influence the system's trajectory, permitting greater precision as well as accurate assessments of its behavior under various circumstances.



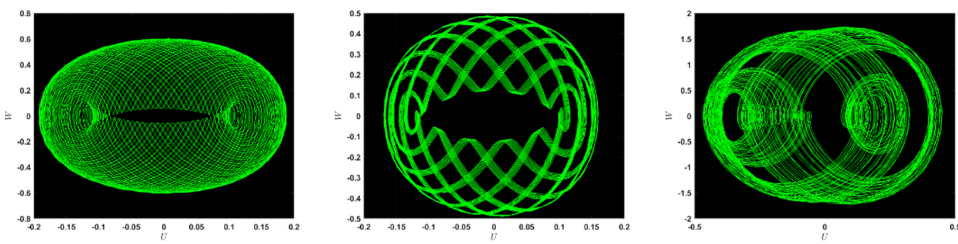
(a) (b) (c)

Fig. 2. Exhibition of the chaotic character of the selected model with suitable value of the parameters and $\Upsilon_2 = \frac{\pi}{2}$.



(a) (b) (c)

Fig. 3. Exhibition of the chaotic character of the selected model along with proper value of the parameters and $\Upsilon_2 = \pi$.



(a) (b) (c)

Fig. 4. Exhibition of the chaotic character of the selected model together with appropriate value of the parameters and $\Upsilon_2 = 2\pi$.

4.3. *Stability analysis of BE*

A system is regarded to be stable whenever any slight alteration has no significant effect on its functionality. Assessing the stability of a time delay system intends to discover the area of the delay parameter space wherein the system remains stable. In this segment, we scrutinize the stability of the BE by formulating the equation associated with the momentum along with the Hamiltonian mechanics as follows:

$$\wp = \frac{1}{2}\int_{-\infty}^{\infty} \hbar^2 d\varphi, \tag{28}$$

where $\wp$ and $\hbar$ represent the momentum and the system's overall power or energy. The closed-form traveling wave solution would become stable if

$$\frac{\partial \wp}{\partial \vartheta} > 0, \tag{29}$$

in which ϑ stands for the frequency. Combining Eq. (28) and u_{5_7}, we have

$$\wp = \frac{1}{2}\int_{-10}^{10}\left[a_0 + \frac{\phi}{A}\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}(x - vt)\right)\right\}\right]^2 dx. \tag{30}$$

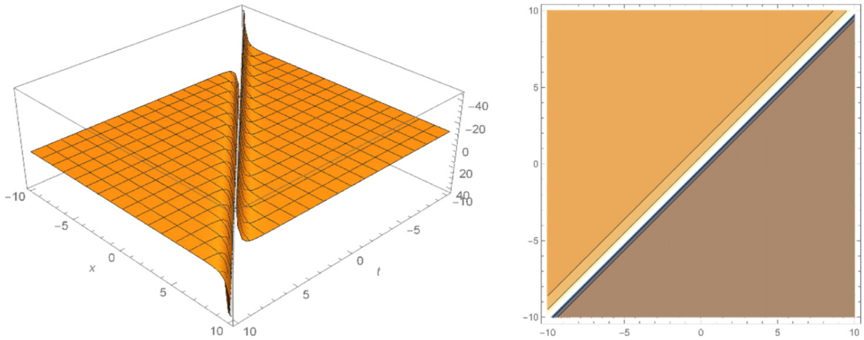
After simplification and using the stabilization rule, one may conclude the following:

$$\frac{\sqrt{P}t\left[\sqrt{P}\left\{\operatorname{sech}^2\left(\frac{\sqrt{P}(tv-10)}{A}\right) - \operatorname{sech}^2\left(\frac{\sqrt{P}(tv+10)}{A}\right)\right\} + 2(d\phi + Aa_0)\left\{\tanh\left(\frac{\sqrt{P}(tv-10)}{A}\right) - \tanh\left(\frac{\sqrt{P}(tv+10)}{A}\right)\right\}\right]}{2A^2} > 0. \tag{31}$$

The inequality in Eqs. (29) and (31) confirms that our selected model, BE, has nonlinear stability.

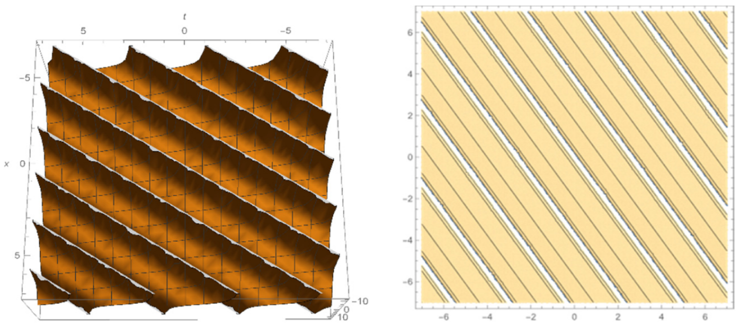
5. Graphical Representation with Rationalizations

In this fragment, we have to focus on the physical interpretation of the wave solutions of the BLE and the BE through the graphical representations. Graphics is a useful tool for identifying problems and analyzing solutions properly. In both the problem demonstration and the occurrence interpretation processes, it is an essential component. Additionally, it reveals the true nature of the consequences that have already been attained. Additionally, they play a crucial role in facilitating communication throughout several fields, analyzing data, and making informed decisions based on data-driven insights in fields such as public policy, education, business, and science. Various visual representations, namely 3D and contour illustrations, were part of our exhibition. We will go over what happened here; for the sake of conciseness, we will just cover six solutions. First, we draw the closed-form solution u_{11} in two formats: 3D and contour shapes with $A = 2, B = -1, C = 1, E = 1$, $a_0 = -1, a = -1, b = 1.5, c = -1, k = 1$, and $x, t \in [-10, 10]$, and then, we have singular kink shape soliton, as placed in Fig. 5. Subsequently, we sketch the closed-form solution u_{12} in two formats: 3D and contour shapes with $A = -4$, $B = -1, C = -2, E = 3, a_0 = 0.2, a = 1, b = -2, c = 1, k = 1$, and $x, t \in [-7, 7]$, and then, we get singular periodic shape soliton, as positioned in Fig. 6. The closed-form solution u_{22} displays the kink shape together with the parameters $A = 4, B = 6$,



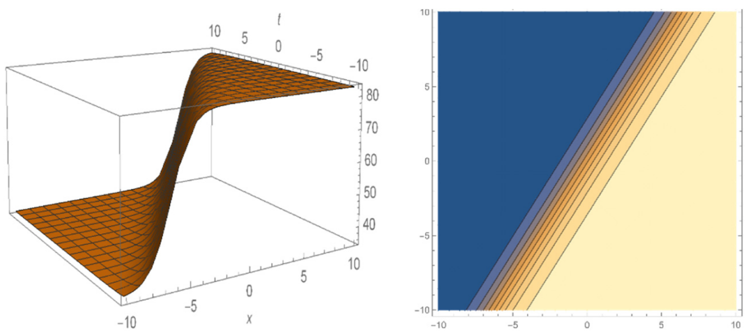
(a) (b)

Fig. 5. Singular kink shape soliton solution of u_{11} having suitable value of the parameters (a) 3D structure and (b) contour structure.



(a) (b)

Fig. 6. Singular periodic shape soliton solution of u_{12} having suitable values of the parameters (a) 3D composition and (b) contour composition.

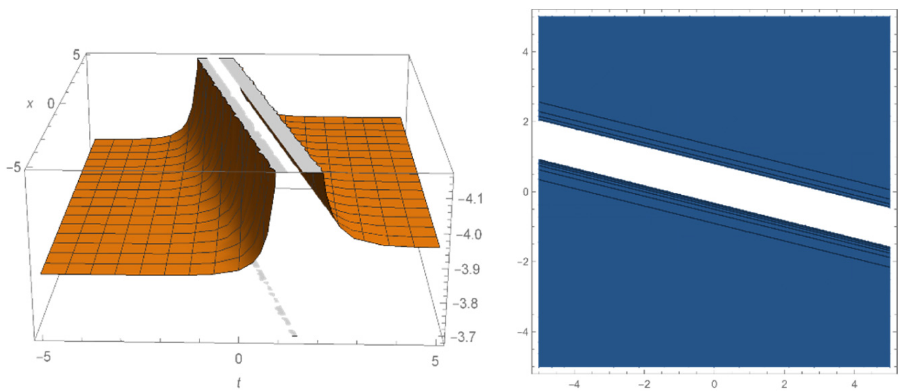


(a) (b)

Fig. 7. Kink shape soliton solution of u_{22} having suitable values of the parameters (a) 3D configuration and (b) contour configuration.

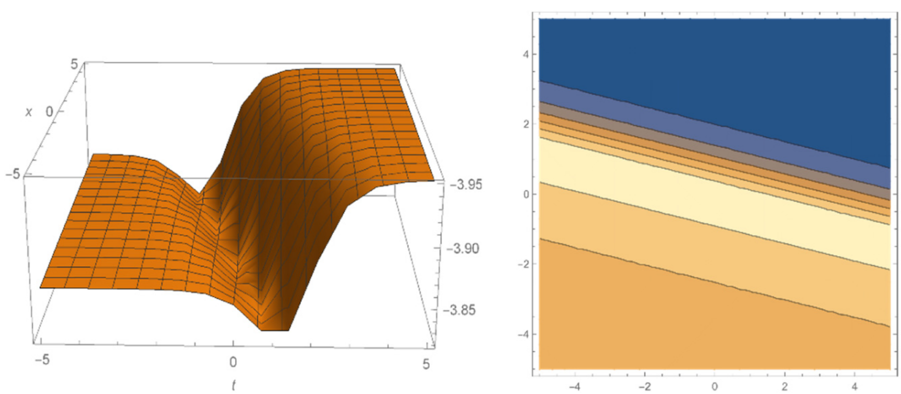
$C = 1, E = -1, a_0 = 1, a = 1, b = 4, c = 1, d = 1, k = 1$, and $x, t \in [-10, 10]$, as ordered in Fig. 7 with two formats: 3D and contour shape.

Figure 8 displays the singular bell shape for the closed-form solution u_{46} with $A = 4, C = 5, E = -1, a_0 = 2$, $x, t \in [-5, 5]$ for 3D, and $x, t \in [-10, 10]$, that is placed in two formats: 3D and contour shape. Additionally, the anti-bell shape soliton is attained for the closed-form solution u_{47} in 3D and contour shapes along with $A = 4, C = 5, E = -1, a_0 = 2$, and $x, t \in [-5, 5]$, as shown in Fig. 9. Finally, we illustrate the closed-from solution u_{52} with $A = 2, B = -1, C = 1, E = 1, d = 1, a_0 = -0.1$, and $x, t \in [-10, 10]$ in two formats: 3D and contour shape, we have the



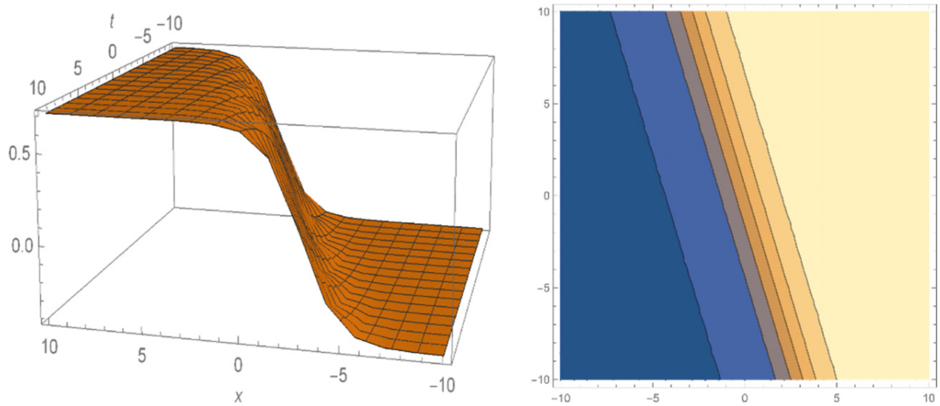
(a) (b)

Fig. 8. Singular bell shape soliton solution of u_{46} having suitable values of the parameters (a) 3D construction and (b) contour construction.



(a) (b)

Fig. 9. Anti-bell shape soliton solution of u_{47} having suitable values of the parameters (a) 3D form and (b) contour form.



(a) (b)

Fig. 10. The anti-kink shape soliton solution of u_{52} having suitable values of the parameters (a) 3D configuration and (b) contour configuration.

anti-kink shape soliton, as shown in Fig. 10. Currently, we establish that the results of the above equations are relatively novel and have not been examined in the previous literature.

6. Conclusion

In modern times, one of the passionate topics is the closed-form wave solution of the NLEEs, which have numerous uses in contemporary science, having the domain of weakly nonlinear long water waves, solid-state physics, plasma physics, chemical kinematics, electromagnetic interaction, fluid dynamics, communication systems, signal transmission, nonlinear optics, and others. In this paper, we have successfully performed to explore different novel and further general closed-form wave solutions of the BLE and the BE that have so many interesting appliances in fluid dynamics, shallow water waves, water wave theory, communication systems, nonlinear optics, plasma physics, seismic waves, nonlinear acoustics, nonlinear wave propagation, traffic flow, nonlinear acoustics, gas dynamics, mathematical biology, and population dynamics. The bifurcation and chaotic analyses of the BLE and the stability analysis of the BE were additionally considered to discover the nature of the findings that have been noted. From the bifurcation analysis of BLE, we have observed the saddle point and center point of the dynamical system in its equilibrium position. Moreover, ringlet dynamics, complex dynamics, periodic dynamics, and scroll dynamics have been identified by chaotic analysis for the dynamical system of BLE. Correspondingly, this research demonstrates that our specified BE exhibits nonlinear stability. We have magnificently attained broader classes of closed-form wave solutions via the new generalized (G'/G)-expansion technique with a multiplicity of individual physical structures such as singular kink, singular periodic, kink, singular bell, anti-bell, and anti-kink-shaped solitons. Additionally, the outcomes of the BLE maintain one more condition $a - b = \mathfrak{B} - \frac{1}{3}$. Figures 5–10 displayed the 3D and contour shapes of some of the established solutions with fixed values of the parameters. The obtained solutions of these equations, which have not been published yet in the form of rational, hyperbolic, and trigonometric functions, have many resourceful inflictions in modern engineering, applied mathematics, and mathematical physics. The attained solutions might be beneficial in miscellaneous naturalistic applications. The pivotal privilege of this implemented method in opposition to other methods is that the method fits out further general and massive amounts of novel wave solutions that validate the superiority of this method. In the future, the proposed method could be utilized to examine various NLPDEs, such as the Allen–Cahn equation, the Lotka–Volterra equation, the nonlinear versions of Maxwell's equation, the Burgers– Huxley equation, the Klein–Gordon equation, the nonlinear diffusion equation, the Einstein field equation, and others seen in recent engineering and mathematical physics fields.

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