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ORIGINAL PAPER

Interaction phenomena between a lump and other multi-solitons for the (2 + 1)-dimensional BLMP and Ito equations

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Abstract In this paper, "new" interaction solutions between a lump solution and other multi-soliton (kinky or stripe) solutions are studied through developing a "new" direct method based on the Hirota bilinear form for the (2 + 1)-dimensional BLMP equation and the (2 + 1)-dimensional Ito equation. Interaction solutions degenerate into lump (or kinky/stripe) solutions while the involved exponential function (or quadratic function) disappears. The interaction phenomena in the presented solutions show that a lump can be drowned or swallowed by other multi-solitary waves (kinky or stripe waves), and such interactions are very rare non-elastic collisions. What is more, we find that the positions of the interaction between a lump and three or four kinky waves are different while we choose different parameters, and the collisions may be at the bottom, middle, top or other positions. The dynamic characteristics of the constructed interaction solutions are illustrated by sequences of interesting figures plotted with the help of Maple.

Keywords Lump solution · Soliton solution · Interaction solution · The (2 + 1)-dimensional BLMP equation · The (2 + 1)-dimensional Ito equation

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1 Introduction

In soliton theory, lump solutions of the nonlinear partial differential equations have attracted more and more attention, which are a kind of rational function solutions localized in all directions of the space. Lump solutions have been widely studied since they were first discovered [1] because of their physical significance [2,3]. There are so many approaches to investigating lump solutions, such as the inverse scattering transformation [4], the Hirota bilinear method [5], and Darboux transformation method [6]. Recently, Ma gave a symbolic computation method to obtain lump solutions for a class of nonlinear equations based on the Harota bilinear form [7]. This approach is simpler and more effective compared with other methods. Lump solutions can be expressed by rational quadratic functions, and other elementary functions have been applied to the construction of other type exact solutions (e.g., traveling wave solutions, solitary solutions and periodic solutions) to some nonlinear partial differential equations [8–10]. In particular, many studies show that many exact solutions can be presented by exponential functions. Later, by the generalization of Ma's method by combining with exponential functions, interaction solutions between lump solutions and other soliton solutions have been found for many nonlinear soliton equations, for instance, $(2+1)$-dimensional Ito equation [11], $(2+1)$-dimensional BLMP equation and $(3+1)$-dimensional nonlinear evolution equation [12], Sawada–Kotera equation [13], Jimbo–Miwa equation [14], reduced generalized $(3+1)$-dimensional shallow water wave equation [15], $(2+1)$-dimensional third-order evolution equation [16], dimensionally reduced generalized KP equation [17] and so on. The interaction between lumps and other solitons has become a hotspot of soliton research.

It is known that lump solutions of some integrable nonlinear equations restore their shapes, amplitudes, velocities after the collisions with other solitons, which means the interaction can be considered completely elastic [18–20]. However, for some equations, the interactions prove to be completely inelastic under some constraint conditions [11,21,22]. In fact, similar phenomena have been observed in many fields of nonlinear science, such as the laser and optical physics [23], plasma physics [24], nuclear physics [25], hydrodynamics [26], gas dynamics [27], passive random walker dynamics and electromagnetics [28,29]. Therefore, it is very important to discuss non-elastic interactions among the solitary waves in certain integrable or non-integrable systems under strong physical backgrounds. Furthermore, such studies may provide a theoretical tool for understanding and supporting the relevant dynamical behaviors.

Based on the Hirota bilinear form, by combining the rational quadratic functions with one or two exponential functions, interaction solutions between lump solutions and one or two other soliton solutions have been found for many nonlinear soliton equations and the rich dynamical features have been analyzed in detail [11–17]. What will happen if the interaction solutions consist of the rational quadratic functions and more exponential functions? These problems are worth studying. In this paper, we will study the "new" interaction phenomena between a lump and other three even four other solitons.

We have studied interaction phenomena between a lump and two kinky solitons of the $(2+1)$-dimensional BLMP equation in [12] and lump solutions and interaction solutions between a lump and one stripe soliton of the $(2+1)$-dimensional Ito equation in [11], respectively. In addition, lump solutions and interaction solutions between a lump and one kinky soliton of the $(2+1)$-dimensional BLMP equation [30] and interactions between a lump and two stripe solitons of the $(2+1)$-dimensional Ito equation [31] have been investigated by other authors. Based these research work, in the paper, we will do further research on "new" interaction phenomena between a lump and three even four other solitons of the $(2+1)$-dimensional BLMP equation and the $(2+1)$-dimensional Ito equation, respectively.

The rest of the paper is organized as follows. In Sect. 2, the basic steps of a new direct method are given. In Sects. 3 and 4, new interaction solutions of the $(2+1)$-dimensional BLMP equation and the $(2+1)$-dimensional Ito equation are studied, respectively. In Sect. 5, a few conclusions are given.

2 A new direct method to study interactions between a lump and other multi-solitons

To study new interaction phenomena between a lump and other multiple solitons, we develop a new direct method by combining the rational quadratic functions with multiple exponential functions based on the Hirota bilinear form. We consider a general form of a $(2+1)$-dimensional nonlinear partial differential equation

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \ldots) = 0, \tag{1}$$

where $u = u(x, y, t)$, and F is a polynomial about u and its derivatives.

The fundamental steps of the direct method can be expressed as follows.

Step 1 We take a transformation as

$$u = T(f), \tag{2}$$

where $f = f(x, y, t)$ is a new unknown function. Then, Eq. (1) can be transformed into a Hirota bilinear form

$$G(D_t, D_x, D_y; f, f) = 0, \tag{3}$$

where the D-operators are Hirota's bilinear operators [32].

Step 2 To study interactions of Eq. (1), we assume that the equation has an interaction solution in the following form

$$f = g^2 + h^2 + \sum_{i=1}^{n} c_i \exp(l_i) + a_7, \tag{4}$$

with

$$g = a_1 x + a_2 y + a_3 t,$$
$$h = a_4 x + a_5 y + a_6 t,$$
$$l_i = m_i x + p_i y + q_i t,$$

where $a_j, (1 \leq j \leq 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, \ldots, n)$ are real parameters to be determined later. Substituting Eq. (4) into Eq. (3), collecting the coefficients of different polynomials of $x, y, t, \exp(k), (k$ is the polynomials consisting of $l_i, i = 1, 2, \ldots, n)$, and then equating the coefficients of these terms to zero, we finally get a set of algebraic equations in the parameters.

Step 3 Solving the set of algebraic equations defined by step 2 with the help of Maple, we can find the relations among $a_j, (1 \leq j \leq 7)$ and c_i, m_i, p_i, q_i, $(i = 1, 2, \ldots, n)$. Substituting the identified values of $a_j, (1 \leq j \leq 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, \ldots, n)$ into Eqs. (4) and (2), we obtain an abundance of exact interaction solutions between a lump and other solitons of Eq. (1).

3 Interaction solutions to the (2 + 1)-dimensional BLMP equation

The (2 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0 \tag{5}$$

was systematically studied by Gilson [33], where $u = u(x, y, t)$, and the subscripts denote partial derivatives. Much research work has been done in Eq. (5). By using the modified Clarkson–Kruskal (CK) direct method, a Bäcklund transformation of Eq. (5) was constructed and some new solutions were gained [34]. New periodic-wave solutions of other (2 + 1)- and (3 + 1)-dimensional BLMP equations were investigated [35]. Recently, the lump solution and the interaction process between a lump and one kinky soliton have been studied in [30] (see Fig. 1a, b). Later, the interaction phenomena between a lump and two kinky solitons have been investigated [12] (see Fig. 1c).

In the following two subsections, we will discuss the interaction phenomena between a lump and three as well as four kinky solitons for Eq. (5) by the above direct method in Sect. 2.

By the dependent variable transformation

$$u = -2(\ln f)_x, \tag{6}$$

where $f = f(x, y, t)$ is an unknown real function, we obtain the Hirota bilinear form of Eq. (5)

$$(D_t D_y + D_y D_x^3) f \cdot f = 0, \tag{7}$$

that is

$$(f_{ty} f - f_t f_y) + (f_{xxxy} f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{xxx}) = 0. \tag{8}$$

3.1 Interaction solutions between a lump and three kinky solitons

To seek interaction solutions between a lump and three kinky solitons of Eq. (5), we suppose $n = 3$ in Eq. (4) and f is expressed by the following form

$$f = g^2 + h^2 + \sum_{i=1}^{3} c_i \exp(l_i) + a_7, \tag{9}$$

with

$$g = a_1 x + a_2 y + a_3 t,$$
$$h = a_4 x + a_5 y + a_6 t,$$
$$l_i = m_i x + p_i y + q_i t,$$

where $a_j, (j = 1, 2, \ldots, 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, 3)$ are real constants to be determined later. Substituting Eq. (9) into Eq. (8), collecting all the coefficients of $x, y, t, \exp(l_1 + l_2), \exp(l_1 + l_3), \exp(l_2 + l_3), \exp(l_i), (i = 1, 2, 3)$, and then equating coefficients of these terms to zero, we get a set of algebraic equations in $a_j, (j = 1, 2, \ldots, 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, 3)$. Solving the algebraic equations, we can obtain the following relations among these parameters.

Case 1

$$a_j = 0 \ (j = 2, 3, 4, 6), \ \ a_j = a_j \ (j = 1, 5, 7),$$
$$p_i = 0, \ \ q_i = -m_i^3, \ \ m_i = m_i,$$
$$c_i = c_i, \ \ (i = 1, 2, 3) \tag{10}$$

Case 2

$$a_1 = -\frac{a_4 a_5}{a_2}, \ \ a_3 = 0, \ \ a_6 = 0,$$
$$a_j = a_j \ (j = 2, 4, 5, 7),$$
$$p_i = 0, \ \ q_i = -m_i^3, \ \ m_i = m_i,$$
$$c_i = c_i, \ \ (i = 1, 2, 3) \tag{11}$$

where $a_2 \neq 0$.

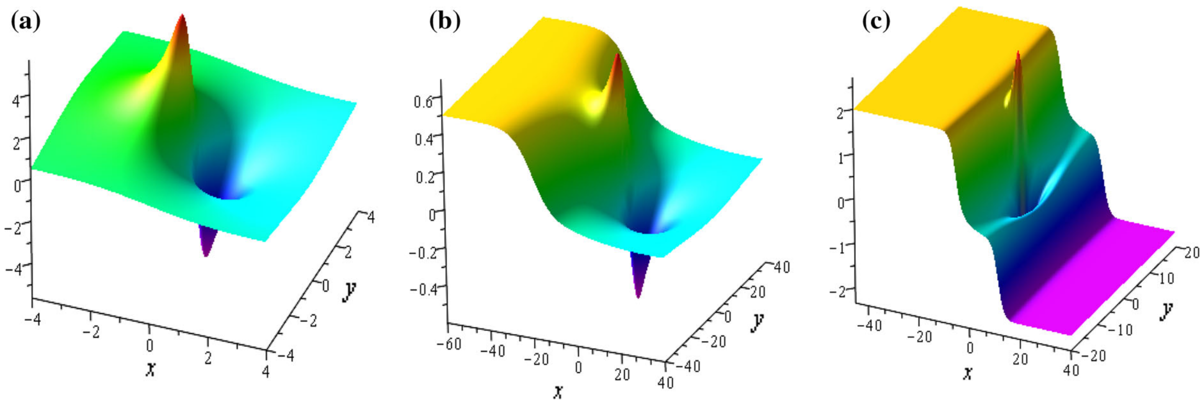

Fig. 1 Lump solution and interaction solution of (2+1)-dimensional BLMP equation. **a** Lump solution; **b** interaction solution between a lump and one kinky soliton; **c** interaction solution between a lump and two kinky solitons

Substituting Eqs. (10) and (11) into Eq. (9), we can get a class of solutions to the bilinear Eq. (7). Then, through the transformation (6), we obtain the exact interaction solutions of Eq. (5)

$$u = -2(\ln f)_x = -\frac{2f_x}{f} = -\frac{2\left(2a_1 g + 2a_4 h + \sum_{i=1}^{3} m_i c_i \exp(l_i)\right)}{g^2 + h^2 + \sum_{i=1}^{3} c_i \exp(l_i) + a_7}. \tag{12}$$

Thus, the resulting solutions are as follows:

$$u_1 = -\frac{2\left(2a_1^2 x + \sum_{i=1}^{3} c_i m_i e^{-m_i(m_i^2 t - x)}\right)}{a_1^2 x^2 + a_5^2 y^2 + a_7 + \sum_{i=1}^{3} c_i e^{-m_i(m_i^2 t - x)}}, \tag{13}$$

Figures 2 and 3 illustrate the different interaction phenomena between a lump and three kinky waves when selecting different parameters.

It is seen that there is an interaction between a lump and three kinky waves at the bottom when $a_1 = 20, a_5 = 20, a_7 = 1, m_1 = -4, m_2 = -6, m_3 = -3, c_1 = c_2 = c_3 = 1$ in Fig. 2. The lump keeps the original position not moving. As time goes by, firstly, there is an interaction between a lump and one kinky wave; then, one kinky wave gradually splits into two, and then into three waves so that there are collisions between a lump and two or three kinky waves; finally, the three waves slowly fuse to two, and then to one wave, but the lump is gradually swallowed by the kinky waves.

$$u_2 = -\frac{2\left(2a_4^2\left(a_2^2 + a_5^2\right)x + a_2^2 \sum_{i=1}^{3} c_i m_i e^{-m_i(m_i^2 t - x)}\right)}{\left(a_2^2 + a_5^2\right)\left(a_2^2 y^2 + a_4^2 x^2\right) + a_7 a_2^2 + a_2^2 \sum_{i=1}^{3} c_i e^{-m_i(m_i^2 t - x)}}. \tag{14}$$

The properties of the lump and the kinky waves which have the interaction process depend on the rational quadratic function and exponential function of u, respectively. Clearly, the forms of u_1 and u_2 are similar. Next, we will mainly discuss the dynamic behaviors of u_1.

From Eqs. (10) and (13), we find that the part of rational quadratic function of u_1 is irrelevant to t for $a_3 = a_6 = 0$, which keeps the original lump not moving. Well, it is $m_i \neq 0, (i = 1, 2, 3)$ that contribute to the collision and $n_1 = n_2 = n_3 = 0$ make the kinky waves walk along the X axis.

There is a collision between a lump and three kinky waves at the middle when $a_1 = 20, a_5 = 20, a_7 = 1, m_1 = -2, m_2 = -4, m_3 = 1, c_1 = c_2 = c_3 = 1$ in Fig. 3. The lump keeps the original position not moving but experiences from the state of being swallowed by one side of the kinky waves, and begins to separate slowly from this side of the kinky waves as the kinky waves walk along the X axis. After reaching the maximum separation, the lump is gradually swallowed by the other side of the kinky waves. During the process, the amplitude of the lump gradually increases from the zero start, reaches its maximum, and then gradually

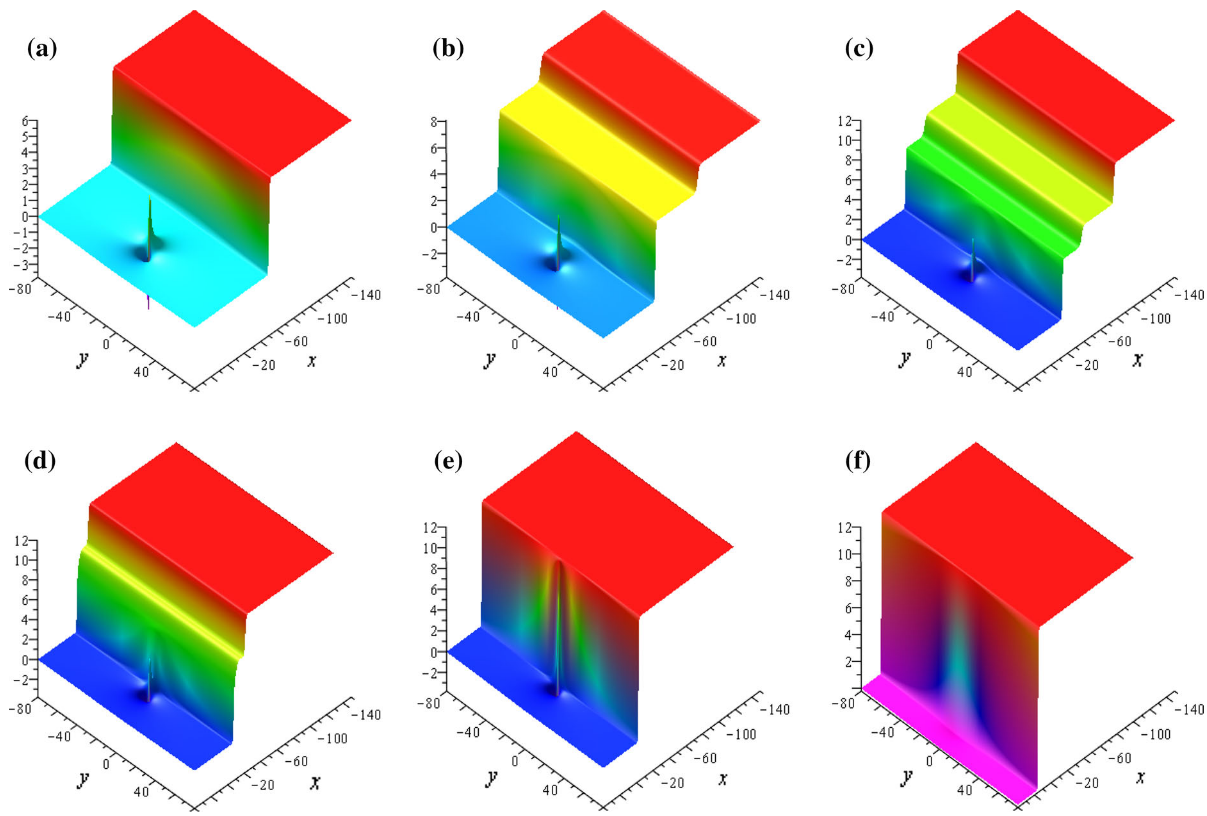

Fig. 2 Plots of interaction solution u_1 via Eq. (13) between a lump and three kinky solitons with parameters $a_1 = 20, a_5 = 20, a_7 = 1, m_1 = -4, m_2 = -6, m_3 = -3, c_1 = c_2 = c_3 = 1$. **a** $t = -5$, **b** $t = -2$, **c** $t = -1$, **d** $t = -0.28$, **e** $t = 0$ and **f** $t = 0.5$

decreases to zero as time goes by. Additionally, there are interactions among the three kinky waves. one kinky wave gradually splits into two and then into three kinky waves; later, the three kinky waves slowly fuse to two, and then to one kinky wave.

Remark 1 In this subsection, we see that the obtained exact solutions of (2+1)-dimensional BLMP equation are all mixed exponential-algebraic solitary wave solutions. They present the completely non-elastic interaction between a lump and three kinky waves. We find that the positions of the interaction are different while we choose different parameters, and the major factors are the sign and value of the parameters $m_i (i = 1, 2, 3)$. The positions of the collision between a lump and three kinky waves locate at the bottom when $m_1 = -4, m_2 = -6, m_3 = -3$ (see Fig. 2) and at the middle when $m_1 = -2, m_2 = -4, m_3 = 1$ (see Fig. 3). Besides these positions, there may be interaction at other positions if we change the sign and value of the parameters $m_i (i = 1, 2, 3)$.

3.2 Interaction solutions between a lump and four kinky solitons

Similarly, to seek interaction solutions between a lump and four kinky solitons of Eq. (5), we assume $n = 4$ in Eq. (4) and f is expressed in the form

$$f = g^2 + h^2 + \sum_{i=1}^{4} c_i \exp(l_i) + a_7, \tag{15}$$

with

$$g = a_1 x + a_2 y + a_3 t,$$
$$h = a_4 x + a_5 y + a_6 t,$$
$$l_i = m_i x + p_i y + q_i t,$$

where $a_j, (j = 1, 2, \ldots, 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, 3, 4)$ are real parameters. By a similar computing process as the above subsection, we can get the relations among these parameters.

Case 1

$$\begin{aligned} &a_j = 0 \ (j = 2, 3, 4, 6), \ \ a_j = a_j \ (j = 1, 5, 7), \\ &p_i = 0, \ \ q_i = -m_i^3, \ \ m_i = m_i, \ \ c_i = c_i, \ \ (i = 1, 2, 3, 4) \end{aligned} \tag{16}$$

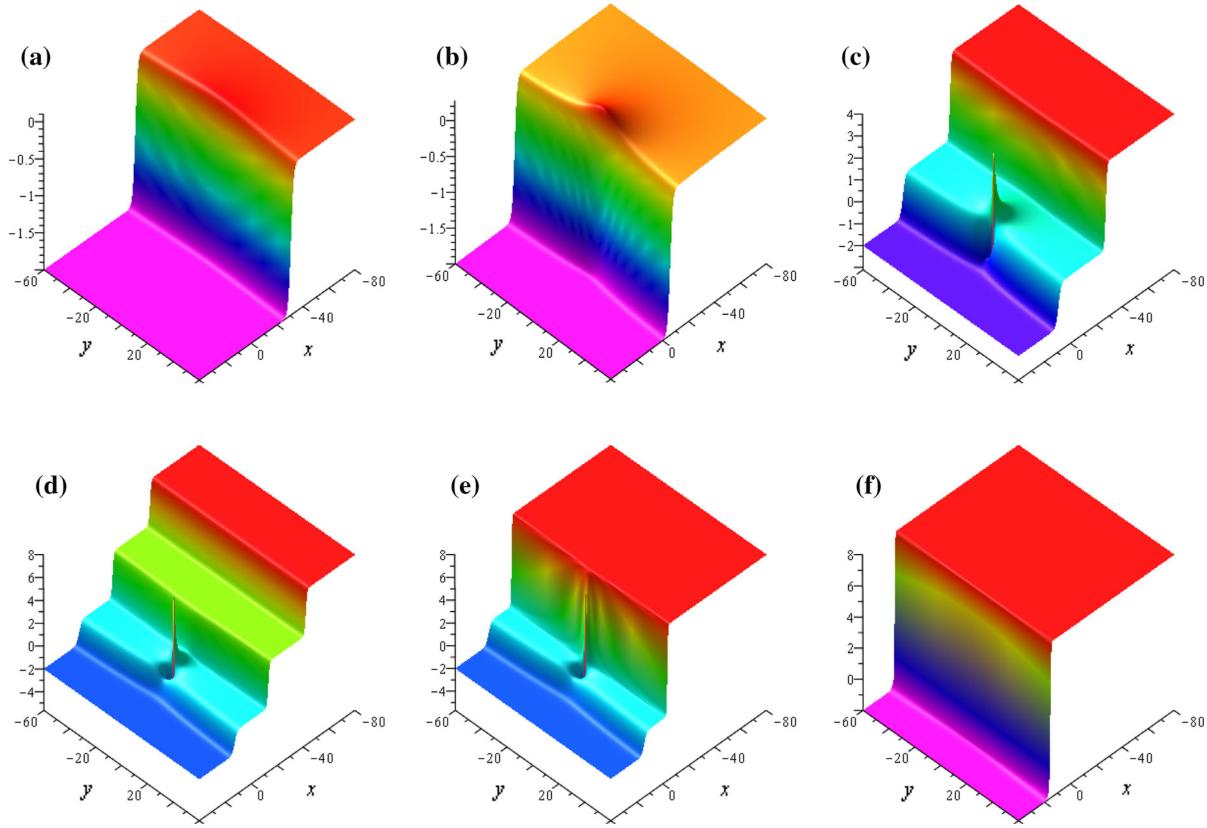

Fig. 3 Plots of interaction solution u_1 via Eq. (13) between a lump and three kinky solitons with parameters $a_1 = 20, a_5 = 20, a_7 = 1, m_1 = -2, m_2 = -4, m_3 = 1, c_1 = c_2 = c_3 = 1$. **a** $t = -45$, **b** $t = -20$, **c** $t = -5$, **d** $t = -1.5$, **e** $t = 0$ and **f** $t = 1.2$

Case 2

$$a_1 = -\frac{a_4 a_5}{a_2},\ a_3 = 0,\ a_6 = 0,\ a_j = a_j\ (j = 2, 4, 5, 7),\quad p_i = 0,\ q_i = -m_i^3,\ m_i = m_i,\ c_i = c_i,\ (i = 1, 2, 3, 4) \tag{17}$$

where $a_2 \neq 0$.

Substituting Eqs. (16) and (17) into Eq. (15), we get a set of solutions to the bilinear Eq. (7). Then, through the transformation (6), we obtain the exact interaction solutions for Eq. (5)

$$u = -\frac{2\left(2a_1 g + 2a_4 h + \sum_{i=1}^{4} m_i c_i \exp(l_i)\right)}{g^2 + h^2 + \sum_{i=1}^{4} c_i \exp(l_i) + a_7}. \tag{18}$$

The resulting solutions for case 1 and case 2 are as follows, respectively,

Figures 4 and 5 display the different interaction between a lump and four kinky waves while choosing different parameters for Eq. (20).

From Fig. 4, we see that the interaction process is similar to the process between a lump and three kinky waves in Fig. 3 in Sect. 3.1 except different positions of collision. The lump is also gradually drowned or swallowed by the kinky waves as time goes by, and during the process there also are collisions among the four kinky waves.

In Fig. 5, there is an interaction between a lump and the four kinky waves at the top and the dynamical process is contrary to that in Fig. 2. As time goes by,

$$u_1 = -\frac{2(2a_1^2 x + \sum_{i=1}^{4} c_i m_i e^{-m_i(m_i^2 t - x)})}{a_1^2 x^2 + a_5^2 y^2 + a_7 + \sum_{i=1}^{4} c_i e^{-m_i(m_i^2 t - x)}}, \tag{19}$$

$$u_2 = -\frac{2\left(2a_4^2\left(a_2^2 + a_5^2\right)x + a_2^2 \sum_{i=1}^{4} c_i m_i e^{-m_i(m_i^2 t - x)}\right)}{\left(a_2^2 + a_5^2\right)\left(a_2^2 y^2 + a_4^2 x^2\right) + a_7 a_2^2 + a_2^2 \sum_{i=1}^{4} c_i e^{-m_i(m_i^2 t - x)}}. \tag{20}$$

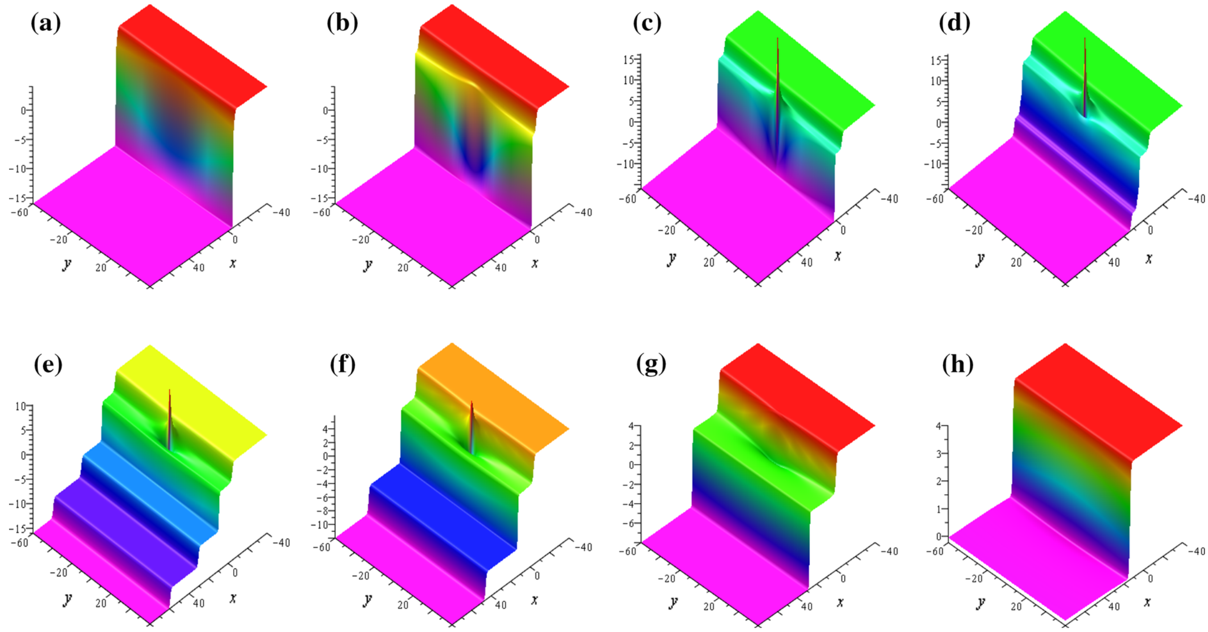

Fig. 4 Plots of interaction solution u_2 via Eq. (20) between a lump and four kinky solitons with parameters $a_2 = 1, a_4 = 5, a_5 = 5, a_7 = 1, m_1 = -2, m_2 = 4, m_3 = 6, m_4 = 8, c_1 = c_2 = c_3 = c_4 = 1$. **a** $t = -0.1$, **b** $t = -0.05$, **c** $t = 0$, **d** $t = 0.085$, **e** $t = 0.4$, **f** $t = 0.6$, **g** $t = 1.5$ and **h** $t = 5$

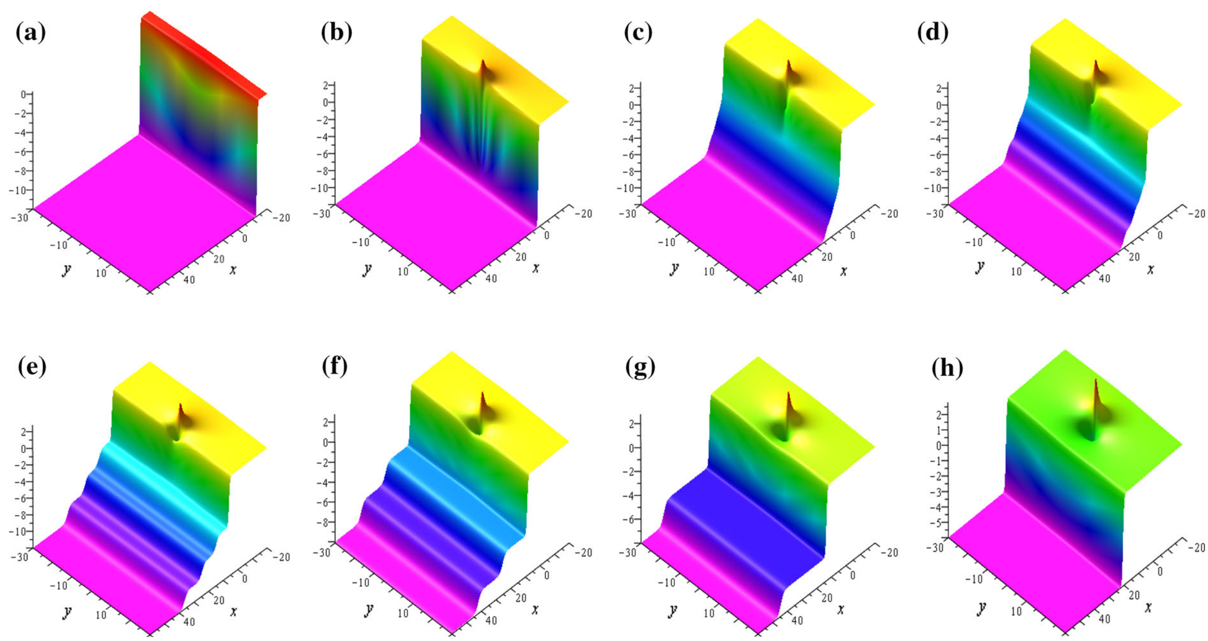

Fig. 5 Plots of interaction solution u_2 via Eq. (20) between a lump and four kinky solitons with parameters $a_2 = 1, a_4 = 1, a_5 = 1, a_7 = 1, m_1 = 3, m_2 = 4, m_3 = 5, m_4 = 6, c_1 = c_2 = c_3 = c_4 = 1$. **a** $t = -0.4$, **b** $t = 0$, **c** $t = 0.15$, **d** $t = 0.22$, **e** $t = 0.4$, **f** $t = 0.69$, **g** $t = 1.2$ and **h** $t = 2$

firstly, there is one kinky wave; then, a lump gradually appears so that there are collisions at the top between a lump and one kinky wave; next, one kinky wave gradually splits into two, and then into three even four kinky waves so that there are collisions between a lump and two or three even four kinky waves; finally, the four kinky waves slowly fuse to three, to two, and then to one kinky wave, but during the process the lump is not swallowed by the kinky waves, and it is always at the top.

Remark 2 In this subsection, the obtained interaction solutions of (2 + 1)-dimensional BLMP equation present the completely non-elastic interaction between a lump and four kinky waves. We see that the positions of the interaction are different while we choose different parameters, and the major factors are the sign and value of the parameters $m_i (i = 1, 2, 3, 4)$. The positions of the collision between a lump and four kinky waves locate at the middle when $m_1 = -2, m_2 = 4, m_3 = 6, m_4 = 8$ (see Fig. 4) and at the top when $m_1 = 3, m_2 = 4, m_3 = 5, m_4 = 6$ (see Fig. 5). Besides these positions, the interaction may be also at other positions if we change the sign and value of the parameters $m_i (i = 1, 2, 3, 4)$.

4 Interaction solutions to the (2 + 1)-dimensional Ito equation

The (2 + 1)-dimensional Ito equation is written as

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + u u_{xt}) + 3u_{xx} \int_{-\infty}^{x} u_t \mathrm{d}x' + \alpha u_{yt} + \beta u_{xt} = 0, \tag{21}$$

which was established by Ito [36], where $u = u(x, y, t)$, α and β are arbitrary constants. And Eq. (21) is reduced to (1 + 1)-dimensional Ito equation when $\alpha = 0, \beta = 0$.

In recent years, there are many studies concerning the (2+1)-dimensional Ito equation. In 2008, Wazwaz investigated its periodic solutions and soliton solutions by the tanh-coth method and the Hirota bilinear method [37]. Recently, Lump solutions and interaction solutions between a lump and one stripe soliton have been studied for Ito equation in [11] (see Fig. 6a, b). Later interaction phenomena between a lump and two line solitons have been found [31](see Fig. 6c).

Next, we will study the interaction solutions between a lump and three as well as four stripe solitons for Eq. (21) by the direct method in Sect. 2.

By the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{22}$$

where $f = f(x, y, t)$ is an unknown real function, we get the Hirota bilinear form of Eq. (21)

$$D_t[D_t + D_x^3 + \alpha D_y + \beta D_x] f \cdot f = 0, \tag{23}$$

that is

$$\begin{aligned} &(f_{tt} f - f_t^2) + (f_{xxxt} f + 3 f_{xt} f_{xx} \\ &- 3 f_{xxt} f_x - f_{xxx} f_t) \\ &+ \alpha(f_{ty} f - f_t f_y) + \beta(f_{tx} f - f_t f_x) = 0 \end{aligned} . \tag{24}$$

4.1 Interaction solutions between a lump and three stripe solitons

To study interaction process between a lump and three stripe solitons of Eq. (21), we suppose $n = 3$ in Eq. (4) and f is written as

$$f = g^2 + h^2 + \sum_{i=1}^{3} c_i \exp(l_i) + a_7, \tag{25}$$

with

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 t, \\ h &= a_4 x + a_5 y + a_6 t, \\ l_i &= m_i x + p_i y + q_i t, \end{aligned}$$

where $a_j, (j = 1, 2, \ldots, 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, 3)$ are real constants to be determined later. We can obtain the exact interaction solutions between a lump and three stripe solitons of Eq. (21) by the similar approach as Sect. 3.1.

Case 1

$$\begin{aligned} &a_2 = -a_1\beta\alpha,\ a_3 = 0,\ a_4 = 0,\ a_6 = -\alpha a_5,\ a_j = a_j, \\ &p_i = -\tfrac{m_i(m_i^2+\beta)}{\alpha},\ q_i = 0,\ m_i = m_i,\ c_i = c_i, \\ &(j = 1, 5, 7,\ i = 1, 2, 3) \end{aligned} \tag{26}$$

where $\alpha \neq 0$.

Case 2

$$\begin{aligned} &a_1 = \tfrac{a_4(\alpha a_5+\beta a_4)}{a_3},\ a_2 = -\tfrac{(\alpha\beta a_4 a_5+\beta^2 a_4^2+a_3^2)}{\alpha a_3}, \\ &a_6 = -\alpha a_5 - \beta a_4,\ a_j = a_j, \\ &p_i = -\tfrac{m_i(m_i^2+\beta)}{\alpha},\ q_i = 0,\ m_i = m_i,\ c_i = c_i, \\ &(j = 3, 4, 5, 7,\ i = 1, 2, 3) \end{aligned} \tag{27}$$

where $\alpha a_3 \neq 0$.

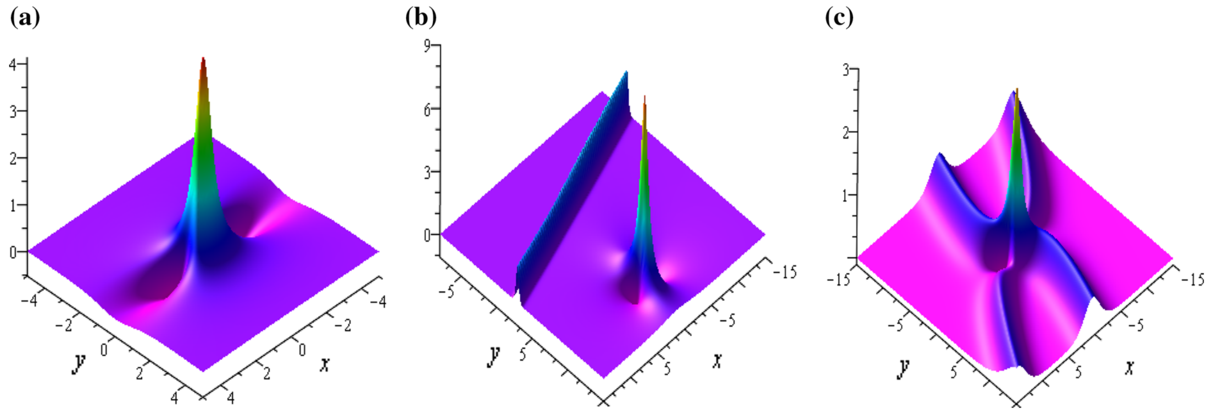

Fig. 6 Lump solution and interaction solution of (2 + 1)-dimensional Ito equation. **a** Lump solution; **b** interaction solution between a lump and one stripe soliton; **c** interaction solution between a lump and two stripe solitons

Substituting Eqs. (26) and (27) into Eq. (25), we get a set of solutions to the bilinear Eq. (23). Then, through the transformation (22), we obtain the exact interaction solutions for Eq. (21).

As for case 1, we get

$$\begin{aligned} u_1 &= 2\frac{f_{1xx}}{f_1} - 2\left(\frac{f_{1x}}{f_1}\right)^2, \\ f_1 &= \left(a_1 x - \frac{\beta a_1}{\alpha} y\right)^2 + (a_5 y - \alpha a_5 t)^2 \\ &\quad + a_7 + \sum_{i=1}^{3} c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\ f_{1x} &= 2a_1\left(a_1 x - \frac{\beta a_1}{\alpha} y\right) + \sum_{i=1}^{3} m_i c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\ f_{1xx} &= 2a_1^2 + \sum_{i=1}^{3} m_i^2 c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}. \end{aligned} \tag{28}$$

As for case 2, we obtain

$$\begin{aligned} u_2 &= 2\frac{f_{2xx}}{f_2} - 2\left(\frac{f_{2x}}{f_2}\right)^2, \\ f_2 &= \left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3} x - \frac{(\alpha\beta a_4 a_5 + \beta^2 a_4^2 + a_3^2)}{\alpha a_3} y + a_3 t\right)^2 \\ &\quad + (a_4 x + a_5 y + (-\alpha a_5 - \beta a_4) t)^2 + a7 \\ &\quad + \sum_{i=1}^{3} c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\ f_{2x} &= 2\frac{a_4(\alpha a_5 + \beta a_4)}{a_3}\left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3} x - \frac{(\alpha\beta a_4 a_5 + \beta^2 a_4^2 + a_3^2)}{\alpha a_3} y + a_3 t\right) \\ &\quad + 2a_4\,(a_4 x + a_5 y + (-\alpha a_5 - \beta a_4) t) \\ &\quad + \sum_{i=1}^{3} m_i c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\ f_{2xx} &= 2\left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3}\right)^2 \\ &\quad + 2a_4^2 + \sum_{i=1}^{3} m_i^2 c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}. \end{aligned} \tag{29}$$

Clearly, Eqs. (28) and (29) have a similar structure. Next, we will mainly discuss the properties of Eq. (28). As seen from (28) and (26), we know that the part of exponential function of u_1 is irrelevant to t for $q_i = 0$, $(i = 1, 2, 3)$. To gain the collision phenomena, $\alpha a_5 \neq 0$ is essential. Based on this, the asymptotic behavior of u_1 can be obtained, the solution $u_1 \to 0$ as $|t| \to \infty$. The asymptotic behavior shows that the lump is finally drowned or swallowed up by the stripe solitons along with the change of time. According to the expression of u_1, we know it is a mixed exponential-algebraic solitary wave solution. It presents a completely non-elastic interaction between a lump and three stripe solitons.

Figure 7 illustrates the typical phenomena in the interaction between a lump and three stripe via Eq. (28) with the parameters $\alpha = 1, \beta = -1, a_1 = 1, a_5 = 2, a_7 = 1, m_1 = \frac{1}{2}, m_2 = 1, m_3 = -\frac{1}{2}, c_1 = 1, c_2 =$

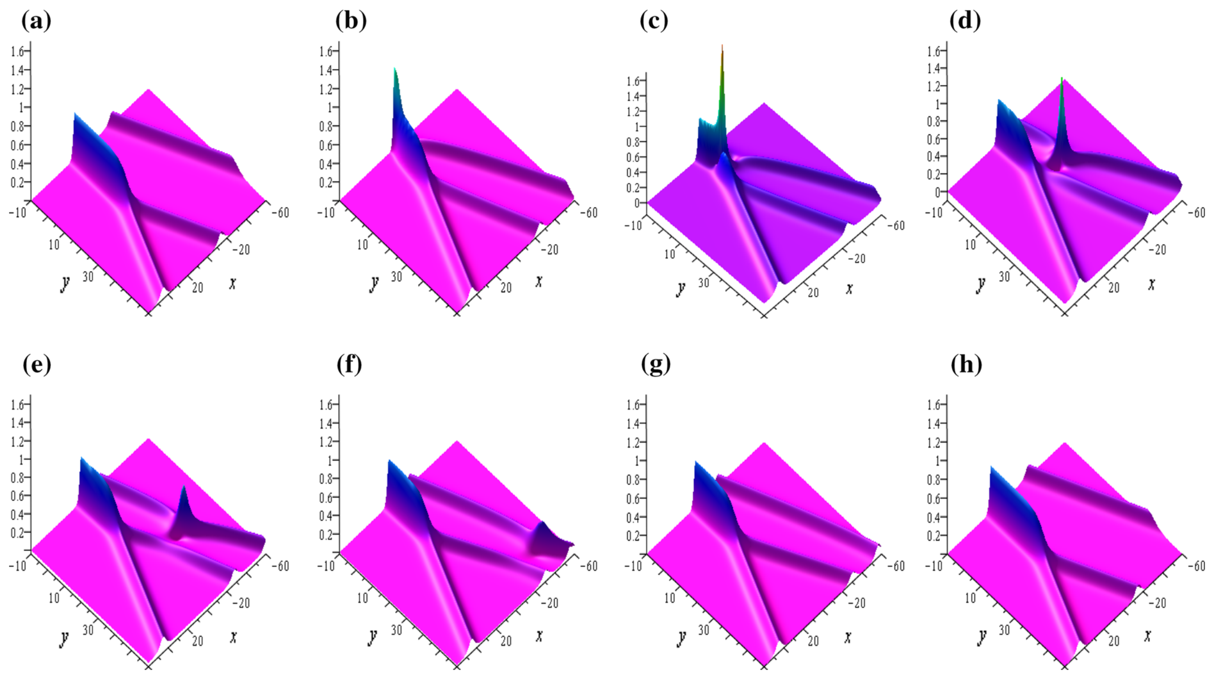

Fig. 7 Plots of interaction solution u_1 via Eq. (28) between a lump and three stripe solitons with parameters $\alpha = 1, \beta = -1, a_1 = 1, a_5 = 2, a_7 = 1, m_1 = \frac{1}{2}, m_2 = 1, m_3 = -\frac{1}{2}, c_1 = 1, c_2 = 1, c_3 = 1$. **a** $t = -1000$, **b** $t = -10$, **c** $t = 0$, **d** $t = 15$, **e** $t = 30$, **f** $t = 50$, **g** $t = 70$ and **h** $t = 1000$

$1, c_3 = 1$. As time goes by, firstly, there is an interaction among three stripe solitons; then, a lump appears and there is a collision between the lump and three stripe solitons; finally, the lump is gradually swallowed up by the stripe solitons after reaching the maximum amplitude of interaction until the lump completely disappears and there is an interaction among three stripe solitons left in the end. "Figure 9 in Appendix" is the corresponding density plots.

4.2 Interaction solutions between a lump and four stripe solitons

Similarly, to discuss interaction solutions between a lump and four stripe solitons of Eq. (21), we assume $n = 4$ in Eq. (4) and

$$f = g^2 + h^2 + \sum_{i=1}^{4} c_i \exp(l_i) + a_7, \tag{30}$$

with

$$g = a_1x + a_2y + a_3t,$$
$$h = a_4x + a_5y + a_6t,$$
$$l_i = m_ix + p_iy + q_it,$$

where $a_j, (j = 1, 2, \ldots, 7)$ and $c_i, m_i, p_i, q_i, (i = 1, 2, 3, 4)$ are real parameters. We get the exact interaction solutions between a lump and four stripe solitons of Eq. (21) by the similar process as Sect. 3.2.

Case 1

$$\begin{aligned} &a_2 = -\frac{\beta a_1}{\alpha},\ a_3 = 0,\ a_4 = 0,\ a_6 = -\alpha a_5,\ a_j = a_j, \\ &p_i = -\frac{m_i(m_i^2+\beta)}{\alpha},\ q_i = 0,\ m_i = m_i,\ c_i = c_i, \\ &(j = 1, 5, 7,\ i = 1, 2, 3, 4) \end{aligned} \tag{31}$$

where $\alpha \neq 0$.

Case 2

$$\begin{aligned} &a_1 = \frac{a_4(\alpha a_5 + \beta a_4)}{a_3},\ a_2 = -\frac{(\alpha\beta a_4 a_5 + \beta^2 a_4^2 + a_3^2)}{\alpha a_3}, \\ &a_6 = -\alpha a_5 - \beta a_4,\ a_j = a_j, \\ &p_i = -\frac{m_i(m_i^2+\beta)}{\alpha},\ q_i = 0,\ m_i = m_i,\ c_i = c_i, \\ &(j = 3, 4, 5, 7,\ i = 1, 2, 3, 4) \end{aligned} \tag{32}$$

where $\alpha a_3 \neq 0$.

As for case 1, we have

$$u_1 = \frac{2f_{1xx}}{f_1} - 2\left(\frac{f_{1x}}{f_1}\right)^2,$$

$$f_1 = \left(a_1x - \frac{\beta a_1}{\alpha}y\right)^2 + (a_5y - \alpha a_5t)^2$$
$$+ a_7 + \sum_{i=1}^{4} c_i e^{m_ix - \frac{m_i(m_i^2+\beta)}{\alpha}y},$$

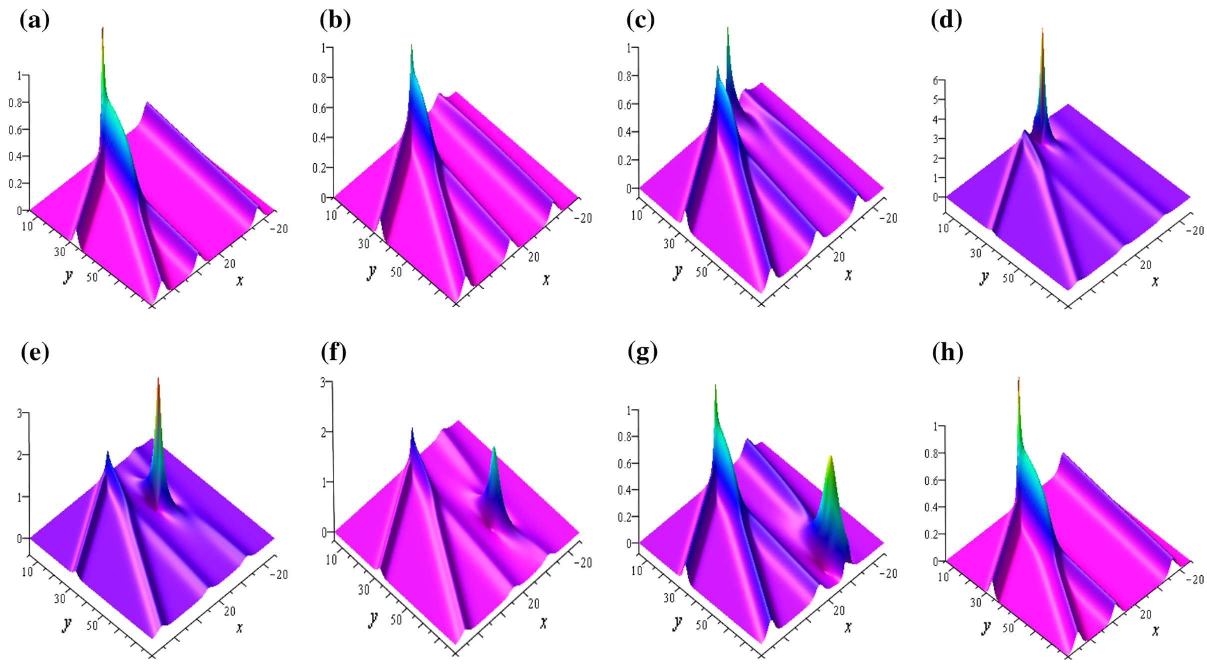

Fig. 8 Plots of interaction solution u_2 via Eq. (34) between a lump and four stripe solitons with parameters $\alpha = 2, \beta = \frac{1}{30}, a_3 = 1, a_4 = 1, a_5 = 1, a_7 = 1, m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}, m_3 = 1, m_4 = \frac{3}{2}, c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1$. **a** $t = -300$, **b** $t = -10$, **c** $t = 0$, **d** $t = 5$, **e** $t = 15$, **f** $t = 25$, **g** $t = 35$ and **h** $t = 300$

$$
\begin{aligned}
f_{1x} &= 2a_1\left(a_1 x - \frac{\beta a_1}{\alpha} y\right) + \sum_{i=1}^{4} m_i c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\
f_{1xx} &= 2a_1^2 + \sum_{i=1}^{4} m_i^2 c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}.
\end{aligned} \tag{33}
$$

As for case 2, we have

$$
\begin{aligned}
u_2 &= 2\frac{f_{2xx}}{f_2} - 2\left(\frac{f_{2x}}{f_2}\right)^2, \\
f_2 &= \left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3} x - \frac{(\alpha\beta a_4 a_5 + \beta^2 a_4^2 + a_3^2)}{\alpha a_3} y + a_3 t\right)^2 \\
&\quad + (a_4 x + a_5 y + (-\alpha a_5 - \beta a_4)t)^2 + a7 \\
&\quad + \sum_{i=1}^{4} c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\
f_{2x} &= 2\frac{a_4(\alpha a_5 + \beta a_4)}{a_3}\left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3} x - \frac{(\alpha\beta a_4 a_5 + \beta^2 a_4^2 + a_3^2)}{\alpha a_3} y + a_3 t\right) \\
&\quad + 2a_4(a_4 x + a_5 y + (-\alpha a_5 - \beta a_4)t) \\
&\quad + \sum_{i=1}^{4} m_i c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}, \\
f_{2xx} &= 2\left(\frac{a_4(\alpha a_5 + \beta a_4)}{a_3}\right)^2 + 2a_4^2 + \sum_{i=1}^{4} m_i^2 c_i e^{m_i x - \frac{m_i(m_i^2+\beta)}{\alpha} y}.
\end{aligned} \tag{34}
$$

Equations (33) and (34) have similar forms. Figure 8 illustrates the interaction phenomena between a lump and four stripe solitons via Eq. (34) with the parameters $\alpha = 2, \beta = \frac{1}{30}, a_3 = 1, a_4 = 1, a_5 = 1, a_7 = 1, m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}, m_3 = 1, m_4 = \frac{3}{2}, c_1 = 1, c_2 = 1, c_3 = 1$. "Figure 10 in Appendix" is the corresponding density plots. It presents a completely non-elastic collision between a lump soliton and four stripe solitons. The interaction process is similar to the process between a lump and three stripe waves in Sect. 4.1, the lump is gradually drowned by the four stripe waves as time goes by.

5 Conclusions

In this paper, through developing a "new" direct method based on the Hirota bilinear form, we studied the "new" interaction phenomena between a lump and three or four kinky solitons of the (2 + 1)-dimensional BLMP equation, as well as we investigated the "new" interaction phenomena between a lump and three or four stripe solitons of the (2 + 1)-dimensional Ito equation. We found that the interaction solutions reduce to the lump (or kinky/stripe) solutions while the part of exponential function (or the quadratic function) disappears. The new interaction phenomena of the two considered equations showed that the lump can be drowned or swallowed by the kinky or stripe solitons, and the involved interactions are the non-elastic collisions. What is more, we found that the positions of the interaction between a lump and three or four kinky waves are different while we choose different parameters, and the major factors are the sign and value of the parameters $m_i (i = 1, 2, 3, 4)$. The collisions may be at the bottom, middle, top or other positions. The dynamical behaviors are richer and more interesting, and the results might be helpful in understanding the propagation processes for nonlinear waves in fluid mechanics. The collisions among lumps and more solitary waves even period waves for the nonlinear evolution equations will be further investigated in future.

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Appendix

See Figs. 9 and 10.

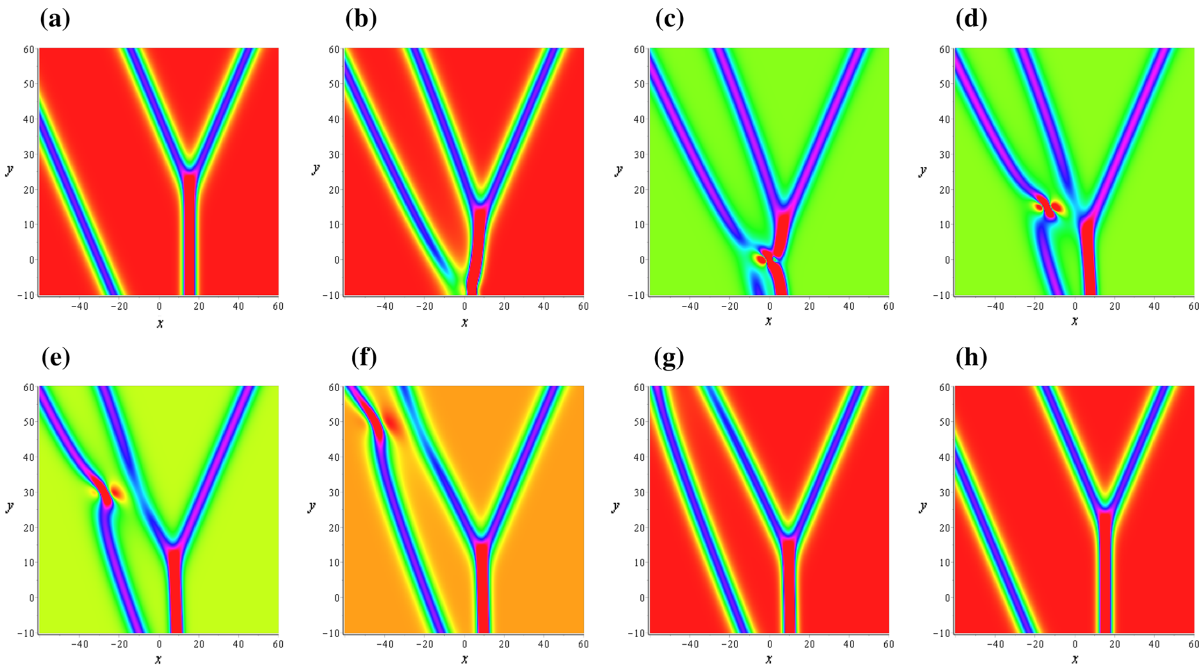

Fig. 9 The corresponding density plots of Fig. 7. **a** $t = -1000$, **b** $t = -10$, **c** $t = 0$, **d** $t = 15$, **e** $t = 30$, **f** $t = 50$, **g** $t = 70$ and **h** $t = 1000$

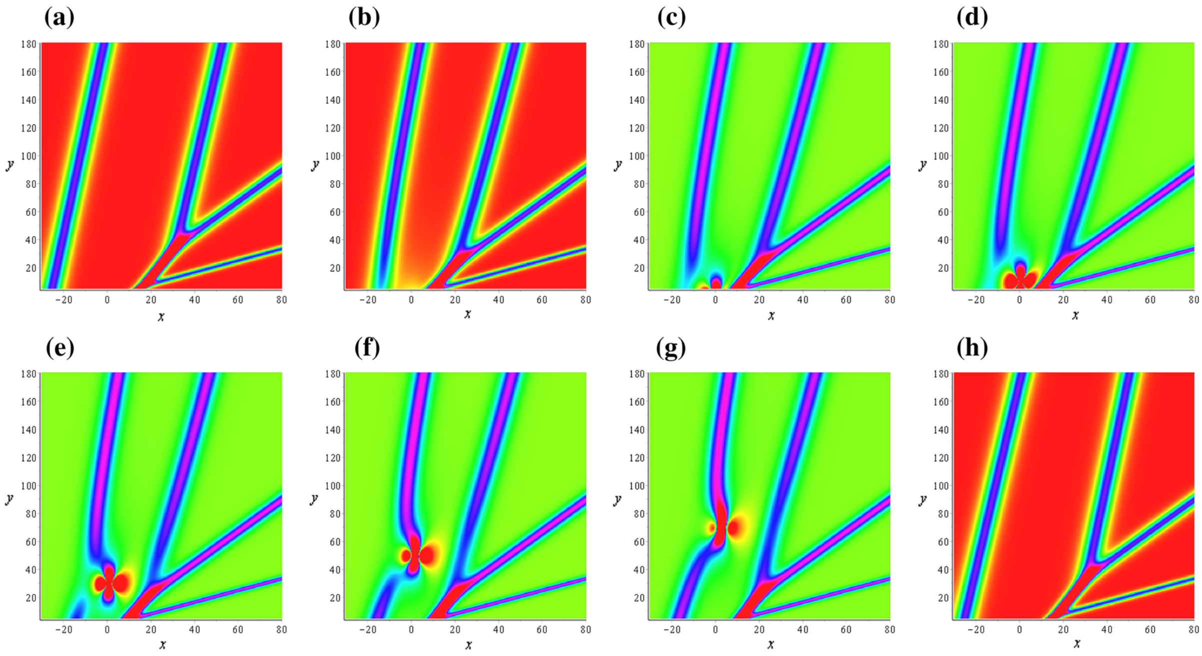

Fig. 10 The corresponding density plots of Fig. 8. **a** $t = -300$, **b** $t = -10$, **c** $t = 0$, **d** $t = 5$, **e** $t = 15$, **f** $t = 25$, **g** $t = 35$ and **h** $t = 300$

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