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Nonlocal symmetries and the nth finite symmetry transformation or AKNS system

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In this paper, by introduction of pseudopotentials, the nonlocal symmetry is obtained for the Ablowitz–Kaup–Newell–Segur system, which is used to describe many physical phenomena in different applications. Together with some auxiliary variables, this kind of nonlocal symmetry can be localized to Lie point symmetry and the corresponding once finite symmetry transformation is calculated for both the original system and the prolonged system. Furthermore, the nth finite symmetry transformation represented in terms of determinant and exact solutions are derived.

Keywords: Nonlocal symmetries; pseudopotentials; Lie point symmetries; the nth finite symmetry transformation.

1. Introduction

The symmetry analysis of the nonlinear differential equations counts mainly on the invariance under a transformation of both independent and dependent variables. This symmetry transformation can map old solutions to new ones, permit to enlarge the diversity of the possible solutions and reduce the number of independent variables of the original system.[1–5] Through conservation laws, some auxiliary

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variables such as potentials and pseudopotentials related to the original system can be used to look for symmetry properties by virtue of symmetry analysis. The corresponding symmetries expressed by these new auxiliary variables are called nonlocal symmetries.

Calculating the once finite symmetry transformation starting from the nonlocal symmetry is an easy calculation to complete and nonlocal symmetries of many special types have been considered in a number of publications,[6–16] while it is difficult to calculate the nth finite symmetry transformation. Recently, Lou[16,17] obtained the nth finite symmetry transformations for the classical KdV equation from Lie point symmetry approach via localization of the residual symmetry and the square spectral function symmetry.

Our basic aim of this paper is to study nonlocal symmetry theory and to find the once finite symmetry transformation as well as the nth finite symmetry transformation for the Ablowitz–Kaup–Newell–Segur (AKNS) system[18–22]

$$\begin{aligned} u_t &= iu_{xx} + 2iu^2 v\,, \\ v_t &= -iv_{xx} - 2iuv^2\,, \end{aligned} \tag{1}$$

by localizing the nonlocal symmetries which are related with pseudopotentials.

2. Refining the Nonlocal Symmetry

Consider the nonlocal symmetry associated to a differential equation

$$u_t + K(x, t, u, u_x, \ldots, u_{x\ldots x}) = 0\,, \tag{2}$$

involving two independent variables $(x, t) \in R^2$ and n dependent variables $u = (u_1, \ldots, u_n) \in R^n$. The generator of the classical Lie point symmetry is of the form

$$v = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta^{u_i}\frac{\partial}{\partial u_i}\,,$$

where ξ, τ and η^{u_i} are functions of x, t and u. The symmetry transformation is equivalent to the solution of the following initial value problem:

$$\begin{aligned} \frac{dX}{d\epsilon} &= \xi(X, T, U_i), \quad X(\epsilon = 0) = x\,, \\ \frac{dT}{d\epsilon} &= \tau(X, T, U_i), \quad T(\epsilon = 0) = t\,, \\ \frac{dU_i}{d\epsilon} &= \eta^{u_i}(X, T, U_i), \quad U_i(\epsilon = 0) = u_i\,. \end{aligned}$$

Let us suppose that the evolution equation (2) admits conservation law of this type

$$\frac{\partial}{\partial t}[G_i(x, t, u, u_x, \ldots, u_{x,\ldots,x})] - \frac{\partial}{\partial x}[H_i(x, t, u, u_x, \ldots, u_{x,\ldots,x})] = 0\,,$$

then, we can introduce a nonlocal potential variable p_i such that

$$p_{i,x} = G_i, \quad p_{i,t} = H_i\,. \tag{3}$$

In a similar way, one can introduce higher potentials from conservation law of the prolonged systems (2) and (3).

Nonlocal Lie Bäcklund operator is of the form

$$v = \eta^{u_i}(x, t, u, u_x, \ldots, u_{x\cdots x}, p)\frac{\partial}{\partial u_i}\,,$$

where p denotes a collection of potentials. The prolonged Lie Bäcklund operator $v_{pr} = \eta^{u_i}\frac{\partial}{\partial u_i} + \eta^{u_{ix}}\frac{\partial}{\partial u_{ix}} + \cdots + \eta^{p_i}\frac{\partial}{\partial p_i} + \cdots$ is generated from the invariance requirement of the equations $u_{i,t} = D_t(u_i), \ldots$ and $p_{i,x} = G_i, p_{i,t} = H_i, \ldots$. Note that in general the prolongation does not close, so the finite symmetry transformation can not be computed directly.

As we know, Lie point symmetries can be applied to construct symmetry transformations, likewise, the similar calculation seems to be invalid for nonlocal symmetries. So it is possible to apply suitably the nonlocal symmetries to the local ones, especially into Lie point symmetries.

Krasil'shchik and Vinogradov[8] proposed a generalization of the concept for nonlocal symmetries by including pseudopotentials of Eq. (2). In contrast to the case of the potential variables, pseudopotential variables can be defined through the implicit equations

$$p_{i,x} = G_i(x, t, u, u_x, \ldots, u_{x,\ldots,x}, p_i, \ldots, p_j)\,,$$

$$p_{i,t} = H_i(x, t, u, u_x, \ldots, u_{x,\ldots,x}, p_i, \ldots, p_j)\,,$$

where the pseudopotential variables are included by G_i and H_i, such that the compatibility condition $p_{i,xt} - p_{i,tx} = \frac{\partial}{\partial t}G_i - \frac{\partial}{\partial x}H_i = 0$ are fulfilled for all solutions of Eq. (2). In the following, we consider nonlocal symmetries of this sort and extend the original system to a closed prolonged system by introducing some additional dependent variables.

3. Nonlocal Symmetry with Pseudopotentials

In this section, we recall the nonlocal symmetries determined by pseudopotentials of AKNS system.[6] The AKNS system has two well-known pseudopotentials p and q, which satisfy the following relations:

$$p_x = -i\lambda p + vq, \tag{4}$$

$$p_t = (-iuv + 2i\lambda^2)p + (-iv_x - 2\lambda v)q\,, \tag{5}$$

$$q_x = -up + i\lambda q\,, \tag{6}$$

$$q_t = (-iu_x + 2\lambda u)p + (iuv - 2i\lambda^2)q\,. \tag{7}$$

The compatibility conditions of this linear system can yield (1). The inclusion of p and q leads to the nonlocal symmetries

$$\sigma^u = -q^2, \quad \sigma^v = p^2\,, \tag{8}$$

of system (1). Obviously, the nonlocal symmetries σ^u and σ^v satisfy the following linearized equations:

$$\sigma^u_t - i\sigma^u_{xx} - 4iuv\sigma^u - 2iu^2\sigma^v = 0\,,$$
$$\sigma^v_t + i\sigma^v_{xx} + 4iuv\sigma^v + 2iv^2\sigma^u = 0\,.$$

Thus, to compute the initial value problem

$$\frac{dU(\epsilon)}{d\epsilon} = -Q(\epsilon)^2, \quad U(0) = u\,,$$
$$\frac{dV(\epsilon)}{d\epsilon} = P(\epsilon)^2, \quad V(0) = v\,,$$

we have to localize the nonlocal symmetries (8). Following from formulas (4)–(7), another potential f should satisfy

$$f_x = pq, \quad f_t = -i(up^2 + vq^2 - 4i\lambda pq)\,, \tag{9}$$

then the nonlocal symmetries become Lie point symmetries

$$\begin{gathered} \sigma^u = -q^2, \quad \sigma^v = p^2, \quad \sigma^p = pf\,, \\ \sigma^q = qf, \quad \sigma^f = f^2\,, \end{gathered} \tag{10}$$

for the prolonged systems (1), (4)–(7) and (9). The obtained Lie point symmetries (10) are consistent with the results in Ref. 6, while only the once finite transformation is obtained in Ref. 6.

The symmetries (10) are solutions of the following linearized system:

$$\sigma^u_t - i\sigma^u_{xx} - 4iuv\sigma^u - 2iu^2\sigma^v = 0\,,$$
$$\sigma^v_t + i\sigma^v_{xx} + 4iuv\sigma^v + 2iv^2\sigma^u = 0\,,$$
$$\sigma^p_x + i\lambda\sigma^p - q\sigma^v - v\sigma^q = 0\,,$$
$$\sigma^p_t + i(\sigma^u v + u\sigma^v)p - i(2\lambda^2 - uv)\sigma^p + (i\sigma^v_x + 2\lambda\sigma^v)q + (iv_x + 2\lambda v)\sigma^q = 0\,,$$
$$\sigma^q_x + \sigma^u p + \sigma^p u - i\lambda\sigma^q = 0\,,$$
$$\sigma^q_t - (2\lambda\sigma^u - i\sigma^u_x)p - (2\lambda u - iu_x)\sigma^p - i(\sigma^u v + u\sigma^v)q - i(uv - 2\lambda^2)\sigma^q = 0\,,$$
$$\sigma^f_x - \sigma^p q - \sigma^q p = 0\,,$$
$$\sigma^f_t + i(\sigma^u p^2 + \sigma^v q^2 + (2up - 4i\lambda q)\sigma^p + (2vq - 4i\lambda p)\sigma^q) = 0\,.$$

After solving the initial value problem

$$\frac{dU(\epsilon)}{d\epsilon} = -Q(\epsilon)^2, \quad U(0) = u\,,$$
$$\frac{dV(\epsilon)}{d\epsilon} = P(\epsilon)^2, \quad V(0) = v\,,$$
$$\frac{dP(\epsilon)}{d\epsilon} = P(\epsilon)F(\epsilon), \quad P(0) = p\,,$$

$$\frac{dQ(\epsilon)}{d\epsilon} = Q(\epsilon)F(\epsilon), \quad Q(0) = q\,,$$
$$\frac{dF(\epsilon)}{d\epsilon} = F(\epsilon)^2, \quad F(0) = f\,,$$

we get the once finite symmetry transformation theorem.

Theorem 1. *If* $\{u, v, p, q, f\}$ *is a solution of the prolonged AKNS systems* (1), (4)–(7) *and* (9), *so is* $\{U(\epsilon), V(\epsilon), P(\epsilon), Q(\epsilon), F(\epsilon)\}$ *with*

$$U(\epsilon) = u + \frac{q^2\epsilon}{\epsilon f - 1}, \quad V(\epsilon) = v - \frac{p^2\epsilon}{\epsilon f - 1}, \quad P(\epsilon) = -\frac{p}{\epsilon f - 1}\,,$$
$$Q(\epsilon) = -\frac{q}{\epsilon f - 1}, \quad F(\epsilon) = -\frac{f}{\epsilon f - 1}\,.$$

4. The nth Finite Symmetry Transformation

Now we shall consider the problem of calculating the nth finite symmetry transformation. Because the parameter λ in Eqs. (4)–(7) is arbitrary, we have infinitely many nonlocal symmetries

$$\sigma_n^u = -\sum_{j=1}^{n} c_j q_j^2, \quad \sigma_n^v = \sum_{j=1}^{n} c_j p_j^2\,, \tag{11}$$

where p_j and $q_j, j = 1, \ldots, n$, are pseudopotentials in Eqs. (4)–(7) with different parameters $\lambda_j \neq \lambda_k, \forall j \neq k$.

To solve the initial value problem related to the nonlocal symmetries of Eq. (11) for any fixed n,

$$\frac{dU(\epsilon)}{d\epsilon} = -\sum_{j=1}^{n} c_j Q_j(\epsilon)^2, \quad U(0) = u\,,$$
$$\frac{dV(\epsilon)}{d\epsilon} = \sum_{j=1}^{n} c_j P_j(\epsilon)^2, \quad V(0) = v\,,$$

we introduce the prolonged system

$$u_t = iu_{xx} + 2iu^2 v\,, \tag{12}$$
$$v_t = -iv_{xx} - 2iuv^2\,, \tag{13}$$
$$p_{j,x} = -i\lambda_j p_j + vq_j\,, \tag{14}$$
$$p_{j,t} = (-iuv + 2i\lambda_j^2)p_j + (-iv_x - 2\lambda_j v)q_j\,, \tag{15}$$
$$q_{j,x} = -up_j + i\lambda_j q_j\,, \tag{16}$$
$$q_{j,t} = (-iu_x + 2\lambda_j u)p_j + (iuv - 2i\lambda_j^2)q_j\,, \tag{17}$$
$$f_{j,x} = p_j q_j\,, \tag{18}$$

$$f_{j,t} = -i(up_j^2 + vq_j^2 - 4i\lambda_j p_j q_j)\,. \tag{19}$$

Thus the nonlocal symmetries (11) become the following Lie point symmetries of the above prolonged system

$$\sigma^u = -\sum_{j=1}^{n} c_j q_j^2\,, \tag{20}$$

$$\sigma^v = \sum_{j=1}^{n} c_j p_j^2\,, \tag{21}$$

$$\sigma^{p_j} = c_j p_j f_j + i\sum_{k\neq j}^{n} \frac{c_k p_k (p_j q_k - p_k q_j)}{2(\lambda_j - \lambda_k)}\,, \tag{22}$$

$$\sigma^{q_j} = c_j q_j f_j + i\sum_{k\neq j}^{n} \frac{c_k q_k (p_j q_k - p_k q_j)}{2(\lambda_j - \lambda_k)}\,, \tag{23}$$

$$\sigma^{f_j} = c_j f_j^2 - \sum_{k\neq j}^{n} \frac{c_k (p_j q_k - p_k q_j)^2}{4(\lambda_j - \lambda_k)^2}\,, \tag{24}$$

which are the solutions of the linearized system

$$\sigma_t^u - i\sigma_{xx}^u - 4iuv\sigma^u - 2iu^2\sigma^v = 0\,, \tag{25}$$

$$\sigma_t^v + i\sigma_{xx}^v + 4iuv\sigma^v + 2iv^2\sigma^u = 0\,, \tag{26}$$

$$\sigma_x^{p_j} + i\lambda_j\sigma^{p_j} - q_j\sigma^v - v\sigma^{q_j} = 0\,, \tag{27}$$

$$\begin{aligned}\sigma_t^{p_j} &+ i(\sigma^u v + u\sigma^v)p_j - i(2\lambda_j^2 - uv)\sigma^{p_j} + (i\sigma_x^v + 2\lambda_j\sigma^v)q_j \\ &+ (iv_x + 2\lambda_j v)\sigma^{q_j} = 0\,,\end{aligned} \tag{28}$$

$$\sigma_x^{q_j} + \sigma^u p_j + \sigma^{p_j} u - i\lambda_j\sigma^{q_j} = 0\,, \tag{29}$$

$$\begin{aligned}\sigma_t^{q_j} &- (2\lambda_j\sigma^u - i\sigma_x^u)p_j - (2\lambda_j u - iu_x)\sigma^{p_j} - i(\sigma^u v + u\sigma^v)q_j \\ &- i(uv - 2\lambda_j^2)\sigma^{q_j} = 0\,,\end{aligned} \tag{30}$$

$$\sigma_x^{f_j} - \sigma^{p_j} q_j - \sigma^{q_j} p_j = 0\,, \tag{31}$$

$$\sigma_t^{f_j} + i(\sigma^u p_j^2 + \sigma^v q_j^2 + (2up_j - 4i\lambda_j q_j)\sigma^{p_j} + (2vq_j - 4i\lambda_j p_j)\sigma^{q_j}) = 0\,, \tag{32}$$

with $j = 1, \ldots, n$.

Proof. For any fixed $c_j \neq 0, c_k = 0, k \neq j$, it is apparent that

$$\sigma^u = -c_j q_j^2, \quad \sigma^v = c_j p_j^2\,, \tag{33}$$

are the solutions of Eqs. (25) and (26). Substituting (33) into Eqs. (27) and (29), we get

$$\sigma_x^{p_j} + i\lambda_j\sigma^{p_j} - c_j p_j^2 q_j - v\sigma^{q_j} = 0\,,$$

$$\sigma^{q_j}_x - c_j q_j^2 p_j + \sigma^{p_j} u - i\lambda_j \sigma^{q_j} = 0\,,$$

then eliminating v and u together with Eqs. (14) and (16), it is easy to verify that

$$\sigma^{p_j} = c_j p_j f_j, \quad \sigma^{q_j} = c_j q_j f_j\,. \tag{34}$$

Substituting (34) back into Eq. (31), we obtain

$$\sigma^f_j = c_j f_j^2\,.$$

To find σ^{p_k} and $\sigma^{q_k}, k \neq j$ from Eqs. (27) and (29) with (33), we eliminate v and u via

$$v = \frac{p_{kx} + i\lambda_k p_k}{q_k}, \quad u = \frac{i\lambda_k q_k - q_{kx}}{p_k}\,.$$

Then we get

$$\sigma^{p_k}_x + i\lambda_k \sigma^{p_k} - c_j p_j^2 q_k - \frac{\sigma^{q_k}(p_{kx} + i\lambda_k p_k)}{q_k} = 0\,,$$

$$\sigma^{q_k} - i\lambda_k \sigma^{q_k} - c_j q_j^2 p_k - \frac{\sigma^{p_k}(q_{kx} - i\lambda_k q_k)}{p_k} = 0\,.$$

By using the relations

$$p_{kx} = \frac{(p_{jx} + i\lambda_j p_j)q_k}{q_j} - i\lambda_k p_k\,, \tag{35}$$

$$q_{kx} = -\frac{(-q_{jx} + i\lambda_j q_j)p_k}{p_j} + i\lambda_k q_k\,, \tag{36}$$

we can verify the results

$$\sigma^{p_k} = \frac{ic_j p_j (p_j q_k - p_k q_j)}{2(\lambda_j - \lambda_k)}\,, \tag{37}$$

$$\sigma^{q_k} = \frac{ic_j q_j (p_j q_k - p_k q_j)}{2(\lambda_j - \lambda_k)}\,. \tag{38}$$

From Eq. (31), for $j = k$, we have

$$\sigma^{f_k}_x - \sigma^{p_k} q_k - \sigma^{q_k} p_k = 0\,,$$

which further yields

$$\sigma^{f_k} = -\frac{c_j (p_k q_j - p_j q_k)^2}{4(\lambda_j - \lambda_k)^2}\,,$$

with Eqs. (35)–(38). Thus, we have proved the above statement. □

Therefore, the corresponding initial value problem for Lie point symmetries (20)–(24) has the form

$$\frac{dU(\epsilon)}{d\epsilon} = -\sum_{j=1}^{n} c_j Q_j(\epsilon)^2\,,$$

$$\begin{aligned}
\frac{dV(\epsilon)}{d\epsilon} &= \sum_{j=1}^{n} c_j P_j(\epsilon)^2 \,, \\
\frac{dP_j(\epsilon)}{d\epsilon} &= c_j P_j(\epsilon) F_j(\epsilon) + i \sum_{k\neq j}^{n} \frac{c_k P_k(\epsilon)(P_j(\epsilon)Q_k(\epsilon) - P_k(\epsilon)Q_j(\epsilon))}{2(\lambda_j - \lambda_k)} \,, \\
\frac{dQ_j(\epsilon)}{d\epsilon} &= c_j Q_j(\epsilon) F_j(\epsilon) + i \sum_{k\neq j}^{n} \frac{c_k Q_k(\epsilon)(P_j(\epsilon)Q_k(\epsilon) - P_k(\epsilon)Q_j(\epsilon))}{2(\lambda_j - \lambda_k)} \,, \\
\frac{dF_j(\epsilon)}{d\epsilon} &= c_j F_j(\epsilon)^2 - \sum_{k\neq j}^{n} \frac{c_k (P_j(\epsilon)Q_k(\epsilon) - P_k(\epsilon)Q_j(\epsilon))^2}{4(\lambda_j - \lambda_k)^2} \,, \\
& U(0) = u, \quad V(0) = v, \quad P_j(0) = p_j, \quad Q_j(0) = q_j \,, \\
& F_j(0) = f_j, \quad j = 1, \ldots, n \,.
\end{aligned}$$

Theorem 2. *If* $\{u, v, p_j, q_j, f_j\}$ *is a solution of the prolonged AKNS system* (12)–(19), *so is* $\{U(\epsilon), V(\epsilon), P_j(\epsilon), Q_j(\epsilon), F_j(\epsilon)\}$ *with*

$$U(\epsilon) = -\partial_\epsilon^{-1} \sum_{j=1}^{n} \frac{c_j \Omega_j^2}{\Delta^2} + C_u \,,$$

$$V(\epsilon) = \partial_\epsilon^{-1} \sum_{j=1}^{n} \frac{c_j \Delta_j^2}{\Delta^2} + C_v \,,$$

$$P_j(\epsilon) = -\frac{\Delta_j}{\Delta}, \quad Q_j(\epsilon) = -\frac{\Omega_j}{\Delta} \,, \quad F_j(\epsilon) = -\frac{\Gamma_j}{\Delta} \,.$$

Here, C_u *and* C_v *are functions and are determined by the initial conditions, namely,* $\{U(0) = u, V(0) = v\}$. $\Delta, \Gamma_j, \Delta_j$ *and* Ω_j *are determinants of the matrices* M, M_j, N_j *and* K_j,

$$M = \begin{pmatrix}
c_1\epsilon f_1 - 1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1j} & \cdots & c_1\epsilon w_{1n} \\
c_2\epsilon w_{12} & c_2\epsilon f_2 - 1 & \cdots & c_2\epsilon w_{2j} & \cdots & c_2\epsilon w_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_j\epsilon w_{1j} & c_1\epsilon w_{2j} & \cdots & c_j\epsilon f_j - 1 & \cdots & c_j\epsilon w_{jn} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{jn} & \cdots & c_n\epsilon f_n - 1
\end{pmatrix},$$

$$M_j = \begin{pmatrix} c_1\epsilon f_1 - 1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,j-1} & c_1\epsilon w_{1j} & c_1\epsilon w_{1,j+1} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 1 & \cdots & c_2\epsilon w_{2,j-1} & c_2\epsilon w_{2j} & c_2\epsilon w_{2,j+1} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{j-1}\epsilon w_{1,j-1} & c_{j-1}\epsilon w_{2,j-1} & \cdots & c_{j-1}\epsilon f_{j-1} - 1 & c_{j-1}\epsilon w_{j-1,j} & c_{j-1}\epsilon w_{j-1,j+1} & \cdots & c_{j-1}\epsilon w_{j-1,n} \\ w_{1j} & w_{2j} & \cdots & w_{j-1,j} & f_j & w_{j,j+1} & \cdots & w_{jn} \\ c_{j+1}\epsilon w_{1,j+1} & c_{j+1}\epsilon w_{2,j+1} & \cdots & c_{j+1}\epsilon w_{j-1,j+1} & c_{j+1}\epsilon w_{j,j+1} & c_{j+1}\epsilon f_{j+1} - 1 & \cdots & c_{j+1}\epsilon w_{j+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{j-1,n} & c_n\epsilon w_{jn} & c_n\epsilon w_{j+1,n} & \cdots & c_n\epsilon f_n - 1 \end{pmatrix},$$

$$N_j = \begin{pmatrix} c_1\epsilon f_1 - 1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,j-1} & c_1\epsilon w_{1j} & c_1\epsilon w_{1,j+1} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 1 & \cdots & c_2\epsilon w_{2,j-1} & c_2\epsilon w_{2j} & c_2\epsilon w_{2,j+1} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{j-1}\epsilon w_{1,j-1} & c_{j-1}\epsilon w_{2,j-1} & \cdots & c_{j-1}\epsilon f_{j-1} - 1 & c_{j-1}\epsilon w_{j-1,j} & c_{j-1}\epsilon w_{j-1,j+1} & \cdots & c_{j-1}\epsilon w_{j-1,n} \\ p_1 & p_2 & \cdots & p_{j-1} & p_j & p_{j+1} & \cdots & p_n \\ c_{j+1}\epsilon w_{1,j+1} & c_{j+1}\epsilon w_{2,j+1} & \cdots & c_{j+1}\epsilon w_{j-1,j+1} & c_{j+1}\epsilon w_{j,j+1} & c_{j+1}\epsilon f_{j+1} - 1 & \cdots & c_{j+1}\epsilon w_{j+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{j-1,n} & c_n\epsilon w_{jn} & c_n\epsilon w_{j+1,n} & \cdots & c_n\epsilon f_n - 1 \end{pmatrix},$$

$$K_j = \begin{pmatrix} c_1\epsilon f_1 - 1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,j-1} & c_1\epsilon w_{1j} & c_1\epsilon w_{1,j+1} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 1 & \cdots & c_2\epsilon w_{2,j-1} & c_2\epsilon w_{2j} & c_2\epsilon w_{2,j+1} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{j-1}\epsilon w_{1,j-1} & c_{j-1}\epsilon w_{2,j-1} & \cdots & c_{j-1}\epsilon f_{j-1} - 1 & c_{j-1}\epsilon w_{j-1,j} & c_{j-1}\epsilon w_{j-1,j+1} & \cdots & c_{j-1}\epsilon w_{j-1,n} \\ q_1 & q_2 & \cdots & q_{j-1} & q_j & q_{j+1} & \cdots & q_n \\ c_{j+1}\epsilon w_{1,j+1} & c_{j+1}\epsilon w_{2,j+1} & \cdots & c_{j+1}\epsilon w_{j-1,j+1} & c_{j+1}\epsilon w_{j,j+1} & c_{j+1}\epsilon f_{j+1} - 1 & \cdots & c_{j+1}\epsilon w_{j+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{j-1,n} & c_n\epsilon w_{jn} & c_n\epsilon w_{j+1,n} & \cdots & c_n\epsilon f_n - 1 \end{pmatrix},$$

with

$$w_{jk} = i\frac{p_j q_k - p_k q_j}{2(\lambda_j - \lambda_k)}, \quad \lambda_j \neq \lambda_k\,.$$

We shall now show that the solutions of AKNS system (1) arise from Theorem 2. For a given trivial solution of AKNS system

$$u = 0, \quad v = 0, \tag{39}$$

functions p_j, q_j, f_j can be solved recursively

$$p_j = \exp(-i\lambda_j(x - 2\lambda_j t)), \quad q_j = \exp(i\lambda_j(x - 2\lambda_j t)), \quad f_j = x - 4\lambda_j t\,, \tag{40}$$

through Eqs. (14)–(19) and w_{jk} can be expressed as

$$w_{jk} = \frac{\sin((\lambda_j - \lambda_k)(2(\lambda_j + \lambda_k)t - x))}{\lambda_k - \lambda_j}\,. \tag{41}$$

Let us first consider the case of $n = 1$, in which case, $U(\epsilon)$ and $V(\epsilon)$ satisfy

$$U(\epsilon) = -\int \frac{c_1\Omega_1^2}{\Delta^2}d\epsilon + C_u, \quad V(\epsilon) = \int \frac{c_1\Delta_1^2}{\Delta^2}d\epsilon + C_v\,. \tag{42}$$

Here, $\Delta, \Omega_1, \Delta_1$ are determinants of matrices M, K_1, N_1,

$$\Delta = |M| = c_1\epsilon f_1 - 1, \quad \Omega_1 = |K_1| = q_1, \quad \Delta_1 = |N_1| = p_1\,, \tag{43}$$

respectively. By requiring initial condition $\{U(0) = u, V(0) = v\}$, we obtain C_u and C_v as

$$C_u = u + \frac{q_1^2}{f_1}, \quad C_v = v - \frac{p_1^2}{f_1}\,. \tag{44}$$

Further, imposing $j = 1$ to (40),

$$p_1 = \exp(-i\lambda_1(x - 2\lambda_1 t)), \quad q_1 = \exp(i\lambda_1(x - 2\lambda_1 t)), \quad f_1 = x - 4\lambda_1 t\,, \tag{45}$$

and substituting these expressions with (39), (43), (44) into (42), one can express the nontrivial solution of AKNS system as

$$U(\epsilon) = -\frac{c_1\epsilon\exp(2\lambda_1 i(x - 2\lambda_1 t))}{4c_1\lambda_1 t\epsilon - c_1 x\epsilon + 1}, \quad V(\epsilon) = \frac{c_1\epsilon\exp(2\lambda_1 i(2\lambda_1 t - x))}{4c_1\lambda_1 t\epsilon - c_1 x\epsilon + 1}\,.$$

We continue in the same way to calculation for the case $n = 2$, which is essentially the same as the case of $n = 1$, and the resulting determinants take the form

$$\begin{aligned}
\Delta &= |M| = (c_1\epsilon f_1 - 1)(c_2\epsilon f_2 - 1) - c_1c_2w_{12}^2\epsilon^2\,,\\
\Omega_1 &= |K_1| = q_1(c_2\epsilon f_2 - 1) - q_2c_2\epsilon w_{12}\,,\\
\Omega_2 &= |K_2| = q_2(c_1\epsilon f_1 - 1) - q_1c_1\epsilon w_{12}\,,\\
\Delta_1 &= |N_1| = p_1(c_2\epsilon f_2 - 1) - p_2c_2\epsilon w_{12}\,,\\
\Delta_2 &= |N_2| = p_2(c_1\epsilon f_1 - 1) - p_1c_1\epsilon w_{12}\,,
\end{aligned} \tag{46}$$

where $p_1, p_2, q_1, q_2, f_1, f_2$ and w_{12} satisfy the equations

$$\begin{aligned}
&p_1 = \exp(-i\lambda_1(x - 2\lambda_1 t)), \quad q_1 = \exp(i\lambda_1(x - 2\lambda_1 t)), \quad f_1 = x - 4\lambda_1 t\,,\\
&p_2 = \exp(-i\lambda_2(x - 2\lambda_2 t)), \quad q_2 = \exp(i\lambda_2(x - 2\lambda_2 t)), \quad f_2 = x - 4\lambda_2 t\,,\\
&w_{12} = \frac{\sin((\lambda_1 - \lambda_2)(2(\lambda_1 + \lambda_2)t - x))}{\lambda_2 - \lambda_1}\,.
\end{aligned} \tag{47}$$

From this, we proceed to calculate

$$U(\epsilon) = -\int\left(\frac{c_1\Omega_1^2}{\Delta^2} + \frac{c_2\Omega_2^2}{\Delta^2}\right)d\epsilon + C_u, \quad V(\epsilon) = \int\left(\frac{c_1\Delta_1^2}{\Delta^2} + \frac{c_2\Delta_2^2}{\Delta^2}\right)d\epsilon + C_v\,. \tag{48}$$

The integral functions

$$C_u = u + \frac{q_1^2 f_2 + q_2^2 f_1 - 2q_1q_2w_{12}}{f_1f_2 - w_{12}^2}, \quad C_v = v - \frac{p_1^2 f_2 + p_2^2 f_1 - 2p_1p_2w_{12}}{f_1f_2 - w_{12}^2}\,, \tag{49}$$

are solved through the initial condition $\{U(0) = u, V(0) = v\}$.

Insertion of these expressions (39), (46), (47) and (49) in (48) yields another exact solution of AKNS system

$$\begin{aligned}
U(\epsilon) = {} & ((\lambda_1-\lambda_2)^2(c_1(c_2\epsilon(4\lambda_2 t - x)+1)\epsilon\exp(i\lambda_1(x-2\lambda_1 t))^2 \\
& + c_2(c_1\epsilon(4\lambda_1 t - x)+1)\epsilon\exp(i\lambda_2(x-2\lambda_2 t))^2) \\
& - 2c_1c_2(\lambda_1-\lambda_2)\epsilon^2\sin((\lambda_1-\lambda_2)(2(\lambda_1+\lambda_2)t-x))\exp(i(\lambda_1+\lambda_2)x \\
& - 2i(\lambda_1^2+\lambda_2^2)t))/(c_1c_2(\sin((\lambda_1-\lambda_2)(2(\lambda_1+\lambda_2)t-x))^2 \\
& - (\lambda_1-\lambda_2)^2(4\lambda_2 t-x)(4\lambda_1 t-x))\epsilon^2 - (\lambda_1-\lambda_2)^2(4(c_1\lambda_1+c_2\lambda_2)t \\
& - (c_1+c_2)x)\epsilon - (\lambda_1-\lambda_2)^2), \\
V(\epsilon) = {} & ((\lambda_1-\lambda_2)^2(-c_1(c_2\epsilon(4\lambda_2 t - x)+1)\epsilon\exp(-i\lambda_1(x-2\lambda_1 t))^2 \\
& - c_2(c_1\epsilon(4\lambda_1 t - x)+1)\epsilon\exp(-i\lambda_2(x-2\lambda_2 t))^2) \\
& + 2c_1c_2(\lambda_1-\lambda_2)\epsilon^2\sin((\lambda_1-\lambda_2)(2(\lambda_1+\lambda_2)t-x))\exp(-i(\lambda_1+\lambda_2)x \\
& + 2i(\lambda_1^2+\lambda_2^2)t))/(c_1c_2(\sin((\lambda_1-\lambda_2)(2(\lambda_1+\lambda_2)t-x))^2 \\
& - (\lambda_1-\lambda_2)^2(4\lambda_2 t-x)(4\lambda_1 t-x))\epsilon^2 - (\lambda_1-\lambda_2)^2(4(c_1\lambda_1+c_2\lambda_2)t \\
& - (c_1+c_2)x)\epsilon - (\lambda_1-\lambda_2)^2)\,,
\end{aligned}$$

for any value of the spectral parameters $\{\lambda_1, \lambda_2\}$ and infinitesimal parameter ϵ.

For each value of n, the process of constructing solution is similar. In principle, Theorem 2 provides us with a way to construct infinitely multiple new solutions through any given solution.

5. Summary

The nonlocal symmetries for AKNS system, which are generated by including pseudopotentials, are equivalent to Lie point symmetries of the prolonged system. Analogous to the once finite symmetry transformation, the nth finite symmetry transformation is further derived.

It would be interesting to investigate whether AKNS system possesses other finite symmetry transformations starting from different nonlocal symmetries. We will further study this problem and investigate the nth finite symmetry transformation for other integrable differential equations in the near future.

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