Chinese Journal of Physics 96 (2025) 577–602

Contents lists available at ScienceDirect

Chinese Journal of Physics

journal homepage: www.elsevier.com/locate/cjph

Research paper

Inverse scattering transform for the fourth-order nonlinear Schrödinger equation with fully asymmetric non-zero boundary conditions

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ARTICLE INFO

Keywords:
Inverse scattering transform
The fourth-order nonlinear Schrödinger equation
Fully asymmetric non-zero boundary conditions
Right and left Marchenko equations
Riemann–Hilbert problem

ABSTRACT

The inverse scattering transform is utilized to address the initial-value problem for the fourth-order nonlinear Schrödinger equation characterized by fully asymmetric non-zero boundary conditions. This work considers the fully asymmetric scenario for both asymptotic amplitudes and phases. The direct problem demonstrates the establishment of the corresponding analytic properties of eigenfunctions and scattering data. The inverse scattering problem is approached using both (left and right) Marchenko integral equations and is also formulated as the Riemann–Hilbert problem on a single sheet of the scattering variable. The temporal evolution of the scattering coefficients is subsequently deduced, revealing that in contrast to solutions with uniform amplitudes, both reflection and transmission coefficients exhibit a nontrivial time dependency here. The findings of this paper are expected to be pivotal for exploring the long-time asymptotic behavior of the fourth-order nonlinear Schrödinger solutions with significant boundary conditions.

1. Introduction

Since the inception of the inverse scattering transform (IST) for the solution of the Korteweg–de Vries (KdV) equation utilizing a Lax pair [1], this methodological framework has been progressively extended to encompass a broader spectrum of physically significant nonlinear wave equations that also admit Lax pairs [2,3]. Prominent among these are the nonlinear Schrödinger (NLS) equation [4], the modified Korteweg–de Vries (mKdV) equation [5], the Benjamin–Ono equation [6], the Kadomtsev–Petviashvili equation [7], and the Davey–Stewartson equation [8], among others. The NLS equation and its various extensions have emerged as particularly intriguing models within the nonlinear science domain, finding applications across diverse fields such as shallow water waves [9], Bose–Einstein condensates [10,11], deep ocean dynamics [12], and even in financial modeling.

The IST stands as a powerful and versatile analytical tool in the study of integrable systems, facilitating the derivation of soliton solutions that are pivotal in understanding the behavior of these systems [13]. Its application has been prolific in exploring a multitude of integrable nonlinear wave equations. Beyond the foundational NLS equation, the scope of IST has been broadened to

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https://doi.org/10.1016/j.cjph.2025.05.021
Received 20 February 2025; Received in revised form 8 May 2025; Accepted 16 May 2025
Available online 2 June 2025

include the Sasa–Satsuma equation [14,15], the nonlocal mKdV equation [16] and the derivative NLS equation [17,18], thereby enriching our comprehension of soliton dynamics across different physical contexts. This expansion underscores the adaptability and profundity of IST as a cornerstone technique in the theoretical exploration of nonlinear phenomena [19,20].

The Riemann–Hilbert (RH) method as a modern variant of the IST has streamlined the computational process of IST and has garnered extensive applications. In recent years, the study of soliton solutions for integrable systems under both zero and non-zero boundary conditions (NZBCs) via the RH method has attracted significant attention [21,22]. Notable achievements have been made using the RH method, including the Sasa–Satsuma equation [23], the matrix KdV equation [24], the modified NLS equation [25], the three-component coupled NLS equation [26] and the Gerdjikov–Ivanov equation [27], among others. The procedure of the RH method can be summarized as follows: Initially, initial values in the Schwartz space are considered, and the associated scattering data is obtained through spectral analysis. Subsequently, the scattering data related to time t is acquired through temporal evolution, and a relationship between the solutions of the original equation and the solutions of the RH problem is established. Ultimately, the general form of exact solutions corresponding to simple and second-order zeros under the reflectionless condition is obtained, and the propagation behaviors of different solutions with specific parameters are analyzed.

The fourth-order NLS equation, a fundamental model in the study of nonlinear optics, deep ocean and Bose–Einstein condensates has been a subject of intensive research due to its complex dynamics and practical significance [28–30]. The higher-order effects play an important role in the wave propagations of ultrashort (e.g. subpicosecond or femtosecond) light pulses in optical fibers. This study aims to investigate the characteristics of the fourth-order NLS equation [28–30]:

$$\mathrm{i}q_t + \sigma_{11}(q_{xx} + 2\,|q|^2\,q) + \mathrm{i}\sigma_{12}(q_{xxx} + 6|q|^2 q_x) + \sigma_{13}(q_{xxxx} + 4|q_x|^2 q + 6q_x^2 q^* + 8q_{xx}|q|^2 + 2q^2 q_{xx}^* + 6|q|^4 q) = 0, \tag{1}$$

where $q = q(x,t)$ denotes a scalar function and the real constants σ_{11}, σ_{12} and σ_{13} correspond to the second-order, third-order and fourth-order dispersions, respectively. When σ_{12} and $\sigma_{13} = 0$, Eq. (1) simplifies to the NLS equation [4], a classical equation that describes the plane self-focusing and one-dimensional self-modulation of waves in nonlinear dispersive media. As $\sigma_{11} = \sigma_{13} = 0$, Eq. (1) becomes the complex mKdV equation [31]. Given that $\sigma_{13} = 0$, Eq. (1) reduces to the Hirota equation [32,33], which was used to describe the propagation of ultra-short light pulses in optical fibers. The Hirota equation first introduced by Hirota in 1973 has since become a focal point in the realm of nonlinear science. Under the assumption that $\sigma_{12} = 0$, Eq. (1) becomes the Lakshmanan–Porsezian–Daniel (LPD) equation [34], which is equivalent to the lowest order continuum limit of the Heisenberg spin chain. The integrability of the fourth-order NLS equation stems from its composition as the sum of LPD equation and complex mKdV equation [31], which are partial differential equations belonging to the same hierarchy, along with the integrable commutative flows derived from the NLS equation [4]. Discussions on the soliton solutions of systems with NZBCs have been presented, as well as studies on rogue waves and rational solutions of the fourth-order NLS equation [30]. Whether dealing with nonlinear integrable systems with zero or non-zero backgrounds, exact solutions can be obtained through various approaches, such as the IST [35] and the Darboux transformation [36].

The objective of this research is to elaborate the IST for Eq. (1), particularly in the context of fully asymmetric NZBCs:

$$q(x,t) \to q_{L/R}(t) = B_{L/R}\mathrm{e}^{(2\mathrm{i}\sigma_{11}B_{L/R}^2 + 6\mathrm{i}\sigma_{13}B_{L/R}^4)t + \mathrm{i}\delta_{L/R}}, \quad x \to \mp\infty, \tag{2}$$

where $B_R \geq B_L > 0$ and $0 \leq \delta_{L/R} < 2\pi$ are arbitrary constants. The focusing and defocusing NLS equations [37,38] with fully asymmetric NZBCs have been studied and correspondingly extended to the focusing and defocusing mKdV equations [39] in few-cycle pulses, the focusing and defocusing Hirota equations [40,41] and the defocusing LPD equation [42]. When the amplitudes of the solutions diverge as $x \to \pm\infty$, it becomes infeasible to define a uniformizing variable in the spectral domain. Such a variable would otherwise facilitate the mapping of the multi-sheeted Riemann surface associated with the scattering parameter, onto a singular complex plane. From a practical standpoint in physics, fully asymmetric NZBCs hold particular importance for theoretically exploring rogue waves and perturbed soliton solutions within microstructured optical fiber systems. These systems feature distinct background amplitudes at the fiber's termini. Moreover, this research bears relevance to elucidating the function of soliton solutions in the nonlinear progression of modulation instability within such contexts.

The structure of the paper is summarized as follows. Section 2 focuses on the direct scattering problem. We begin by defining the Jost solutions and elucidating their continuity and analyticity properties. The symmetries linking eigenfunctions and scattering data are also explored. Moreover, we investigate the properties of the discrete spectrum and the trace formula. Finally, a systematic analysis of the behavior of eigenfunctions and scattering data for large values of g is conducted, where g represents the conventional complex scattering parameter utilized within the IST framework. In Section 3, we develop two triangular representations for the fundamental eigenfunctions. Furthermore, the inverse problem will be addressed using both (left and right) Marchenko integral equations and formulated as the RH problem on a unified sheet of the scattering variable $\beta_{L/R} = \sqrt{g^2 + B_{L/R}^2}$, which involves reconstructing the eigenfunctions and the potential based on the scattering data. Section 4 offers an alternative approach to the inverse problem, formulated as the RH problem for the eigenfunctions, where the discontinuities are described using the scattering data. Additionally, this section addresses the temporal evolution of the eigenfunctions and scattering coefficients. Section 5 contains concluding remarks. Additional technical proofs are compiled in Appendices A and B.

2. Direct scattering problem

2.1. Lax pair and the fundamental eigensolutions

The fourth-order NLS equation (1) is commonly linked to the following Lax pair:

$$\psi_x = \mathbf{X}\psi, \quad \psi_t = \mathbf{T}\psi, \tag{3}$$

where $\psi = \psi(x,t)$ and $\mathbf{X} = \mathbf{X}(g;x,t) = -\mathrm{i}g\sigma_3 + \mathbf{Q}$, with the matrix denoted by $\mathbf{T}$ is shown below:

$$\begin{aligned}\mathbf{T} = \mathbf{T}(g;x,t) &= \sigma_{11}[2g\mathbf{Q} - 2\mathrm{i}g^2\sigma_3 + \mathrm{i}\sigma_3(\mathbf{Q}_x - \mathbf{Q}^2)] + \sigma_{12}[4g^2\mathbf{Q} - 4\mathrm{i}g^3\sigma_3 + 2\mathrm{i}g\sigma_3(\mathbf{Q}_x - \mathbf{Q}^2) + 2\mathbf{Q}^3 \\ &- \mathbf{Q}_{xx} + \mathbf{Q}_x\mathbf{Q} - \mathbf{Q}\mathbf{Q}_x] + \sigma_{13}[8\mathrm{i}g^4\sigma_3 - 8g^3\mathbf{Q} - 4\mathrm{i}g^2\sigma_3(\mathbf{Q}_x - \mathbf{Q}^2) - 6\mathrm{i}\sigma_3\mathbf{Q}^2\mathbf{Q}_x \\ &- g(4\mathbf{Q}^3 - 2\mathbf{Q}_{xx} + 2\mathbf{Q}_x\mathbf{Q} - 2\mathbf{Q}\mathbf{Q}_x) + \mathrm{i}\sigma_3(3\mathbf{Q}^4 + \mathbf{Q}_{xxx} + \mathbf{Q}_x^2 - \mathbf{Q}_{xx}\mathbf{Q} - \mathbf{Q}\mathbf{Q}_{xx})],\end{aligned} \tag{4}$$

where the expressions for the matrices σ_2, σ_3 and $\mathbf{Q} = \mathbf{Q}(x,t)$ are delineated below:

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{5}$$

We will now examine potentials with nontrivial boundary conditions, assuming that $B_R \geq B_L > 0$. It should be noted that although the asymptotic amplitudes $B_{L/R}$ can be considered time-independent, the asymptotic phases evolve according to the following expressions: $\delta_{L/R}(t) = (2\sigma_{11}B_{L/R}^2 + 6\sigma_{13}B_{L/R}^4)t + \delta_{L/R}$. Consequently, unlike the case of equal-amplitude boundary conditions where $B_R = B_L$, it is unfeasible to remove the background and make both boundary conditions time-independent.

This section explicitly disregards temporal dependence. Integrability condition is assumed whether $t = 0$ or for any $t > 0$:

$$(\boldsymbol{B}_j): \quad \int_0^{\infty} (1+|x|)^j [|q(-x) - q_L| + |q(x) - q_R|]\,\mathrm{d}x < +\infty, \tag{6}$$

the assumption of $j = 0, 1, 2$ applies to all $t \geq 0$. Note that the condition $(\boldsymbol{B}_j)$ is equivalent to assuming $(q(x) - q_{L/R}) \in L^{1,j}(\mathbb{R}^{\mp})$. We use $\mathbf{Q}_R(t)$ to represent the limit of $\mathbf{Q}(x,t)$ as $x \to +\infty$, and $\mathbf{Q}_L(t)$ to represent the limit of $\mathbf{Q}(x,t)$ as $x \to -\infty$. Let the matrix $\mathbf{Q}_a(x,t)$ be defined as follows:

$$\mathbf{Q}_a(x,t) = \mathbf{Q}_R(t)\mu(x) + \mathbf{Q}_L(t)\mu(-x), \tag{7}$$

where $\mu(x)$ is the Heaviside function. It is beneficial to introduce the operators in the limit $x \to \mp\infty$:

$$\mathbf{G}_{L/R}(g) = -\mathrm{i}g\sigma_3 + \mathbf{Q}_{L/R}, \quad \mathbf{G}(x,g) = -\mathrm{i}g\sigma_3 + \mathbf{Q}_a(x) = \mathbf{G}_R(g)\mu(x) + \mathbf{G}_L(g)\mu(-x), \tag{8}$$

and introduce the fundamental eigensolutions $\widetilde{\mathbf{U}}(x,g)$ and $\widetilde{\mathbf{V}}(x,g)$ that satisfy the asymptotic conditions:

$$\widetilde{\mathbf{U}}(x,g) = \mathrm{e}^{x\mathbf{G}_R(g)}[\mathbf{I}_2 + o(1)], \quad x \to +\infty, \tag{9a}$$

$$\widetilde{\mathbf{V}}(x,g) = \mathrm{e}^{x\mathbf{G}_L(g)}[\mathbf{I}_2 + o(1)], \quad x \to -\infty. \tag{9b}$$

The exponential function $\mathrm{e}^{x\mathbf{G}_{L/R}(g)}$ remains bounded for all $x \in \mathbb{R}$ if and only if $\mathbf{G}_{L/R}(g)$ is diagonalizable and its eigenvalues are either zero or purely imaginary. This condition holds if and only if $g \in \mathbb{R} \cup (-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R})$. For $g = \pm\mathrm{i}B_{L/R}$, the norm of the group $\mathrm{e}^{x\mathbf{G}_{L/R}(g)}$ increases linearly with x as $x \to \mp\infty$. Appendix A provides detailed norm estimates, as well as proofs for a proposition and two theorems discussed in this section.

Proposition 1. *Assume the potential satisfies* $(\boldsymbol{B}_0)$. *Then the fundamental eigensolution* $\widetilde{\mathbf{U}}(x,g)$ *which exhibits asymptotic behavior* (9a) *can be uniquely determined as*

$$\widetilde{\mathbf{U}}(x,g) = \mathrm{e}^{x\mathbf{G}_R(g)} - \int_x^{\infty} \mathrm{e}^{(x-y)\mathbf{G}_R(g)}[\mathbf{Q}(y) - \mathbf{Q}_R]\widetilde{\mathbf{U}}(y,g)\,\mathrm{d}y, \quad g \in \mathbb{R} \cup (-\mathrm{i}B_R, \mathrm{i}B_R). \tag{10}$$

Furthermore, the function $\widetilde{\mathbf{U}}(x,g)$ *exhibits continuity for all* $x_0 \leq x$, *where* x_0 *is any finite value, and it remains continuous as a function of* g *across the entire domain* $g \in \mathbb{R} \cup (-\mathrm{i}B_R, \mathrm{i}B_R)$. *Likewise, the fundamental eigensolution* $\widetilde{\mathbf{V}}(x,g)$ *which exhibits asymptotic behavior* (9b) *can be uniquely determined as*

$$\widetilde{\mathbf{V}}(x,g) = \mathrm{e}^{x\mathbf{G}_L(g)} + \int_{-\infty}^{x} \mathrm{e}^{(x-y)\mathbf{G}_L(g)}[\mathbf{Q}(y) - \mathbf{Q}_L]\widetilde{\mathbf{V}}(y,g)\,\mathrm{d}y, \quad g \in \mathbb{R} \cup (-\mathrm{i}B_L, \mathrm{i}B_L). \tag{11}$$

Furthermore, the function $\widetilde{\mathbf{V}}(x,g)$ *exhibits continuity for all* $x \leq x_0$ *and it remains continuous across the entire domain* $g \in \mathbb{R} \cup (-\mathrm{i}B_L, \mathrm{i}B_L)$. *Moreover, under the assumption* $(\boldsymbol{B}_1)$, $\widetilde{\mathbf{U}}(x,g)$ (10) *[resp.* $\widetilde{\mathbf{V}}(x,g)$ (11)*] has a unique and continuous solution for* $g \in [-\mathrm{i}B_R, \mathrm{i}B_R]$ *[resp.* $g \in [-\mathrm{i}B_L, \mathrm{i}B_L]$*]. The findings can be extended to the respective branch points.*

Assuming $(\boldsymbol{B}_1)$, it is possible to substitute the integral equations (10) and (11) with alternative formulations. Let the matrix $\mathbf{W}(g;x,y)$ in the following manner:

$$\mathbf{W}(g;x,y) = \mu(x)\mu(y)\mathrm{e}^{(x-y)\mathbf{G}_R(g)} + \mu(-x)\mu(-y)\mathrm{e}^{(x-y)\mathbf{G}_L(g)} + \mu(x)\mu(-y)\mathrm{e}^{x\mathbf{G}_R(g)}\mathrm{e}^{-y\mathbf{G}_L(g)} + \mu(-x)\mu(y)\mathrm{e}^{x\mathbf{G}_L(g)}\mathrm{e}^{-y\mathbf{G}_R(g)}. \tag{12}$$

The matrix function $\mathbf{W}(g;x,y)$ is continuous with respect to $(g;x,y) \in \mathbb{R}^2 \times \mathbb{C}$ and fulfills the associated initial value problems:

$$\frac{\partial \mathbf{W}(g;x,y)}{\partial x} = \mathbf{G}(x,g)\mathbf{W}(g;x,y), \quad \mathbf{W}(g;y,y) = \mathbf{I}_2, \tag{13a}$$

$$\frac{\partial \mathbf{W}(g;x,y)}{\partial y} = -\mathbf{W}(g;x,y)\mathbf{G}(y,g), \quad \mathbf{W}(g;x,x) = \mathbf{I}_2. \tag{13b}$$

By employing Eqs. (13a) and (13b), it can be readily verified that the fundamental eigenfunctions also adhere to the integral equations:

$$\widetilde{\mathbf{U}}(x,g)=\mathbf{W}(g;x,0)-\int_x^{\infty}\mathbf{W}(g;x,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{U}}(y,g)\,\mathrm{d}y, \tag{14a}$$

$$\widetilde{\mathbf{V}}(x,g)=\mathbf{W}(g;x,0)+\int_{-\infty}^{x}\mathbf{W}(g;x,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{V}}(y,g)\,\mathrm{d}y, \tag{14b}$$

where $\mathbf{W}(g;x,0)=\mu(x)\mathrm{e}^{x\mathbf{G}_R(g)}+\mu(-x)\mathrm{e}^{x\mathbf{G}_L(g)}$ as specified in Eq. (12). It is important to observe that Eq. (14a) aligns with Eq. (10) for $x\geq 0$, and Eq. (14b) aligns with Eq. (11) for $x\leq 0$. Furthermore, by applying Eq. (12), we obtain

$$\widetilde{\mathbf{U}}(x,g)=\mathrm{e}^{x\mathbf{G}_L(g)}\left[\mathbf{I}_2-\int_x^{\infty}\mathbf{W}(g;0,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{U}}(y,g)\,\mathrm{d}y\right],\quad x\leq 0, \tag{15a}$$

$$\widetilde{\mathbf{V}}(x,g)=\mathrm{e}^{x\mathbf{G}_R(g)}\left[\mathbf{I}_2+\int_{-\infty}^{x}\mathbf{W}(g;0,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{V}}(y,g)\,\mathrm{d}y\right],\quad x\geq 0, \tag{15b}$$

where the integrals on the right-hand sides converge absolutely as $x\to\mp\infty$. It should be noted that the fundamental matrix $\mathbf{W}(g;x,y)$ depends on both groups $\mathrm{e}^{x\mathbf{G}_{L/R}(g)}$. Consequently, the integral equations (14a) and (14b) are applicable only for defining the fundamental eigenfunctions $\widetilde{\mathbf{U}}(x,g)$ and $\widetilde{\mathbf{V}}(x,g)$ when $g\in\mathbb{R}\cup[-\mathrm{i}B_L,\mathrm{i}B_L]$. The condition $\frac{\partial}{\partial x}\det\psi=\mathrm{tr}\mathbf{X}\det\psi=0$ is used to determine that $\det\widetilde{\mathbf{U}}(x,g)=\det\widetilde{\mathbf{V}}(x,g)=1$ and considering the asymptotic behavior (9a) and (9b).

2.2. Jost solutions

Given that the asymptotic scattering operators $\mathbf{G}_{L/R}(g)$ are traceless and obey the relation $\mathbf{G}^2_{L/R}(g)=-(g^2+B^2_{L/R})\mathbf{I}_2$, it is reasonable to introduce the conformal mappings $\beta_{L/R}=\sqrt{g^2+B^2_{L/R}}$ with branch cuts defined along the imaginary intervals $\Xi_{L/R}=[-\mathrm{i}B_{L/R},\mathrm{i}B_{L/R}]$. By employing suitable local polar coordinates, we proceed to define

$$\beta_R=\sqrt{\lambda_1\lambda_2}\mathrm{e}^{\mathrm{i}(\delta_1+\delta_2)/2},\quad \beta_L=\sqrt{\lambda_3\lambda_4}\mathrm{e}^{\mathrm{i}(\delta_3+\delta_4)/2}, \tag{16}$$

where $\lambda_j\geq 0$ and $-\pi/2\leq\delta_j<3\pi/2$ for $j=1,2,3,4$ as indicated in Fig. 1. We shall examine a single sheet of the complex plane for g, and let $\mathbf{\Lambda}_{L/R}$ represent the plane with cuts along the segments $\Xi_{L/R}$ on the imaginary axis. Let $\mathbb{C}^{\pm}$ denote the open upper/lower complex half-planes, respectively, and let $\mathbf{\Lambda}^{\pm}_{L/R}$ represent the corresponding open upper/lower complex half-planes with cuts along $\Xi_{L/R}$. β_R establishes a bijective mapping across the ensuing domains:

- $g\in\mathbf{\Lambda}^+_R=\mathbb{C}^+\backslash(0,\mathrm{i}B_R]$ and $\beta_R\in\mathbb{C}^+$.
- $g\in\partial\mathbf{\Lambda}^+_R=\mathbb{R}\cup\{\mathrm{i}h-0^+:0<h<B_R\}\cup\{\mathrm{i}B_R\}\cup\{\mathrm{i}h+0^+:0<h<B_R\}$ and $\beta_R\in\mathbb{R}$.
- $g\in\mathbf{\Lambda}^-_R=\mathbb{C}^-\backslash[-\mathrm{i}B_R,0)$ and $\beta_R\in\mathbb{C}^-$.
- $g\in\partial\mathbf{\Lambda}^-_R=\mathbb{R}\cup\{\mathrm{i}h-0^+:-B_R<h<0\}\cup\{-\mathrm{i}B_R\}\cup\{\mathrm{i}h+0^+:-B_R<h<0\}$ and $\beta_R\in\mathbb{R}$.

Similarly, β_L establishes a bijective mapping among the subsequent domains:

- $g\in\mathbf{\Lambda}^+_L=\mathbb{C}^+\backslash(0,\mathrm{i}B_L]$ and $\beta_L\in\mathbb{C}^+$.
- $g\in\partial\mathbf{\Lambda}^+_L=\mathbb{R}\cup\{\mathrm{i}h-0^+:0<h<B_L\}\cup\{\mathrm{i}B_L\}\cup\{\mathrm{i}h+0^+:0<h<B_L\}$ and $\beta_L\in\mathbb{R}$.
- $g\in\mathbf{\Lambda}^-_L=\mathbb{C}^-\backslash[-\mathrm{i}B_L,0)$ and $\beta_L\in\mathbb{C}^-$.
- $g\in\partial\mathbf{\Lambda}^-_L=\mathbb{R}\cup\{\mathrm{i}h-0^+:-B_L<h<0\}\cup\{-\mathrm{i}B_L\}\cup\{\mathrm{i}h+0^+:-B_L<h<0\}$ and $\beta_L\in\mathbb{R}$.

It is important to note that with this selection of branch cuts, $\beta_R\sim\beta_L\sim g$ as $g\to\infty$ throughout the entire plane (see Eq. (156)). Subsequently, $\beta^{\pm}_L(g)$ [resp. $\beta^{\pm}_R(g)$] will represent the boundary values of $\beta_L(g)$ [resp. $\beta_R(g)$] for $g\in\Xi_L$ [resp. $g\in\Xi_R$], taken from the right/left edge of the cut, with

$$\beta^{\pm}_L(g)=\pm\sqrt{B^2_L-|g|^2},\quad g=\mathrm{i}h\pm 0^+,\quad |h|\leq B_L, \tag{17a}$$

$$\beta^{\pm}_R(g)=\pm\sqrt{B^2_R-|g|^2},\quad g=\mathrm{i}h\pm 0^+,\quad |h|\leq B_R, \tag{17b}$$

on the right/left edge. It is evident that $\pm\mathrm{i}\beta_{L/R}$ are the eigenvalues of $\mathbf{G}_{L/R}(g)$, and the corresponding eigenvector matrices $\mathbf{E}_{L/R}(g)$ satisfy

$$\mathbf{G}_{L/R}(g)\mathbf{E}_{L/R}(g)=-\mathrm{i}\beta_{L/R}\mathbf{E}_{L/R}(g)\sigma_3, \tag{18}$$

where

$$\mathbf{E}_{L/R}(g)=\mathbf{I}_2-\frac{\mathrm{i}\sigma_3\mathbf{Q}_{L/R}}{\beta_{L/R}+g}. \tag{19}$$

We may subsequently define the Jost solutions utilizing the fundamental eigensolutions in the following manner:

$$\mathbf{U}(x,g)=(\widetilde{\mathbf{u}}(x,g),\mathbf{u}(x,g)):=\widetilde{\mathbf{U}}(x,g)\mathbf{E}_R(g), \tag{20a}$$

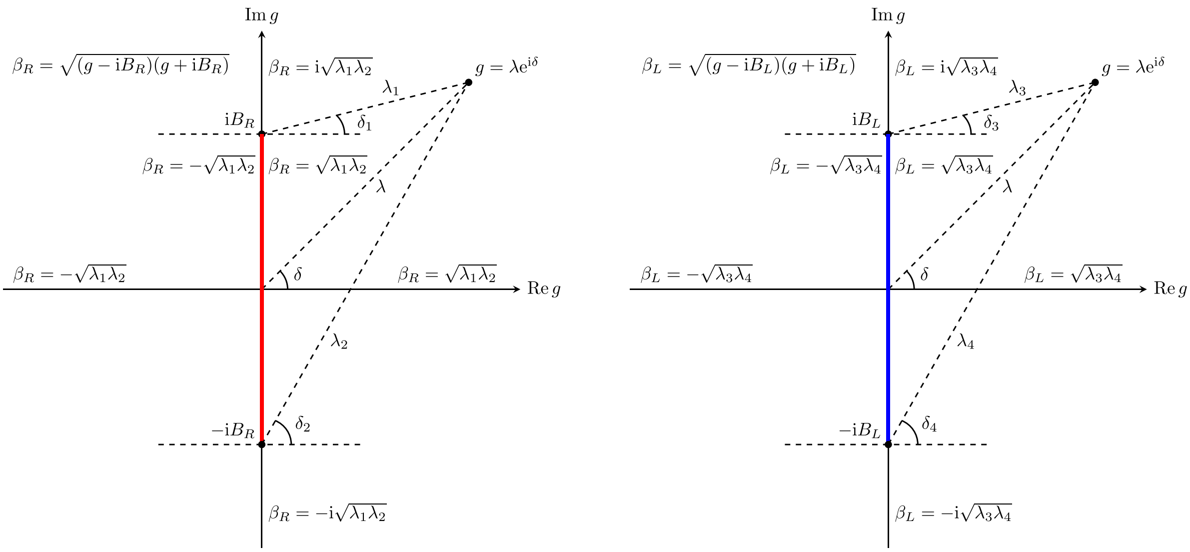

Fig. 1. The branch cuts for $\beta_R = \sqrt{g^2 + B_R^2}$ and $\beta_L = \sqrt{g^2 + B_L^2}$: we define $\beta_R = \sqrt{\lambda_1\lambda_2}e^{i(\delta_1+\delta_2)/2}$ with $\lambda_1 = |g - iB_R|$, $\lambda_2 = |g + iB_R|$ and angles $-\pi/2 < \delta_1, \delta_2 \le 3\pi/2$. Similarly, $\beta_L = \sqrt{\lambda_3\lambda_4}e^{i(\delta_3+\delta_4)/2}$ with $\lambda_3 = |g - iB_L|$, $\lambda_4 = |g + iB_L|$ and angles $-\pi/2 < \delta_3, \delta_4 \le 3\pi/2$.

$$\mathbf{V}(x,g) = (\mathbf{v}(x,g), \widetilde{\mathbf{v}}(x,g)) := \widetilde{\mathbf{V}}(x,g)\mathbf{E}_L(g). \tag{20b}$$

Alternatively, they can be characterized as solutions to the scattering problem exhibiting the following asymptotic behavior:

$$\mathbf{V}(x,g) \sim \mathbf{E}_L(g)e^{-i\beta_L x\sigma_3} \quad x \to -\infty, \tag{21a}$$

$$\mathbf{U}(x,g) \sim \mathbf{E}_R(g)e^{-i\beta_R x\sigma_3} \quad x \to +\infty. \tag{21b}$$

The Jost solutions $\widetilde{\mathbf{u}}(x,g)$ and $\mathbf{u}(x,g)$ [resp. $\mathbf{v}(x,g)$ and $\widetilde{\mathbf{v}}(x,g)$] are defined for $\beta_R \in \mathbb{R}$ [resp. $\beta_L \in \mathbb{R}$], where g lies on the boundary of $\partial\mathbf{\Lambda}_R^+ \cup \partial\mathbf{\Lambda}_R^-$ [resp. $\partial\mathbf{\Lambda}_L^+ \cup \partial\mathbf{\Lambda}_L^-$]. Specifically, when $g = ih \in [-iB_R, iB_R]$ [resp. $g = ih \in [-iB_L, iB_L]$], the solutions on the right/left edge of the cut in both half-planes are distinguished by the superscripts $\pm$, respectively.

$$\mathbf{U}^\pm(x,ih) = (\widetilde{\mathbf{u}}^\pm(x,ih), \mathbf{u}^\pm(x,ih)) := \widetilde{\mathbf{U}}(x,ih)\mathbf{E}_R(ih \pm 0^+), \quad |h| \le B_R, \tag{22a}$$

$$\mathbf{V}^\pm(x,ih) = (\mathbf{v}^\pm(x,ih), \widetilde{\mathbf{v}}^\pm(x,ih)) := \widetilde{\mathbf{V}}(x,ih)\mathbf{E}_L(ih \pm 0^+), \quad |h| \le B_L. \tag{22b}$$

Given that $\widetilde{\mathbf{V}}(x,g)$ [resp. $\widetilde{\mathbf{U}}(x,g)$] remains single-valued across the cut, and $\mathbf{E}_L(g)$ [resp. $\mathbf{E}_R(g)$] is characterized by right/left limits as specified in Eq. (17a) [resp. Eq. (17b)]. The subsequent theorems delineate the analytic characteristics of the Jost solutions with respect to g. The conventional over-bar notation is employed to signify the closure of a set.

Theorem 1. *Assuming* $(\boldsymbol{B}_1)$ *is satisfied for* $x \in \mathbb{R}$, $\mathbf{u}(x,g)$ *[resp.* $\widetilde{\mathbf{u}}(x,g)$*] extends to a function that is continuous on* $g \in \overline{\mathbf{\Lambda}_R^+} \cup \partial\mathbf{\Lambda}_R^-$ *[resp.* $g \in \overline{\mathbf{\Lambda}_R^-} \cup \partial\mathbf{\Lambda}_R^+$*] and analytic within* $g \in \mathbf{\Lambda}_R^+$ *[resp.* $g \in \mathbf{\Lambda}_R^-$*]. Likewise,* $\mathbf{v}(x,g)$ *[resp.* $\widetilde{\mathbf{v}}(x,g)$*] extends to a function that is continuous on* $g \in \overline{\mathbf{\Lambda}_L^+} \cup \partial\mathbf{\Lambda}_L^-$ *[resp.* $g \in \overline{\mathbf{\Lambda}_L^-} \cup \partial\mathbf{\Lambda}_L^+$*] and analytic within* $g \in \mathbf{\Lambda}_L^+$ *[resp.* $g \in \mathbf{\Lambda}_L^-$*].*

Theorem 2. *Assuming* $(\boldsymbol{B}_2)$ *is satisfied for* $x \in \mathbb{R}$*, the derivative* $\partial_g\mathbf{u}(x,g)$ *[resp.* $\partial_g\widetilde{\mathbf{u}}(x,g)$*] extends to a function that is continuous on* $g \in \mathbf{\Lambda}_R^+ \cup \partial\mathbf{\Lambda}_R^- \backslash \{-iB_R\}$ *[resp.* $g \in \mathbf{\Lambda}_R^- \cup \partial\mathbf{\Lambda}_R^+ \backslash \{iB_R\}$*] and analytic within* $g \in \mathbf{\Lambda}_R^-$ *[resp.* $g \in \mathbf{\Lambda}_R^+$*]; the derivative* $\partial_g\mathbf{v}(x,g)$ *[resp.* $\partial_g\widetilde{\mathbf{v}}(x,g)$*] extends to a function that is continuous on* $g \in \mathbf{\Lambda}_L^+ \cup \partial\mathbf{\Lambda}_L^- \backslash \{-iB_L\}$ *[resp.* $g \in \mathbf{\Lambda}_L^- \cup \partial\mathbf{\Lambda}_L^+ \backslash \{iB_L\}$*] and analytic within* $g \in \mathbf{\Lambda}_L^-$ *[resp.* $g \in \mathbf{\Lambda}_L^+$*].*

The entities $\mathbf{\Lambda}_{L/R}^\pm$ are to be understood as analytic manifolds. The continuity of the Jost solutions across the cuts is defined in terms of the existence of right/left continuous limits, which are only considered within the domains where the branch cut constitutes a boundary of the analytic manifold. In regions of the half-planes where analytic continuation is not possible near the branch cut, the functions $\mathbf{u}^\pm(x,g)$ and $\widetilde{\mathbf{u}}^\pm(x,g)$ [resp. $\mathbf{v}^\pm(x,g)$ and $\widetilde{\mathbf{v}}^\pm(x,g)$] are defined according to the two possible values of $\beta_R^\pm$ [resp. $\beta_L^\pm$]. These functions can be uniquely determined by solving the associated Volterra integral equations.

2.3. Scattering matrix and its coefficients

By examining the integral equations for the fundamental matrices given in Eqs. (15a) and (15b), it is straightforward to deduce that

$$\widetilde{\mathbf{U}}(x,g) = e^{x\mathbf{G}_L(g)}[\mathbf{A}_R(g) + o(1)], \quad x \to -\infty, \tag{23a}$$

$$\widetilde{\mathbf{V}}(x,g)=\mathrm{e}^{x\mathbf{G}_R(g)}[\mathbf{A}_L(g)+o(1)],\quad x\to+\infty, \tag{23b}$$

where

$$\mathbf{A}_R(g)=\mathbf{I}_2-\int_{-\infty}^{\infty}\mathbf{W}(g;0,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{U}}(y,g)\,\mathrm{d}y, \tag{24a}$$

$$\mathbf{A}_L(g)=\mathbf{I}_2+\int_{-\infty}^{\infty}\mathbf{W}(g;0,y)[\mathbf{Q}(y)-\mathbf{Q}_a(y)]\widetilde{\mathbf{V}}(y,g)\,\mathrm{d}y, \tag{24b}$$

with the matrices $\mathbf{A}_R(g)$ and $\mathbf{A}_L(g)$ are mutual inverses. The assumptions of Proposition 1, in conjunction Eqs. (23a), (23b) with (144), lead to the conclusion that $\det\mathbf{A}_R(g)=\det\mathbf{A}_L(g)=1$ for $g\in\mathbb{R}\cup[-\mathrm{i}B_L,\mathrm{i}B_L]$. The scattering matrices $\mathbf{C}(g)$ and $\mathbf{H}(g)$ can subsequently be represented as follows:

$$(\mathbf{v}(x,g),\widetilde{\mathbf{v}}(x,g))=(\widetilde{\mathbf{u}}(x,g),\mathbf{u}(x,g))\mathbf{C}(g), \tag{25a}$$

$$(\widetilde{\mathbf{u}}(x,g),\mathbf{u}(x,g))=(\mathbf{v}(x,g),\widetilde{\mathbf{v}}(x,g))\mathbf{H}(g), \tag{25b}$$

with the scattering matrices $\mathbf{C}(g)=(c_{ij}(g))$ and $\mathbf{H}(g)=(h_{ij}(g))$ are clearly mutual inverses, and they are defined as

$$\mathbf{C}(g)=\mathbf{E}_R^{-1}(g)\mathbf{A}_L(g)\mathbf{E}_L(g),\quad \mathbf{H}(g)=\mathbf{E}_L^{-1}(g)\mathbf{A}_R(g)\mathbf{E}_R(g). \tag{26}$$

It is apparent that $\mathbf{C}(g)$ [resp. $\mathbf{H}(g)$] is generally defined wherever all four Jost solutions exist, specifically for $g\in\partial\mathbf{\Lambda}_L^-\cup\partial\mathbf{\Lambda}_L^+$ [resp. $g\in\partial\mathbf{\Lambda}_L^-\cup\partial\mathbf{\Lambda}_L^+\backslash\{\pm\mathrm{i}B_L\}$, the branch points are omitted due to the second condition]. When $g\in[-\mathrm{i}B_L,\mathrm{i}B_L]$, the scattering matrices and its coefficients are determined by the values on the right/left boundary of the cut, and are denoted with superscripts $\pm$ as explained below. Given that $\det\mathbf{V}(x,g)$ and $\det\mathbf{U}(x,g)$ are independent with respect to x, it is straightforward to confirm that

$$\det\mathbf{C}(g)=\frac{\det\mathbf{E}_L(g)}{\det\mathbf{E}_R(g)}=\frac{\beta_L(\beta_R+g)}{\beta_R(\beta_L+g)},\quad \det\mathbf{H}(g)=\frac{\det\mathbf{E}_R(g)}{\det\mathbf{E}_L(g)}=\frac{\beta_R(\beta_L+g)}{\beta_L(\beta_R+g)},\quad g\in\mathbb{R}, \tag{27a}$$

$$\det\mathbf{C}^{\pm}(g)=\frac{\beta_L^+(g)[\beta_R^{\pm}(g)+g]}{\beta_R^+(g)[\beta_L^{\pm}(g)+g]},\quad \det\mathbf{H}^{\pm}(g)=\frac{\beta_R^+(g)[\beta_L^{\pm}(g)+g]}{\beta_L^+(g)[\beta_R^{\pm}(g)+g]},\quad g\in(-\mathrm{i}B_L,\mathrm{i}B_L). \tag{27b}$$

Eqs. (25a) and (25b) provide the determinant expressions for the subsequent scattering coefficients:

$$c_{11}(g)=\frac{\det(\mathbf{v},\mathbf{u})}{\det(\widetilde{\mathbf{u}},\mathbf{u})}=\frac{\beta_R+g}{2\beta_R}\det(\mathbf{v},\mathbf{u}),\quad h_{22}(g)=\frac{\beta_L+g}{2\beta_L}\det(\mathbf{v},\mathbf{u})=\frac{\beta_R(\beta_L+g)}{\beta_L(\beta_R+g)}c_{11}(g), \tag{28a}$$

$$c_{22}(g)=\frac{\det(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})}{\det(\widetilde{\mathbf{u}},\mathbf{u})}=\frac{\beta_R+g}{2\beta_R}\det(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}),\quad h_{11}(g)=\frac{\beta_L+g}{2\beta_L}\det(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})=\frac{\beta_R(\beta_L+g)}{\beta_L(\beta_R+g)}c_{22}(g), \tag{28b}$$

$$c_{21}(g)=\frac{\det(\widetilde{\mathbf{u}},\mathbf{v})}{\det(\widetilde{\mathbf{u}},\mathbf{u})}=\frac{\beta_R+g}{2\beta_R}\det(\widetilde{\mathbf{u}},\mathbf{v}),\quad h_{12}(g)=\frac{\beta_L+g}{2\beta_L}\det(\mathbf{u},\widetilde{\mathbf{v}})=-\frac{\beta_R(\beta_L+g)}{\beta_L(\beta_R+g)}c_{12}(g), \tag{28c}$$

$$c_{12}(g)=\frac{\det(\widetilde{\mathbf{v}},\mathbf{u})}{\det(\widetilde{\mathbf{u}},\mathbf{u})}=\frac{\beta_R+g}{2\beta_R}\det(\widetilde{\mathbf{v}},\mathbf{u}),\quad h_{21}(g)=\frac{\beta_L+g}{2\beta_L}\det(\mathbf{v},\widetilde{\mathbf{u}})=-\frac{\beta_R(\beta_L+g)}{\beta_L(\beta_R+g)}c_{21}(g), \tag{28d}$$

where the arguments (x,g) of the Jost solutions are omitted for the sake of brevity. Observe that the aforementioned determinant representations facilitate the definition of the scattering coefficients' values from the right/left edge of the cuts $\Xi_{L/R}$. Specifically, we have

$$c_{11}^{\pm}(g)=\frac{\beta_R^{\pm}(g)+g}{2\beta_R^{\pm}(g)}\det(\mathbf{v}^{\pm}(x,g),\mathbf{u}^{\pm}(x,g)),\quad g\in(-\mathrm{i}B_L,\mathrm{i}B_R), \tag{29a}$$

$$c_{22}^{\pm}(g)=\frac{\beta_R^{\pm}(g)+g}{2\beta_R^{\pm}(g)}\det(\widetilde{\mathbf{u}}^{\pm}(x,g),\widetilde{\mathbf{v}}^{\pm}(x,g)),\quad g\in(-\mathrm{i}B_R,\mathrm{i}B_L), \tag{29b}$$

$$c_{21}^{\pm}(g)=\frac{\beta_R^{\pm}(g)+g}{2\beta_R^{\pm}(g)}\det(\widetilde{\mathbf{u}}^{\pm}(x,g),\mathbf{v}^{\pm}(x,g)),\quad g\in(-\mathrm{i}B_L,\mathrm{i}B_R), \tag{29c}$$

$$c_{12}^{\pm}(g)=\frac{\beta_R^{\pm}(g)+g}{2\beta_R^{\pm}(g)}\det(\widetilde{\mathbf{v}}^{\pm}(x,g),\mathbf{u}^{\pm}(x,g)),\quad g\in(-\mathrm{i}B_R,\mathrm{i}B_L), \tag{29d}$$

and analogously for the scattering coefficients derived from the left.

Then Eqs. (28a), (28b), (28c) and (28d) enable the extension of certain scattering coefficients under the assumption ($\boldsymbol{B}_1$). Indeed, this is corroborated by Theorem 1, which indicates that:

- $c_{11}(g)$ [resp. $c_{22}(g)$] is continuous for $g\in\overline{\mathbf{\Lambda}_R^+}\cup\partial\mathbf{\Lambda}_L^-\backslash\{\mathrm{i}B_R\}$ [resp. $g\in\overline{\mathbf{\Lambda}_R^-}\cup\partial\mathbf{\Lambda}_L^+\backslash\{-\mathrm{i}B_R\}$] (with values across the cut denoted as $c_{11}^{\pm}(g)$ [resp. $c_{22}^{\pm}(g)$]), and analytic in $g\in\mathbf{\Lambda}_R^+$ [resp. $g\in\mathbf{\Lambda}_R^-$], while

$$c_{11}(g)\sim\frac{\mathrm{i}B_R}{2\beta_R}\det(\mathbf{v}(x,\mathrm{i}B_R),\mathbf{u}(x,\mathrm{i}B_R)),\qquad g\to\mathrm{i}B_R, \tag{30a}$$

$$c_{22}(g)\sim-\frac{\mathrm{i}B_R}{2\beta_R}\det(\widetilde{\mathbf{u}}(x,-\mathrm{i}B_R),\widetilde{\mathbf{v}}(x,-\mathrm{i}B_R)),\qquad g\to-\mathrm{i}B_R. \tag{30b}$$

- $c_{21}(g)$ [resp. $c_{12}(g)$] is continuous for $\partial\mathbf{\Lambda}_R^+\cup\partial\mathbf{\Lambda}_L^-\backslash\{\mathrm{i}B_R\}$ [resp. $\partial\mathbf{\Lambda}_R^-\cup\partial\mathbf{\Lambda}_L^+\backslash\{-\mathrm{i}B_R\}$] (with values across the cut denoted as $c_{21}^{\pm}(g)$ [resp. $c_{12}^{\pm}(g)$]), while

$$c_{21}^{\pm}(g)\sim\frac{\mathrm{i}B_R}{2\beta_R}\det(\widetilde{\mathbf{u}}(x,\mathrm{i}B_R),\mathbf{v}(x,\mathrm{i}B_R)),\qquad g\to\mathrm{i}B_R, \tag{31a}$$

$$c_{12}^{\pm}(g)\sim-\frac{\mathrm{i}B_R}{2\beta_R}\det(\widetilde{\mathbf{v}}(x,-\mathrm{i}B_R),\mathbf{u}(x,-\mathrm{i}B_R)),\qquad g\to-\mathrm{i}B_R. \tag{31b}$$

Corresponding results can be derived for the remaining four scattering coefficients and their properties can also be deduced from the previous ones using Eqs. (28c) and (28d). Eqs. (30a), (30b), (31a) and (31b) demonstrate that the scattering coefficients typically exhibit singularities at the branch points $g=\pm\mathrm{i}B_R$, specifically when $\beta_R=0$.

For ease of future reference, the reflection coefficients from the right and left are defined as follows:

$$\gamma(g)=\frac{c_{21}(g)}{c_{11}(g)},\quad g\in\mathbb{R},\quad \gamma^{\pm}(g)=\frac{c_{21}^{\pm}(g)}{c_{11}^{\pm}(g)},\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_R), \tag{32a}$$

$$\widetilde{\gamma}(g)=\frac{c_{12}(g)}{c_{22}(g)},\quad g\in\mathbb{R},\quad \widetilde{\gamma}^{\pm}(g)=\frac{c_{12}^{\pm}(g)}{c_{22}^{\pm}(g)},\quad g\in(-\mathrm{i}B_R,\mathrm{i}B_L], \tag{32b}$$

$$\alpha(g)=\frac{h_{12}(g)}{h_{22}(g)}=-\frac{c_{12}(g)}{c_{11}(g)},\quad g\in\mathbb{R},\quad \alpha^{\pm}(g)=\frac{h_{12}^{\pm}(g)}{h_{22}^{\pm}(g)}=-\frac{c_{12}^{\pm}(g)}{c_{11}^{\pm}(g)},\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_L], \tag{32c}$$

$$\widetilde{\alpha}(g)=\frac{h_{21}(g)}{h_{11}(g)}=-\frac{c_{21}(g)}{c_{22}(g)},\quad g\in\mathbb{R},\quad \widetilde{\alpha}^{\pm}(g)=\frac{h_{21}^{\pm}(g)}{h_{11}^{\pm}(g)}=-\frac{c_{21}^{\pm}(g)}{c_{22}^{\pm}(g)},\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_L]. \tag{32d}$$

The coefficients $1/c_{11}(g)$ [resp. $1/c_{22}(g)$] for $g\in\mathbf{\Lambda}_R^+$ [resp. for $g\in\mathbf{\Lambda}_R^-$] and $1/c_{11}^{\pm}(g)$ [resp. $1/c_{22}^{\pm}(g)$] for $g\in[-\mathrm{i}B_L,\mathrm{i}B_R)$ [resp. $g\in(-\mathrm{i}B_R,\mathrm{i}B_L]$] are commonly designated as (right) transmission coefficients. Analogous definitions can be readily established as (left) transmission coefficients, namely $1/h_{22}(g)$ and $1/h_{11}(g)$. Likewise, we can derive the scattering matrix:

$$\mathbf{C}(g)=\int_0^{\infty}\mathrm{e}^{\mathrm{i}\beta_R y\sigma_3}\mathbf{E}_R^{-1}(g)[\mathbf{Q}(y)-\mathbf{Q}_R]\mathbf{V}(y,g)\,\mathrm{d}y+\mathbf{E}_R^{-1}(g)\mathbf{E}_L(g)\left[\mathbf{I}_2+\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\beta_L y\sigma_3}\mathbf{E}_L^{-1}(g)[\mathbf{Q}(y)-\mathbf{Q}_L]\mathbf{V}(y,g)\,\mathrm{d}y\right], \tag{33}$$

this integral representation offers an alternative to the determinant formulations for analytically continuing the scattering coefficients $c_{11}(g)$ and $c_{22}(g)$ within their respective half-planes.

2.4. Symmetries of eigenfunctions and scattering data

The scattering problem (3) exhibits two involutions: $(g,\beta_{L/R})\to(g^*,\beta_{L/R}^*)$ and $(g,\beta_{L/R})\to(g,-\beta_{L/R})$. Consequently, the eigenfunctions and scattering data adhere to two distinct sets of symmetry relations. In the asymmetric scenario addressed here, which involves four branch points and two distinct branch cuts, it is crucial to differentiate between the situation where both β_R and β_L are discontinuous, then for $g\in[-\mathrm{i}B_L,\mathrm{i}B_L]$, and the case where only one is discontinuous (here β_R, due to the selection $B_L\le B_R$), which corresponds to $g\in[-\mathrm{i}B_R,\mathrm{i}B_L]\cup[\mathrm{i}B_L,\mathrm{i}B_R]$.

First symmetry: On the single sheet of g under consideration, the involution $g\to g^*$ entails that $\beta_{L/R}\to\beta_{L/R}^*$. It is straightforward to verify that if $\chi(x,g)=(\chi_1(x,g),\chi_2(x,g))^T$ is a solution to the scattering problem (3), then $\widetilde{\chi}(x,g)=-\mathrm{i}\sigma_2^*\chi^*(x,g^*)$ also constitutes a solution to the same scattering problem (3), where T denotes matrix transpose. Considering the boundary conditions (21a) and (21b), the symmetries of the Jost solutions are as follows:

$$\widetilde{\mathbf{u}}^*(x,g^*)=\mathrm{i}\sigma_2\mathbf{u}(x,g),\quad g\in\mathbf{\Lambda}_R^+\cup\mathbb{R},\quad [\widetilde{\mathbf{u}}^{\pm}(x,g^*)]^*=\mathrm{i}\sigma_2\mathbf{u}^{\pm}(x,g),\quad g\in[0,\mathrm{i}B_R], \tag{34a}$$

$$\mathbf{u}^*(x,g^*)=-\mathrm{i}\sigma_2\widetilde{\mathbf{u}}(x,g),\quad g\in\mathbf{\Lambda}_R^-\cup\mathbb{R},\quad [\mathbf{u}^{\pm}(x,g^*)]^*=-\mathrm{i}\sigma_2\widetilde{\mathbf{u}}^{\pm}(x,g),\quad g\in[-\mathrm{i}B_R,0], \tag{34b}$$

$$\mathbf{v}^*(x,g^*)=\mathrm{i}\sigma_2\widetilde{\mathbf{v}}(x,g),\quad g\in\mathbf{\Lambda}_L^-\cup\mathbb{R},\quad [\mathbf{v}^{\pm}(x,g^*)]^*=\mathrm{i}\sigma_2\widetilde{\mathbf{v}}^{\pm}(x,g),\quad g\in[-\mathrm{i}B_L,0], \tag{34c}$$

$$\widetilde{\mathbf{v}}^*(x,g^*)=-\mathrm{i}\sigma_2\mathbf{v}(x,g),\quad g\in\mathbf{\Lambda}_L^+\cup\mathbb{R},\quad [\widetilde{\mathbf{v}}^{\pm}(x,g^*)]^*=-\mathrm{i}\sigma_2\mathbf{v}^{\pm}(x,g),\quad g\in[0,\mathrm{i}B_L]. \tag{34d}$$

From Eqs. (25a) and (25b), we derive that $\mathbf{C}^*(g^*)=\sigma_2\mathbf{C}(g)\sigma_2$. Specifically, assuming $(\boldsymbol{B}_1)$ for the potential, the symmetry relations of the scattering coefficients can be expressed as

$$c_{22}^*(g^*)=c_{11}(g),\quad g\in\mathbf{\Lambda}_L^+\cup\mathbb{R},\quad [c_{22}^{\pm}(g^*)]^*=c_{11}^{\pm}(g),\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_R), \tag{35a}$$

$$c_{12}^*(g)=-c_{21}(g),\quad g\in\mathbb{R},\quad [c_{12}^{\pm}(g^*)]^*=-c_{21}^{\pm}(g),\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_R). \tag{35b}$$

It is important to note that the aforementioned symmetries connect the values of the scattering coefficients in the upper/lower half-planes of g, and from the same side of the cuts. With this in mind, it is straightforward to determine the symmetry relations obeyed by the reflection coefficients:

$$\widetilde{\gamma}^*(g)=-\gamma(g),\quad g\in\mathbb{R},\quad [\widetilde{\gamma}^{\pm}(g^*)]^*=-\gamma^{\pm}(g),\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_R), \tag{36a}$$

$$\widetilde{\alpha}^*(g)=-\alpha(g),\quad g\in\mathbb{R},\quad [\widetilde{\alpha}^{\pm}(g^*)]^*=-\alpha^{\pm}(g),\quad g\in[-\mathrm{i}B_L,\mathrm{i}B_R). \tag{36b}$$

Second symmetry: When employing a single sheet for the Riemann surface of the functions $\beta^2_{L/R} = g^2 + B^2_{L/R}$, the involution $(g, \beta_{L/R}) \to (g, -\beta_{L/R})$ is only valid across the cuts. Thus, this second involution connects the values of eigenfunctions and scattering coefficients for the same g value from opposite sides of the cut. On the innermost cut, where both β_L and β_R exhibit discontinuity, specifically for $g \in \Xi_L$, we have

$$\widetilde{\mathbf{u}}^{\mp}(x,g) = \frac{\beta_R^{\pm} + g}{-\mathrm{i}q_R}\mathbf{u}^{\pm}(x,g), \quad \widetilde{\mathbf{v}}^{\mp}(x,g) = \frac{\beta_L^{\pm} + g}{-\mathrm{i}q_L^*}\mathbf{v}^{\pm}(x,g), \quad g \in [-\mathrm{i}B_L, \mathrm{i}B_L]. \tag{37}$$

For $g \in \Xi_R \backslash \Xi_L$, the symmetries for $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ remain unchanged as previously stated. Meanwhile, $\mathbf{v}^+(x,g) = \mathbf{v}^-(x,g)$ for $g \in [\mathrm{i}B_L, \mathrm{i}B_R]$ and $\widetilde{\mathbf{v}}^+(x,g) = \widetilde{\mathbf{v}}^-(x,g)$ for $g \in [-\mathrm{i}B_R, -\mathrm{i}B_L]$. Additionally, considering that $q_{L/R} = B_{L/R}\mathrm{e}^{\mathrm{i}\delta_{L/R}}$, the aforementioned symmetry relations also lead to

$$\widetilde{\mathbf{u}}(x, \pm\mathrm{i}B_R) = \mp\mathrm{e}^{-\mathrm{i}\delta_R}\mathbf{u}(x, \pm\mathrm{i}B_R), \quad \widetilde{\mathbf{v}}(x, \pm\mathrm{i}B_L) = \mp\mathrm{e}^{\mathrm{i}\delta_L}\mathbf{v}(x, \pm\mathrm{i}B_L). \tag{38}$$

By leveraging the symmetries in the determinant representations of the scattering coefficients, one can derive for $g \in \Xi_L$:

$$c_{11}^{\pm}(g) = \frac{q_L^* q_R}{[\beta_L^{\pm}(g) + g][\beta_R^{\pm}(g) - g]}c_{22}^{\mp}(g) = \frac{q_R[\beta_L^{\pm}(g) - g]}{q_L[\beta_R^{\pm}(g) - g]}c_{22}^{\mp}(g), \tag{39a}$$

$$c_{21}^{\pm}(g) = \frac{q_L^*[\beta_R^{\pm}(g) + g]}{q_R[\beta_L^{\pm}(g) + g]}c_{12}^{\mp}(g) = \frac{q_R^*[\beta_L^{\pm}(g) - g]}{q_L[\beta_R^{\pm}(g) - g]}c_{12}^{\mp}(g). \tag{39b}$$

Conversely, for $g \in \Xi_R \backslash \Xi_L$, the symmetry relations transform into:

$$c_{11}^{\pm}(g) = \frac{\beta_R^{\mp}(g) - g}{\mathrm{i}q_R^*}c_{21}^{\mp}(g), \quad g \in [\mathrm{i}B_L, \mathrm{i}B_R], \quad c_{22}^{\mp}(g) = \frac{\beta_R^{\mp}(g) + g}{-\mathrm{i}q_R}c_{12}^{\pm}(g), \quad g \in [-\mathrm{i}B_R, -\mathrm{i}B_L]. \tag{40}$$

Moreover, for $g \in \Xi_R \backslash \Xi_L$, the symmetric relationship between the scattering coefficients and the reflection coefficients are expressed by the following expression:

$$\gamma^{\pm}(g) = \frac{\mathrm{i}q_R^* c_{11}^{\mp}(g)}{c_{11}^{\pm}(g)[\beta_R^{\pm}(g) - g]}, \quad \gamma^+(g)\gamma^-(g) = \frac{q_R^*}{q_R}, \quad g \in [\mathrm{i}B_L, \mathrm{i}B_R], \tag{41a}$$

$$\widetilde{\gamma}^{\pm}(g) = \frac{-\mathrm{i}q_R c_{22}^{\mp}(g)}{c_{22}^{\pm}(g)[\beta_R^{\mp}(g) + g]}, \quad \widetilde{\gamma}^+(g)\widetilde{\gamma}^-(g) = \frac{q_R}{q_R^*}, \quad g \in [-\mathrm{i}B_R, -\mathrm{i}B_L], \tag{41b}$$

$$h_{22}^{\pm}(g) = -\frac{\mathrm{i}q_R h_{21}^{\mp}(g)}{\beta_R^{\pm}(g) + g}, \quad g \in [\mathrm{i}B_L, \mathrm{i}B_R], \quad h_{11}^{\pm}(g) = \frac{\beta_R^{\mp}(g) + g}{-\mathrm{i}q_R}h_{12}^{\mp}(g), \quad g \in [-\mathrm{i}B_R, -\mathrm{i}B_L]. \tag{41c}$$

2.5. Discrete eigenvalues and asymptotic behavior

A discrete eigenvalue is defined as a value $g \in \mathbf{\Lambda}_R^+ \cup \mathbf{\Lambda}_R^-$ [associated with $\beta_R, \beta_L \in \mathbb{C}\backslash\mathbb{R}$] for which a nontrivial solution χ to Eq. (3) exists, with its components residing in the space $L^2(\mathbb{R})$. For each discrete eigenvalue $g_m \in \mathbf{\Lambda}_R^+$, where $m = 1, \ldots, M$ (M is a finite number), the eigenfunctions $\mathbf{v}(x, g_m)$ and $\mathbf{u}(x, g_m)$ are linearly dependent. Specifically, there exists a complex constant d_m such that $\mathbf{v}(x, g_m) = d_m\mathbf{u}(x, g_m)$. Subsequently, let ω_m represent the residue of $1/c_{11}(g)$ at the simple pole $\beta_R = \beta_R(g_m)$, we can then express this as

$$\lim_{g \to g_m}[\beta_R(g) - \beta_R(g_m)]\frac{\mathbf{v}(x,g)}{c_{11}(g)} = F_m\mathbf{u}(x, g_m), \quad F_m = d_m\omega_m, \tag{42}$$

where F_m is designated as the norming constant corresponding to g_m. Likewise, for $g_1^*, \ldots, g_M^*$ within $\mathbf{\Lambda}_R^-$, the eigenfunctions $\widetilde{\mathbf{v}}(x, g_m^*)$ and $\widetilde{\mathbf{u}}(x, g_m^*)$ are linearly dependent. Hence, there exist a complex constant $\widetilde{d}_m$ such that $\widetilde{\mathbf{v}}(x, g_m^*) = \widetilde{d}_m\widetilde{\mathbf{u}}(x, g_m^*)$. Subsequently, let $\widetilde{\omega}_m$ denote the residue of $1/c_{22}(g)$ at the pole $\beta_R = \beta_R(g_m^*)$, we can then express this as

$$\lim_{g \to g_m^*}[\beta_R(g) - \beta_R(g_m^*)]\frac{\widetilde{\mathbf{v}}(x,g)}{c_{22}(g)} = \widetilde{F}_m\widetilde{\mathbf{u}}(x, g_m^*), \quad \widetilde{F}_m = \widetilde{d}_m\widetilde{\omega}_m, \tag{43}$$

with $\widetilde{F}_m$ is termed the norming constant corresponding to the discrete eigenvalue g_m^*. By employing the symmetry relations and the definitions (42) and (43), we obtain $\widetilde{\omega}_m = \omega_m^*$, $\widetilde{d}_m = -d_m^*$, and $\widetilde{F}_m = -F_m^*$.

We aim to derive the asymptotic behavior as g becomes large, utilizing the Volterra integral equations (154a), (154b), (154c) and (154d). For convenience, we introduce the modified eigenfunction $\mathbf{\Phi}(x,g) = \mathbf{u}(x,g)\mathrm{e}^{-\mathrm{i}\beta_R x}$, and label its jth component with $j = 1, 2$. The components of $\mathbf{\Phi}(x,g)$ are represented as follows:

$$\begin{aligned}\mathbf{\Phi}_1(x,g) = &-\frac{\mathrm{i}q_R}{\beta_R + g} - \frac{\beta_R + g}{2\beta_R}\int_x^{\infty}[q(y) - q_R]\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\mathbf{\Phi}_2(y,g)\,\mathrm{d}y - \frac{\beta_R - g}{2\beta_R}\int_x^{\infty}[q(y) - q_R]\mathbf{\Phi}_2(y,g)\,\mathrm{d}y \\ &- \frac{\mathrm{i}q_R}{2\beta_R}\int_x^{\infty}[q^*(y) - q_R^*]\mathbf{\Phi}_1(y,g)\,\mathrm{d}y + \frac{\mathrm{i}q_R}{2\beta_R}\int_x^{\infty}[q^*(y) - q_R^*]\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\mathbf{\Phi}_1(y,g)\,\mathrm{d}y,\end{aligned} \tag{44}$$

and

$$\begin{aligned}\mathbf{\Phi}_2(x,g) = 1 + \frac{\beta_R - g}{2\beta_R}\int_x^\infty [q^*(y) - q_R^*]\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\mathbf{\Phi}_1(y,g)\,\mathrm{d}y + \frac{\beta_R + g}{2\beta_R}\int_x^\infty [q^*(y) - q_R^*]\mathbf{\Phi}_1(y,g)\,\mathrm{d}y \\ - \frac{\mathrm{i}q_R^*}{2\beta_R}\int_x^\infty [q(y) - q_R]\mathbf{\Phi}_2(y,g)\,\mathrm{d}y + \frac{\mathrm{i}q_R^*}{2\beta_R}\int_x^\infty [q(y) - q_R]\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\mathbf{\Phi}_2(y,g)\,\mathrm{d}y.\end{aligned} \tag{45}$$

Consequently, based on Theorem 1, the functions $\mathbf{\Phi}_1(x,g)$ and $\mathbf{\Phi}_2(x,g) - 1$ are uniformly bounded for $(x,g) \in [x_0, +\infty) \times [\mathbf{\Lambda}_R^+ \cup \partial\mathbf{\Lambda}_R^+ \cup \partial\mathbf{\Lambda}_R^-]$ in the aforementioned integral equations. Additionally, the iteration of the integral equations converges uniformly for (x,g) within the same set. Assuming the potential satisfies $\partial_x q \in L^1(\mathbb{R})$, we can express

$$\int_x^\infty [q(y) - q_R]\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\,\mathrm{d}y = \frac{\mathrm{i}}{2\beta_R}\left[q(x) - q_R + \int_x^\infty \frac{\partial}{\partial y}q(y)\mathrm{e}^{2\mathrm{i}\beta_R(y-x)}\,\mathrm{d}y\right] = \frac{\mathrm{i}[q(x) - q_R]}{2g} + o(g^{-1}). \tag{46}$$

By iterating for $\mathbf{\Phi}_1(x,g)$ [resp. $\mathbf{\Phi}_2(x,g)$] once with respect to the other unknown $\mathbf{\Phi}_2(x,g)$ [resp. $\mathbf{\Phi}_1(x,g)$], we obtain the formulas in powers of g^{-1} for the inhomogeneous terms:

$$\mathbf{\Phi}_1^{\mathrm{inh}}(x,g) = -\frac{\mathrm{i}q_R}{2g} - \frac{\mathrm{i}[q(x) - q_R]}{2g} + o(g^{-1}) = -\frac{\mathrm{i}q(x)}{2g} + o(g^{-1}), \quad \mathbf{\Phi}_2^{\mathrm{inh}}(x,g) = 1 - \frac{\mathrm{i}}{2g}\int_x^\infty [q^*(y) - q_R^*]q(y)\,\mathrm{d}y + o(g^{-1}). \tag{47}$$

Upon substituting these expressions into the integral equations and calculating their first iterates, it becomes evident that only the penultimate term in the right-hand side of the second integral equation results in an additional contribution of order g^{-1}, specifically,

$$-\frac{\mathrm{i}q_R^*}{2g}\int_x^\infty [q(y) - q_R]\,\mathrm{d}y. \tag{48}$$

The iterations do not yield any further terms of $O(1)$ or $O(g^{-1})$. Consequently, we derive

$$\mathbf{\Phi}_1(x,g) = -\frac{\mathrm{i}q(x)}{2g} + o(g^{-1}), \tag{49a}$$

$$\mathbf{\Phi}_2(x,g) = 1 - \frac{\mathrm{i}q_R^*}{2g}\int_x^\infty [q(y) - q_R]\,\mathrm{d}y - \frac{\mathrm{i}}{2g}\int_x^\infty [q^*(y) - q_R^*]q(y)\,\mathrm{d}y + o(g^{-1}) = 1 - \frac{\mathrm{i}}{2g}\int_x^\infty [|q(y)|^2 - B_R^2]\,\mathrm{d}y + o(g^{-1}). \tag{49b}$$

In summary, the Volterra integral equations provide the following asymptotic behaviors for the eigenfunctions as $|g| \to \infty$ within the relevant half-planes:

$$\mathbf{U}(x,g)\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x} = \left[\mathbf{I}_2 + \frac{\mathrm{i}\mathbf{Q}(x)\sigma_3}{2g}\right][1 + o(1)], \quad \mathbf{V}(x,g)\mathrm{e}^{\mathrm{i}\beta_L\sigma_3 x} = \left[\mathbf{I}_2 + \frac{\mathrm{i}\mathbf{Q}(x)\sigma_3}{2g}\right][1 + o(1)]. \tag{50}$$

For future convenience, we also note that

$$\partial_x\left[\mathbf{U}(x,g)\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x}\right] = \frac{\mathrm{i}\partial_x\mathbf{Q}(x)\sigma_3}{2g}[1 + o(1)]. \tag{51}$$

Based on the determinant representations of the scattering coefficients, and considering once more that $\beta_R \sim \beta_L \sim g$ as $g \to \infty$, we subsequently derive the asymptotic behavior:

$$c_{11}(g) = \frac{\beta_R + g}{2\beta_R}\det(\mathbf{v}(x,g), \mathbf{u}(x,g)) \sim 1, \quad |g| \to \infty, \quad g \in \mathbf{\Lambda}_R^+ \cup \mathbb{R}, \tag{52a}$$

$$c_{22}(g) = -\frac{\beta_R + g}{2\beta_R}\det(\widetilde{\mathbf{v}}(x,g), \widetilde{\mathbf{u}}(x,g)) \sim 1, \quad |g| \to \infty, \quad g \in \mathbf{\Lambda}_R^- \cup \mathbb{R}, \tag{52b}$$

$$c_{21}(g) = O(g^{-2}), \quad \gamma(g) = O(g^{-2}), \quad \widetilde{\gamma}(g) = O(g^{-2}), \quad |g| \to \infty, \quad g \in \mathbb{R}, \tag{52c}$$

$$c_{12}(g) = O(g^{-2}), \quad \alpha(g) = O(g^{-2}), \quad \widetilde{\alpha}(g) = O(g^{-2}), \quad |g| \to \infty, \quad g \in \mathbb{R}. \tag{52d}$$

2.6. Trace formula

To establish a representation for the scattering coefficient $c_{11}(g)$ in terms of discrete eigenvalues and reflection coefficients, commonly known as the trace formula, we begin with the quasi-unitarity of the scattering matrix $\mathbf{C}(g)$. Considering the symmetries (35) and (36), Eqs. (27a) and (27b) for $\det \mathbf{C}(g)$ transform into

$$|c_{11}(g)|^2 + |c_{21}(g)|^2 = \frac{\beta_L(\beta_R + g)}{\beta_R(\beta_L + g)}, \quad g \in \mathbb{R}, \tag{53a}$$

$$c_{11}^{\pm}(g)[c_{11}^{\pm}(g^*)]^* + c_{21}^{\pm}(g)[c_{21}^{\pm}(g^*)]^* = \frac{\beta_L^+(\beta_R^{\pm} + g)}{\beta_R^+(\beta_L^{\pm} + g)}, \quad g \in \varXi_L. \tag{53b}$$

Conversely, the aforementioned equations can be reformulated utilizing the reflection coefficients in the following manner:

$$|c_{11}(g)|^2 = \frac{\beta_L(\beta_R + g)}{\beta_R(\beta_L + g)[1 + |\gamma(g)|^2]}, \quad g \in \mathbb{R}, \tag{54a}$$

$$c_{11}^{\pm}(g)[c_{11}^{\pm}(g^*)]^* = \frac{\beta_L^+(\beta_R^\pm + g)}{\beta_R^+(\beta_L^\pm + g)[1+\gamma^\pm(g)[\gamma^\pm(g^*)]^*]}, \quad g \in \Xi_L. \tag{54b}$$

Given that $c_{11}(g)$ [resp. $c_{22}(g)$] exhibits analyticity within $\mathbf{\Lambda}_R^+$ [resp. $\mathbf{\Lambda}_R^-$] and continuity across $\overline{\mathbf{\Lambda}_R^+}$ [resp. $\overline{\mathbf{\Lambda}_R^-}$], converges to 1 as $g \to \infty$, and features (simple) zeros at $g = g_m$ [resp. $g = g_m^*$], where $m = 1, \ldots, M$, we proceed to introduce

$$\rho(g) = c_{11}(g)\prod_{m=1}^{M}\frac{g-g_m^*}{g-g_m}, \quad \widetilde{\rho}(g) = c_{22}(g)\prod_{m=1}^{M}\frac{g-g_m}{g-g_m^*}. \tag{55}$$

Owing to the analytic characteristics of $\rho(g)$ and $\widetilde{\rho}(g)$, the application of Cauchy's integral formula for $g \in \mathbf{\Lambda}_R^+$ results in

$$\begin{aligned}\ln\rho(g) = \frac{1}{2\pi\mathrm{i}}\int_{D_+}\frac{\ln\rho(\xi)}{\xi-g}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\Bigg[&\int_{\mathbb{R}}\frac{\ln\rho(\xi)}{\xi-g}\mathrm{d}\xi - \int_{-\mathrm{i}B_L}^{0}\frac{\ln\rho^-(\xi)}{\xi-g}\mathrm{d}\xi + \int_{-\mathrm{i}B_L}^{0}\frac{\ln\rho^+(\xi)}{\xi-g}\mathrm{d}\xi \\ &+ \int_0^{\mathrm{i}B_L}\frac{\ln\rho^-(\xi)}{\xi-g}\mathrm{d}\xi + \int_{\mathrm{i}B_L}^{\mathrm{i}B_R}\frac{\ln\rho^-(\xi)}{\xi-g}\mathrm{d}\xi - \int_{\mathrm{i}B_L}^{\mathrm{i}B_R}\frac{\ln\rho^+(\xi)}{\xi-g}\mathrm{d}\xi - \int_0^{\mathrm{i}B_L}\frac{\ln\rho^+(\xi)}{\xi-g}\mathrm{d}\xi\Bigg],\end{aligned} \tag{56}$$

and

$$\begin{aligned}0 = \frac{1}{2\pi\mathrm{i}}\int_{D_-}\frac{\ln\widetilde{\rho}(\xi)}{\xi-g}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\Bigg[&\int_{\mathbb{R}}\frac{\ln\widetilde{\rho}(\xi)}{\xi-g}\mathrm{d}\xi - \int_{-\mathrm{i}B_L}^{0}\frac{\ln\widetilde{\rho}^-(\xi)}{\xi-g}\mathrm{d}\xi - \int_{-\mathrm{i}B_R}^{-\mathrm{i}B_L}\frac{\ln\widetilde{\rho}^-(\xi)}{\xi-g}\mathrm{d}\xi \\ &+ \int_{-\mathrm{i}B_R}^{-\mathrm{i}B_L}\frac{\ln\widetilde{\rho}^+(\xi)}{\xi-g}\mathrm{d}\xi + \int_{-\mathrm{i}B_L}^{0}\frac{\ln\widetilde{\rho}^+(\xi)}{\xi-g}\mathrm{d}\xi + \int_0^{\mathrm{i}B_L}\frac{\ln\widetilde{\rho}^-(\xi)}{\xi-g}\mathrm{d}\xi - \int_0^{\mathrm{i}B_L}\frac{\ln\widetilde{\rho}^+(\xi)}{\xi-g}\mathrm{d}\xi\Bigg],\end{aligned} \tag{57}$$

where $D_\pm$ denote the oriented contours, with the superscripts $\pm$ in $\rho(g)$ and $\widetilde{\rho}(g)$ selected based on whether the integration is conducted along the right or left boundary of each respective cut.

Upon summing the two equations and invoking $\rho(g)$ and $\widetilde{\rho}(g)$, the resultant expression is

$$\begin{aligned}\ln c_{11}(g) = \sum_{m=1}^{M}\ln\left[\frac{g-g_m}{g-g_m^*}\right] + \frac{1}{2\pi\mathrm{i}}\Bigg\{&\int_{\mathbb{R}}\frac{\ln[c_{11}(\xi)c_{22}(\xi)]}{\xi-g}\mathrm{d}\xi + \int_0^{\mathrm{i}B_L}\frac{\ln\left[\frac{c_{11}^-(\xi)c_{22}^-(\xi)}{c_{11}^+(\xi)c_{22}^+(\xi)}\right]}{\xi-g}\mathrm{d}\xi \\ &+ \int_{-\mathrm{i}B_L}^{0}\frac{\ln\left[\frac{c_{11}^+(\xi)c_{22}^+(\xi)}{c_{11}^-(\xi)c_{22}^-(\xi)}\right]}{\xi-g}\mathrm{d}\xi + \int_{\mathrm{i}B_L}^{\mathrm{i}B_R}\frac{\ln\left[\frac{c_{11}^-(\xi)}{c_{11}^+(\xi)}\right]}{\xi-g}\mathrm{d}\xi + \int_{-\mathrm{i}B_R}^{-\mathrm{i}B_L}\frac{\ln\left[\frac{c_{22}^+(\xi)}{c_{22}^-(\xi)}\right]}{\xi-g}\mathrm{d}\xi\Bigg\}.\end{aligned} \tag{58}$$

Utilizing the aforementioned, we subsequently derive what is commonly referred to as the trace formula:

$$\begin{aligned}c_{11}(g) = \prod_{m=1}^{M}\left[\frac{g-g_m}{g-g_m^*}\right]\exp\Bigg\{&-\frac{1}{2\pi\mathrm{i}}\int_{\Xi}\frac{\ln\left[\frac{\beta_R(\beta_L+\xi)}{\beta_L(\beta_R+\xi)}[1+\gamma(\xi)\gamma^*(\xi^*)]\right]}{\xi-g}\mathrm{d}\xi \\ &+\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}B_L}^{\mathrm{i}B_R}\frac{\ln\left[\frac{|\beta_R(\xi)|-\xi}{\mathrm{i}q_R^*}\gamma^+(\xi)\right]}{\xi-g}\mathrm{d}\xi + \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}B_R}^{-\mathrm{i}B_L}\frac{\ln\left[-\frac{|\beta_R(\xi)|+\xi}{\mathrm{i}q_R}\widetilde{\gamma}^-(\xi)\right]}{\xi-g}\mathrm{d}\xi\Bigg\},\end{aligned} \tag{59}$$

where $\Xi = \mathbb{R}\cup[-\mathrm{i}B_L,0]_{\mathrm{L}}\cup[-\mathrm{i}B_L,0]_{\mathrm{R}}\cup[0,\mathrm{i}B_L]_{\mathrm{L}}\cup[0,\mathrm{i}B_L]_{\mathrm{R}}$. It is noteworthy that the term within the square brackets in the numerator of the first integral in (59) transforms into $[1+|\gamma(\xi)|^2]$ for $\xi \in \mathbb{R}$, and $[1+\gamma^\pm(\xi)[\gamma^\pm(\xi^*)]^*]$ for $\xi \in [-\mathrm{i}B_L, \mathrm{i}B_L]$. Eq. (59) elucidates that $c_{11}(g)$ is fully ascertainable for $g \in \mathbf{\Lambda}_R^+$ based on the following:

- its zeros, identified as discrete eigenvalues $g_m \in \mathbf{\Lambda}_R^+$;
- the reflection coefficient $\gamma(g)$ for $g \in \mathbb{R}$, and $\gamma^\pm(g)$ for $g \in \Xi_L$;
- $\gamma^+(k)$ for $g \in [\mathrm{i}B_L, \mathrm{i}B_R]$, and $\widetilde{\gamma}^-(g)$ for $g \in [-\mathrm{i}B_R, -\mathrm{i}B_L]$.

3. Inverse scattering problem

Within the framework of the IST, the inverse scattering problem initially involves reconstructing the eigenfunctions using the scattering data, followed by deducing the potential, which corresponds to the solution of the NLS equation, in terms of these eigenfunctions. For example, when formulating the inverse problem from the right, the following scattering data are necessary:

- the reflection coefficient $\gamma(g)$ for $g \in \mathbb{R}$, along with its values $\gamma^{\pm}(g)$ for $g \in [-\mathrm{i}B_L, \mathrm{i}B_L]$ on the boundaries of the cut (this component represents the continuous spectrum of the scattering operator and functions analogously to the direct Fourier transform of the initial condition in solving the initial-value problem for a linear partial differential equation (PDE) via Fourier transform; it should be noted that the reflection coefficient $\widetilde{\gamma}(g)$ is interconnected with $\gamma(g)$ through the symmetry expressed in Eq. (36a));
- the discrete eigenvalues $g_m \in \mathbf{\Lambda}_R^+$, together with their corresponding norming constants F_m for $m = 1, \ldots, M$ (it should be observed that discrete eigenvalues within $\mathbf{\Lambda}_R^-$ and their associated norming constants are not considered independent, as they can be derived from the aforementioned through conjugation symmetry);
- supplementary scattering data $\gamma^+(g)$ for $g \in [\mathrm{i}B_L, \mathrm{i}B_R]$ and $\widetilde{\gamma}^-(g)$ for $g \in [-\mathrm{i}B_R, \mathrm{i}B_L]$ (it is noteworthy that $\gamma^-(g)$ and $\widetilde{\gamma}^+(g)$ are interconnected with the former by symmetrical relationships; additionally, in accordance with the trace formula delineated in (59), the values of the transmission coefficient $1/c_{11}(g)$ for all $g \in \mathbf{\Lambda}_R^+$, and $1/c_{11}^{\pm}(g)$ for $g \in \Xi_R^+$ can be ascertained from the scattering data previously mentioned).

3.1. Triangular representations for the eigenfunctions

In the present section, we introduce the following pair of triangular representations pertaining to the fundamental eigenfunctions:

$$\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)} = \mathbf{I}_2 + \int_x^{\infty} \mathbf{Z}(x,h)\mathrm{e}^{(h-x)\mathbf{G}_R(g)}\,\mathrm{d}h, \tag{60a}$$

$$\widetilde{\mathbf{V}}(x,g)\mathrm{e}^{-x\mathbf{G}_L(g)} = \mathbf{I}_2 + \int_{-\infty}^{x} \mathbf{P}(x,h)\mathrm{e}^{(h-x)\mathbf{G}_L(g)}\,\mathrm{d}h, \tag{60b}$$

with the kernels $\mathbf{Z}(x,h) = [\mathbf{Z}_{ij}(x,h)]_{i,j=1,2}$ and $\mathbf{P}(x,h) = [\mathbf{P}_{ij}(x,h)]_{i,j=1,2}$ are designated as "triangular" kernels, characterized by the property that $\mathbf{Z}(x,h) = 0$ when $x > h$, and $\mathbf{P}(x,h) = 0$ when $x < h$. It is observed that Eqs. (60a) and (60b) lead to analogous triangular representations for the Jost solutions delineated in Eqs. (20a) and (20b):

$$\mathbf{U}(x,g) = \mathbf{E}_R(g)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3 x} + \int_x^{\infty} \mathbf{Z}(x,h)\mathbf{E}_R(g)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3 h}\,\mathrm{d}h, \tag{61a}$$

$$\mathbf{V}(x,g) = \mathbf{E}_L(g)\mathrm{e}^{-\mathrm{i}\beta_L\sigma_3 x} + \int_{-\infty}^{x} \mathbf{P}(x,h)\mathbf{E}_L(g)\mathrm{e}^{-\mathrm{i}\beta_L\sigma_3 h}\,\mathrm{d}h. \tag{61b}$$

Proposition 2. *Considering the explicit form of the groups $\mathrm{e}^{x\mathbf{G}_{L/R}(g)}$ as presented in Eq. (144), and acknowledging that the transformation $g \to -g$ results in $\beta_{L/R} \to -\beta_{L/R}$ [cf. Eq. (16)], it is possible to derive from Eq. (60a) for $x \in \mathbb{R}$ and $g \in \mathbb{R} \cup [-\mathrm{i}B_R, \mathrm{i}B_R]$,*

$$\begin{aligned}\mathbf{Z}(x,h) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\beta_R(h-x)} \Big\{ &\left[\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)} - \mathbf{I}_2\right]\left[\mathbf{I}_2 - \frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R - \frac{\beta_R}{g}\sigma_3\right] \\ &+ \left[\widetilde{\mathbf{U}}(x,-g)\mathrm{e}^{-x\mathbf{G}_R(-g)} - \mathbf{I}_2\right]\left[\mathbf{I}_2 + \frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R + \frac{\beta_R}{g}\sigma_3\right]\Big\}\,\mathrm{d}\beta_R.\end{aligned} \tag{62}$$

Analogously, from the second equation of Eq. (60b), the following can be deduced:

$$\begin{aligned}\mathbf{P}(x,h) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\beta_L(h-x)} \Big\{ &\left[\widetilde{\mathbf{V}}(x,g)\mathrm{e}^{-x\mathbf{G}_L(g)} - \mathbf{I}_2\right]\left[\mathbf{I}_2 - \frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_L - \frac{\beta_L}{g}\sigma_3\right] \\ &+ \left[\widetilde{\mathbf{V}}(x,-g)\mathrm{e}^{-x\mathbf{G}_L(-g)} - \mathbf{I}_2\right]\left[\mathbf{I}_2 + \frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_L + \frac{\beta_L}{g}\sigma_3\right]\Big\}\,\mathrm{d}\beta_L.\end{aligned} \tag{63}$$

Proposition 2 will be proven in Appendix B. We are now in a position to establish the existence of the kernels $\mathbf{Z}(x,h)$ and $\mathbf{P}(x,h)$ as Fourier transforms, as stipulated by Eqs. (62) and (63). To substantiate this claim, it is necessary to posit that the potential $q(x)$ adheres to condition $(\boldsymbol{B}_2)$ and that $\partial_x q$ is an element of $L^1(\mathbb{R})$ (which entails that $q(x)$ is continuous for $x \in \mathbb{R}$, and approaches $q_{L/R}$ as $x \to \mp\infty$). Should Eq. (62) define $\mathbf{Z}(x,h)$, as a function of x, as the Fourier transform of a matrix function whose entries reside in $L^2(\mathbb{R}, \mathrm{d}\beta_R)$, the existence of the kernel $\mathbf{Z}(x,h)$ can be affirmed. Initially, it is noted that under the assumptions of $(\boldsymbol{B}_1)$ and $\partial_x q \in L^1(\mathbb{R})$, the expression can be formulated as:

$$\begin{aligned}\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)} - \mathbf{I}_2 &= \frac{\mathrm{i}}{2\beta_R}\sigma_3\mathbf{Q}_R - \frac{\beta_R - g}{2\beta_R}\mathbf{I}_2 + \frac{1}{g}\left[\mathbf{\Phi}^{(1)}(x) + o(1)\right]\mathbf{E}_R^{-1}(g) \\ &= \frac{\mathrm{i}}{2\beta_R}\sigma_3\mathbf{Q}_R + \frac{1}{g}\left[\mathbf{\Phi}^{(1)}(x) + o(1)\right] + O(g^{-2}).\end{aligned} \tag{64}$$

Herein, $\mathbf{\Phi}(x,g) = \mathbf{U}(x,g)\mathrm{e}^{\mathrm{i}\beta_R x\sigma_3}$ and $\mathbf{\Phi}^{(1)}(x) = (\widetilde{\Phi}^{(1)}(x), \Phi^{(1)}(x))$ represents the $O(g^{-1})$ term in the asymptotic expansion of $\mathbf{\Phi}(x,g)$ for large g. In a like manner,

$$\begin{aligned}\widetilde{\mathbf{U}}(x,-g)\mathrm{e}^{-x\mathbf{G}_R(-g)} - \mathbf{I}_2 &= -\frac{\mathrm{i}}{2\beta_R}\sigma_3\mathbf{Q}_R - \frac{\beta_R - g}{2\beta_R}\mathbf{I}_2 - \frac{1}{g}\left[\mathbf{\Phi}^{(1)}(x) + o(1)\right]\mathbf{E}_R^{-1}(-g) \\ &= -\frac{\mathrm{i}}{2\beta_R}\sigma_3\mathbf{Q}_R - \frac{1}{g}\left[\mathbf{\Phi}^{(1)}(x) + o(1)\right] + O(g^{-2}).\end{aligned} \tag{65}$$

We are now in a position to demonstrate that $\mathbf{Z}(x,h)$ is an element of $L^2(x,+\infty)$ with respect to h. To achieve this, it is necessary to establish that

$$\int_{-\infty}^{\infty}\left\|\left[\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)}-\mathbf{I}_2\right]\left[\mathbf{I}_2-\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R-\frac{\beta_R}{g}\sigma_3\right]+\left[\widetilde{\mathbf{U}}(x,-g)\mathrm{e}^{-x\mathbf{G}_R(-g)}-\mathbf{I}_2\right]\left[\mathbf{I}_2+\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R+\frac{\beta_R}{g}\sigma_3\right]\right\|^2\mathrm{d}\beta_R, \tag{66}$$

is finite for every $x\in\mathbb{R}$. To circumvent potential singularities at $g=0$ [i.e., $\beta_R=\pm B_R$] and $g=\pm\mathrm{i}B_R$ [i.e., $\beta_R=0$], we partition the integral into two segments. Assuming condition $(\boldsymbol{B}_1)$ holds, the integral with respect to β_R over any interval $\beta_R\in[-\sigma_4,\sigma_4]$, where $0<\sigma_4<B_R$, circumvents the point $g=0$. The integrand within this range is continuous in β_R, thereby ensuring the integral is well-defined and finite. The integral concerning β_R across the remaining segment of the real β_R-axis encompasses a domain that does not circumvent $\beta_R=\pm B_R$, that is $g=0$. Nevertheless, by substituting this integral with one in terms of g, and leveraging the relation $\mathrm{d}\beta_R=(g/\beta_R)\mathrm{d}g$, under the assumption of $(\boldsymbol{B}_2)$, the existence of the limit can be established:

$$\lim_{g\to0}\frac{\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)}-\widetilde{\mathbf{U}}(x,-g)\mathrm{e}^{-x\mathbf{G}_R(-g)}}{2g}=\left[\frac{\partial}{\partial g}\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)}\right]_{g=0}, \tag{67}$$

ensures the integral is well-behaved in the vicinity of $g=0$. Consequently, the sole aspect remaining for consideration as $g\to\pm\infty$ [resp. as $\beta_R\to\pm\infty$]. It is necessary to demonstrate that

$$\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)\left\|\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)}-\mathbf{I}_2\right\|^2\mathrm{d}g,\quad x\in\mathbb{R}. \tag{68}$$

This conclusion is substantiated by the large g expansion, given the premises $(\boldsymbol{B}_1)$ and $\partial_x q\in L^1(\mathbb{R})$, the most significant term as $g\to\pm\infty$ is of the order $O(g^{-1})$. Hence, we have established the existence of $\mathbf{Z}(x,h)$ in h for $x\in\mathbb{R}$, fulfilling the condition that

$$\int_x^{\infty}\|\mathbf{Z}(x,h)\|^2\,\mathrm{d}h<+\infty. \tag{69}$$

As a result, when considered as a function of h, $\mathbf{Z}(x,h)$ represents the Fourier transform of an L^2 matrix function and is therefore an L^2 matrix function itself, uniformly for $x\geq x_0$ for every $x_0\in\mathbb{R}$. A parallel reasoning applies to $\mathbf{P}(x,h)$. Ultimately, the subsequent finding enables the reconstruction of the potential in terms of the kernels $\mathbf{Z}(x,h)$ and $\mathbf{P}(x,h)$. Theorem 3 will be proven in Appendix B.

Theorem 3. *Assuming that condition $(\boldsymbol{B}_2)$ is satisfied, and furthermore that $\partial_x q\in L^1(\mathbb{R})$, it follows that:*

$$\mathbf{Q}(x)-\mathbf{Q}_R=2\sigma_3\mathbf{Z}(x,x)\sigma_3,\quad \mathbf{Q}(x)-\mathbf{Q}_L=2\sigma_3\mathbf{P}(x,x)\sigma_3. \tag{70}$$

Specifically, Eq. (70) indicates that the diagonal elements of both Marchenko kernels $\mathbf{Z}(x,x)$ and $\mathbf{P}(x,x)$ are zero, whereas the off-diagonal elements adhere to the subsequent relationship:

$$q(x)=q_R-2\mathbf{Z}_{12}(x,x)=q_L-2\mathbf{P}_{12}(x,x),\quad q^*(x)=q_R^*+2\mathbf{Z}_{21}(x,x)=q_L^*+2\mathbf{P}_{21}(x,x). \tag{71}$$

The condition $q\in C^1(\mathbb{R})$ assures that Eqs. (70) and (71) are defined everywhere, not just almost everywhere. Time dependency is excluded here for conciseness. Assuming that all conditions on the potential hold for all $t\geq0$, the inclusion of time dependency in the Jost and fundamental eigenfunctions gives rise to a t-parametric dependency in the Marchenko kernels. The reconstruction formulas (71) for the potential, applicable for all $t\geq0$, are expressed as:

$$q(x,t)=q_R(t)-2\mathbf{Z}_{12}(x,x;t)=q_L(t)-2\mathbf{P}_{12}(x,x;t). \tag{72}$$

It should be noted that the presence of the Marchenko kernels is associated with the subsequent Goursat problem:

$$(\partial_x+\partial_h)\begin{pmatrix}\mathbf{Z}_{11}(x,h)\\ \mathbf{Z}_{22}(x,h)\end{pmatrix}=\begin{pmatrix}-q_R^* & q(x)\\ -q^*(x) & q_R\end{pmatrix}\begin{pmatrix}\mathbf{Z}_{12}(x,h)\\ \mathbf{Z}_{21}(x,h)\end{pmatrix}, \tag{73a}$$

$$(\partial_x-\partial_h)\begin{pmatrix}\mathbf{Z}_{12}(x,h)\\ \mathbf{Z}_{21}(x,h)\end{pmatrix}=\begin{pmatrix}-q_R & q(x)\\ -q^*(x) & q_R^*\end{pmatrix}\begin{pmatrix}\mathbf{Z}_{11}(x,h)\\ \mathbf{Z}_{22}(x,h)\end{pmatrix}, \tag{73b}$$

with boundary conditions:

$$q(x)=q_R-2\mathbf{Z}_{12}(x,x)=q_R+2\mathbf{Z}_{12}^*(x,x),\quad \lim_{h\to\infty}\mathbf{Z}_{ij}(x,h)=0,\quad i,j=1,2, \tag{74}$$

this derivation, contingent upon supplementary conditions regarding the potential, stems from the scattering problem. Nevertheless, we have furnished a direct inversion formula for the kernels integral to the inverse problem, formulated explicitly in terms of the eigenfunctions from the direct scattering problem. Additionally, Plancherel's theorem was employed to ascertain an explicit representation for $\mathbf{Z}(x,h)$.

3.2. Right Marchenko equations

We develop the (left and right) Marchenko integral equations as an approach to tackle the inverse problem. This involves the reconstruction of eigenfunctions and subsequently the potential, based on the scattering data provided. We shall explicitly articulate the Right Marchenko equations as follows:

$$\frac{\mathbf{v}(x,g)}{c_{11}(g)} = \widetilde{\mathbf{u}}(x,g) + \gamma(g)\mathbf{u}(x,g), \quad g \in \mathbb{R}, \tag{75a}$$

$$\frac{\mathbf{v}^{\pm}(x,g)}{c_{11}^{\pm}(g)} = \widetilde{\mathbf{u}}^{\pm}(x,g) + \gamma^{\pm}(g)\mathbf{u}^{\pm}(x,g), \quad g \in [-\mathrm{i}B_L, \mathrm{i}B_R], \tag{75b}$$

$$\frac{\widetilde{\mathbf{v}}(x,g)}{c_{22}(g)} = \mathbf{u}(x,g) + \widetilde{\gamma}(g)\widetilde{\mathbf{u}}(x,g), \quad g \in \mathbb{R}, \tag{75c}$$

$$\frac{\widetilde{\mathbf{v}}^{\pm}(x,g)}{c_{22}^{\pm}(g)} = \mathbf{u}^{\pm}(x,g) + \widetilde{\gamma}^{\pm}(g)\widetilde{\mathbf{u}}^{\pm}(x,g), \quad g \in (-\mathrm{i}B_R, \mathrm{i}B_L], \tag{75d}$$

where $\gamma(g)$, $\gamma^{\pm}(g)$ and $\widetilde{\gamma}(g)$, $\widetilde{\gamma}^{\pm}(g)$ are specified by Eqs. (32a) and (32b), respectively. It is noteworthy that although Eq. (25a) is originally defined for $g \in \mathbb{R} \cup [-\mathrm{i}B_L, \mathrm{i}B_L]$, Eq. (75b) indicates that the first column of Eq. (25a) can be analytically continued to $g \in [\mathrm{i}B_L, \mathrm{i}B_R]$. Similarly, Eq. (75b) extends the domain of the second column of Eq. (25a) to $g \in [-\mathrm{i}B_R, -\mathrm{i}B_L]$. Subsequently, we shall proceed under the assumptions aligned with the preceding discourse, namely:

- the absence of spectral singularities;
- the simplicity of all discrete eigenvalues;
- at $g = \mathrm{i}B_R$, the $\det(\mathbf{v}, \mathbf{u})$ lacks multiple zeros (it is recalled that due to symmetries, an analogous condition applies at $g = -\mathrm{i}B_R$ for $\det(\widetilde{\mathbf{v}}, \widetilde{\mathbf{u}})$);
- condition $(\boldsymbol{B}_1)$ applies in the generic scenario, whereas $(\boldsymbol{B}_2)$ pertains in the exceptional scenario.

Multiplying Eq. (75a) by $\mathrm{e}^{\mathrm{i}\beta_R y}$ for $y > x$, and then substituting the triangular representations given by Eq. (61a), we derive the following result:

$$\begin{aligned}\left[\frac{\mathrm{e}^{\mathrm{i}\beta_R x}\mathbf{v}(x,g)}{c_{11}(g)} - \mathbf{E}_{R,1}(g)\right]\mathrm{e}^{\mathrm{i}\beta_R(y-x)} &= \int_x^{\infty} \mathbf{Z}(x,h)\mathbf{E}_{R,1}(g)\mathrm{e}^{\mathrm{i}\beta_R(y-h)}\,\mathrm{d}h \\ &\quad + \gamma(g)\left[\mathrm{e}^{\mathrm{i}\beta_R(x+y)}\mathbf{E}_{R,2}(g) + \int_x^{\infty} \mathbf{Z}(x,h)\mathbf{E}_{R,2}(g)\mathrm{e}^{\mathrm{i}\beta_R(y+h)}\,\mathrm{d}h\right],\end{aligned} \tag{76}$$

where $\mathbf{E}_{R,j}(g)$ represents the jth column of the eigenvector matrix $\mathbf{E}_R(g)$ as defined in Eq. (19). It is noted that β_R approaches β_L as $|g| \to \infty$, implying that the term on the left-hand side diminishes as $|g| \to \infty$ within $\boldsymbol{\Lambda}_R^+ \cup \mathbb{R}$. It is advantageous to regard the eigenfunctions as functions of β_R, that is, considering g as a function of β_R, specifically $g = g(\beta_R) \equiv \sqrt{\beta_R^2 - B_R^2}$. Observe that $\beta_R \in \mathbb{R}$ corresponds uniquely to either $g \in \Theta_R^+$ or $g \in \Theta_R^-$. For the subsequent discussion, we shall consider $g \in \Theta_R^+$ for $\mathbf{u}(x,g)$ (which is analytic for $g \in \boldsymbol{\Lambda}_R^+$), and $g \in \Theta_R^-$ for $\widetilde{\mathbf{u}}(x,g)$ (analytic for $g \in \boldsymbol{\Lambda}_R^-$). Subsequently, we formally integrate equation (76) with respect to β_R, interchange the order of integration and proceed with the evaluation:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}1\\ -\frac{\mathrm{i}q_R^*}{\beta_R+g}\end{pmatrix}\mathrm{e}^{\mathrm{i}\beta_R(y-h)}\,\mathrm{d}h = \begin{pmatrix}\theta(y-h)\\ 0\end{pmatrix}, \tag{77}$$

with

$$\theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}hx}\,\mathrm{d}h, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\beta_R(y-s)}\,\mathrm{d}\beta_R = \theta(y-s), \tag{78}$$

to obtain

$$\mathbf{J} = \mathbf{Z}(x,y)\begin{pmatrix}1\\0\end{pmatrix} + K(x+y) + \int_x^{\infty}\mathbf{Z}(x,h)K(h+y)\,\mathrm{d}h, \tag{79}$$

where

$$\mathbf{J} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\mathrm{e}^{\mathrm{i}\beta_R x}\mathbf{v}(x,g)}{c_{11}(g)} - \mathbf{E}_{R,1}(g)\right]\mathrm{e}^{\mathrm{i}\beta_R(y-x)}\,\mathrm{d}\beta_R, \quad K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma(g)\mathbf{E}_{R,2}(g)\mathrm{e}^{\mathrm{i}\beta_R x}\,\mathrm{d}\beta_R. \tag{80}$$

As previously detailed, in the integrals mentioned, $g = g(\beta_R)$ with $g \in \Theta_R^+$. The subsequent objective is to articulate $\mathbf{J}$ in relation to the Marchenko kernel $\mathbf{Z}(x,h)$. It should be recalled that we have posited the discrete eigenvalues $g_1, \ldots, g_M$, associated with the zeros of $c_{11}(g)$ in $\boldsymbol{\Lambda}_R^+$ to be simple. Thereafter, by contemplating the function $k(\beta_R)$ derived from the integrand of $\mathbf{J}$ by excising its poles and considering Eq. (42), we arrive at the following:

$$k(\beta_R) = \mathrm{e}^{\mathrm{i}\beta_R(y-x)}\left[\frac{\mathrm{e}^{\mathrm{i}\beta_R x}\mathbf{v}(x,g)}{c_{11}(g)} - \mathbf{E}_{R,1}(g)\right] - \sum_{m=1}^{M}\frac{\mathrm{e}^{\mathrm{i}\beta_R(g_m)y}F_m\mathbf{u}(x,g_m)}{\beta_R - \beta_R(g_m)}. \tag{81}$$

Given that $k(\beta_R)$ is analytic for $\beta_R \in \mathbb{C}^+$, the application of the residue theorem in conjunction with Jordan's Lemma results in:

$$\mathbf{J} = \mathrm{i}\sum_{m=1}^{M} \mathrm{e}^{\mathrm{i}\beta_R(g_m)y} F_m \mathbf{u}(x, g_m). \tag{82}$$

Considering the triangular representation given by Eq. (61a), the following is obtained:

$$\mathbf{J} = K_a(x+y) + \int_x^\infty \mathbf{Z}(x,h)K_a(h+y)\,\mathrm{d}h, \quad K_a(x) = \mathrm{i}\sum_{m=1}^{M} \mathrm{e}^{\mathrm{i}\beta_R(g_m)x} F_m \mathbf{E}_{R,2}(g_m). \tag{83}$$

Inserting the aforementioned formulation of $\mathbf{J}$ into Eq. (79), we subsequently derive the right Marchenko integral equation:

$$\mathbf{Z}(x,y)\begin{pmatrix}1\\0\end{pmatrix} + \Psi_R(x+y) + \int_x^\infty \mathbf{Z}(x,h)\Psi_R(h+y)\,\mathrm{d}h = \begin{pmatrix}0\\0\end{pmatrix}, \tag{84}$$

where

$$\Psi_R(x) = K(x) - K_a(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma(g)\mathbf{E}_{R,2}(g)\mathrm{e}^{\mathrm{i}\beta_R x}\,\mathrm{d}\beta_R - \mathrm{i}\sum_{m=1}^{M} \mathrm{e}^{\mathrm{i}\beta_R(g_m)x} F_m \mathbf{E}_{R,2}(g_m). \tag{85}$$

It should be noted that in the integral of Eq. (85) $\beta_R \in \mathbb{R}$ and accordingly $g = g(\beta_R) \in \Theta_R^+$.

Subsequently, we multiply Eq. (75c) by $\mathrm{e}^{-\mathrm{i}\beta_R y}$ for $y > x$ and insert Eqs. (61a) and (61b). This leads to the following result:

$$\begin{aligned}\left[\frac{\mathrm{e}^{-\mathrm{i}\beta_R x}\widetilde{\mathbf{v}}(x,g)}{c_{22}(g)} - \mathbf{E}_{R,2}(g)\right]\mathrm{e}^{\mathrm{i}\beta_R(x-y)} &= \int_x^\infty \mathbf{Z}(x,h)\mathbf{E}_{R,2}(g)\mathrm{e}^{\mathrm{i}\beta_R(h-y)}\,\mathrm{d}h \\ &+ \widetilde{\gamma}(g)\left[\mathrm{e}^{-\mathrm{i}\beta_R(x+y)}\mathbf{E}_{R,1}(g) + \int_x^\infty \mathbf{Z}(x,h)\mathbf{E}_{R,1}(g)\mathrm{e}^{-\mathrm{i}\beta_R(y+h)}\,\mathrm{d}h\right].\end{aligned} \tag{86}$$

By formally integrating with respect to β_R and following the previous procedure, we arrive at the subsequent outcome:

$$\widetilde{\mathbf{J}} = \mathbf{Z}(x,y)\begin{pmatrix}0\\1\end{pmatrix} + \widetilde{K}(x+y) + \int_x^\infty \mathbf{Z}(x,h)\widetilde{K}(h+y)\,\mathrm{d}h, \tag{87}$$

where

$$\widetilde{\mathbf{J}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{i}\beta_R x}\widetilde{\mathbf{v}}(x,g)}{c_{22}(g)} - \mathbf{E}_{R,2}(g)\right]\mathrm{e}^{-\mathrm{i}\beta_R(y-x)}\,\mathrm{d}\beta_R, \quad \widetilde{K}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\gamma}(g)\mathbf{E}_{R,1}(g)\mathrm{e}^{-\mathrm{i}\beta_R x}\,\mathrm{d}\beta_R. \tag{88}$$

For the integrals mentioned above, it is assumed that $g = g(\beta_R) \in \Theta_R^-$. Considering Eq. (43), in a manner analogous to the previous case, $\widetilde{\mathbf{J}}$ can be represented in terms of the Marchenko kernel:

$$\widetilde{\mathbf{J}} = \widetilde{K}_a(x+y) + \int_x^\infty \mathbf{Z}(x,h)\widetilde{K}_a(h+y)\,\mathrm{d}h, \quad \widetilde{K}_a(x) = -\mathrm{i}\sum_{m=1}^{M} \mathrm{e}^{-\mathrm{i}\beta_R(g_m^*)x}\widetilde{F}_m \mathbf{E}_{R,1}(g_m^*), \tag{89}$$

we have

$$\mathbf{Z}(x,y)\begin{pmatrix}0\\1\end{pmatrix} + \widetilde{\Psi}_R(x+y) + \int_x^\infty \mathbf{Z}(x,h)\widetilde{\Psi}_R(h+y)\,\mathrm{d}h = \begin{pmatrix}0\\0\end{pmatrix}, \tag{90}$$

where

$$\widetilde{\Psi}_R(x) = \widetilde{K}(x) - \widetilde{K}_a(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\gamma}(g)\mathbf{E}_{R,1}(g)\mathrm{e}^{-\mathrm{i}\beta_R x}\,\mathrm{d}\beta_R + \mathrm{i}\sum_{m=1}^{M} \mathrm{e}^{-\mathrm{i}\beta_R(g_m^*)x}\widetilde{F}_m \mathbf{E}_{R,1}(g_m^*). \tag{91}$$

Observe that within the integral of Eq. (91), $\beta_R \in \mathbb{R}$ and accordingly $g = g(\beta_R) \in \Theta_R^-$.

Consequently, due to the symmetries inherent in the scattering data, it follows that: $K^*(x) = \mathrm{i}\sigma_2\widetilde{K}(x)$, $K_a^*(x) = \mathrm{i}\sigma_2\widetilde{K}_a(x)$ and $\Psi_R^*(x) = \mathrm{i}\sigma_2\widetilde{\Psi}_R(x)$. In summary, the Marchenko equations (84) and (90) can be consolidated into a singular 2×2 Marchenko equation, featuring a 2×2 Marchenko kernel presented as follows:

$$\mathbf{Z}(x,y) + \mathbf{\Psi}_R(x+y) + \int_x^\infty \mathbf{Z}(x,h)\mathbf{\Psi}_R(h+y)\,\mathrm{d}h = \mathbf{0}_{2\times 2}, \tag{92}$$

where $\mathbf{\Psi}_R(x) = (\Psi_R(x), \widetilde{\Psi}_R(x))$, with $\Psi_R(x)$, $\widetilde{\Psi}_R(x)$ are defined by Eqs. (85) and (91), respectively, and they adhere to the relationship $\widetilde{\Psi}_R(x) = -\mathrm{i}\sigma_2\Psi_R^*(x)$. It is observed that $\mathbf{\Psi}_R^*(x) = \sigma_2\mathbf{\Psi}_R(x)\sigma_2$, consistent with $\mathbf{Z}^*(x,h) = \sigma_2\mathbf{Z}(x,h)\sigma_2$.

3.3. Left Marchenko equations

To derive the left Marchenko equations, we explicitly express Eq. (25b) as follows:

$$\frac{\widetilde{\mathbf{u}}(x,g)}{h_{11}(g)} = \mathbf{v}(x,g) + \widetilde{\alpha}(g)\widetilde{\mathbf{v}}(x,g), \quad g \in \mathbb{R}, \tag{93a}$$

$$\frac{\widetilde{\mathbf{u}}^{\pm}(x,g)}{h_{11}^{\pm}(g)} = \mathbf{v}^{\pm}(x,g) + \widetilde{\alpha}^{\pm}(g)\widetilde{\mathbf{v}}^{\pm}(x,g), \quad g \in (-\mathrm{i}B_L, \mathrm{i}B_L), \tag{93b}$$

$$\frac{\mathbf{u}(x,g)}{h_{22}(g)} = \widetilde{\mathbf{v}}(x,g) + \alpha(g)\mathbf{v}(x,g), \quad g \in \mathbb{R}, \tag{93c}$$

$$\frac{\mathbf{u}^{\pm}(x,g)}{h_{22}^{\pm}(g)} = \widetilde{\mathbf{v}}^{\pm}(x,g) + \alpha^{\pm}(g)\mathbf{v}^{\pm}(x,g), \quad g \in (-\mathrm{i}B_L, \mathrm{i}B_L), \tag{93d}$$

where $\alpha(g)$, $\alpha^{\pm}(g)$ and $\widetilde{\alpha}(g)$, $\widetilde{\alpha}^{\pm}(g)$ are specified by Eqs. (32c) and (32d), respectively. With analogous assumptions concerning the potential and the discrete spectrum, and by considering the eigenfunctions as functions of β_L, where $g = g(\beta_L) = \sqrt{\beta_L^2 - B_L^2}$. We proceed by multiplying Eq. (93c) by $\mathrm{e}^{-\mathrm{i}\beta_L y}$ for $y < x$ and substituting the triangular representations as given in Eq. (61b). Subsequently, we formally integrate with respect to β_L and interchange the order of integration, leading to the following result:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{i}\beta_L x}\mathbf{u}(x,g)}{h_{22}(g)} - \mathbf{E}_{L,2}(g)\right]\mathrm{e}^{\mathrm{i}\beta_L(x-y)}\,\mathrm{d}\beta_L &= \frac{1}{2\pi}\int_{-\infty}^{x}\mathbf{P}(x,h)\,\mathrm{d}h\int_{-\infty}^{\infty}\mathbf{E}_{L,2}(g)\mathrm{e}^{\mathrm{i}\beta_L(h-y)}\,\mathrm{d}\beta_L \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha(g)\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L(x+y)}\,\mathrm{d}\beta_L &+ \frac{1}{2\pi}\int_{-\infty}^{x}\mathbf{P}(x,h)\,\mathrm{d}h\int_{-\infty}^{\infty}\alpha(g)\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L(h+y)}\,\mathrm{d}\beta_L,\end{aligned} \tag{94}$$

where $\mathbf{E}_{L,j}(g)$ represents the jth column of the matrix of asymptotic eigenvectors $\mathbf{E}_L(g)$. As previously mentioned, $\beta_L \in \mathbb{R}$ corresponds uniquely to either $g \in \Theta_L^+$ or $g \in \Theta_L^-$. We consider $g \in \Theta_L^+$ for $\mathbf{v}(x,g)$ [which is analytic for $g \in \mathbf{\Lambda}_L^+$], and $g \in \Theta_L^-$ for $\widetilde{\mathbf{v}}(x,g)$ [analytic for $g \in \mathbf{\Lambda}_L^-$]. As in the previous case, we can simplify the identity (94) to:

$$\widehat{\mathbf{J}} = \mathbf{P}(x,y)\begin{pmatrix}0\\1\end{pmatrix} + N(x+y) + \int_{-\infty}^{x}\mathbf{P}(x,h)N(h+y)\,\mathrm{d}h, \tag{95}$$

where

$$\widehat{\mathbf{J}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{i}\beta_L x}\mathbf{u}(x,g)}{h_{22}(g)} - \mathbf{E}_{L,2}(g)\right]\mathrm{e}^{\mathrm{i}\beta_L(x-y)}\,\mathrm{d}\beta_L, \quad N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha(g)\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L x}\,\mathrm{d}\beta_L. \tag{96}$$

To calculate $\widehat{\mathbf{J}}$ and represent it in terms of the Marchenko kernel $\mathbf{P}(x,y)$, it is necessary to close the contour at infinity in the upper half-plane of β_L. Contrary to the scenario with the right Marchenko equations, closing the contour here involves accounting for the additional branch cut, which in the g-plane corresponds to $\Xi_R \backslash \Xi_L$. Towards this objective, for $0 < \varepsilon < T < +\infty$, let us examine the closed contour $\Theta(T,\varepsilon)$ composed of the following segments, with the indicated orientation: (1) the interval $[-T,-\varepsilon]$; (2) the segment from $[-\varepsilon + \mathrm{i}0, -\varepsilon + \mathrm{i}\Gamma]$ along the imaginary β_L axis; (3) the semicircle $\{\mathrm{i}\Gamma + \varepsilon \mathrm{e}^{\mathrm{i}[\pi-\delta]} : 0 \le \delta \le \pi\}$ oriented clockwise; (4) the segment from $[\varepsilon + \mathrm{i}0, \varepsilon + \mathrm{i}\Gamma]$ along the imaginary β_L axis; (5) the interval $[\varepsilon, T]$; (6) the semicircle $\{T\mathrm{e}^{\mathrm{i}\delta} : 0 \le \delta \le \pi\}$ oriented counterclockwise. We establish the notation $\Gamma = \sqrt{B_R^2 - B_L^2}$. It is presumed that T is sufficiently large and ε is sufficiently small such that all discrete eigenvalues $g_m \in \mathbf{\Lambda}_L^+$ for $m = 1, 2, \ldots, M$, and their corresponding $\beta_L(g_m)$ lie within the interior of the specified contour. Given that $\mathbf{u}(x,g)$ and $1/h_{22}(g)$ approach finite limits as $g \to \mathrm{i}B_R$, the contribution to the integral defining $\widehat{\mathbf{J}}$ from the semicircle encircling the branch point vanishes as $\varepsilon \to 0^+$. Due to Jordan's lemma, the integral that defines $\widehat{\mathbf{J}}$ over the large semicircle (6) also yields no contribution as $T \to +\infty$ (note that $\beta_L \sim \beta_R$ as $g \to \infty$, which ensures the integrand tends to zero as $g \to \infty$). Consequently, there are two significant contributions to the integral $\widehat{\mathbf{J}} = \widehat{\mathbf{J}}_1 + \widehat{\mathbf{J}}_2$: the contribution $\widehat{\mathbf{J}}_1$ related to the residues of the integrand at the poles $g \in \mathbf{\Lambda}_R^+$, and the contribution $\widehat{\mathbf{J}}_2$ associated with the integral around $\beta_L \in [0, \mathrm{i}\Gamma]$ in the upper-half β_L-plane. These two contributions will be assessed individually.

Given our assumption that the discrete eigenvalues g_m in $\mathbf{\Lambda}_R^+$ are simple poles of $1/h_{22}(g)$, and considering that the reflection and transmission coefficients are continuous for $g \in \partial\mathbf{\Lambda}_R^+$, along with the fact that $\mathbf{u}(x,g_m) = \mathbf{v}(x,g_m)/d_m$, we derive the following:

$$\widehat{\mathbf{J}}_1 = \mathrm{i}\sum_{m=1}^{M}\mathrm{e}^{-\mathrm{i}\beta_L(g_m)y}\widetilde{F}_m\mathbf{v}(x,g_m), \quad \widetilde{F}_m = \frac{\widetilde{\omega}_m}{d_m}, \tag{97}$$

where $\widetilde{\omega}_m$ denotes the residue of $1/h_{22}(g)$ at $\beta_L = \beta_L(g_m)$ and $\widetilde{F}_m$ is the corresponding norming constant. It is noted that this implies a relationship between the residues $\widetilde{\omega}_m$ and ω_m, and consequently between the norming constants $\widetilde{F}_m$ and F_m as follows:

$$\widetilde{\omega}_m = \frac{\beta_R(g_m) + g_m}{\beta_L(g_m) + g_m}\omega_m, \quad \widetilde{F}_m F_m = \omega_m^2\frac{\beta_R(g_m) + g_m}{\beta_L(g_m) + g_m}. \tag{98}$$

Consequently, we arrive at the following conclusion:

$$\widehat{\mathbf{J}}_1 = N_1(x+y) + \int_{-\infty}^{x}\mathbf{P}(x,h)N_1(h+y)\,\mathrm{d}h, \quad N_1(x) = \mathrm{i}\sum_{m=1}^{M}\mathrm{e}^{-\mathrm{i}\beta_L(g_m)x}\widetilde{F}_m\mathbf{E}_{L,1}(g_m). \tag{99}$$

We now turn our attention to the second contribution $\widehat{\mathbf{J}}_2$, which emerges for $\beta_L \in [0, \mathrm{i}\Gamma]$, corresponding to $g \in [\mathrm{i}B_L, \mathrm{i}B_R]$ and $\beta_R \in \mathbb{R}$.

$$\widehat{\mathbf{J}}_2 = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\left(\int_{\mathrm{i}0-\varepsilon}^{\mathrm{i}\Gamma-\varepsilon} - \int_{\mathrm{i}0+\varepsilon}^{\mathrm{i}\Gamma+\varepsilon}\right)\left[\frac{\mathrm{e}^{-\mathrm{i}\beta_L x}\mathbf{u}(x,g)}{h_{22}(g)} - \mathbf{E}_{L,2}(g)\right]\mathrm{e}^{\mathrm{i}\beta_L(x-y)}\,\mathrm{d}\beta_L = \frac{1}{2\pi}\int_0^{\mathrm{i}\Gamma}\left[\frac{\mathbf{u}^-(x,g)}{h_{22}^-(g)} - \frac{\mathbf{u}^+(x,g)}{h_{22}^+(g)}\right]\mathrm{e}^{-\mathrm{i}\beta_L y}\,\mathrm{d}\beta_L. \tag{100}$$

In the integral along the cut $[0, \mathrm{i}\Gamma]$ on the positive imaginary axis, the superscripts $\pm$ conventionally represent the limiting values from the right/left side of the cut, respectively. It is also noted that β_L and consequently $\mathbf{E}_{L,1}$ are continuous across this cut. Utilizing Eq. (28c), $\widehat{\mathbf{J}}_2$ can be expressed as follows:

$$\widehat{\mathbf{J}}_2 = \frac{1}{2\pi}\int_0^{\mathrm{i}\Gamma}\left[\frac{|\beta_L| - g}{|\beta_L|}\frac{\mathbf{u}^-(x,g)}{c_{11}^-(g)} - \frac{|\beta_R| + g}{|\beta_R|}\frac{\mathbf{u}^+(x,g)}{c_{11}^+(g)}\right]\frac{\beta_L}{\beta_L + g}\mathrm{e}^{-\mathrm{i}\beta_L y}\,\mathrm{d}\beta_L, \tag{101}$$

the symmetry relations given by Eqs. (37) and (40) facilitate the expression of:

$$\frac{(|\beta_R|-g)\mathbf{u}^-(x,g)}{c_{11}^-(g)}=-\frac{(|\beta_R|+g)\widetilde{\mathbf{u}}^+(x,g)}{c_{21}^+(g)}. \tag{102}$$

Consequently,

$$\widehat{\mathbf{J}}_2=-\frac{1}{2\pi}\int_0^{\mathrm{i}\Gamma}\frac{\beta_L(|\beta_R|+g)}{|\beta_R|(\beta_L+g)}\left[\frac{\widetilde{\mathbf{u}}^+(x,g)}{c_{21}^+(g)}+\frac{\mathbf{u}^+(x,g)}{c_{11}^+(g)}\right]\mathrm{e}^{-\mathrm{i}\beta_L y}\,\mathrm{d}\beta_L. \tag{103}$$

Initially employing the scattering equation (75b), and subsequently reapplying the symmetry relation (40), we ultimately arrive at:

$$\widehat{\mathbf{J}}_2=-\frac{1}{2\pi}\int_0^{\mathrm{i}\Gamma}\frac{\beta_L(|\beta_R|+g)\mathbf{v}^+(x,g)\mathrm{e}^{-\mathrm{i}\beta_L y}}{|\beta_R|(\beta_L+g)c_{11}^+(g)c_{21}^+(g)}\,\mathrm{d}\beta_L=\frac{\mathrm{i}q_R}{2\pi}\int_0^{\mathrm{i}\Gamma}\frac{\beta_L\mathbf{v}^+(x,g)\mathrm{e}^{-\mathrm{i}\beta_L y}}{|\beta_R|(\beta_L+g)c_{11}^+(g)c_{11}^-(g)}\,\mathrm{d}\beta_L. \tag{104}$$

Substituting the triangular representation (61a) into the aforementioned expression yields:

$$\widehat{\mathbf{J}}_2=N_2(x+y)+\int_{-\infty}^{x}\mathbf{P}(x,h)N_2(h+y)\,\mathrm{d}h,\quad N_2(x)=\frac{\mathrm{i}q_R}{2\pi}\int_0^{\mathrm{i}\Gamma}\frac{\beta_L\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L x}}{|\beta_R|(\beta_L+g)c_{11}^+(g)c_{11}^-(g)}\,\mathrm{d}\beta_L. \tag{105}$$

Upon defining

$$\begin{aligned}\Psi_L(x)=N(x)-N_1(x)-N_2(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha(g)\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L x}\,\mathrm{d}\beta_L-\mathrm{i}\sum_{m=1}^{M}\mathrm{e}^{-\mathrm{i}\beta_L(g_m)x}\widetilde{F}_m\mathbf{E}_{L,1}(g_m)\\-\frac{\mathrm{i}q_R}{2\pi}\int_0^{\mathrm{i}\Gamma}\frac{\beta_L\mathbf{E}_{L,1}(g)\mathrm{e}^{-\mathrm{i}\beta_L x}}{|\beta_R|(\beta_L+g)c_{11}^+(g)c_{11}^-(g)}\,\mathrm{d}\beta_L,\end{aligned} \tag{106}$$

in the first integral, where $\beta_L\in\mathbb{R}$, it follows that $g=g(\beta_L)\in\Theta_L^+$. Utilizing Eqs. (99) and (105) to evaluate $\widehat{\mathbf{J}}=\widehat{\mathbf{J}}_1+\widehat{\mathbf{J}}_2$ and incorporating it into Eq. (95), we ultimately derive the left Marchenko integral equation:

$$\mathbf{P}(x,y)\begin{pmatrix}0\\1\end{pmatrix}+\Psi_L(x+y)+\int_{-\infty}^{x}\mathbf{P}(x,h)\Psi_L(h+y)\,\mathrm{d}h=\begin{pmatrix}0\\0\end{pmatrix}. \tag{107}$$

Following a similar procedure, commencing from Eq. (93c), the "adjoint" left Marchenko equation can be deduced:

$$\mathbf{P}(x,y)\begin{pmatrix}1\\0\end{pmatrix}+\widetilde{\Psi}_L(x+y)+\int_{-\infty}^{x}\mathbf{P}(x,h)\widetilde{\Psi}_L(h+y)\,\mathrm{d}h=\begin{pmatrix}0\\0\end{pmatrix}, \tag{108}$$

where $\widetilde{\Psi}_L(x)=\mathrm{i}\sigma_2\Psi_L^*(x)$. The pair of Marchenko equations can be succinctly expressed in matrix notation as follows:

$$\mathbf{P}(x,y)+\mathbf{\Psi}_L(x+y)+\int_{-\infty}^{x}\mathbf{P}(x,h)\mathbf{\Psi}_L(h+y)\,\mathrm{d}h=\mathbf{0}_{2\times2}, \tag{109}$$

where $\mathbf{\Psi}_L(x)=(\widetilde{\Psi}_L(x),\Psi_L(x))$ and $\mathbf{\Psi}_L^*(x)=\sigma_2\mathbf{\Psi}_L(x)\sigma_2$.

It is pertinent to offer some observations regarding the left and right Marchenko integral equations (92) and (109) obtained above. Initially, it should be noted that the disparity between the left and right Marchenko equations stems from the asymmetry of the boundary conditions, specifically due to the selection $B_R\geq B_L$ (with the roles of the two integral equations being reversed should $B_R\leq B_L$ be chosen). In the left Marchenko integral equations, $\mathbf{\Psi}_L(x)$ in Eq. (106) comprises three distinct contributions: the first from the discrete spectrum, the second from the reflection coefficients from the left, $\alpha(g)$ and $\widetilde{\alpha}(g)$, integrated over g in the continuous spectrum, that is $g\in\mathbb{R}\cup\Xi_L$ and a third contribution (often termed the dispersive shock wave contribution) that involves an integral over imaginary values of β_L where the product of transmission coefficients $1/c_{11}^+(g)c_{11}^-(g)$ is present. In contrast, $\mathbf{\Psi}_R(x)$ in the right Marchenko integral equations (see Eq. (85)) consists of only two contributions: one from the discrete spectrum and the other from the reflection coefficients from the right, $\gamma(g)$ and $\widetilde{\gamma}(g)$. It is important to observe that in the latter the reflection coefficients are integrated over the entire range of $\beta_R\in\mathbb{R}$. This implies that the integral encompasses not only the continuous spectrum $\mathbb{R}\cup\Xi_L$ but also contributions from $\Xi_R\backslash\Xi_L$. Furthermore, the integrand over $\Xi_R\backslash\Xi_L$ is never identically zero. According to Eq. (41a), in the absence of spectral singularities, $\gamma^\pm(g)\neq0$ for all $g\in[\mathrm{i}B_L,\mathrm{i}B_R]$, and $\widetilde{\gamma}^\pm(g)\neq0$ for all $g\in[-\mathrm{i}B_R,-\mathrm{i}B_L]$. Consequently, when $\Xi_R\backslash\Xi_L\neq\emptyset$ (that is, in scenarios involving asymmetric boundary conditions where $B_R\neq B_L$), pure soliton solutions cannot exist.

4. Riemann-Hilbert problem and time evolution

4.1. Riemann-Hilbert problem

The objective of this section is to present an alternative approach to the inverse problem, formulated as the RH problem for the eigenfunctions, where the discontinuities are characterized by the scattering data. Upon solving the RH problem, the asymptotic expansion of the eigenfunctions for large g facilitates the reconstruction of the potential. We examine the following matrix composed of eigenfunctions:

$$\mathbf{M}(x,g)=\begin{cases}\left[\frac{\mathbf{v}(x,g)}{c_{11}(g)}\mathrm{e}^{\mathrm{i}\beta_L x},\mathbf{u}(x,g)\mathrm{e}^{-\mathrm{i}\beta_R x}\right], & g\in\mathbf{\Lambda}_R^+,\\ \left[\widetilde{\mathbf{u}}(x,g)\mathrm{e}^{\mathrm{i}\beta_R x},\frac{\widetilde{\mathbf{v}}(x,g)}{c_{22}(g)}\mathrm{e}^{-\mathrm{i}\beta_L x}\right], & g\in\mathbf{\Lambda}_R^-,\end{cases} \tag{110}$$

the matrix $\mathbf{M}(x,g)$ is constructed such that it approaches the 2×2 identity matrix $\mathbf{I}_2$ as g tends to infinity. The inverse problem is then formulated as the RH problem for the sectionally meromorphic matrix $\mathbf{M}(x,g)$ across $\partial\mathbf{\Lambda}_R^+\cup\mathbf{\Lambda}_R^-$. Specifically, we identify the five jump matrices: $\mathbf{O}_0$ represents the jump matrix across the real axis of the complex g-plane; $\mathbf{O}_1$ is across $\Xi_L^+=[0,\mathrm{i}B_L]$; $\mathbf{O}_2$ is across $\Xi_L^-=[-\mathrm{i}B_L,0)$; $\mathbf{O}_3$ is across $\Xi_R^+\backslash\Xi_L^+=(\mathrm{i}B_L,\mathrm{i}B_R]$, and $\mathbf{O}_4$ is across $\Xi_R^-\backslash\Xi_L^-=[-\mathrm{i}B_R,-\mathrm{i}B_L)$. Each of the jump matrices is dependent on g along the respective contour in the complex plane, and also, in a parametric sense, on $(x,t)\in\mathbb{R}\times\mathbb{R}^+$ (where the dependence on x is explicit, whereas the dependency on time is implicitly contained within the corresponding reflection coefficients).

The RH problem defined across $g\in\mathbb{R}$ is represented as: $\mathbf{M}^+(x,g)=\mathbf{M}^-(x,g)\mathbf{O}_0(x,g)$ which signifies that:

$$\left[\frac{\mathbf{v}^+(x,g)}{c_{11}^+(g)}\mathrm{e}^{\mathrm{i}\beta_L x},\mathbf{u}^+(x,g)\mathrm{e}^{-\mathrm{i}\beta_R x}\right]=\left[\widetilde{\mathbf{u}}^-(x,g)\mathrm{e}^{\mathrm{i}\beta_R x},\frac{\widetilde{\mathbf{v}}^-(x,g)}{c_{22}^-(g)}\mathrm{e}^{-\mathrm{i}\beta_L x}\right]\mathbf{O}_0(x,g),\quad g\in\mathbb{R},\tag{111}$$

where the superscripts $\pm$ represent the limiting values from the upper/lower halves of the complex plane, respectively. The jump matrix across the real axis can be readily determined from Eq. (25a) and is expressed as:

$$\mathbf{O}_0(x,g)=\begin{pmatrix}[1-\gamma(g)\widetilde{\gamma}(g)]\mathrm{e}^{\mathrm{i}(\beta_L-\beta_R)x} & [-\widetilde{\gamma}(g)\mathrm{e}^{-2\mathrm{i}\beta_R x}]\\ \gamma(g)\mathrm{e}^{2\mathrm{i}\beta_L x} & \mathrm{e}^{\mathrm{i}(\beta_L-\beta_R)x}\end{pmatrix}.\tag{112}$$

The RH problem across Ξ_L^+ is formulated as: $\mathbf{M}^+(x,g)=\mathbf{M}^-(x,g)\mathbf{O}_1(x,g)$ for $g\in\mathbb{C}^+$, where the superscripts $\pm$ now indicate the limiting values from the right/left side of the cut across Ξ_L^+ (Ξ_L in the upper half-plane). Considering that across Ξ_L both β_L and β_R change signs, and employing the notation $\beta_{L/R}^+=-\beta_{L/R}^-=\beta_{L/R}$, we obtain:

$$\left[\frac{\mathbf{v}^+(x,g)}{c_{11}^+(g)}\mathrm{e}^{\mathrm{i}\beta_L^+ x},\mathbf{u}^+(x,g)\mathrm{e}^{-\mathrm{i}\beta_R^+ x}\right]=\left[\frac{\mathbf{v}^-(x,g)}{c_{11}^-(g)}\mathrm{e}^{\mathrm{i}\beta_L^- x},\mathbf{u}^-(x,g)\mathrm{e}^{-\mathrm{i}\beta_R^- x}\right]\mathbf{O}_1(x,g),\tag{113}$$

where the jump matrix $\mathbf{O}_1(x,g)$ can be readily calculated utilizing Eq. (25a) along with the symmetry relations given by Eq. (37), resulting in:

$$\mathbf{O}_1(x,g)=-\frac{\mathrm{i}q_R}{\beta_R+g}\begin{pmatrix}\gamma^+(g)\mathrm{e}^{2\mathrm{i}\beta_L x} & \mathrm{e}^{\mathrm{i}(\beta_L-\beta_R)x}\\ \left[\frac{q_R^*}{q_R}-\gamma^+(g)\gamma^-(g)\right]\mathrm{e}^{\mathrm{i}(\beta_L-\beta_R)x} & -\gamma^-(g)\mathrm{e}^{-2\mathrm{i}\beta_R x}\end{pmatrix}.\tag{114}$$

The RH problem across Ξ_L^- is expressed as $\mathbf{M}^+(x,g)=\mathbf{M}^-(x,g)\mathbf{O}_2(x,g)$ for $g\in\mathbb{C}^-$, where the superscripts $\pm$ indicate the non-tangential limits from the right/left side of the cut across Ξ_L^-, that is, Ξ_L in the lower half-plane. Specifically, we have:

$$\left[\widetilde{\mathbf{u}}^+(x,g)\mathrm{e}^{\mathrm{i}\beta_R^+ x},\frac{\widetilde{\mathbf{v}}^+(x,g)}{c_{22}^+(g)}\mathrm{e}^{-\mathrm{i}\beta_L^+ x}\right]=\left[\widetilde{\mathbf{u}}^-(x,g)\mathrm{e}^{\mathrm{i}\beta_R^- x},\frac{\widetilde{\mathbf{v}}^-(x,g)}{c_{22}^-(g)}\mathrm{e}^{-\mathrm{i}\beta_L^- x}\right]\mathbf{O}_2(x,g).\tag{115}$$

Similar to the previous cases, the jump matrix can be ascertained by utilizing Eqs. (25a) and (37), and is presented as follows:

$$\mathbf{O}_2(x,g)=-\frac{\mathrm{i}q_R^*}{\beta_R+g}\begin{pmatrix}-\widetilde{\gamma}^-(g)\mathrm{e}^{2\mathrm{i}\beta_R x} & \left[\frac{q_R}{q_R^*}-\widetilde{\gamma}^+(g)\widetilde{\gamma}^-(g)\right]\mathrm{e}^{\mathrm{i}(\beta_R-\beta_L)x}\\ \mathrm{e}^{\mathrm{i}(\beta_R-\beta_L)x} & \widetilde{\gamma}^+(g)\mathrm{e}^{-2\mathrm{i}\beta_L x}\end{pmatrix}.\tag{116}$$

The RH problem across $\Xi_R^+\backslash\Xi_L^+$ is formulated as $\mathbf{M}^+(x,g)=\mathbf{M}^-(x,g)\mathbf{O}_3(x,g)$ for $g\in\mathbb{C}^+$:

$$\left[\frac{\mathbf{v}^+(x,g)}{c_{11}^+(g)}\mathrm{e}^{\mathrm{i}\beta_L x},\mathbf{u}^+(x,g)\mathrm{e}^{-\mathrm{i}\beta_R^+ x}\right]=\left[\frac{\mathbf{v}^-(x,g)}{c_{11}^-(g)}\mathrm{e}^{\mathrm{i}\beta_L x},\mathbf{u}^-(x,g)\mathrm{e}^{-\mathrm{i}\beta_R^- x}\right]\mathbf{O}_3(x,g).\tag{117}$$

Considering that β_R changes sign while β_L and $\mathbf{v}(x,g)$ remain continuous, by applying Eqs. (25a) and (37), the following is derived:

$$\mathbf{O}_3(x,g)=-\frac{\mathrm{i}q_R}{\beta_R+g}\begin{pmatrix}\gamma^+(g) & \mathrm{e}^{-\mathrm{i}(\beta_L+\beta_R)x}\\ \left[\frac{q_R^*}{q_R}-\gamma^+(g)\gamma^-(g)\right]\mathrm{e}^{\mathrm{i}(\beta_L-\beta_R)x} & -\gamma^-(g)\mathrm{e}^{-2\mathrm{i}\beta_R x}\end{pmatrix}.\tag{118}$$

The symmetry given by Eq. (41a) ultimately leads to:

$$\mathbf{O}_3(x,g)=-\frac{\mathrm{i}q_R}{\beta_R+g}\begin{pmatrix}\gamma^+(g) & \mathrm{e}^{-\mathrm{i}(\beta_L+\beta_R)x}\\ 0 & -\frac{q_R^*}{q_R\gamma^+(g)}\mathrm{e}^{-2\mathrm{i}\beta_R x}\end{pmatrix}.\tag{119}$$

Lastly, the RH problem across $\Xi_R^-\backslash\Xi_L^-$, where β_R changes sign while β_L and $\widetilde{\mathbf{v}}(x,g)$ remain continuous, is expressed as $\mathbf{M}^+(x,g)=\mathbf{M}^-(x,g)\mathbf{O}_4(x,g)$ for $g\in\mathbb{C}^-$:

$$\left[\widetilde{\mathbf{u}}^+(x,g)\mathrm{e}^{\mathrm{i}\beta_R^+ x},\frac{\widetilde{\mathbf{v}}^+(x,g)}{c_{22}^+(g)}\mathrm{e}^{-\mathrm{i}\beta_L x}\right]=\left[\widetilde{\mathbf{u}}^-(x,g)\mathrm{e}^{\mathrm{i}\beta_R^- x},\frac{\widetilde{\mathbf{v}}^-(x,g)}{c_{22}^-(g)}\mathrm{e}^{-\mathrm{i}\beta_L x}\right]\mathbf{O}_4(x,g),\tag{120}$$

where

$$\mathbf{O}_4(x,g)=-\frac{\mathrm{i}q_R^*}{\beta_R+g}\begin{pmatrix}-\widetilde{\gamma}^-(g)\mathrm{e}^{2\mathrm{i}\beta_R x} & 0\\ \mathrm{e}^{\mathrm{i}(\beta_L+\beta_R)x} & \frac{q_R}{q_R^*\widetilde{\gamma}^-(g)}\end{pmatrix}.\tag{121}$$

It should be noted that the jump matrices adhere to the following symmetry in the upper/lower half-planes: $\mathbf{O}_2(x,g)=\sigma_2\mathbf{O}_1^*(x,g^*)\sigma_2$ and $\mathbf{O}_4(x,g)=\sigma_2\mathbf{O}_3^*(x,g^*)\sigma_2$.

Addressing the inverse problem as the RH problem, which includes poles corresponding to the zeros of $c_{11}(g)$ and $c_{22}(g)$ in the upper/lower half-planes, entails calculating the sectionally meromorphic matrix $\mathbf{M}(x,g)$ with the specified discontinuities and normalized to the identity matrix as g approaches infinity. Specifically, the problem can be formulated as $\mathbf{M}^{+}(x,g) = \mathbf{M}^{-}(x,g) + [\mathbf{O}(x,g) - \mathbf{I}_2]\mathbf{M}^{-}(x,g)$, where $\mathbf{O}(x,g) = \mathbf{O}_j(x,g)$ for $j = 0,\dots,4$ depending on the segment of the contour under consideration, and the superscripts $\pm$ indicate non-tangential limits from either side of the contour. Subsequently, by subtracting the asymptotic behavior as $g \to \infty$ and the residues of $\mathbf{M}^{\pm}$ at the poles in $\boldsymbol{\Lambda}_R^{\pm}$ from both sides, we derive:

$$\mathbf{M}^{+} - \mathbf{I}_2 - \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m} \mathbf{M}^{+}}{g - g_m} - \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m^*} \mathbf{M}^{-}}{g - g_m^*} = \mathbf{M}^{-} - \mathbf{I}_2 - \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m} \mathbf{M}^{+}}{g - g_m} - \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m^*} \mathbf{M}^{-}}{g - g_m^*} + (\mathbf{O} - \mathbf{I}_2)\mathbf{M}^{-}. \tag{122}$$

The left-hand side of the aforementioned equation is now analytic within $\boldsymbol{\Lambda}_R^{+}$ and is of order $O(g^{-1})$ as g approaches infinity therein. Meanwhile, the sum of all terms except the last on the right-hand side is analytic in $\boldsymbol{\Lambda}_R^{-}$ and is also of order $O(g^{-1})$ as g tends to infinity in that region. Consequently, we introduce projectors $A_{\pm}$ over $\Theta_R^{\pm} = \mathbb{R} \cup \Xi_R^{\pm}$:

$$A_{\pm}[m](g) = \frac{1}{2\pi \mathrm{i}} \int_{\Theta_R^{\pm}} \frac{m(\xi)}{\xi - g}\,\mathrm{d}\xi, \tag{123}$$

where $\int_{\Theta_R^{+}}$ [resp. $\int_{\Theta_R^{-}}$] signifies the integral along the oriented contours, and when $g \in \Theta_R^{\pm} \cap \mathbb{R}$, the limit is taken from above/below respectively. It can be readily demonstrated that if $m^{\pm}$ are analytic in $\boldsymbol{\Lambda}_R^{\pm}$ and are $O(g^{-1})$ as $g \to \infty$, then the following relations hold: $A_{\pm}m^{\pm} = \pm m^{\pm}$ and $A_{+}m^{-} = A_{-}m^{+} = 0$. Subsequently, by applying $A_{\pm}$ to both sides of Eq. (122), we obtain:

$$\mathbf{M}(g) = \mathbf{I}_2 + \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m} \mathbf{M}^{+}}{g - g_m} + \sum_{m=1}^{M} \frac{\operatorname{Res}_{g_m^*} \mathbf{M}^{-}}{g - g_m^*} + \frac{1}{2\pi \mathrm{i}} \int_{\Theta_R^{\pm}} \frac{\mathbf{M}^{-}(\xi)}{\xi - g}[\mathbf{O}(\xi) - \mathbf{I}_2]\,\mathrm{d}\xi, \quad g \in \mathbb{C}^{\pm} \backslash \Xi_R, \tag{124}$$

in the expressions for the eigenfunctions and jump matrices, the x-dependence has been omitted for conciseness. Considering that the second column of $\operatorname{Res}_{g_m} \mathbf{M}^{+}$ is zero for all m, while the first column is proportional to the second column of $\mathbf{M}^{+}(x, g_m)$, and conversely, the first column of $\operatorname{Res}_{g_m^*} \mathbf{M}^{-}$ is zero for all m, while the second column is proportional to the second column of $\mathbf{M}^{-}(x, g_m^*)$ as per Eqs. (42) and (43), the integral/algebraic system can be completed by evaluating it at each $g = g_m$ and $g = g_m^*$. The potential is subsequently reconstructed through the large g expansion of the resulting expressions, since

$$\mathbf{M}(x,g) = \left[\mathbf{I}_2 + \frac{\mathrm{i}}{2g}\mathbf{Q}(x)\sigma_3\right][1 + o(1)]. \tag{125}$$

It should be noted that, in contrast to the scenario with equal amplitudes, the aforementioned system cannot be simplified to a purely algebraic form. Even though the reflection coefficients can be set to zero across the continuous spectrum, that is, for $g \in \mathbb{R} \cup \Xi_L$, the integrals on the right-hand side of as per Eq. (124) always have a non-zero contribution from the contours $\Xi_R^{\pm} \backslash \Xi_L^{\pm}$. Specifically, this indicates that pure soliton solutions do not exist, and solitons are invariably accompanied by some form of radiative contribution. Nevertheless, one could approach the solution of the system iteratively, under the assumption that the reflection coefficients are minor for $g \in \Xi_L^{\pm}$ (and/or for $g \in \Xi_R^{\pm} \backslash \Xi_L^{\pm}$). This method would yield solutions that consist of solitons overlaid with minimal radiation. The temporal aspect within the system is straightforwardly incorporated through the time-varying nature of the scattering coefficients.

The RH problem can similarly be posed utilizing the left scattering data, by introducing a sectionally meromorphic matrix composed of eigenfunctions:

$$\widetilde{\mathbf{M}}(x,g) = \begin{cases} \left[\mathbf{v}(x,g)\mathrm{e}^{\mathrm{i}\beta_L x}, \frac{\mathbf{u}(x,g)}{h_{22}(g)}\mathrm{e}^{-\mathrm{i}\beta_R x}\right], & g \in \boldsymbol{\Lambda}_R^{+}, \\ \left[\frac{\widetilde{\mathbf{u}}(x,g)}{h_{11}(g)}\mathrm{e}^{\mathrm{i}\beta_R x}, \widetilde{\mathbf{v}}(x,g)\mathrm{e}^{-\mathrm{i}\beta_L x}\right], & g \in \boldsymbol{\Lambda}_R^{-}. \end{cases} \tag{126}$$

The RH problem defined across $g \in \mathbb{R}$ is represented as: $\widetilde{\mathbf{M}}^{+}(x,g) = \widetilde{\mathbf{M}}^{-}(x,g)\widetilde{\mathbf{O}}_0(x,g)$ which applies for $g \in \mathbb{R}$, indicating that:

$$\left[\mathbf{v}^{+}(x,g)\mathrm{e}^{\mathrm{i}\beta_L x}, \frac{\mathbf{u}^{+}(x,g)}{h_{22}^{+}(g)}\mathrm{e}^{-\mathrm{i}\beta_R x}\right] = \left[\frac{\widetilde{\mathbf{u}}^{-}(x,g)}{h_{11}^{-}(g)}\mathrm{e}^{\mathrm{i}\beta_R x}, \widetilde{\mathbf{v}}^{-}(x,g)\mathrm{e}^{-\mathrm{i}\beta_L x}\right]\widetilde{\mathbf{O}}_0(x,g), \tag{127}$$

where the superscripts $\pm$ represent the limiting values from the upper/lower halves of the complex plane, respectively. The jump matrix across $g \in \mathbb{R}$ can be readily determined from Eq. (25b) and is expressed as:

$$\widetilde{\mathbf{O}}_0(x,g) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\beta_L - \beta_R)x} & \alpha(g)\mathrm{e}^{-2\mathrm{i}\beta_R x} \\ -\widetilde{\alpha}(g)\mathrm{e}^{2\mathrm{i}\beta_L x} & [1 - \alpha(g)\widetilde{\alpha}(g)]\mathrm{e}^{\mathrm{i}(\beta_L - \beta_R)x} \end{pmatrix}. \tag{128}$$

Similarly, the jump matrices $\widetilde{\mathbf{O}}_1(x,g)$ across Ξ_L^{+} and $\widetilde{\mathbf{O}}_2(x,g)$ across Ξ_L^{-} can be derived, respectively, by:

$$\widetilde{\mathbf{O}}_1(x,g) = -\frac{\mathrm{i}q_L^*}{\beta_L + g}\begin{pmatrix} -\alpha^{-}(g)\mathrm{e}^{2\mathrm{i}\beta_L x} & \left[\frac{q_L}{q_L^*} - \alpha^{+}(g)\alpha^{-}(g)\right]\mathrm{e}^{\mathrm{i}(\beta_L - \beta_R)x} \\ \mathrm{e}^{\mathrm{i}(\beta_L - \beta_R)x} & \alpha^{+}(g)\mathrm{e}^{-2\mathrm{i}\beta_R x} \end{pmatrix}, \quad \text{across} \quad \Xi_L^{+}, \tag{129a}$$

$$\widetilde{\mathbf{O}}_2(x,g) = -\frac{\mathrm{i}q_L}{\beta_L + g}\begin{pmatrix} \widetilde{\alpha}^{+}(g)\mathrm{e}^{2\mathrm{i}\beta_R x} & \mathrm{e}^{\mathrm{i}(\beta_R - \beta_L)x} \\ \left[\frac{q_L^*}{q_L} - \widetilde{\alpha}^{+}(g)\widetilde{\alpha}^{-}(g)\right]\mathrm{e}^{\mathrm{i}(\beta_R - \beta_L)x} & -\widetilde{\alpha}^{-}(g)\mathrm{e}^{-2\mathrm{i}\beta_L x} \end{pmatrix}, \quad \text{across} \quad \Xi_L^{-}. \tag{129b}$$

Regarding the RH problem formulated with scattering coefficients from the right, the jump matrices $\widetilde{\mathbf{O}}_1(x,g)$ and $\widetilde{\mathbf{O}}_2(x,g)$ adhere to the following symmetry in the upper/lower half-planes: $\widetilde{\mathbf{O}}_2(x,g)=\sigma_2\widetilde{\mathbf{O}}_1^*(x,g^*)\sigma_2$. In addressing the RH problem across $\Xi_R^+\backslash\Xi_L^+$, the situation is as follows:

$$\widetilde{\mathbf{M}}^+(x,g)=\left[\mathbf{v}^+(x,g)\mathrm{e}^{\mathrm{i}\beta_L x},\frac{\mathbf{u}^+(x,g)}{h_{22}^+(g)}\mathrm{e}^{-\mathrm{i}\beta_R^+ x}\right],\quad \widetilde{\mathbf{M}}^-(x,g)=\left[\mathbf{v}^-(x,g)\mathrm{e}^{\mathrm{i}\beta_L x},\frac{\mathbf{u}^-(x,g)}{h_{22}^-(g)}\mathrm{e}^{-\mathrm{i}\beta_R^- x}\right]. \tag{130}$$

It should be noted that in contrast to the RH problem from the right, the jump cannot be determined here in the same manner. The same limitation applies to the RH problem on $\Xi_R^-\backslash\Xi_L^-$. In fact, the right-hand sides are only defined simultaneously for $g\in\mathbb{R}\cup\Xi_L$ and cannot be extended to either $\Xi_R^+\backslash\Xi_L^+$ or $\Xi_R^-\backslash\Xi_L^-$. This distinction is also apparent when considering that, unlike $\gamma^\pm(g)$ and $\widetilde{\gamma}^\pm(g)$, which can be analytically continued onto $\Xi_R^+\backslash\Xi_L^+$ or $\Xi_R^-\backslash\Xi_L^-$ respectively, the reflection coefficients from the left, $\alpha^\pm(g)$ and $\widetilde{\alpha}^\pm(g)$ are typically defined only on the continuous spectrum, that is, for $g\in\mathbb{R}\cup\Xi_L$.

To properly define the RH problem from the left on $\Xi_R\backslash\Xi_L$, it is necessary to consider both segments of the cut $\Xi_R^+\backslash\Xi_L^+$ and $\Xi_R^-\backslash\Xi_L^-$ concurrently, and to account for the following:

- β_R changes sign across $\Xi_R^+\backslash\Xi_L^+$ and $\Xi_R^-\backslash\Xi_L^-$;
- $\mathbf{v}^+(x,g)=\mathbf{v}^-(x,g)$ for $g\in\Xi_R^+\backslash\Xi_L^+$, and $\widetilde{\mathbf{v}}^+(x,g)=\widetilde{\mathbf{v}}^-(x,g)$ for $g\in\Xi_R^-\backslash\Xi_L^-$;
- $\mathbf{u}^\pm(x,g)/h_{22}^\pm(g)$ and $\widetilde{\mathbf{u}}^\pm(x,g)/h_{11}^\pm(g)$ are interconnected through the symmetry relations given by Eqs. (37) and (41c), that is:

$$\frac{\mathbf{u}^\pm(x,g)}{h_{22}^\pm(g)}=\frac{\widetilde{\mathbf{u}}^\mp(x,g)}{h_{21}^\mp(g)},\quad g\in\Xi_R^+\backslash\Xi_L^+,\qquad \frac{\widetilde{\mathbf{u}}^\pm(x,g)}{h_{11}^\pm(g)}=\frac{\mathbf{u}^\mp(x,g)}{h_{12}^\mp(g)},\quad g\in\Xi_R^-\backslash\Xi_L^-. \tag{131}$$

Addressing the RH problem from the left, which includes poles corresponding to the zeros of $h_{22}(g)$ and $h_{11}(g)$ in the upper/lower half-planes (identical to those from the right), involves calculating the sectionally meromorphic matrix $\widetilde{\mathbf{M}}(x,g)$ with the specified jumps $\widetilde{\mathbf{O}}_j$, and normalized to the identity matrix as g approaches infinity. Subsequently, the potential is reconstructed through the large g expansion of this matrix, since

$$\widetilde{\mathbf{M}}(x,g)=\left[\mathbf{I}_2+\frac{\mathrm{i}}{2g}\mathbf{Q}(x)\sigma_3\right][1+o(1)]. \tag{132}$$

4.2. Time evolution of the scattering data

In accordance with the second part of Eq. (3), the temporal evolution is described by $\psi_t=\mathbf{T}\psi$. Asymptotically, considering $q(x,t)\to q_{L/R}(t)=B_{L/R}\mathrm{e}^{\mathrm{i}\delta_{L/R}(t)}$ as $x\to\mp\infty$, we have $\psi_t\simeq\widetilde{\mathbf{T}}(g,t)\psi$ and $\widetilde{\mathbf{T}}(g,t)=(\widetilde{T}_{ij}(g,t))$, where

$$\widetilde{T}_{11}(g,t)=(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(\mathrm{i}B_{L/R}^2-2\mathrm{i}g^2)+3\mathrm{i}\sigma_{13}B_{L/R}^4, \tag{133a}$$

$$\widetilde{T}_{12}(g,t)=[2g\sigma_{11}+2\sigma_{12}(2g^2-B_{L/R}^2)+4\sigma_{13}(gB_{L/R}^2-2g^3)]q_{L/R}(t), \tag{133b}$$

$$\widetilde{T}_{21}(g,t)=[2\sigma_{12}(B_{L/R}^2-2g^2)-2g\sigma_{11}+4\sigma_{13}(2g^3-gB_{L/R}^2)]q_{L/R}^*(t), \tag{133c}$$

$$\widetilde{T}_{22}(g,t)=(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(2\mathrm{i}g^2-\mathrm{i}B_{L/R}^2)-3\mathrm{i}\sigma_{13}B_{L/R}^4. \tag{133d}$$

The scattering problem outlined in Eq. (3) yields as $x\to\mp\infty$, the following for the two components of any eigenfunction $\psi(x,t)$:

$$q_{L/R}^*(t)\psi^{(1)}\simeq-\psi_x^{(2)}+\mathrm{i}g\psi^{(2)},\quad q_{L/R}(t)\psi^{(2)}\simeq\psi_x^{(1)}+\mathrm{i}g\psi^{(1)}. \tag{134}$$

Substituting these equations into Eqs. (133a), (133b), (133c) and (133d), we derive the following as $x\to\mp\infty$:

$$\psi_t^{(1)}\simeq[2g\sigma_{11}+2\sigma_{12}(2g^2-B_{L/R}^2)+4\sigma_{13}(gB_{L/R}^2-2g^3)]\psi_x^{(1)}+\mathrm{i}\sigma_{11}B_{L/R}^2\psi^{(1)}+3\mathrm{i}\sigma_{13}B_{L/R}^4\psi^{(1)}, \tag{135a}$$

$$\psi_t^{(2)}\simeq[2g\sigma_{11}+2\sigma_{12}(2g^2-B_{L/R}^2)+4\sigma_{13}(gB_{L/R}^2-2g^3)]\psi_x^{(2)}-\mathrm{i}\sigma_{11}B_{L/R}^2\psi^{(2)}-3\mathrm{i}\sigma_{13}B_{L/R}^4\psi^{(2)}. \tag{135b}$$

The Jost solutions with boundary values as $x\to\mp\infty$ specified by Eqs. (21a) and (21b), do not align with the aforementioned temporal evolution. Consequently, we introduce time-dependent eigenfunctions that serve as solutions to the evolution equation. For example, consider $\phi(x,g,t)=\mathrm{e}^{\mathrm{i}B_\infty t}\mathbf{v}(x,g,t)$, such that

$$\phi_t=\mathrm{i}B_\infty\phi+\mathrm{e}^{\mathrm{i}B_\infty t}\mathbf{v}_t. \tag{136}$$

Considering that the components of ϕ asymptotically fulfill the systems (135a) and (135b) as $x\to-\infty$ and acknowledging that $\mathbf{V}(x,g,t)=(\mathbf{v}(x,g,t),\widetilde{\mathbf{v}}(x,g,t))\sim\mathbf{E}_L(g)\mathrm{e}^{-\mathrm{i}\beta_L x\sigma_3}$, we have

$$\mathbf{v}(x,g,t)\simeq\begin{pmatrix}1\\ \frac{-\mathrm{i}q_L^*(t)}{\beta_L+g}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\beta_L x},\quad \mathbf{v}_t(x,g,t)\simeq\begin{pmatrix}0\\ -\frac{\dot{\delta}_L(t)q_L^*(t)}{\beta_L+g}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\beta_L x},\quad \mathbf{v}_x(x,g,t)\simeq-\mathrm{i}\beta_L\begin{pmatrix}1\\ \frac{-\mathrm{i}q_L^*(t)}{\beta_L+g}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\beta_L x},\quad x\to-\infty, \tag{137}$$

where the dot signifies differentiation with respect to time. Substituting into Eq. (136), the first component results in $B_\infty=\sigma_{11}B_L^2+3\sigma_{13}B_L^4-\beta_L[2g\sigma_{11}+2\sigma_{12}(2g^2-B_L^2)+4\sigma_{13}(gB_L^2-2g^3)]$ and from the second component, we derive: $\dot{\delta}_L(t)=2\sigma_{11}B_L^2+6\sigma_{13}B_L^4$, leading to $\delta_L(t)=2\sigma_{11}B_L^2t+6\sigma_{13}B_L^4t+\delta_L(0)$. Similarly, the evolution of the asymptotic phase as $x\to+\infty$ can be determined:

$\delta_R(t) = 2\sigma_{11}B_R^2 t + 6\sigma_{13}B_R^4 t + \delta_R(0)$, as well as the temporal evolution of the other Jost solutions. This yields for $\mathbf{V} = (\mathbf{v}, \widetilde{\mathbf{v}})$ and $\mathbf{U} = (\widetilde{\mathbf{u}}, \mathbf{u})$,

$$\partial_t \mathbf{V} = \mathbf{TV} - \mathrm{i}\left[\sigma_{11}B_L^2 + 3\sigma_{13}B_L^4 - \beta_L[2g\sigma_{11} + 2\sigma_{12}(2g^2 - B_L^2) + 4\sigma_{13}(gB_L^2 - 2g^3)]\right]\mathbf{V}\sigma_3, \tag{138a}$$

$$\partial_t \mathbf{U} = \mathbf{TU} - \mathrm{i}\left[\sigma_{11}B_R^2 + 3\sigma_{13}B_R^4 - \beta_R[2g\sigma_{11} + 2\sigma_{12}(2g^2 - B_R^2) + 4\sigma_{13}(gB_R^2 - 2g^3)]\right]\mathbf{U}\sigma_3. \tag{138b}$$

By differentiating Eq. (25a) with respect to t and considering the temporal evolution of the Jost solutions as given by Eqs. (138a) and (138b), we derive the scattering matrix:

$$\begin{aligned}\partial_t \mathbf{C} = {} & \mathrm{i}\left[\sigma_{11}B_R^2 + 3\sigma_{13}B_R^4 - \beta_R[2g\sigma_{11} + 2\sigma_{12}(2g^2 - B_R^2) + 4\sigma_{13}(gB_R^2 - 2g^3)]\right]\sigma_3\mathbf{C} \\ & - \mathrm{i}\left[\sigma_{11}B_L^2 + 3\sigma_{13}B_L^4 - \beta_L[2g\sigma_{11} + 2\sigma_{12}(2g^2 - B_L^2) + 4\sigma_{13}(gB_L^2 - 2g^3)]\right]\mathbf{C}\sigma_3.\end{aligned} \tag{139}$$

Specifically, this results in the following expressions for the time evolution of the scattering coefficients $c_{11}(g,t)$ and $c_{21}(g,t)$, as well as the reflection coefficient from the right $\gamma(g,t)$:

$$c_{11}(g,t) = c_{11}(g,0)\mathrm{e}^{2\mathrm{i}g(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(\beta_L-\beta_R)t+\mathrm{i}\sigma_{11}(B_R^2-B_L^2)t+3\mathrm{i}\sigma_{13}(B_R^4-B_L^4)t+(2\mathrm{i}\sigma_{12}-4\mathrm{i}g\sigma_{13})(\beta_R B_R^2-\beta_L B_L^2)t}, \tag{140a}$$

$$c_{21}(g,t) = c_{21}(g,0)\mathrm{e}^{2\mathrm{i}g(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(\beta_L+\beta_R)t-\mathrm{i}\sigma_{11}(B_R^2+B_L^2)t-3\mathrm{i}\sigma_{13}(B_R^4+B_L^4)t+(4\mathrm{i}g\sigma_{13}-2\mathrm{i}\sigma_{12})(\beta_R B_R^2+\beta_L B_L^2)t}, \tag{140b}$$

$$\gamma(g,t) = \gamma(g,0)\mathrm{e}^{4\mathrm{i}g\beta_R(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})t-2\mathrm{i}\sigma_{11}B_R^2 t-6\mathrm{i}\sigma_{13}B_R^4 t+\beta_R B_R^2(8\mathrm{i}g\sigma_{13}-4\mathrm{i}\sigma_{12})t}. \tag{140c}$$

The initial equation indicates that the discrete eigenvalues g_m are invariant with respect to time and are determined by the zeros of $c_{11}(g,0)$. It is noted that in the symmetric scenario where $B_L = B_R$, it follows that $c_{11}(g,t) = c_{11}(g,0)$, meaning the transmission coefficient remains constant over time. Furthermore, regarding the behavior of $c_{11}(g,t)$ for large g, consistent with $c_{11}(g,t) \sim 1$ as $|g| \to \infty$ for $g \in \mathbf{\Lambda}_R^+ \cup \mathbb{R}$ and for all $t \geq 0$; this ensures the inverse problem is well-posed. Similarly, the evolution of the other scattering coefficients such as $h_{22}(g,t)$, $h_{12}(g,t)$ etc., can be determined, as well as the reflection coefficient from the left $\alpha(g,t)$:

$$h_{12}(g,t) = h_{12}(g,0)\mathrm{e}^{\mathrm{i}\sigma_{11}(B_R^2+B_L^2)t-2\mathrm{i}g(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(\beta_L+\beta_R)t+3\mathrm{i}\sigma_{13}(B_R^4+B_L^4)t+(2\mathrm{i}\sigma_{12}-4\mathrm{i}g\sigma_{13})(\beta_R B_R^2+\beta_L B_L^2)t}, \tag{141a}$$

$$h_{22}(g,t) = h_{22}(g,0)\mathrm{e}^{\mathrm{i}\sigma_{11}(B_R^2-B_L^2)t+2\mathrm{i}g(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})(\beta_L-\beta_R)t+3\mathrm{i}\sigma_{13}(B_R^4-B_L^4)t+(2\mathrm{i}\sigma_{12}-4\mathrm{i}g\sigma_{13})(\beta_R B_R^2-\beta_L B_L^2)t}, \tag{141b}$$

$$\alpha(g,t) = \alpha(g,0)\mathrm{e}^{2\mathrm{i}\sigma_{11}B_L^2 t+6\mathrm{i}\sigma_{13}B_L^4 t-4\mathrm{i}g\beta_L(\sigma_{11}+2g\sigma_{12}-4g^2\sigma_{13})t+2\beta_L B_L^2(2\mathrm{i}\sigma_{12}-4\mathrm{i}g\sigma_{13})t}. \tag{141c}$$

Lastly, it is necessary to ascertain the temporal dependence of the norming constants. By differentiating $\mathbf{v}(x, g_m) = d_m\mathbf{u}(x, g_m)$ with respect to time and assessing the first and second columns at $g = g_m$, we obtain:

$$\begin{aligned}d_m(t) = {} & d_m(0)\exp[2\mathrm{i}g_m(\sigma_{11} + 2g_m\sigma_{12} - 4g_m^2\sigma_{13})(\beta_L(g_m) + \beta_R(g_m))t - \mathrm{i}\sigma_{11}(B_R^2 + B_L^2)t \\ & - 3\mathrm{i}\sigma_{13}(B_R^4 + B_L^4)t + (4\mathrm{i}g_m\sigma_{13} - 2\mathrm{i}\sigma_{12})(\beta_R(g_m)B_R^2 + \beta_L(g_m)B_L^2)t],\end{aligned} \tag{142}$$

where $m = 1, \ldots, M$. Subsequently, utilizing the definition of F_m as given in Eq. (42), we derive:

$$F_m(t) = F_m(0)\mathrm{e}^{4\mathrm{i}g_m\beta_R(g_m)(\sigma_{11}+2g_m\sigma_{12}-4g_m^2\sigma_{13})t-2\mathrm{i}\sigma_{11}B_R^2 t-6\mathrm{i}\sigma_{13}B_R^4 t+\beta_R(g_m)B_R^2(8\mathrm{i}g_m\sigma_{13}-4\mathrm{i}\sigma_{12})t}. \tag{143}$$

5. Discussion and final remarks

We have advanced the IST for the fourth-order NLS equation with fully asymmetric NZBCs as $x \to \pm\infty$. This represents a significant extension of the scenario where the amplitudes of the background field are equal at both spatial infinities, entailing the management of supplementary technical complexities. The most critical of these is the inability to introduce a uniformization variable in the spectral domain when the amplitudes of the soliton solutions differ as $x \to \pm\infty$. This is because such a variable would be necessary to map the multi-sheeted Riemann surface for the scattering parameter onto a single complex plane, a step that becomes infeasible under these asymmetric conditions. Significant distinctions from the symmetric case also emerge in the inverse problem. In addition to solitons where correspond to the discrete eigenvalues of the scattering problem and radiation which corresponds to the continuous spectrum of the scattering operator and is represented in the inverse problem by the reflection coefficients for $g \in \mathbb{R} \cup (-\mathrm{i}B_L, \mathrm{i}B_L)$, there is also a notable contribution from the transmission coefficients for $g \in (-\mathrm{i}B_R, -\mathrm{i}B_L) \cup (\mathrm{i}B_L, \mathrm{i}B_R)$. This is evidenced by the last term in Eq. (106), which contributes to the left Marchenko equations. Accordingly, Eqs. (41a) and (41b) indicate that in the right Marchenko equations, there is always a significant contribution from the integral terms in Eqs. (85) and (91), since $\gamma(g)$ [resp. $\widetilde{\gamma}(g)$] does not vanish for $g \in (\mathrm{i}B_L, \mathrm{i}B_R)$ [resp. $g \in (-\mathrm{i}B_R, -\mathrm{i}B_L)$]. Specifically, this suggests that pure soliton solutions are not feasible, and solitons are invariably coupled with some form of radiative contribution. Consequently, unlike in the symmetric scenario, an explicit solution cannot be derived merely by simplifying the inverse problem into a system of algebraic equations.

The advancements presented in this paper regarding the IST for the fourth-order NLS equation with fully asymmetric NZBCs as $x \to \pm\infty$ open new avenues for future research in mathematical physics. This work not only extends the understanding of the fourth-order NLS equation under asymmetric conditions but also provides a robust framework for analyzing the long-time asymptotic behavior of solutions. The results are expected to facilitate the application of the nonlinear steepest descent method to study the evolution of solutions over extended periods, similar to what has been achieved for the focusing NLS [43–48] and mKdV [49,50] equation with step-like initial conditions. The Marchenko integral equations derived in this study offer an alternative approach to investigate the long-time behavior of solutions through matched asymptotics, akin to recent developments in the study of the KdV equation [51]. This methodological advancement is significant as it allows for a more nuanced understanding of the interplay between solitons and radiation in the context of NZBCs.

Furthermore, the insights gained from this research will be instrumental in exploring the dynamics of solitons and their interactions with the background field under asymmetric conditions [52–56]. The inability to introduce a uniformization variable due to differing amplitudes at spatial infinities presents a unique challenge that this study begins to address, paving the way for further exploration into the complex behavior of nonlinear waves in diverse physical settings [57–60]. In summary, the findings of this paper are expected to stimulate further research into the long-time asymptotic behavior of soliton solutions with nontrivial boundary conditions, potentially leading to new theoretical developments and practical applications in the field of mathematical physics.

CRediT authorship contribution statement

Peng-Fei Han: Writing – review & editing, Writing – original draft, Software, Data curation, Conceptualization. **Kun Zhu:** Software, Project administration, Investigation. **Feng Zhang:** Visualization, Validation, Supervision, Formal analysis. **Wen-Xiu Ma:** Visualization, Resources, Project administration, Investigation, Funding acquisition. **Yi Zhang:** Visualization, Resources, Investigation, Funding acquisition.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yi Zhang reports financial support was provided by National Natural Science Foundation of China. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We appreciate the efforts of the reviewers in providing detailed and helpful comments that have contributed to the enhancement of our research. Zhang's and Ma's works were partially supported by the National Natural Science Foundation of China (Grant Nos. 11371326 and 12271488). Ma's work was also partially supported by the Ministry of Science and Technology of China (G2021016032L and G2023016011L). Zhu's work was partially supported by the Natural Science Foundation of Jiangxi Province (Nos. 20224BAB201015 and 20232BAB202034).

Appendix A. Direct scattering problem

Given that $\mathbf{G}^2_{L/R}(g) = -\beta^2_{L/R}\mathbf{I}_2$, the explicit form of $e^{x\mathbf{G}_{L/R}(g)}$ can be easily derived from its series expansion:

$$e^{x\mathbf{G}_{L/R}(g)} = \cos(\beta_{L/R}x)\mathbf{I}_2 + \frac{\sin(\beta_{L/R}x)}{\beta_{L/R}}[\mathbf{Q}_{L/R} - \mathrm{i}g\sigma_3]. \tag{144}$$

The Hilbert–Schmidt norm of a matrix $\mathbf{L}$ is given by $\|\mathbf{L}\|^2_{\mathrm{HS}} = \mathrm{tr}(\mathbf{L}^\dagger\mathbf{L})$, and the spectral norm is the square root of the largest singular value of $\mathbf{L}^\dagger\mathbf{L}$. Then, $\det e^{x\mathbf{G}_{L/R}(g)} = 1$ for any $g \in \mathbb{C}$ and $\beta_{L/R} = \sqrt{g^2 + B^2_{L/R}} \in \mathbb{R}$. For $g \in \mathbb{R} \cup [-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R}]$, we have

$$\|e^{x\mathbf{G}_{L/R}(g)}\|^2_{\mathrm{HS}} =: 2Y(x,g), \quad Y(x,g) = \cos^2(\beta_{L/R}x) + \frac{|g|^2 + B^2_{L/R}}{\beta^2_{L/R}}\sin^2(\beta_{L/R}x). \tag{145}$$

Since $|g|^2 + B^2_{L/R} = g^2 + B^2_{L/R} = \beta^2_{L/R}$ for all $g \in \mathbb{R}$, it follows that $Y(x,g) = 1$. Taking the limit of $Y(x,g)$ as $\beta_{L/R} \to 0$ in Eq. (145) results in $Y(x, \pm\mathrm{i}B_{L/R}) = 1 + 2B^2_{L/R}x^2$. Additionally, for $g \in (-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R})$, we have $|g|^2 = -g^2$ and $\beta^2_{L/R} \le B^2_{L/R}$. Therefore, for $g \in (-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R})$, we find that

$$Y(x,g) = \cos^2(\beta_{L/R}x) + \frac{|g|^2 + B^2_{L/R}}{\beta^2_{L/R}}\sin^2(\beta_{L/R}x) = 1 + \frac{2(B^2_{L/R} - \beta^2_{L/R})}{\beta^2_{L/R}}\sin^2(\beta_{L/R}x) \ge 1. \tag{146}$$

Moreover, for $g \in [-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R}]$, it follows that $Y(x,g) \le 1 + 2B^2_{L/R}x^2$.

For $g \in \mathbb{R}$, the boundedness of $Y(x,g) \le 1 + 2B^2_{L/R}x^2$ is clearly established. Using the identity

$$\|\mathbf{L}\|^2 = \frac{1}{2}\left[\|\mathbf{L}\|^2_{\mathrm{HS}} + \sqrt{\|\mathbf{L}\|^4_{\mathrm{HS}} - 4|\det \mathbf{L}|^2}\right], \tag{147}$$

we then obtain

$$\|e^{x\mathbf{G}_{L/R}(g)}\|^2 = Y(x,g) + \sqrt{Y^2(x,g) - 1}, \quad g \in \mathbb{R} \cup (-\mathrm{i}B_{L/R}, \mathrm{i}B_{L/R}), \tag{148}$$

yielding

$$D_{L/R}(g) = \sup_{x\in\mathbb{R}} \|e^{x\mathbf{G}_{L/R}(g)}\| = \begin{cases} 1, & g\in\mathbb{R}, \\ \sqrt{Y(g)+\sqrt{Y^2(g)-1}}, & g\in(-iB_{L/R}, iB_{L/R}), \end{cases} \tag{149}$$

where

$$Y(g) = 1 + \frac{2(B_{L/R}^2-\beta_{L/R}^2)}{\beta_{L/R}^2} = 1 + \frac{2|g|^2}{\beta_{L/R}^2}, \quad g\in(-iB_{L/R}, iB_{L/R}). \tag{150}$$

Ultimately, for all $x\in\mathbb{R}$ and $g\in[-iB_{L/R}, iB_{L/R}]\cup\mathbb{R}$, from $Y(x,g)\le 1+2B_{L/R}^2x^2$ we find that

$$\|e^{x\mathbf{G}_{L/R}(\pm iB_{L/R})}\|^2 = 1+2B_{L/R}^2x^2+\sqrt{(1+2B_{L/R}^2x^2)^2-1} \le \widetilde{D}_{L/R}^2(1+|x|)^2. \tag{151}$$

Here, $\widetilde{D}_{L/R}$ is a positive constant that does not depend on $x\in\mathbb{R}$.

Proof of Proposition 1. For all $x\in\mathbb{R}$ and $g\in\mathbb{R}\cup(-iB_{L/R}, iB_{L/R})$, the estimate (149) demonstrates that $\|e^{x\mathbf{G}_R(g)}\|\le D_R(g)$. Applying $\widetilde{\mathbf{U}}(x,g)$ (10) and Gronwall's inequality with $\alpha_1(x,g)=D_R(g)$, $\alpha_2(y,g)=D_R(g)\|\mathbf{Q}(y)-\mathbf{Q}_R\|$ yield the desired result:

$$\|\widetilde{\mathbf{U}}(x,g)\| \le D_R(g)e^{D_R(g)\int_x^\infty \|\mathbf{Q}(y)-\mathbf{Q}_R\|\,dy}. \tag{152}$$

From the estimate (151), we deduce that there is a constant $\widetilde{D}_R$ such that $\|e^{x\mathbf{G}_R(g)}\|\le\widetilde{D}_R(1+|x|)$ for all $x\in\mathbb{R}$ and $g\in\mathbb{R}\cup[-iB_R, iB_R]$. Applying Gronwall's inequality leads to

$$\|\widetilde{\mathbf{U}}(x,g)\| \le \widetilde{D}_R(1+|x|)e^{\widetilde{D}_R\int_x^\infty(1+|y-x|)\|\mathbf{Q}(y)-\mathbf{Q}_R\|\,dy} \le \widetilde{D}_R(x_0)(1+|x|)e^{\widetilde{D}_R(x_0)\int_x^\infty(1+|y|)\|\mathbf{Q}(y)-\mathbf{Q}_R\|\,dy}, \tag{153}$$

with $\widetilde{D}_R(x_0)=\widetilde{D}_R(1+\max(0,-x_0))$ and $x\ge x_0$. □

Proof of Theorem 1. By multiplying Eqs. (10) and (11) from the right by the respective appropriate columns of the matrices $\mathbf{E}_{L/R}(g)$ and utilizing the explicit forms given in Eq. (144) for $e^{(x-y)\mathbf{G}_{L/R}(g)}$, we derive the subsequent Volterra integral equations for the Jost solutions:

$$e^{i\beta_R x}\widetilde{\mathbf{u}}(x,g) = \mathbf{E}_{R,1}(g) - \int_x^\infty \mathbf{\Omega}_R^-(y-x,g)[\mathbf{Q}(y)-\mathbf{Q}_R]e^{i\beta_R y}\widetilde{\mathbf{u}}(y,g)\,dy, \tag{154a}$$

$$e^{-i\beta_R x}\mathbf{u}(x,g) = \mathbf{E}_{R,2}(g) - \int_x^\infty \mathbf{\Omega}_R^+(y-x,g)[\mathbf{Q}(y)-\mathbf{Q}_R]e^{-i\beta_R y}\mathbf{u}(y,g)\,dy, \tag{154b}$$

$$e^{i\beta_L x}\mathbf{v}(x,g) = \mathbf{E}_{L,1}(g) + \int_{-\infty}^x \mathbf{\Omega}_L^+(x-y,g)[\mathbf{Q}(y)-\mathbf{Q}_L]e^{i\beta_L y}\mathbf{v}(y,g)\,dy, \tag{154c}$$

$$e^{-i\beta_L x}\widetilde{\mathbf{v}}(x,g) = \mathbf{E}_{L,2}(g) + \int_{-\infty}^x \mathbf{\Omega}_L^-(x-y,g)[\mathbf{Q}(y)-\mathbf{Q}_L]e^{-i\beta_L y}\widetilde{\mathbf{v}}(y,g)\,dy, \tag{154d}$$

where the subscripts $j=1,2$ in the matrices $\mathbf{E}_{L/R}(g)$ indicate their jth column, with

$$\mathbf{\Omega}_R^-(x,g) = \begin{pmatrix} 1+\frac{\beta_R-g}{2\beta_R}[e^{-2i\beta_R x}-1] & -\frac{iq_R}{2\beta_R}[e^{-2i\beta_R x}-1] \\ \frac{iq_R^*}{2\beta_R}[e^{-2i\beta_R x}-1] & e^{-2i\beta_R x}-\frac{\beta_R-g}{2\beta_R}[e^{-2i\beta_R x}-1] \end{pmatrix}, \tag{155a}$$

$$\mathbf{\Omega}_R^+(x,g) = \begin{pmatrix} e^{2i\beta_R x}-\frac{\beta_R-g}{2\beta_R}[e^{2i\beta_R x}-1] & \frac{iq_R}{2\beta_R}[e^{2i\beta_R x}-1] \\ -\frac{iq_R^*}{2\beta_R}[e^{2i\beta_R x}-1] & 1+\frac{\beta_R-g}{2\beta_R}[e^{2i\beta_R x}-1] \end{pmatrix}, \tag{155b}$$

$$\mathbf{\Omega}_L^+(x,g) = \begin{pmatrix} 1+\frac{\beta_L-g}{2\beta_L}[e^{2i\beta_L x}-1] & -\frac{iq_L}{2\beta_L}[e^{2i\beta_L x}-1] \\ \frac{iq_L^*}{2\beta_L}[e^{2i\beta_L x}-1] & e^{2i\beta_L x}-\frac{\beta_L-g}{2\beta_L}[e^{2i\beta_L x}-1] \end{pmatrix}, \tag{155c}$$

$$\mathbf{\Omega}_L^-(x,g) = \begin{pmatrix} e^{-2i\beta_L x}-\frac{\beta_L-g}{2\beta_L}[e^{-2i\beta_L x}-1] & \frac{iq_L}{2\beta_L}[e^{-2i\beta_L x}-1] \\ -\frac{iq_L^*}{2\beta_L}[e^{-2i\beta_L x}-1] & 1+\frac{\beta_L-g}{2\beta_L}[e^{-2i\beta_L x}-1] \end{pmatrix}. \tag{155d}$$

Given the chosen branch cuts, it is straightforward to deduce the following expressions that describe the behavior of $\beta_{L/R}$ as $g\to\infty$:

$$\beta_R - g = \frac{B_R^2}{2g}[1+O(g^{-2})], \quad \beta_L - g = \frac{B_L^2}{2g}[1+O(g^{-2})]. \tag{156}$$

Furthermore, when $x\ge 0$, we observe that

$$\left|\frac{e^{\pm 2i\beta_R x}-1}{2\beta_R}\right| \equiv \left|\int_0^x e^{\pm 2i\beta_R h}\,dh\right| \le \min\left(x, \frac{1}{|\beta_R|}\right), \quad g\in\mathbf{\Lambda}_R^\pm\cup\partial\mathbf{\Lambda}_R^\pm, \tag{157}$$

Similarly, this applies to the quantities with the L-subscript. Additionally, applying the maximum modulus principle, we obtain

$$\left\|\left(1,-\frac{\mathrm{i}q_R}{\beta_R+g}\right)^T\right\| = \left\|\left(1,-\frac{\mathrm{i}(\beta_R-g)}{q_R^*}\right)^T\right\| = \left[1+\left(\frac{1}{|q_R|}\max_{g\in\partial\boldsymbol{\Lambda}_R^\pm}|\beta_R-g|\right)^2\right]^{\frac{1}{2}} = \sqrt{2},\quad g\in\boldsymbol{\Lambda}_R^\pm\cup\partial\boldsymbol{\Lambda}_R^\pm. \tag{158}$$

Then, utilizing Eqs. (156), (157) and (158) for estimation, we find that

$$\|\boldsymbol{\Omega}_R^-(x,g)\| \le 1+2|q_R|\min\left(x,\frac{1}{|\beta_R|}\right),\quad g\in\boldsymbol{\Lambda}_R^-\cup\partial\boldsymbol{\Lambda}_R^-. \tag{159}$$

Employing Gronwall's inequality, we arrive at the following result:

$$\|\mathrm{e}^{\mathrm{i}\beta_R x}\widetilde{\mathbf{u}}(x,g)\| \le \sqrt{2}\mathrm{e}^{\int_x^\infty[1+2|q_R|(y-x)]\|\mathbf{Q}(y)-\mathbf{Q}_R\|\,\mathrm{d}y}, \tag{160}$$

with the estimate is uniformly valid for $(x,g)\in[x_0,+\infty)\times[\boldsymbol{\Lambda}_R^-\cup\partial\boldsymbol{\Lambda}_R^-\cup\partial\boldsymbol{\Lambda}_R^+]$, provided that the hypothesis $(\boldsymbol{B}_1)$ holds. Consequently, assuming $(\boldsymbol{B}_1)$ holds and for any $x\in\mathbb{R}$, the Jost solution $\widetilde{\mathbf{u}}(x,g)$ is continuous for $g\in\boldsymbol{\Lambda}_R^-\cup\partial\boldsymbol{\Lambda}_R^-\cup\partial\boldsymbol{\Lambda}_R^+$ and analytic for $g\in\boldsymbol{\Lambda}_R^-$. Similarly, we derive analogous estimates for the remaining three Jost solutions, thereby establishing the continuity and analyticity properties previously discussed. □

Proof of Theorem 2. Upon differentiating Eq. (154a) with respect to g, we arrive at the following integral equation:

$$\begin{aligned}\frac{\partial\left[\mathrm{e}^{\mathrm{i}\beta_R x}\widetilde{\mathbf{u}}(x,g)-\mathbf{E}_{R,1}(g)\right]}{\partial g} &= -\frac{\partial}{\partial g}\int_x^\infty\boldsymbol{\Omega}_R^-(y-x,g)[\mathbf{Q}(y)-\mathbf{Q}_R]\mathbf{E}_{R,1}(g)\,\mathrm{d}y\\ &\quad-\int_x^\infty\frac{\partial\boldsymbol{\Omega}_R^-(y-x,g)}{\partial g}[\mathbf{Q}(y)-\mathbf{Q}_R]\left[\mathrm{e}^{\mathrm{i}\beta_R y}\widetilde{\mathbf{u}}(y,g)-\mathbf{E}_{R,1}(g)\right]\mathrm{d}y\\ &\quad-\int_x^\infty\boldsymbol{\Omega}_R^-(y-x,g)[\mathbf{Q}(y)-\mathbf{Q}_R]\frac{\partial\left[\mathrm{e}^{\mathrm{i}\beta_R y}\widetilde{\mathbf{u}}(y,g)-\mathbf{E}_{R,1}(g)\right]}{\partial g}\,\mathrm{d}y,\end{aligned} \tag{161}$$

where

$$\begin{aligned}\frac{\partial\boldsymbol{\Omega}_R^-(y-x,g)}{\partial g} &= -\frac{2\mathrm{i}g}{\beta_R}(y-x)\mathrm{e}^{-2\mathrm{i}\beta_R(y-x)}\begin{pmatrix}0&0\\0&1\end{pmatrix}-\frac{\beta_R-g}{2\beta_R^2}(\mathrm{e}^{-2\mathrm{i}\beta_R(y-x)}-1)\sigma_3\\ &\quad-\frac{g}{2\beta_R^2}\begin{pmatrix}\beta_R-g&-\mathrm{i}q_R\\\mathrm{i}q_R^*&g-\beta_R\end{pmatrix}\left[\frac{\mathrm{e}^{-2\mathrm{i}\beta_R(y-x)}-1}{\beta_R}+2\mathrm{i}(y-x)\mathrm{e}^{-2\mathrm{i}\beta_R(y-x)}\right].\end{aligned} \tag{162}$$

Applying Eq. (159), we derive the following result:

$$\left\|\frac{\partial\boldsymbol{\Omega}_R^-(y-x,g)}{\partial g}\right\| \le 2\frac{|g|}{|\beta_R|}(y-x)\left[1+\frac{|\beta_R-g|+|q_R|}{|\beta_R|}\right]+\frac{|\beta_R-g|}{|\beta_R|}(y-x). \tag{163}$$

A straightforward application of Gronwall's inequality allows us to estimate the solution, provided that $g\neq\pm\mathrm{i}B_R$. An analogous conclusion applies to $\mathbf{u}(x,g)$ and the other two Jost solutions, under the condition that $g\neq\pm\mathrm{i}B_L$. □

Appendix B. Inverse scattering problem

Proof of Proposition 2. Let us examine the representation given by Eq. (60a), which may be expressed as:

$$\begin{aligned}\widetilde{\mathbf{U}}(x,g) &= \mathrm{e}^{x\mathbf{G}_R(g)}+\int_x^\infty\mathbf{Z}(x,h)\mathrm{e}^{h\mathbf{G}_R(g)}\,\mathrm{d}h = \int_x^\infty\mathbf{Z}(x,h)\left[\cos(\beta_R h)+\frac{\sin(\beta_R h)}{\beta_R}\mathbf{Q}_R\right]\mathrm{d}h\\ &\quad+\cos(\beta_R x)\mathbf{I}_2+\frac{\sin(\beta_R x)}{\beta_R}[\mathbf{Q}_R-\mathrm{i}g\sigma_3]-\mathrm{i}g\int_x^\infty\mathbf{Z}(x,h)\frac{\sin(\beta_R h)}{\beta_R}\sigma_3\,\mathrm{d}h\\ &= \cos(\beta_R x)\mathbf{I}_2+\frac{\sin(\beta_R x)}{\beta_R}[\mathbf{Q}_R-\mathrm{i}g\sigma_3]+\mathbf{U}_a(x,\beta_R)+\mathbf{U}_b(x,\beta_R)[\mathbf{Q}_R-\mathrm{i}g\sigma_3],\end{aligned} \tag{164}$$

where

$$\mathbf{U}_a(x,\beta_R) = \int_x^\infty\mathbf{Z}(x,h)\cos(\beta_R h)\,\mathrm{d}h,\quad \mathbf{U}_b(x,\beta_R) = \int_x^\infty\mathbf{Z}(x,h)\frac{\sin(\beta_R h)}{\beta_R}\,\mathrm{d}h. \tag{165}$$

By isolating the components that are even and odd with respect to β_R, we arrive at:

$$\frac{\widetilde{\mathbf{U}}(x,g)+\widetilde{\mathbf{U}}(x,-g)}{2} = \cos(\beta_R x)\mathbf{I}_2+\frac{\sin(\beta_R x)}{\beta_R}\mathbf{Q}_R+\mathbf{U}_a(x,\beta_R)+\mathbf{U}_b(x,\beta_R)\mathbf{Q}_R, \tag{166a}$$

$$\frac{\widetilde{\mathbf{U}}(x,g)-\widetilde{\mathbf{U}}(x,-g)}{2} = -\mathrm{i}g\frac{\sin(\beta_R x)}{\beta_R}\sigma_3-\mathrm{i}g\mathbf{U}_b(x,\beta_R)\sigma_3. \tag{166b}$$

Consequently,

$$\mathbf{U}_b(x,\beta_R)=\frac{\widetilde{\mathbf{U}}(x,-g)-\widetilde{\mathbf{U}}(x,g)}{2\mathrm{i}g}\sigma_3-\frac{\sin(\beta_R x)}{\beta_R}\mathbf{I}_2, \tag{167a}$$

$$\mathbf{U}_a(x,\beta_R)=\frac{\widetilde{\mathbf{U}}(x,g)+\widetilde{\mathbf{U}}(x,-g)}{2}-\cos(\beta_R x)\mathbf{I}_2+\frac{\widetilde{\mathbf{U}}(x,g)-\widetilde{\mathbf{U}}(x,-g)}{2\mathrm{i}g}\sigma_3\mathbf{Q}_R. \tag{167b}$$

Proceeding from the identity,

$$\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{\mathrm{i}\beta_R h}\,\mathrm{d}h=\mathbf{U}_a(x,\beta_R)+\mathrm{i}\beta_R\mathbf{U}_b(x,\beta_R), \tag{168}$$

one may express it as:

$$\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{\mathrm{i}\beta_R h}\,\mathrm{d}h=\frac{1}{2}\widetilde{\mathbf{U}}(x,g)\left[\mathbf{I}_2-\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R-\frac{\beta_R}{g}\sigma_3\right]+\frac{1}{2}\widetilde{\mathbf{U}}(x,-g)\left[\mathbf{I}_2+\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R+\frac{\beta_R}{g}\sigma_3\right]-\mathrm{e}^{\mathrm{i}\beta_R x}\mathbf{I}_2. \tag{169}$$

Subsequently, the verification of the following identities can be readily undertaken:

$$\mathrm{e}^{x\mathbf{G}_R(g)}\left[\mathbf{I}_2-\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R-\frac{\beta_R}{g}\sigma_3\right]=\mathrm{e}^{\mathrm{i}\beta_R x}\left[\mathbf{I}_2-\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R-\frac{\beta_R}{g}\sigma_3\right], \tag{170a}$$

$$\mathrm{e}^{x\mathbf{G}_R(-g)}\left[\mathbf{I}_2+\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R+\frac{\beta_R}{g}\sigma_3\right]=\mathrm{e}^{\mathrm{i}\beta_R x}\left[\mathbf{I}_2+\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R+\frac{\beta_R}{g}\sigma_3\right]. \tag{170b}$$

Upon multiplying both sides of Eq. (169) by $\mathrm{e}^{-\mathrm{i}\beta_R x}$, and employing the aforementioned identities, we ascertain that:

$$\begin{aligned}\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{\mathrm{i}\beta_R(h-x)}\,\mathrm{d}h=&\frac{1}{2}\left[\widetilde{\mathbf{U}}(x,g)\mathrm{e}^{-x\mathbf{G}_R(g)}-\mathbf{I}_2\right]\left[\mathbf{I}_2-\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R-\frac{\beta_R}{g}\sigma_3\right]\\&+\frac{1}{2}\left[\widetilde{\mathbf{U}}(x,-g)\mathrm{e}^{-x\mathbf{G}_R(-g)}-\mathbf{I}_2\right]\left[\mathbf{I}_2+\frac{\mathrm{i}}{g}\sigma_3\mathbf{Q}_R+\frac{\beta_R}{g}\sigma_3\right],\end{aligned} \tag{171}$$

Hence, $\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{\mathrm{i}\beta_R(h-x)}\,\mathrm{d}h=\int_{-\infty}^\infty \mathbf{Z}(x,h)\mathrm{e}^{\mathrm{i}\beta_R(h-x)}\,\mathrm{d}h$, given that $\mathbf{Z}(x,h)\equiv 0$ for $x>h$, we consequently derive Eq. (62). Similarly, it can be demonstrated from Eq. (60b) that Eq. (63) holds. □

Proof of Theorem 3. Upon substituting Eq. (61a) into Eq. (3) and subsequently multiplying the resultant equation on the right by $\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x}$, we obtain:

$$\partial_x\left[\mathbf{U}(x,g)\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x}\right]-\mathrm{i}\beta_R\mathbf{U}(x,g)\sigma_3\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x}=[\mathbf{Q}(x)-\mathrm{i}g\sigma_3]\left[\mathbf{E}_R(g)+\int_x^\infty \mathbf{Z}(x,h)\mathbf{E}_R(g)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\right]. \tag{172}$$

Utilizing Eqs. (19), (50) and (51) with the assumption in Eq. (51) that $\partial_x q\in L^1(\mathbb{R})$, we derive the following:

$$\begin{aligned}&\left[\frac{\mathrm{i}\partial_x\mathbf{Q}(x)\sigma_3}{2g}-\mathrm{i}\beta_R\left(\mathbf{I}_2+\frac{\mathrm{i}\mathbf{Q}(x)\sigma_3}{2g}\right)\sigma_3\right][1+o(1)]=[\mathbf{Q}(x)-\mathrm{i}g\sigma_3]\left[\mathbf{I}_2+\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\right]\\&+\frac{\mathrm{i}[\mathbf{Q}(x)-\mathrm{i}g\sigma_3]}{\beta_R+g}\left[\mathbf{Q}_R+\int_x^\infty \mathbf{Z}(x,h)\mathbf{Q}_R\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\right]\sigma_3.\end{aligned} \tag{173}$$

That is to say,

$$\begin{aligned}&\mathrm{i}g\sigma_3+\frac{1}{2}\mathbf{Q}(x)+o(g^{-1})=-\mathrm{i}g\sigma_3+\mathbf{Q}(x)+\mathbf{Q}(x)\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\\&-\mathrm{i}\beta_R\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h+\frac{\mathrm{i}B_R^2}{\beta_R+g}\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\\&+\frac{1}{2}\left[1-\frac{B_R^2}{(\beta_R+g)^2}\right]\sigma_3\mathbf{Q}_R\sigma_3+\frac{\mathrm{i}\mathbf{Q}(x)\mathbf{Q}_R\sigma_3}{\beta_R+g}+\frac{\mathrm{i}[\mathbf{Q}(x)-\mathrm{i}g\sigma_3]}{\beta_R+g}\int_x^\infty \mathbf{Z}(x,h)\mathbf{Q}_R\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\sigma_3\,\mathrm{d}h.\end{aligned} \tag{174}$$

On the right-hand side, the third, fifth, and final terms encompass Fourier integrals of matrix functions, the entries of which are in $L^2(\mathbb{R};\mathrm{d}\beta_R)$, and are multiplied by factors that remain bounded as g becomes large; consequently, these terms tend to zero as $g,\beta_R\to\infty$. Subsequently, for the fourth term, we express it as:

$$\begin{aligned}-\mathrm{i}\beta_R\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h=&-\mathrm{i}\beta_R\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathbf{E}_R(g)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\\&-\frac{\beta_R}{\beta_R+g}\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathbf{Q}_R\sigma_3\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h,\end{aligned} \tag{175}$$

wherein the second term on the right features entries in $L^2(\mathbb{R};\mathrm{d}\beta_R)$. The initial term on the right is articulated as:

$$\begin{aligned}&-\mathrm{i}\beta_R\sigma_3\int_x^\infty \mathbf{Z}(x,h)\mathbf{E}_R(g)\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h=\sigma_3\left\{-\mathbf{Z}(x,x)\mathbf{E}_R(g)+\int_x^\infty\left[\partial_x\mathbf{Z}(x,h)\right]\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h\right.\\&\left.+\frac{\mathrm{i}}{\beta_R+g}\int_x^\infty\left[\partial_x\mathbf{Z}(x,h)\right]\mathbf{Q}_R\sigma_3\mathrm{e}^{-\mathrm{i}\beta_R\sigma_3(h-x)}\,\mathrm{d}h-\partial_x\left[\mathbf{U}(x,g)\mathrm{e}^{\mathrm{i}\beta_R\sigma_3 x}-\mathbf{E}_R(g)\right]\right\}\sigma_3.\end{aligned} \tag{176}$$

Let us now scrutinize the individual terms within the brackets on the right-hand side of the aforementioned identity. The first term simplifies to $-\mathbf{Z}(x,x)+o(1)$. According to Eq. (51), the terminal term dissipates as $g\to\pm\infty$, the penultimate term comprises entries in $L^2(\mathbb{R};\mathrm{d}\beta_R)$ (as it represents the Fourier transform of an L^2 matrix function), and the antepenultimate term of the final component is L^2 scaled by a bounded coefficient. Consequently, dismissing all contributions that tend to zero as $g\to\pm\infty$, and leveraging the fact that $\sigma_3\mathbf{Q}_R\sigma_3=-\mathbf{Q}_R$, we arrive at:

$$-\mathrm{i}g\sigma_3+\frac{1}{2}\mathbf{Q}(x)=-\mathrm{i}g\sigma_3+\mathbf{Q}(x)-\sigma_3\mathbf{Z}(x,x)\sigma_3-\frac{1}{2}\mathbf{Q}_R. \tag{177}$$

That is, the first part of Eq. (70) is established. The second equality in Eq. (70) can be demonstrated through a similar approach. □

Data availability

All data generated or analyzed during this study are included in this published article.

References

[1] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg–de Vries equation, Phys. Rev. Lett. 19 (1967) 1095.
[2] J.D. Gibbon, A.C. Newell, M. Tabor, Y.B. Zeng, Lax pairs, Bäcklund transformations and special solutions for ordinary differential equations, Nonlinearity 1 (1988) 481.
[3] J.C. Pu, Y. Chen, Lax pairs informed neural networks solving integrable systems, J. Comput. Phys. 510 (2024) 113090.
[4] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315.
[5] T. Trogdon, S. Olver, B. Deconinck, Numerical inverse scattering for the Korteweg–de Vries and modified Korteweg–de Vries equations, Phys. D 241 (2012) 1003–1025.
[6] D.J. Kaup, Y. Matsuno, The inverse scattering transform for the Benjamin–Ono equation, Stud. Appl. Math. 101 (1998) 73–98.
[7] M.J. Ablowitz, D.B. Yaacov, A.S. Fokas, On the inverse scattering transform for the Kadomtsev–Petviashvili equation, Stud. Appl. Math. 69 (1983) 135–143.
[8] L.Y. Sung, An inverse scattering transform for the Davey–Stewartson-II equations, I, J. Math. Anal. Appl. 183 (1994) 121–154.
[9] P.F. Han, Y. Zhang, Investigation of shallow water waves near the coast or in lake environments via the KdV-Calogero-Bogoyavlenskii-Schiff equation, Chaos Solitons Fractals 184 (2024) 115008.
[10] S.T. Li, B. Prinari, G. Biondini, Solitons and rogue waves in spinor Bose–Einstein condensates, Phys. Rev. E 97 (2018) 022221.
[11] A. Romero-Ros, G.C. Katsimiga, P.G. Kevrekidis, B. Prinari, G. Biondini, P. Schmelcher, On-demand generation of dark soliton trains in Bose–Einstein condensates, Phys. Rev. A 103 (2021) 023329.
[12] X.Y. Gao, In an ocean or a river: Bilinear auto-Bäcklund transformations and similarity reductions on an extended time-dependent (3+1)-dimensional shallow water wave equation, China Ocean Eng. 39 (2025) 160–165.
[13] W.X. Ma, Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger equations, Proc. Am. Math. Soc. 149 (2021) 251–263.
[14] H. Liu, P.P. Zhou, X.G. Geng, Inverse scattering transform for the Sasa-Satsuma equation: multiple-pole case of N pairs, Z. Angew. Math. Phys. 75 (2024) 193.
[15] L.L. Wen, E.G. Fan, Y. Chen, The Sasa-Satsuma equation on a non-zero background: the inverse scattering transform and multi-soliton solutions, Acta Math. Sci. 43 (2023) 1045–1080.
[16] J.L. Ji, Z.N. Zhu, Soliton solutions of an integrable nonlocal modified Korteweg–de Vries equation through inverse scattering transform, J. Math. Anal. Appl. 453 (2017) 973–984.
[17] X.J. Chen, W.K. Lam, Inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions, Phys. Rev. E 69 (2004) 066604.
[18] S.K. Cui, Z. Wang, Numerical inverse scattering transform for the derivative nonlinear Schrödinger equation, Nonlinearity 37 (2024) 105015.
[19] P.F. Han, R.S. Ye, Y. Zhang, Inverse scattering transform for the coupled Lakshmanan-Porsezian-Daniel equations with non-zero boundary conditions in optical fiber communications, Math. Comput. Simulation 232 (2025) 483–503.
[20] P.F. Han, W.X. Ma, R.S. Ye, Y. Zhang, Inverse scattering transform for the defocusing-defocusing coupled Hirota equations with non-zero boundary conditions: Multiple double-pole solutions, Phys. D 471 (2025) 134434.
[21] A.B.D. Monvel, I. Karpenko, D. Shepelsky, A Riemann-Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions, J. Math. Phys. 61 (2020) 031504.
[22] Y. Xiao, Q.Z. Zhu, X. Wu, Riemann-Hilbert problem and multiple poles solution for an extended modified Korteweg–de Vries equation with zero/nonzero boundary conditions, Nonlinear Dynam. 110 (2022) 1723–1746.
[23] B. Yang, Y. Chen, High-order soliton matrices for Sasa-Satsuma equation via local Riemann-Hilbert problem, Nonlinear Anal.-Real 45 (2019) 918–941.
[24] X.T. Chen, Y. Zhang, J.L. Liang, R. Wang, The N-soliton solutions for the matrix modified Korteweg–de Vries equation via the Riemann-Hilbert approach, Eur. Phys. J. Plus 135 (2020) 574.
[25] Y.L. Yang, E.G. Fan, Riemann-Hilbert approach to the modified nonlinear Schrödinger equation with non-vanishing asymptotic boundary conditions, Phys. D 417 (2021) 132811.
[26] W.Q. Peng, S.F. Tian, X.B. Wang, T.T. Zhang, Y. Fang, Riemann-Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations, J. Geom. Phys. 146 (2019) 103508.
[27] Z.F. Zou, R. Guo, The Riemann-Hilbert approach for the higher-order Gerdjikov-Ivanov equation, soliton interactions and position shift, Commun. Nonlinear Sci. 124 (2023) 107316.
[28] M. Lakshmanan, K. Porsezian, M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, Phys. Lett. A 133 (1988) 483–488.
[29] K. Porsezian, M. Daniel, M. Lakshmanan, On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain, J. Math. Phys. 33 (1992) 1807–1816.
[30] W.F. Weng, G.Q. Zhang, Z.Y. Yan, Strong and weak interactions of rational vector rogue waves and solitons to any n-component nonlinear Schrödinger system with higher-order effects, Proc. R. Soc. A-Math. Phys. 478 (2022) 20210670.
[31] H.Y. Zhang, Y.F. Zhang, Spectral analysis and long-time asymptotics of complex mKdV equation, J. Math. Phys. 63 (2022) 021509.
[32] S.Y. Chen, Z.Y. Yan, B.L. Guo, Long-time asymptotics for the focusing Hirota equation with non-zero boundary conditions at infinity via the Deift-Zhou approach, Math. Phys. Anal. Geom. 24 (2021) 17.
[33] C.J. Wang, J. Zhang, Inverse scattering transform for the focusing Hirota equation with asymmetric boundary conditions, Theoret. Math. Phys. 221 (2024) 2109–2131.

[34] X.H. Wu, Y.T. Gao, X. Yu, C.C. Ding, L.Q. Li, Modified generalized Darboux transformation and solitons for a Lakshmanan-Porsezian-Daniel equation, Chaos Solitons Fractals 162 (2022) 112399.
[35] G.Q. Zhang, S.Y. Chen, Z.Y. Yan, Focusing and defocusing Hirota equations with non-zero boundary conditions: Inverse scattering transforms and soliton solutions, Commun. Nonlinear Sci. 80 (2020) 104927.
[36] H.Q. Zhang, S.S. Yuan, Dark soliton solutions of the defocusing Hirota equation by the binary Darboux transformation, Nonlinear Dynam. 89 (2017) 531–538.
[37] F. Demontis, B. Prinari, C.V.D. Mee, F. Vitale, The inverse scattering transform for the focusing nonlinear Schrödinger equation with asymmetric boundary conditions, J. Math. Phys. 55 (2014) 101505.
[38] G. Biondini, E. Fagerstrom, B. Prinari, Inverse scattering transform for the defocusing nonlinear Schrödinger equation with fully asymmetric non-zero boundary conditions, Phys. D 333 (2016) 117–136.
[39] Y. Zhao, D.H. Zhu, A Riemann-Hilbert approach for the focusing and defocusing mKdV equation with asymmetric boundary conditions in few-cycle pulses, Eur. Phys. J. Plus 139 (2024) 603.
[40] R.S. Ye, P.F. Han, Y. Zhang, Defocusing Hirota equation with fully asymmetric non-zero boundary conditions: the inverse scattering transform, 2024, arxiv preprint arxiv:2401.16684.
[41] C.J. Wang, J. Zhang, Inverse scattering transform for the focusing Hirota equation with asymmetric boundary conditions, Theoret. Math. Phys. 221 (2024) 2109–2131.
[42] J.Y. Ji, X.Y. Xie, Riemann-Hilbert problem for the defocusing Lakshmanan-Porsezian-Daniel equation with fully asymmetric nonzero boundary conditions, Chin. Phys. B 33 (2024) 090201.
[43] A.B.D. Monvel, A. Its, V. Kotlyarov, Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line, Comm. Math. Phys. 290 (2009) 479–522.
[44] A.B.D. Monvel, V.P. Kotlyarov, D. Shepelsky, Focusing NLS equation: long-time dynamics of step-like initial data, Int. Math. Res. Not. IMRN 7 (2011) 1613–1653.
[45] A.B.D. Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics, Comm. Math. Phys. 383 (2021) 893–952.
[46] Y. Rybalko, D. Shepelsky, Long-time asymptotics for the integrable nonlocal focusing nonlinear Schrödinger equation for a family of step-like initial data, Comm. Math. Phys. 382 (2021) 87–121.
[47] A.B.D. Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: the genus 3 sector, Comm. Math. Phys. 390 (2022) 1081–1148.
[48] A.B.D. Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: Asymptotics in a transition zone, J. Differential Equations 429 (2025) 747–801.
[49] A. Minakov, Long-time behavior of the solution to the mKdV equation with step-like initial data, J. Phys. A-Math. Theor. 44 (2011) 085206.
[50] T. Grava, A. Minakov, On the long-time asymptotic behavior of the modified Korteweg–de Vries equation with step-like initial data, SIAM J. Math. Anal. 52 (2020) 5892–5993.
[51] P. Deift, S. Venakides, X. Zhou, The collisionless shock region for the long-time behavior of solutions of the KdV equation, Commun. Pure Appl. Math. 47 (1994) 199–206.
[52] P.F. Han, T. Bao, Hybrid localized wave solutions for a generalized Calogero-Bogoyavlenskii-Konopelchenko-Schiff system in a fluid or plasma, Nonlinear Dynam. 108 (2022) 2513–2530.
[53] X.Y. Gao, Two-layer-liquid and lattice considerations through a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system, Appl. Math. Lett. 152 (2024) 109018.
[54] P.F. Han, Y. Zhang, C.H. Jin, Novel evolutionary behaviors of localized wave solutions and bilinear auto-Bäcklund transformations for the generalized (3+1)-dimensional Kadomtsev–Petviashvili equation, Nonlinear Dynam. 111 (2023) 8617–8636.
[55] X.Y. Gao, Auto-Bäcklund transformation with the solitons and similarity reductions for a generalized nonlinear shallow water wave equation, Qual. Theory Dyn. Syst. 23 (2024) 181.
[56] P.F. Han, Y. Zhang, Linear superposition formula of solutions for the extended (3+1)-dimensional shallow water wave equation, Nonlinear Dynam. 109 (2022) 1019–1032.
[57] X.Y. Gao, Hetero-Bäcklund transformation, bilinear forms and multi-solitons for a (2+1)-dimensional generalized modified dispersive water-wave system for the shallow water, Chinese J. Phys. 92 (2024) 1233–1239.
[58] P.F. Han, Y. Zhang, Superposition behavior of the lump solutions and multiple mixed function solutions for the (3+1)-dimensional Sharma-Tasso-Olver-like equation, Eur. Phys. J. Plus 139 (2024) 157.
[59] X.Y. Gao, In the shallow water: Auto-Bäcklund, hetero-Bäcklund and scaling transformations via a (2+1)-dimensional generalized Broer-Kaup system, Qual. Theory Dyn. Syst. 23 (2024) 184.
[60] X.Y. Gao, In plasma physics and fluid dynamics: Symbolic computation on a (2+1)-dimensional variable-coefficient Sawada-Kotera system, Appl. Math. Lett. 159 (2025) 109262.