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Inverse scattering transform for the defocusing–defocusing coupled Hirota equations with non-zero boundary conditions: Multiple double-pole solutions

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ABSTRACT

The inverse scattering transform for the defocusing–defocusing coupled Hirota equations with non-zero boundary conditions at infinity is thoroughly discussed. We delve into the analytical properties of the Jost eigenfunctions and scrutinize the characteristics of the scattering coefficients. To enhance our investigation of the fundamental eigenfunctions, we have derived additional auxiliary eigenfunctions with the help of the adjoint problem. Two symmetry conditions are studied to constrain the behavior of the eigenfunctions and scattering coefficients. Utilizing these symmetries, we precisely delineate the discrete spectrum and establish the associated symmetries of the scattering data. By framing the inverse problem within the context of the Riemann–Hilbert problem, we develop suitable jump conditions to express the eigenfunctions. Consequently, we have not only derived the pure soliton solutions from the defocusing–defocusing coupled Hirota equations but also provided the multiple double-pole solutions for the first time.

1. Introduction

In the vastness of nature, nonlinear wave phenomena play an important role. Their dynamic behavior is described by nonlinear wave equations, which are a class of evolutionary nonlinear partial differential equations [1]. These equations are the basis for understanding a large number of phenomena in various scientific fields. In particular, the nonlinear Schrödinger (NLS) equation which is expressed in scalar [2], vector [3], and matrix [4] forms, serves as a universal model for the evolution of weakly nonlinear dispersive wave trains. This makes it an indispensable tool in the study of deep-water waves [5] and nonlinear optics. Furthermore, the NLS equations are instrumental in elucidating the complex phenomena of modulational instability [6] and the genesis of rogue waves [7].

In order to solve the mystery of these nonlinear systems, the inverse scattering transform (IST) is a powerful analytical technique. IST was first proposed by Gardner, Greene, Kruskal, and Miura in 1967 [8], aiming to provide an exact solution to the initial value problem of the Korteweg–de-Vries equation by using the Lax pairs [9]. This pioneering method was later extended to many integrable systems characterized by Lax pairs, providing a principled method for solving the initial value problems [10,11]. When applicable, the IST serves as an efficacious tool for dissecting the intricate behavior of solutions [12–14]. Despite its utility, the formulation of the IST remains an open challenge in certain scenarios, indicating that there is still much to explore and understand in this intricate field of study.

The Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy is an important research field that provides a powerful mathematical framework for understanding various nonlinear phenomena. In the case of local group reduction, the AKNS hierarchy can be simplified by applying two local group reductions to obtain a specific instance of the Sasa-Satsuma type matrix integrable hierarchies [15]. From a mathematical perspective, the general coupled Hirota equations are a linear combination of a special case of a reduced AKNS hierarchy with one local group reduction [16].

The general coupled Hirota equations [17–28] offer a comprehensive model for studying the wave propagation of two ultrashort optical fields in optical fibers, accounting for the intricate interplay of nonlinear and dispersive effects that govern the evolution of the pulses as they travel

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through the fiber.

$$\mathrm{i}\widetilde{q}_{1,t}+\widetilde{q}_{1,xx}+2(\sigma_1|\widetilde{q}_1|^2+\sigma_2|\widetilde{q}_2|^2)\widetilde{q}_1+\mathrm{i}\sigma[\widetilde{q}_{1,xxx}+(6\sigma_1|\widetilde{q}_1|^2+3\sigma_2|\widetilde{q}_2|^2)\widetilde{q}_{1,x}+3\sigma_2\widetilde{q}_1\widetilde{q}_2^*\widetilde{q}_{2,x}]=0, \tag{1a}$$

$$\mathrm{i}\widetilde{q}_{2,t}+\widetilde{q}_{2,xx}+2(\sigma_1|\widetilde{q}_1|^2+\sigma_2|\widetilde{q}_2|^2)\widetilde{q}_2+\mathrm{i}\sigma[\widetilde{q}_{2,xxx}+(6\sigma_2|\widetilde{q}_2|^2+3\sigma_1|\widetilde{q}_1|^2)\widetilde{q}_{2,x}+3\sigma_1\widetilde{q}_2\widetilde{q}_1^*\widetilde{q}_{1,x}]=0, \tag{1b}$$

where $\widetilde{q}_1=\widetilde{q}_1(x,t)$ and $\widetilde{q}_2=\widetilde{q}_2(x,t)$ are the two-component electric fields, the parameters σ_1, σ_2 and σ are real constants [17]. Given the positivity of σ_1 and σ_2, the Eqs. (1) are designated as the focusing–focusing coupled Hirota equations, which lead to the energy concentration of the wave during the interaction and the focusing effect. With the assumption that σ_1 and σ_2 are negative, the set of Eqs. (1) is referred to as the defocusing–defocusing coupled Hirota equations, which lead to the energy concentration of the wave during the interaction and the defocusing effect. Supposing σ_1 and σ_2 exhibit opposite signs, the Eqs. (1) are categorized as the mixed coupled Hirota equations [17], which lead to the coexistence of focusing and defocusing of waves, resulting in more complex wave dynamics.

In 1992, Tasgal and Potasek [18] employed the IST to derive soliton solutions for the coupled higher-order NLS equations within a specific parameter regime. This work underscores the integrability of the general coupled Hirota equations, characterized by the presence of the Lax pair, Nth order Darboux transformation and a variety of localized wave solutions, as further elaborated in [17,19]. It has been shown that the general coupled Hirota equations also admit the dark soliton solutions [20], the high-order rational rogue waves and multi-dark soliton structures [21], rogue wave solutions [22,23], semirational solutions [24], analytical solutions [25], dark-bright-rogue wave solutions [26], the interactions between breathers and rogue waves [27], the interactions between dark-bright solitons and rogue waves [27]. Utilizing nonlinear steepest descent techniques, the leading-order asymptotic expressions and consistent error bounds for solutions to the coupled Hirota equations were meticulously examined, as detailed in [28].

Recently, the IST and Riemann-Hilbert (RH) method have been extensively applied to investigate soliton solutions for the Hirota equation, as evidenced by recent studies [29–32]. Furthermore, these methods have yet to be explored in the context of the more complex, general coupled Hirota equations with the non-zero boundary conditions (NZBCs) at infinity. In light of this gap in the literature, the present paper aims to delve into the application of the IST to the defocusing–defocusing coupled Hirota equations, offering a novel perspective on this under-explored area. When $\sigma_1=\sigma_2=-1$, Eqs. (1) can be converted to the defocusing–defocusing coupled Hirota equations:

$$\mathrm{i}\widetilde{\mathbf{q}}_t+\widetilde{\mathbf{q}}_{xx}-2\|\widetilde{\mathbf{q}}\|^2\widetilde{\mathbf{q}}+\mathrm{i}\sigma[\widetilde{\mathbf{q}}_{xxx}-3\|\widetilde{\mathbf{q}}\|^2\widetilde{\mathbf{q}}_x-3(\widetilde{\mathbf{q}}^\dagger\widetilde{\mathbf{q}}_x)\widetilde{\mathbf{q}}]=\mathbf{0}, \tag{2}$$

where $\widetilde{\mathbf{q}}(x,t)=(\widetilde{q}_1(x,t),\widetilde{q}_2(x,t))^T$. Employing the variable transformation $\widetilde{\mathbf{q}}(x,t)=\mathbf{q}(x,t)\mathrm{e}^{-2\mathrm{i}q_0^2t}$, we can derive the defocusing–defocusing coupled Hirota equations with the NZBCs, namely

$$\mathrm{i}\mathbf{q}_t+\mathbf{q}_{xx}-2(\|\mathbf{q}\|^2-q_0^2)\mathbf{q}+\mathrm{i}\sigma[\mathbf{q}_{xxx}-3\|\mathbf{q}\|^2\mathbf{q}_x-3(\mathbf{q}^\dagger\mathbf{q}_x)\mathbf{q}]=\mathbf{0}, \tag{3}$$

and the corresponding NZBCs at infinity are

$$\lim_{x\to\pm\infty}\mathbf{q}(x,t)=\mathbf{q}_\pm=\mathbf{q}_0\mathbf{e}^{\mathrm{i}\delta_\pm}, \tag{4}$$

where $\mathbf{q}=\mathbf{q}(x,t)=(q_1(x,t),q_2(x,t))^T$ and $\mathbf{q}_0$ are two-component vectors, $q_0=\|\mathbf{q}_0\|$, with $\delta_\pm$ are real numbers. For the defocusing–defocusing coupled Hirota equations with the NZBCs $\mathbf{q}_+$ and $\mathbf{q}_-$ at infinity, the scenarios of parallel and non-parallel orientations have not yet been explored. This paper initially focuses on the case where $\mathbf{q}_+$ is parallel to $\mathbf{q}_-$, with the intention to address the non-parallel case of $\mathbf{q}_+$ and $\mathbf{q}_-$ in future research endeavors.

Compared with the defocusing Hirota equation [29], the defocusing–defocusing coupled Hirota equations [17] are associated with a 3×3 matrix Lax pair, which makes the study of spectral analysis very difficult. In the case of the NZBCs, the study of a 3×3 matrix Lax pair usually has the problem that the analytical Jost eigenfunction is not analytical, which will make it more difficult to construct the IST. Specifically, the IST has demonstrated its unique value in the study of nonlinear wave equations with specific boundary conditions [33,34]. For example, when studying the focusing [35] and defocusing [36] Manakov systems with the NZBCs at infinity, the application of IST enables researchers to analyze the dynamic characteristics of dark–dark and dark-bright solitons in detail. In addition, the IST has also proved its effectiveness in solving the coupled Gerdjikov-Ivanov equation [37] with the NZBCs, which not only helps to reveal the existence of the dark–dark solitons, bright–bright and breather–breather, but also provides insight into understanding their interaction. Based on these studies [33–37], it is planned to further explore the application of IST in solving nonlinear equations with more complex boundary conditions [38]. In this paper, we will use the IST method to study the defocusing–defocusing coupled Hirota equations with the parallel NZBCs at infinity in order to more fully understand and predict their analytical and asymptotic properties.

The structure of the remaining sections of this paper is outlined as follows. Section 2 delves into the intricacies of the direct scattering problem. Initially, we delineate the Jost functions associated with the Lax pair, ensuring they adhere to the stipulated boundary conditions. Subsequently, we scrutinize the analytical characteristics of the modified eigenfunctions, leveraging the foundational definitions of the Jost functions. Furthermore, we rigorously establish the analytical properties of the corresponding coefficients of the scattering matrix, based on the precise formulation of the scattering matrix. Ultimately, we address the adjoint problem by delineating the auxiliary eigenfunctions, facilitating the derivation of symmetries for the Jost eigenfunctions, scattering coefficients, and auxiliary eigenfunctions. In Section 3, we delve into the characterization of the discrete spectrum. Additionally, we systematically analyze the asymptotic behavior of the modified Jost eigenfunctions and the scattering matrix elements for $z\to\infty$ and $z\to0$. In Section 4, based on the RH problem to formulate the inverse problem, we construct appropriate jump conditions to express the eigenfunctions. By using the meromorphic matrices, the corresponding residue conditions and norming constants are obtained. We construct the formal solutions of the RH problem and reconstruction formula with the help of the Plemelj's formula. The pure soliton solutions are derived within the framework of reflectionless potentials and comprehensively proven. Subsequently, the discussion delves into the categorization of solitons possessing discrete eigenvalues, both within and beyond the specified circumference. In Section 5, we derive the solutions associated with multiple double zeros of the analytic scattering coefficients, and explicitly present the solutions for multiple double poles. The results are summarized in Section 6.

2. Direct scattering problem

Generally speaking, the IST of integrable nonlinear equations needs to be studied through the formula of their Lax pairs. Our calculations are based on the following 3×3 Lax pair, which corresponds to the defocusing–defocusing coupled Hirota equations (3)

$$\psi_x = \mathbf{X}\psi, \quad \psi_t = \mathbf{T}\psi, \tag{5}$$

where $\psi = \psi(x,t)$, the matrices $\mathbf{X}$ and $\mathbf{T}$ are written as

$$\mathbf{X} = \mathbf{X}(k;x,t) = \mathrm{i}k\mathbf{J} + \mathrm{i}\mathbf{Q}, \tag{6a}$$

$$\mathbf{T} = \mathbf{T}(k;x,t) = 4\mathrm{i}\sigma k^3\mathbf{J} - \mathrm{i}q_0^2\mathbf{J} + k^2(4\mathrm{i}\sigma\mathbf{Q} + 2\mathrm{i}\mathbf{J}) + k(2\mathrm{i}\mathbf{Q} - 2\sigma\mathbf{Q}_x\mathbf{J} - 2\mathrm{i}\sigma\mathbf{J}\mathbf{Q}^2) - 2\mathrm{i}\sigma\mathbf{Q}^3 - \mathrm{i}\mathbf{J}\mathbf{Q}^2 - \mathrm{i}\sigma\mathbf{Q}_{xx} - \mathbf{Q}_x\mathbf{J} + \sigma[\mathbf{Q},\mathbf{Q}_x], \tag{6b}$$

with $[Q_a, Q_b] = Q_aQ_b - Q_bQ_a$, while the definitions of $\mathbf{J}$ and $\mathbf{Q} = \mathbf{Q}(x,t)$ are as follows:

$$\mathbf{J} = \begin{pmatrix} 1 & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & -\mathbf{I}_{2\times 2} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & -\mathbf{q}^\dagger \\ \mathbf{q} & \mathbf{0}_{2\times 2} \end{pmatrix}, \quad \mathbf{J}\mathbf{Q} = -\mathbf{Q}\mathbf{J}, \tag{7}$$

where k is the spectral parameter and $\dagger$ denotes conjugate transpose. The compatibility condition $\psi_{xt} = \psi_{tx}$ is ascertained through the zero-curvature equation $\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X},\mathbf{T}] = \mathbf{0}$.

2.1. Jost solutions and scattering matrix

Taking into account the Jost eigenfunctions as $x \to \pm\infty$, the spatial and temporal evolution of the solutions for the asymptotic Lax pair can be described as follows:

$$\psi_x = \mathbf{X}_\pm\psi, \quad \psi_t = \mathbf{T}_\pm\psi, \tag{8}$$

where

$$\lim_{x\to\pm\infty} \mathbf{X} = \mathbf{X}_\pm = \mathrm{i}k\mathbf{J} + \mathrm{i}\mathbf{Q}_\pm, \tag{9a}$$

$$\lim_{x\to\pm\infty} \mathbf{T} = \mathbf{T}_\pm = 4\mathrm{i}\sigma k^3\mathbf{J} - \mathrm{i}q_0^2\mathbf{J} + k^2(4\mathrm{i}\sigma\mathbf{Q}_\pm + 2\mathrm{i}\mathbf{J}) + k(2\mathrm{i}\mathbf{Q}_\pm - 2\mathrm{i}\sigma\mathbf{J}\mathbf{Q}_\pm^2) - 2\mathrm{i}\sigma\mathbf{Q}_\pm^3 - \mathrm{i}\mathbf{J}\mathbf{Q}_\pm^2. \tag{9b}$$

By the definition of $\mathbf{X}_\pm$ and $\mathbf{T}_\pm$ in (9), the eigenvalues of the corresponding matrix can be derived as follows:

$$\mathbf{X}_{\pm,1} = -\mathrm{i}k, \quad \mathbf{X}_{\pm,2,3} = \pm\mathrm{i}\lambda, \quad \mathbf{T}_{\pm,1} = -\mathrm{i}(\lambda^2 + k^2 + 4\sigma k^3), \quad \mathbf{T}_{\pm,2,3} = \pm 2\mathrm{i}\lambda[k + \sigma(3k^2 - \lambda^2)], \tag{10}$$

where

$$\lambda(k) = \sqrt{k^2 - q_0^2}. \tag{11}$$

Biondini et al. introduced the two-sheeted Riemann surface [36] defined by (11), and $\lambda(k)$ is a single-valued function of k that satisfies $\lambda(\pm q_0) = 0$. Then, the branch points are $k = \pm\mathrm{i}q_0$. We define the uniformization variable

$$z = k + \lambda, \tag{12}$$

whose corresponding inverse map is given by

$$k(z) = \frac{1}{2}\left(z + \frac{q_0^2}{z}\right), \quad \lambda(z) = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right). \tag{13}$$

The relevant theories and property descriptions of the two-sheeted Riemann surface can be referred to in [36]. Therefore, it can be defined that

$$\mathbb{D}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad \mathbb{D}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}. \tag{14}$$

The analytical regions of the eigenfunctions are determined by the sign of $\operatorname{Im}\lambda(z)$. Therefore, all k dependencies will be rewritten as dependencies on z. The continuous spectrum of k is given by $k \in \mathbb{R}\backslash(-q_0, q_0)$. In the complex z-plane, the corresponding set is the whole real axis. Let us define a two-component vector $\mathbf{v} = (v_1, v_2)^T$ and its corresponding orthogonal vector as $\mathbf{v}^\perp = (v_2, -v_1)^\dagger$. Consider the eigenvector matrix of the asymptotic Lax pair (8) as follows:

$$\mathbf{Y}_\pm(z) = \begin{pmatrix} \mathrm{i} & 0 & \frac{q_0}{z} \\ \frac{\mathrm{i}\mathbf{q}_\pm}{z} & \frac{\mathbf{q}_\pm^\perp}{q_0} & \frac{\mathbf{q}_\pm}{q_0} \end{pmatrix}, \quad \det\mathbf{Y}_\pm(z) = \mathrm{i}\rho(z), \quad \rho(z) = 1 - \frac{q_0^2}{z^2}, \tag{15}$$

where

$$\mathbf{Y}_\pm^{-1}(z) = \frac{1}{\mathrm{i}\rho(z)}\begin{pmatrix} 1 & -\frac{\mathbf{q}_\pm^\dagger}{z} \\ 0 & \frac{\mathrm{i}\rho(z)}{q_0}(\mathbf{q}_\pm^\perp)^\dagger \\ -\frac{\mathrm{i}q_0}{z} & \frac{\mathrm{i}\mathbf{q}_\pm^\dagger}{q_0} \end{pmatrix}, \quad \det\mathbf{Y}_\pm^{-1}(z) = \frac{1}{\mathrm{i}\rho(z)}, \tag{16}$$

so that

$$\mathbf{X}_\pm\mathbf{Y}_\pm = \mathrm{i}\mathbf{Y}_\pm\mathbf{\Lambda}_1, \quad \mathbf{T}_\pm\mathbf{Y}_\pm = \mathrm{i}\mathbf{Y}_\pm\mathbf{\Lambda}_2, \tag{17}$$

where $\mathbf{Y}_\pm = \mathbf{Y}_\pm(z)$, the definitions of $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1(z)$ and $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2(z)$ are as follows:

$$\mathbf{\Lambda}_1(z) = \mathrm{diag}\,(\lambda, -k, -\lambda)\,, \tag{18a}$$

$$\mathbf{\Lambda}_2(z) = \mathrm{diag}\left(2\lambda[k+\sigma(3k^2-\lambda^2)], -(\lambda^2+k^2+4\sigma k^3), -2\lambda[k+\sigma(3k^2-\lambda^2)]\right). \tag{18b}$$

Since the NZBCs are constant, the relationship $[\mathbf{X}_\pm, \mathbf{T}_\pm] = \mathbf{0}$ can be calculated using expression (9). Therefore, $\mathbf{X}_\pm$ and $\mathbf{T}_\pm$ have a common eigenvector. Then, the Jost solutions $\psi(z;x,t)$ of the Lax pair (8) on $z \in \mathbb{R}$ satisfying the boundary conditions

$$\psi_\pm(z;x,t) = \mathbf{Y}_\pm(z)\mathrm{e}^{\mathrm{i}\mathbf{\Delta}(z;x,t)} + o(1), \quad x \to \pm\infty, \tag{19}$$

where $\mathbf{\Delta}(z;x,t) = \mathrm{diag}\left(\delta_1, \delta_2, -\delta_1\right)$ and

$$\delta_1 = \delta_1(z;x,t) = \lambda x + 2\lambda[k+\sigma(3k^2-\lambda^2)]t, \quad \delta_2 = \delta_2(z;x,t) = -kx - (\lambda^2+k^2+4\sigma k^3)t. \tag{20}$$

Consistently, we define the modified eigenfunctions

$$v_\pm(z;x,t) = \psi_\pm(z;x,t)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z;x,t)}, \tag{21}$$

so that $\lim_{x\to\pm\infty} v_\pm(z;x,t) = \mathbf{Y}_\pm(z)$. We perform factor decomposition on the asymptotic behavior of the potential and rewrite the Lax pair (8) as follows:

$$(\psi_\pm)_x = \mathbf{X}_\pm\psi_\pm + (\mathbf{X}-\mathbf{X}_\pm)\psi_\pm, \quad (\psi_\pm)_t = \mathbf{T}_\pm\psi_\pm + (\mathbf{T}-\mathbf{T}_\pm)\psi_\pm, \tag{22}$$

where $\psi_\pm = \psi_\pm(z;x,t)$ and the following systems can be exported through the Lax pair (22)

$$(\mathbf{Y}_\pm^{-1}v_\pm)_x = \mathrm{i}\mathbf{\Lambda}_1\mathbf{Y}_\pm^{-1}v_\pm - \mathrm{i}\mathbf{Y}_\pm^{-1}v_\pm\mathbf{\Lambda}_1 + \mathbf{Y}_\pm^{-1}(\mathbf{X}-\mathbf{X}_\pm)v_\pm, \qquad (\mathbf{Y}_\pm^{-1}v_\pm)_t = \mathrm{i}\mathbf{\Lambda}_2\mathbf{Y}_\pm^{-1}v_\pm - \mathrm{i}\mathbf{Y}_\pm^{-1}v_\pm\mathbf{\Lambda}_2 + \mathbf{Y}_\pm^{-1}(\mathbf{T}-\mathbf{T}_\pm)v_\pm, \tag{23}$$

where $\mathbf{Y}_\pm^{-1} = \mathbf{Y}_\pm^{-1}(z)$ and $v_\pm = v_\pm(z;x,t)$. Subsequently, the systems (23) can be expressed in complete differential form

$$\mathrm{d}[\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}}\mathbf{Y}_\pm^{-1}v_\pm\mathrm{e}^{\mathrm{i}\mathbf{\Delta}}] = \mathrm{e}^{-\mathrm{i}\mathbf{\Delta}}[\mathbf{Y}_\pm^{-1}(\mathbf{X}-\mathbf{X}_\pm)v_\pm\,\mathrm{d}x + \mathbf{Y}_\pm^{-1}(\mathbf{T}-\mathbf{T}_\pm)v_\pm\,\mathrm{d}t]\mathrm{e}^{\mathrm{i}\mathbf{\Delta}}, \tag{24}$$

where $\mathbf{\Delta} = \mathbf{\Delta}(z;x,t)$. The choice of integration path is independent of t, it has been confirmed that the spectral problem concerning $v_\pm(z;x,t)$ is equivalent to the Volterra integral equations

$$v_-(z;x,t) = \mathbf{Y}_- + \int_{-\infty}^{x} \mathbf{Y}_-\mathrm{e}^{\mathrm{i}(x-y)\mathbf{\Lambda}_1}[\mathbf{Y}_-^{-1}(\mathbf{X}(z;y,t)-\mathbf{X}_-(z))v_-(z;y,t)]\mathrm{e}^{-\mathrm{i}(x-y)\mathbf{\Lambda}_1}\,\mathrm{d}y, \tag{25a}$$

$$v_+(z;x,t) = \mathbf{Y}_+ - \int_{x}^{\infty} \mathbf{Y}_+\mathrm{e}^{\mathrm{i}(x-y)\mathbf{\Lambda}_1}[\mathbf{Y}_+^{-1}(\mathbf{X}(z;y,t)-\mathbf{X}_+(z))v_+(z;y,t)]\mathrm{e}^{-\mathrm{i}(x-y)\mathbf{\Lambda}_1}\,\mathrm{d}y. \tag{25b}$$

In Appendix A, we provide a detailed proof of the following theorems.

Theorem 1. *Suppose that $\mathbf{q}(\cdot,t)-\mathbf{q}_- \in L^1(-\infty,\sigma_3)$ ($\mathbf{q}(\cdot,t)-\mathbf{q}_+ \in L^1(\sigma_3,\infty)$) for every fixed $\sigma_3 \in \mathbb{R}$, the ensuing columns of $v_-(z;x,t)$ ($v_+(z;x,t)$) fulfill the requisite properties:*

$$v_{-,1}(z;x,t) \text{ and } v_{+,3}(z;x,t) \ :\ z \in \mathbb{D}^-, \qquad v_{-,3}(z;x,t) \text{ and } v_{+,1}(z;x,t) \ :\ z \in \mathbb{D}^+. \tag{26}$$

Eq. (21) suggests that the columns of $\psi_\pm(z;x,t)$ and $v_\pm(z;x,t)$ have the same analytic and bounded characteristics. Assuming $\psi(z;x,t)$ is a solution of the Lax pair (5), it can be obtained that

$$\partial_x[\det\psi_\pm] = \mathrm{tr}\,\mathbf{X}\,\det\psi_\pm = -\mathrm{i}k\,\det\psi_\pm, \qquad \partial_t[\det\psi_\pm] = \mathrm{tr}\,\mathbf{T}\,\det\psi_\pm = -\mathrm{i}(\lambda^2+k^2+4\sigma k^3)\,\det\psi_\pm. \tag{27}$$

From Abel's theorem, it can be inferred that $\partial x[\det v_\pm] = 0$ and $\partial t[\det v_\pm] = 0$. Then (19) implies

$$\det\psi_\pm(z;x,t) = \mathrm{i}\rho(z)\mathrm{e}^{\mathrm{i}\delta_2(z;x,t)}, \quad (x,t) \in \mathbb{R}^2, \quad z \in \mathbb{R}\backslash\{\pm q_0\}. \tag{28}$$

The scattering matrix $\mathbf{H}(z)$ and $\mathbf{S}(z)$ are characterized through the following definition

$$\psi_+(z;x,t) = \psi_-(z;x,t)\mathbf{H}(z), \quad z \in \mathbb{R}\backslash\{\pm q_0\}, \tag{29}$$

where $\mathbf{H}(z) = (h_{ij}(z))$. According to the definition, $\mathbf{H}(z)$ is independent of x and t variables. Using (28) and (29) can indicate that

$$\det\mathbf{H}(z) = 1, \quad z \in \mathbb{R}\backslash\{\pm q_0\}. \tag{30}$$

Similarly, define $\mathbf{S}(z) = \mathbf{H}^{-1}(z) = (s_{ij}(z))$.

Theorem 2. *According to the same assumption in Theorem 1, the scattering coefficients have the following properties:*

$$s_{33}(z) \text{ and } h_{11}(z) \ :\ z \in \mathbb{D}^+, \qquad s_{11}(z) \text{ and } h_{33}(z) \ :\ z \in \mathbb{D}^-. \tag{31}$$

2.2. Adjoint problem

Since the $v_{\pm,2}(z;x,t)$ is not analytic, then solving the inverse problem requires handling this non-analytic term. In order to set up the scattering problem, it is essential to possess a fully analytic function. We now turn our attention to the so-called "adjoint" Lax pair, which is a key component in this context.

$$\widetilde{\psi}_x = \widetilde{\mathbf{X}}\widetilde{\psi}, \quad \widetilde{\psi}_t = \widetilde{\mathbf{T}}\widetilde{\psi}, \tag{32}$$

where

$$\widetilde{\mathbf{X}}=\widetilde{\mathbf{X}}(k;x,t)=-\mathrm{i}k\mathbf{J}-\mathrm{i}\mathbf{Q}^*, \tag{33a}$$

$$\widetilde{\mathbf{T}}=\widetilde{\mathbf{T}}(k;x,t)=-4\mathrm{i}\sigma k^3\mathbf{J}+\mathrm{i}q_0^2\mathbf{J}-k^2(4\mathrm{i}\sigma\mathbf{Q}^*+2\mathrm{i}\mathbf{J})+k(2\mathrm{i}\sigma\mathbf{J}(\mathbf{Q}^*)^2-2\mathrm{i}\mathbf{Q}^*-2\sigma\mathbf{Q}_x^*\mathbf{J})+2\mathrm{i}\sigma(\mathbf{Q}^*)^3+\mathrm{i}\mathbf{J}(\mathbf{Q}^*)^2+\mathrm{i}\sigma\mathbf{Q}_{xx}^*-\mathbf{Q}_x^*\mathbf{J}+\sigma[\mathbf{Q}^*,\mathbf{Q}_x^*], \tag{33b}$$

where $\widetilde{\psi}=\widetilde{\psi}(z,x,t)$, with $\widetilde{\mathbf{X}}=\mathbf{X}^*$ and $\widetilde{\mathbf{T}}=\mathbf{T}^*$ for all $z\in\mathbb{R}$. The following proposition can be directly proven through properties $\mathbf{JQ}=-\mathbf{QJ}$, $\mathbf{JQ}^*=-\mathbf{Q}^*\mathbf{J}$, $\mathbf{JQ}^*=\mathbf{Q}^T\mathbf{J}$ and the identity in [36,37].

Proposition 1. *If $\widetilde{\mathbf{v}}_2(z;x,t)$ and $\widetilde{\mathbf{v}}_3(z;x,t)$ are two arbitrary solutions of the "adjoint" Lax pair (32), while "$\times$" denotes the usual cross product, then*

$$\mathbf{v}_1(z;x,t)=\mathrm{e}^{\mathrm{i}\delta_2(z;x,t)}\mathbf{J}[\widetilde{\mathbf{v}}_2(z;x,t)\times\widetilde{\mathbf{v}}_3(z;x,t)], \tag{34}$$

is a solution of the Lax pair (5).

As $x\to\pm\infty$, the behavior of the solutions derived from the "adjoint" Lax pair (32) will approach an asymptotic state in terms of both spatial and temporal

$$\widetilde{\psi}_x=\widetilde{\mathbf{X}}_\pm\widetilde{\psi},\quad \widetilde{\psi}_t=\widetilde{\mathbf{T}}_\pm\widetilde{\psi}, \tag{35}$$

where

$$\lim_{x\to\pm\infty}\widetilde{\mathbf{X}}=\widetilde{\mathbf{X}}_\pm=-\mathrm{i}k\mathbf{J}-\mathrm{i}\mathbf{Q}_\pm^*, \tag{36a}$$

$$\lim_{x\to\pm\infty}\widetilde{\mathbf{T}}=\widetilde{\mathbf{T}}_\pm=-4\mathrm{i}\sigma k^3\mathbf{J}+\mathrm{i}q_0^2\mathbf{J}-k^2(4\mathrm{i}\sigma\mathbf{Q}_\pm^*+2\mathrm{i}\mathbf{J})+k(2\mathrm{i}\sigma\mathbf{J}(\mathbf{Q}_\pm^*)^2-2\mathrm{i}\mathbf{Q}_\pm^*)+2\mathrm{i}\sigma(\mathbf{Q}_\pm^*)^3+\mathrm{i}\mathbf{J}(\mathbf{Q}_\pm^*)^2. \tag{36b}$$

The eigenvalues of $\widetilde{\mathbf{X}}_\pm$ are $\mathrm{i}k$ and $\pm\mathrm{i}\lambda$, the eigenvalues of $\widetilde{\mathbf{T}}_\pm$ are $\mathrm{i}(\lambda^2+k^2+4\sigma k^3)$ and $\pm2\mathrm{i}\lambda[k+\sigma(3k^2-\lambda^2)]$. Additionally, properties $\widetilde{\mathbf{Y}}_\pm(z)=\mathbf{Y}_\pm^*(z)$ and $\det\widetilde{\mathbf{Y}}_\pm(z)=-\mathrm{i}\rho(z)$ are present. It can be straightforwardly determined that $\widetilde{\mathbf{X}}_\pm$ and $\widetilde{\mathbf{T}}_\pm$ fulfill the subsequent conditions:

$$\widetilde{\mathbf{X}}_\pm\widetilde{\mathbf{Y}}_\pm=-\mathrm{i}\widetilde{\mathbf{Y}}_\pm\mathbf{\Lambda}_1,\quad \widetilde{\mathbf{T}}_\pm\widetilde{\mathbf{Y}}_\pm=-\mathrm{i}\widetilde{\mathbf{Y}}_\pm\mathbf{\Lambda}_2, \tag{37}$$

where $\widetilde{\mathbf{Y}}_\pm=\widetilde{\mathbf{Y}}_\pm(z)$. Similarly, the Jost solutions of the "adjoint" Lax pair (32)

$$\widetilde{\psi}_\pm(z;x,t)=\widetilde{\mathbf{Y}}_\pm(z)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z;x,t)}+o(1),\quad x\to\pm\infty,\quad z\in\mathbb{R}. \tag{38}$$

Introducing the modified Jost solutions

$$\widetilde{v}_\pm(z;x,t)=\widetilde{\psi}_\pm(z;x,t)\mathrm{e}^{\mathrm{i}\mathbf{\Delta}(z;x,t)}, \tag{39}$$

the subsequent columns of the function $\widetilde{v}_\pm(z;x,t)$ adhere to the ensuing properties:

$$\widetilde{v}_{-,3}(z;x,t)\ \text{and}\ \widetilde{v}_{+,1}(z;x,t)\ :\ z\in\mathbb{D}^-,\qquad \widetilde{v}_{-,1}(z;x,t)\ \text{and}\ \widetilde{v}_{+,3}(z;x,t)\ :\ z\in\mathbb{D}^+. \tag{40}$$

The modified Jost solutions imply that the columns of $\widetilde{\psi}_\pm(z;x,t)$ exhibit analogous properties of analyticity and boundedness. In a similar fashion, the "adjoint" scattering matrix can likewise be defined as follows:

$$\widetilde{\psi}_+(z;x,t)=\widetilde{\psi}_-(z;x,t)\widetilde{\mathbf{H}}(z), \tag{41}$$

where $\widetilde{\mathbf{H}}(z)=(\widetilde{h}_{ij}(z))$. Similarly, define $\widetilde{\mathbf{S}}(z)=\widetilde{\mathbf{H}}^{-1}(z)=(\widetilde{s}_{ij}(z))$. The following scattering coefficients satisfy the following properties:

$$\widetilde{s}_{33}(z)\ \text{and}\ \widetilde{h}_{11}(z)\ :\ z\in\mathbb{D}^-,\qquad \widetilde{s}_{11}(z)\ \text{and}\ \widetilde{h}_{33}(z)\ :\ z\in\mathbb{D}^+. \tag{42}$$

In order to fully construct the analytical eigenfunctions, two new solutions for the original Lax pair (5) are defined:

$$\gamma(z)=-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_2(z)}\mathbf{J}[\widetilde{\psi}_{-,3}(z)\times\widetilde{\psi}_{+,1}(z)]}{\rho(z)},\quad z\in\mathbb{D}^-,\qquad \widetilde{\gamma}(z)=-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_2(z)}\mathbf{J}[\widetilde{\psi}_{-,1}(z)\times\widetilde{\psi}_{+,3}(z)]}{\rho(z)},\quad z\in\mathbb{D}^+, \tag{43}$$

where $\gamma(z)=\gamma(z;x,t)$, $\widetilde{\gamma}(z)=\widetilde{\gamma}(z;x,t)$ and $\widetilde{\psi}_{\pm,j}(z)=\widetilde{\psi}_{\pm,j}(z;x,t)$ for $j=1,3$. Then we can directly derive the following three conclusions:

Corollary 1. *For all cyclic indices j, l and m with $z\in\mathbb{R}$,*

$$\psi_{\pm,j}(z)=-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_2(z)}\mathbf{J}[\widetilde{\psi}_{\pm,l}(z)\times\widetilde{\psi}_{\pm,m}(z)]}{\rho_j(z)},\quad \widetilde{\psi}_{\pm,j}(z)=\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\delta_2(z)}\mathbf{J}[\psi_{\pm,l}(z)\times\psi_{\pm,m}(z)]}{\rho_j(z)}, \tag{44}$$

where

$$\rho_1(z)=-1,\quad \rho_2(z)=\rho(z),\quad \rho_3(z)=1. \tag{45}$$

Corollary 2. *The scattering matrix $\mathbf{S}(z)$ and $\widetilde{\mathbf{S}}(z)$ are related as follows:*

$$\widetilde{\mathbf{S}}^{-1}(z)=\mathbf{J}_1(z)\mathbf{S}^T(z)\mathbf{J}_1^{-1}(z),\quad \mathbf{J}_1(z)=\mathrm{diag}\,(-1,\rho(z),1). \tag{46}$$

Corollary 3. *The Jost eigenfunctions exhibit the following decompositions for $z\in\mathbb{R}$*

$$\psi_{-,2}(z)=\frac{s_{32}(z)\psi_{-,3}(z)-\widetilde{\gamma}(z)}{s_{33}(z)}=\frac{s_{12}(z)\psi_{-,1}(z)+\gamma(z)}{s_{11}(z)},\qquad \psi_{+,2}(z)=\frac{h_{12}(z)\psi_{+,1}(z)-\widetilde{\gamma}(z)}{h_{11}(z)}=\frac{h_{32}(z)\psi_{+,3}(z)+\gamma(z)}{h_{33}(z)}, \tag{47}$$

where $\psi_{\pm,j}(z)=\psi_{\pm,j}(z;x,t)$ for $j=1,2,3$.

Furthermore, the modified auxiliary eigenfunctions are delineated as follows:

$$d(z;x,t)=\gamma(z)\mathrm{e}^{-\mathrm{i}\delta_2(z)},\quad z\in\mathbb{D}^-,\qquad \widetilde{d}(z;x,t)=\widetilde{\gamma}(z)\mathrm{e}^{-\mathrm{i}\delta_2(z)},\quad z\in\mathbb{D}^+. \tag{48}$$

2.3. Symmetries

Compared with the equation with the initial value condition of ZBCs, when dealing with the equation with NZBCs, the corresponding symmetry becomes complicated due to the existence of Riemann surface. Hence, it is imperative to recognize that the symmetry inherent in the potential within the Lax pair engenders the corresponding symmetry in the scattering data. To ensure symmetry, it is essential to contemplate the subsequent involutions: $z \mapsto z^*$ and $z \mapsto q_0^2/z$.

2.3.1. First symmetry

Consider the first involution: $z \mapsto z^*$, implying $(k,\lambda) \mapsto (k^*,\lambda^*)$.

Proposition 2. *If $\psi(z;x,t)$ is a non-singular solution of the Lax pair (5), so is*

$$\mathbf{v}_4(z;x,t) = \mathbf{J}[\psi^\dagger(z^*;x,t)]^{-1}. \tag{49}$$

By using property $[\mathrm{e}^{\mathrm{i}\mathbf{\Delta}(z^*;x,t)}]^\dagger = \mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z;x,t)}$, the Jost eigenfunctions exhibit the specific symmetry:

$$\psi_\pm(z;x,t) = \mathbf{J}[\psi_\pm^\dagger(z^*;x,t)]^{-1}\mathbf{J}_2(z), \quad \mathbf{J}_2(z) = \mathrm{diag}\,(\rho(z),-1,-\rho(z)), \quad z\in\mathbb{R}. \tag{50}$$

Subsequently, employing the Schwarz reflection principle, we obtain the subsequent results:

$$\psi_{-,1}^*(z^*) = \frac{\mathrm{i}\mathbf{J}\mathrm{e}^{-\mathrm{i}\delta_2(z)}[\tilde{\gamma}(z)\times\psi_{-,3}(z)]}{s_{33}(z)}, \quad \operatorname{Im} z \ge 0, \tag{51a}$$

$$\psi_{+,1}^*(z^*) = \frac{\mathrm{i}\mathbf{J}\mathrm{e}^{-\mathrm{i}\delta_2(z)}[\gamma(z)\times\psi_{+,3}(z)]}{-h_{33}(z)}, \quad \operatorname{Im} z \le 0, \tag{51b}$$

$$\psi_{-,3}^*(z^*) = \frac{\mathrm{i}\mathbf{J}\mathrm{e}^{-\mathrm{i}\delta_2(z)}[\gamma(z)\times\psi_{-,1}(z)]}{-s_{11}(z)}, \quad \operatorname{Im} z \le 0, \tag{51c}$$

$$\psi_{+,3}^*(z^*) = \frac{\mathrm{i}\mathbf{J}\mathrm{e}^{-\mathrm{i}\delta_2(z)}[\tilde{\gamma}(z)\times\psi_{+,1}(z)]}{h_{11}(z)}, \quad \operatorname{Im} z \ge 0. \tag{51d}$$

In addition, using the scattering relationship (29), (50) and the property $\mathbf{J}_2(z) = -\rho(z)\mathbf{J}_1^{-1}(z)$, the scattering matrices are interrelated as follows:

$$\mathbf{S}^\dagger(z) = \mathbf{J}_1^{-1}(z)\mathbf{H}(z)\mathbf{J}_1(z), \quad z\in\mathbb{R}. \tag{52}$$

Accordingly, it can be deduced that for $z\in\mathbb{R}$

$$h_{11}(z) = s_{11}^*(z), \qquad h_{12}(z) = -\frac{s_{21}^*(z)}{\rho(z)}, \qquad h_{13}(z) = -s_{31}^*(z), \qquad h_{32}(z) = \frac{s_{23}^*(z)}{\rho(z)}, \qquad h_{21}(z) = -\rho(z)s_{12}^*(z), \tag{53a}$$

$$h_{22}(z) = s_{22}^*(z), \qquad h_{23}(z) = \rho(z)s_{32}^*(z), \qquad h_{31}(z) = -s_{13}^*(z), \qquad h_{33}(z) = s_{33}^*(z). \tag{53b}$$

According to the Schwarz reflection principle, we can draw this conclusion:

$$h_{11}(z) = s_{11}^*(z^*), \quad \operatorname{Im} z \ge 0, \qquad h_{33}(z) = s_{33}^*(z^*), \quad \operatorname{Im} z \le 0. \tag{54}$$

The property $\psi_\pm^*(z^*;x,t) = \tilde{\psi}_\pm(z;x,t)$ is obtained, so the following conditions are established

$$\psi_{\pm,1}^*(z^*) = \tilde{\psi}_{\pm,1}(z), \quad \operatorname{Im} z \lessgtr 0, \qquad \psi_{\pm,3}^*(z^*) = \tilde{\psi}_{\pm,3}(z), \quad \operatorname{Im} z \gtrless 0. \tag{55}$$

Through the properties (55) and new auxiliary eigenfunctions (43), we derive the following conclusion:

Corollary 4. *The new auxiliary eigenfunctions (43) adhere to the symmetry relations:*

$$\gamma(z) = -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_2(z)}\mathbf{J}[\psi_{-,3}^*(z^*)\times\psi_{+,1}^*(z^*)]}{\rho(z)}, \quad z\in\mathbb{D}^-, \qquad \tilde{\gamma}(z) = -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_2(z)}\mathbf{J}[\psi_{-,1}^*(z^*)\times\psi_{+,3}^*(z^*)]}{\rho(z)}, \quad z\in\mathbb{D}^+. \tag{56}$$

Furthermore, the symmetrical properties are also present:

$$\psi_{\pm,j}^*(z) = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\delta_2(z)}\mathbf{J}[\psi_{\pm,l}(z)\times\psi_{\pm,m}(z)]}{\rho_j(z)}, \quad z\in\mathbb{R}, \tag{57}$$

where j, l and m are cyclic indices.

2.3.2. Second symmetry

Consider the second involution: $z \mapsto q_0^2/z$, implying $(k,\lambda)\mapsto(k,-\lambda)$.

Proposition 3. *If $\psi(z;x,t)$ is a non-singular solution of the Lax pair (5), so is*

$$\mathbf{v}_5(z;x,t) = \psi(\frac{q_0^2}{z};x,t). \tag{58}$$

The subsequent properties can be derived from the principle of progressiveness

$$\psi_{\pm}(z;x,t)=\psi_{\pm}(\frac{q_0^2}{z};x,t)\mathbf{J}_3(z),\quad \mathbf{J}_3(z)=\begin{pmatrix}0 & 0 & -\frac{\mathrm{i}q_0}{z}\\ 0 & 1 & 0\\ \frac{\mathrm{i}q_0}{z} & 0 & 0\end{pmatrix},\quad z\in\mathbb{R}. \tag{59}$$

Consistent with the previous discussion, the eigenfunctions exhibit the following analytical characteristics:

$$\psi_{\pm,1}(z)=\frac{\mathrm{i}q_0}{z}\psi_{\pm,3}(\frac{q_0^2}{z}),\qquad \mathrm{Im}\, z\gtrless 0, \tag{60a}$$

$$\psi_{\pm,3}(z)=-\frac{\mathrm{i}q_0}{z}\psi_{\pm,1}(\frac{q_0^2}{z}),\qquad \mathrm{Im}\, z\lessgtr 0, \tag{60b}$$

$$\psi_{\pm,2}(z)=\psi_{\pm,2}(\frac{q_0^2}{z}),\qquad z\in\mathbb{R}. \tag{60c}$$

Utilizing the scattering relationships given by Eqs. (29) and (59), the scattering matrices exhibit the following relationship:

$$\mathbf{S}(\frac{q_0^2}{z})=\mathbf{J}_3(z)\mathbf{S}(z)\mathbf{J}_3^{-1}(z),\quad z\in\mathbb{R}. \tag{61}$$

Accordingly, it can be deduced that

$$s_{11}(z)=s_{33}(\frac{q_0^2}{z}),\qquad s_{12}(z)=-\frac{\mathrm{i}z}{q_0}s_{32}(\frac{q_0^2}{z}),\qquad s_{13}(z)=-s_{31}(\frac{q_0^2}{z}),\qquad s_{21}(z)=\frac{\mathrm{i}q_0}{z}s_{23}(\frac{q_0^2}{z}),\qquad s_{22}(z)=s_{22}(\frac{q_0^2}{z}), \tag{62a}$$

$$s_{23}(z)=-\frac{\mathrm{i}q_0}{z}s_{21}(\frac{q_0^2}{z}),\qquad s_{31}(z)=-s_{13}(\frac{q_0^2}{z}),\qquad s_{32}(z)=\frac{\mathrm{i}z}{q_0}s_{12}(\frac{q_0^2}{z}),\qquad s_{33}(z)=s_{11}(\frac{q_0^2}{z}), \tag{62b}$$

the analytical domain of the scattering coefficients

$$s_{11}(z)=s_{33}(\frac{q_0^2}{z}),\quad h_{33}(z)=h_{11}(\frac{q_0^2}{z}),\quad \mathrm{Im}\, z\le 0. \tag{63}$$

The auxiliary eigenfunctions exhibit the following characteristics:

$$\widetilde{\gamma}(z)=-\gamma(\frac{q_0^2}{z}),\quad \mathrm{Im}\, z\ge 0. \tag{64}$$

We introduce the new reflections as follows:

$$\beta_1(z)=\frac{s_{13}(z)}{s_{11}(z)}=-\frac{h_{31}^*(z)}{h_{11}^*(z)},\qquad \beta_1(\frac{q_0^2}{z})=-\frac{s_{31}(z)}{s_{33}(z)}=\frac{h_{13}^*(z)}{h_{33}^*(z)},\qquad \beta_2(z)=\frac{h_{21}(z)}{h_{11}(z)}=-\rho(z)\frac{s_{12}^*(z)}{s_{11}^*(z)},\qquad \beta_2(\frac{q_0^2}{z})=\frac{\mathrm{i}z}{q_0}\frac{h_{23}(z)}{h_{33}(z)}=\frac{\mathrm{i}z\rho(z)}{q_0}\frac{s_{32}^*(z)}{s_{33}^*(z)}. \tag{65}$$

3. Discrete spectrum and asymptotic behavior

A direct link is established between the zeros of the scattering coefficients and the discrete eigenvalues, each signifying the presence of a bound state within the system [36]. C_o is a circle with a radius of q_0 centered at the origin of the complex z-plane. It has been established that discrete eigenvalues are excluded from the continuous spectrum, hence they are confined to the domain within the circle C_o. Furthermore, the self-adjoint property of the scattering problem ensures that the discrete eigenvalues k must be real numbers, and there are no discrete eigenvalues within the continuous spectrum. Consequently, these discrete eigenvalues are found only within the circle C_o on the z-plane.

Proposition 4. *Let* $\mathbf{v}(z;x,t)$ *denote a nontrivial solution to the scattering problem in* (5)*. If* $\mathbf{v}(z;x,t)\in L^2(\mathbb{R})$*, then* $z\in C_o$*.*

In order to fully represent the characteristics of the inverse problem, it is necessary to consider the zeros of the analytical scattering coefficient outside the circle C_o. This view does not conflict with Proposition 4, which states that the zero point of the analytical scattering coefficient outside the circle C_o does not lead to the generation of bound states. The discrete spectrum is the set of all $z\in\mathbb{C}\backslash\mathbb{R}$ such that $h_{11}(z)=0$ or $s_{11}(z^*)=0$, values for which the Jost eigenfunctions belong to $L^2(\mathbb{R})$. Consequently, the existence of zeros for $h_{11}(z)$ within C_o is permissible, and such zeros lead to eigenfunctions that do not exhibit decay towards both spatial infinities.

3.1. Discrete spectrum

To delve into the discrete spectrum, we define two 3×3 matrices

$$\mathbf{\Psi}^+(z)=(\psi_{+,1}(z),-\widetilde{\gamma}(z),\psi_{-,3}(z)),\quad z\in\mathbb{D}^+,\qquad \mathbf{\Psi}^-(z)=(\psi_{-,1}(z),\gamma(z),\psi_{+,3}(z)),\quad z\in\mathbb{D}^-, \tag{66}$$

where $\mathbf{\Psi}^\pm(z)=\mathbf{\Psi}^\pm(z;x,t)$. By the decompositions (47) and taking the determinant, we obtain the following result:

$$\det\mathbf{\Psi}^+(z)=\mathrm{i}e^{\mathrm{i}\delta_2(z)}h_{11}(z)s_{33}(z)\rho(z),\quad \mathrm{Im}\, z\ge 0,\qquad \det\mathbf{\Psi}^-(z)=\mathrm{i}e^{\mathrm{i}\delta_2(z)}s_{11}(z)h_{33}(z)\rho(z),\quad \mathrm{Im}\, z\le 0. \tag{67}$$

Nevertheless, the symmetries inherent in the scattering coefficients imply that these zeros are interrelated and not mutually exclusive.

Proposition 5 (*Off the Circle* C_o)**.** *Suppose that* $h_{11}(z)$ *possesses a zero* θ_g *within the upper half plane of* z*, then*

$$h_{11}(\theta_g)=0\Longleftrightarrow s_{11}(\theta_g^*)=0\Longleftrightarrow s_{33}(\frac{q_0^2}{\theta_g^*})=0\Longleftrightarrow h_{33}(\frac{q_0^2}{\theta_g})=0. \tag{68}$$

Therefore, it can be considered that the discrete eigenvalues z_g on the circle C_o are $\{z_g, z_g^*\}$ and the discrete eigenvalues θ_g off the circle C_o are $\{\theta_g, \theta_g^*, q_0^2/\theta_g, q_0^2/\theta_g^*\}$.

Proposition 6. *If* $\operatorname{Im}\theta_g > 0$ *and* $\theta_g \notin C_o$*, then* $\widetilde{\gamma}(\theta_g; x, t) \neq \mathbf{0}$.

Proposition 7. *Suppose* $\operatorname{Im}\theta_g > 0$*, the following conclusions are equivalent:*

(1) $\widetilde{\gamma}(\theta_g) = \mathbf{0} \Longleftrightarrow \gamma(\theta_g^*) = \mathbf{0} \Longleftrightarrow \gamma(\frac{q_0^2}{\theta_g}) = \mathbf{0} \Longleftrightarrow \widetilde{\gamma}(\frac{q_0^2}{\theta_g^*}) = \mathbf{0}$.

(2) $\psi_{-,3}(\theta_g)$ *and* $\psi_{+,1}(\theta_g)$ *are linearly correlated.* $\psi_{-,1}(\theta_g^*)$ *and* $\psi_{+,3}(\theta_g^*)$ *are linearly correlated.*

(3) $\psi_{-,3}(\frac{q_0^2}{\theta_g^*})$ *and* $\psi_{+,1}(\frac{q_0^2}{\theta_g^*})$ *are linearly correlated.* $\psi_{-,1}(\frac{q_0^2}{\theta_g})$ *and* $\psi_{+,3}(\frac{q_0^2}{\theta_g})$ *are linearly correlated.*

With the premise of simplicity and non-repetition of the discrete eigenvalues, the subsequent two theorems are derived:

Theorem 3. *Let* z_g *be a zero of* $h_{11}(z)$ *in the upper half plane with* $|z_g| = q_0$*, then* $\widetilde{\gamma}(z_g) = \gamma(z_g^*) = \mathbf{0}$*, there exist constants* c_g*, and* $\bar{c}_g$ *such that*

$$\psi_{+,1}(z_g) = c_g \psi_{-,3}(z_g), \quad \psi_{+,3}(z_g^*) = \bar{c}_g \psi_{-,1}(z_g^*). \tag{69}$$

Theorem 4. *Let* θ_g *be a zero of* $h_{11}(z)$ *in the upper half plane with* $|\theta_g| \neq q_0$*, then* $|\theta_g| < q_0$ *and* $s_{33}(\theta_g) \neq 0$*, there exist constants* f_g*,* $\hat{f}_g$*,* $\check{f}_g$ *and* $\bar{f}_g$ *such that*

$$\psi_{+,1}(\theta_g) = \frac{f_g}{s_{33}(\theta_g)} \widetilde{\gamma}(\theta_g), \quad \psi_{+,3}(\frac{q_0^2}{\theta_g}) = \check{f}_g \gamma(\frac{q_0^2}{\theta_g}), \tag{70a}$$

$$\gamma(\theta_g^*) = \bar{f}_g \psi_{-,1}(\theta_g^*), \qquad \widetilde{\gamma}(\frac{q_0^2}{\theta_g^*}) = \hat{f}_g \psi_{-,3}(\frac{q_0^2}{\theta_g^*}). \tag{70b}$$

In Appendix B, we provide a detailed proof of the following two corollaries.

Corollary 5. *Suppose that* $h_{11}(z)$ *has simple zeros* $\{z_g\}_{g=1}^{G_1}$ *on* C_o*, it can be inferred that the norming constants adhere to the symmetry relationship:*

$$\bar{c}_g = -c_g, \quad c_g^* = \frac{s'_{11}(z_g^*)}{h'_{33}(z_g^*)} \bar{c}_g, \quad g = 1, 2, \ldots, G_1. \tag{71}$$

Corollary 6. *Suppose that* $h_{11}(z)$ *has zeros* $\{\theta_g\}_{g=1}^{G_2}$ *off* C_o*, it is known that the norming constants adhere to the symmetry relationship:*

$$\check{f}_g = \frac{\mathrm{i}\theta_g}{q_0 s_{33}(\theta_g)} f_g, \quad \bar{f}_g = -\frac{f_g^*}{\rho(\theta_g^*)}, \quad \hat{f}_g = \frac{\mathrm{i}q_0}{\theta_g^* \rho(\theta_g^*)} f_g^*, \quad g = 1, 2, \ldots, G_2. \tag{72}$$

3.2. Asymptotic behavior

The asymptotic behavior of the modified Jost eigenfunctions for $z \to \infty$ and $z \to 0$ can be analyzed using the Wentzel–Kramers–Brillouin approximation technique. Specifically, when applied to the differential Eqs. (23), it reveals asymptotic characteristics:

Corollary 7. *The asymptotic expansion as* $z \to \infty$ *is delineated as follows:*

$$v_{\pm,1}(z; x, t) = \begin{pmatrix} \mathrm{i} \\ \frac{\mathrm{i}}{z}\mathbf{q}(x,t) \end{pmatrix} + O(\frac{1}{z^2}), \quad v_{\pm,3}(z; x, t) = \begin{pmatrix} \frac{\mathbf{q}_\pm}{q_0 z}\mathbf{q}^\dagger(x,t) \\ \frac{\mathbf{q}_\pm}{q_0} \end{pmatrix} + O(\frac{1}{z^2}). \tag{73}$$

Similarly, the asymptotic expansion as $z \to 0$ *is delineated as follows:*

$$v_{\pm,1}(z; x, t) = \begin{pmatrix} \frac{\mathrm{i}\mathbf{q}_\pm}{q_0^2}\mathbf{q}^\dagger(x,t) \\ \frac{\mathrm{i}\mathbf{q}_\pm}{z} \end{pmatrix} + O(z), \quad v_{\pm,3}(z; x, t) = \begin{pmatrix} \frac{q_0}{z} \\ \frac{1}{q_0}\mathbf{q}(x,t) \end{pmatrix} + O(z). \tag{74}$$

By combining the modified auxiliary eigenfunctions (48) with the asymptotic properties (73) and (74), the following results are obtained:

$$d(z) = \begin{pmatrix} \frac{\mathbf{q}_-^\perp}{q_0 z}\mathbf{q}^\dagger(x,t) \\ \frac{\mathbf{q}_-^\perp}{q_0} \end{pmatrix} + O(\frac{1}{z^2}), \quad \widetilde{d}(z) = \begin{pmatrix} -\frac{\mathbf{q}_+^\perp}{q_0 z}\mathbf{q}^\dagger(x,t) \\ -\frac{\mathbf{q}_+^\perp}{q_0} \end{pmatrix} + O(\frac{1}{z^2}), \quad z \to \infty, \tag{75}$$

and

$$d(z) = \begin{pmatrix} 0 \\ \frac{\mathbf{q}_+^\perp}{q_0} \end{pmatrix} + O(z), \quad \widetilde{d}(z) = \begin{pmatrix} 0 \\ -\frac{\mathbf{q}_-^\perp}{q_0} \end{pmatrix} + O(z), \quad z \to 0. \tag{76}$$

Corollary 8. *The asymptotic behavior of scattering matrix entries as* $z \to \infty$ *is delineated as follows:*

$$s_{11}(z) = 1 + O(\frac{1}{z}), \quad s_{22}(z) = h_{33}(z) = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + O(\frac{1}{z}), \quad s_{32}(z) = \frac{\mathbf{q}_+^\dagger \mathbf{q}_-^\perp}{q_0^2} + O(\frac{1}{z}), \quad s_{23}(z) = \frac{(\mathbf{q}_+^\perp)^\dagger \mathbf{q}_-}{q_0^2} + O(\frac{1}{z}), \tag{77a}$$

$$h_{11}(z) = 1 + O(\frac{1}{z}), \quad h_{22}(z) = s_{33}(z) = \frac{\mathbf{q}_+^\dagger \mathbf{q}_-}{q_0^2} + O(\frac{1}{z}), \quad h_{32}(z) = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+^\perp}{q_0^2} + O(\frac{1}{z}), \quad h_{23}(z) = \frac{(\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+}{q_0^2} + O(\frac{1}{z}), \tag{77b}$$

the asymptotic behavior of other entries as $z \to \infty$ in the scattering matrix is $O(1/z)$. Similarly, one can show that as $z \to 0$

$$s_{11}(z) = h_{22}(z) = \frac{\mathbf{q}_+^\dagger \mathbf{q}_-}{q_0^2} + O(z), \quad s_{21}(z) = \frac{\mathrm{i}(\mathbf{q}_+^\perp)^\dagger \mathbf{q}_-}{q_0 z} + O(1), \quad s_{33}(z) = 1 + O(z), \tag{78a}$$

$$s_{22}(z) = h_{11}(z) = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + O(z), \quad h_{21}(z) = \frac{\mathrm{i}(\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+}{q_0 z} + O(1), \quad h_{33}(z) = 1 + O(z), \tag{78b}$$

the asymptotic behavior of other entries as $z \to 0$ in the scattering matrix is $O(z)$.

3.3. Behavior at the branch points

Next, we will analyze the characteristics of the Jost eigenfunctions and the scattering matrix at the branching points $k = \pm q_0$. At the branching points, the matrices $\mathbf{Y}_\pm(z)$ are degenerate. However, the term $\mathbf{Y}_\pm(z)\mathrm{e}^{\mathrm{i}(x-y)\mathbf{\Lambda}_1}\mathbf{Y}_\pm^{-1}(z)$ remains finite at the branching points.

$$\lim_{z\to\pm q_0} \mathbf{Y}_\pm \mathrm{e}^{\mathrm{i}(x-y)\mathbf{\Lambda}_1}\mathbf{Y}_\pm^{-1} = \begin{pmatrix} 1 \pm \mathrm{i}q_0(x-y) & \mathrm{i}(y-x)\mathbf{q}_\pm^\dagger \\ \mathrm{i}(x-y)\mathbf{q}_\pm & \frac{1}{q_0^2}\left[(1 \mp \mathrm{i}q_0(x-y))\mathbf{q}_\pm \mathbf{q}_\pm^\dagger + \mathrm{e}^{\mp \mathrm{i}q_0(x-y)}\mathbf{q}_\pm^\perp (\mathbf{q}_\pm^\perp)^\dagger\right] \end{pmatrix}. \tag{79}$$

For all $(x,t) \in \mathbb{R}^2$, it can be inferred from expression (28) that $\det \psi_\pm(\pm q_0; x, t) = 0$. Then, the columns of $\psi_\pm(q_0; x, t)$ and $\psi_\pm(-q_0; x, t)$ are linearly dependent. With the help of the analytical properties (60), the following conditions are obtained:

$$\psi_{\pm,1}(q_0; x, t) = \mathrm{i}\psi_{\pm,3}(q_0; x, t), \quad \psi_{\pm,1}(-q_0; x, t) = -\mathrm{i}\psi_{\pm,3}(-q_0; x, t). \tag{80}$$

We examine the characteristics of the scattering matrix $\mathbf{S}(z)$ in the neighborhood of the branch points, which can be expressed in terms of Wronskian determinants. Consequently, we express the scattering coefficients in terms of these Wronskians:

$$s_{jl}(z) = \frac{z^2}{\mathrm{i}(z^2 - q_0^2)} W_{jl}(z)\mathrm{e}^{-\mathrm{i}\delta_2(z)} = \frac{W_{jl}(z)}{\mathrm{i}\rho(z)}\mathrm{e}^{-\mathrm{i}\delta_2(z)}, \tag{81}$$

where

$$W_{jl}(z) = \det(\psi_{-,l}(z), \psi_{+,j+1}(z), \psi_{+,j+2}(z)), \tag{82}$$

where $j+1$ and $j+2$ are calculated modulo 3. The scattering coefficients as $z \to \pm q_0$ are articulated as follows:

$$s_{ij}(z) = \frac{s_{ij,\pm}}{z \mp q_0} + s_{ij,\pm}^{(o)} + O(z \mp q_0), \quad z \in \mathbb{R}\backslash\{\pm q_0\}, \tag{83}$$

where

$$s_{ij,\pm} = \mp\frac{\mathrm{i}q_0}{2} W_{ij}(\pm q_0; x, t)\mathrm{e}^{[\mathrm{i}q_0^2 t \pm \mathrm{i}q_0(x + 4\sigma q_0^2 t)]}, \qquad s_{ij,\pm}^{(o)} = \left[\mp\frac{\mathrm{i}q_0}{2}\frac{\partial}{\partial z} W_{ij}(z; x, t)\Big|_{z=\pm q_0} - \mathrm{i}W_{ij}(\pm q_0; x, t)\right]\mathrm{e}^{[\mathrm{i}q_0^2 t \pm \mathrm{i}q_0(x + 4\sigma q_0^2 t)]}. \tag{84}$$

Subsequently, the asymptotic series for $\mathbf{S}(z)$ in the vicinity of the branch points can be described as follows:

$$\mathbf{S}(z) = \frac{\mathbf{S}_\pm}{z \mp q_0} + \mathbf{S}_\pm^{(o)} + O(z \mp q_0), \tag{85}$$

where

$$\mathbf{S}_\pm^{(o)} = \left(s_{ij,\pm}^{(o)}\right), \qquad \mathbf{S}_\pm = s_{11,\pm}\begin{pmatrix} 1 & 0 & \mp\mathrm{i} \\ 0 & 0 & 0 \\ \mp\mathrm{i} & 0 & -1 \end{pmatrix} + s_{12,\pm}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & \mp\mathrm{i} & 0 \end{pmatrix}. \tag{86}$$

The asymptotic behavior of the reflection coefficients (65) at the branching points can be directly obtained through (86) and the symmetry (53):

$$\lim_{z\to\pm q_0} \beta_1(z) = \mp\mathrm{i}, \quad \lim_{z\to\pm q_0} \beta_2(z) = 0. \tag{87}$$

4. Inverse problem

Generally, the IST is formulated as a suitable RH problem, which allows for the study of its various properties. Thus, the meromorphic eigenfunctions in the upper half z-plane are related to those in the lower half z-plane through an appropriate jump condition.

4.1. Riemann-Hilbert problem

To formulate the matrix RH problem, it is essential to establish appropriate transition conditions that define the behavior of eigenfunctions, which are characterized by their meromorphic nature within the specified domain. Given that certain Jost eigenfunctions lack analytic properties, it is necessary to define new modified meromorphic functions in the corresponding regions.

Lemma 1. *Define the piecewise meromorphic function $\mathbf{R}^\pm(z; x, t) = (\mathbf{r}_1^\pm, \mathbf{r}_2^\pm, \mathbf{r}_3^\pm)$ as follows:*

$$\mathbf{R}^+(z; x, t) = \mathbf{\Psi}^+(z)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z)}\,\mathrm{diag}\left(\frac{1}{h_{11}(z)}, \frac{1}{s_{33}(z)}, 1\right) = \left[\frac{\nu_{+,1}(z)}{h_{11}(z)}, -\frac{\widetilde{d}(z)}{s_{33}(z)}, \nu_{-,3}(z)\right], \quad z \in \mathbb{D}^+, \tag{88a}$$

$$\mathbf{R}^{-}(z;x,t)=\mathbf{\Psi}^{-}(z)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z)}\operatorname{diag}\left(1,\frac{1}{s_{11}(z)},\frac{1}{h_{33}(z)}\right)=\left[v_{-,1}(z),\frac{d(z)}{s_{11}(z)},\frac{v_{+,3}(z)}{h_{33}(z)}\right],\qquad z\in\mathbb{D}^{-},\tag{88b}$$

where $\mathbf{\Psi}^{\pm}(z)=\mathbf{\Psi}^{\pm}(z;x,t)$, $d(z)=d(z;x,t)$, $\widetilde{d}(z)=\widetilde{d}(z;x,t)$ *and* $v_{\pm,j}(z)=v_{\pm,j}(z;x,t)$ *for* $j=1,3$. *The corresponding jump condition is*

$$\mathbf{R}^{+}(z;x,t)=\mathbf{R}^{-}(z;x,t)[\mathbf{I}-\mathrm{e}^{\mathrm{i}\mathbf{\Delta}(z)}\mathbf{L}(z)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z)}],\quad z\in\mathbb{R},\tag{89}$$

and

$$\mathbf{L}(z)=\begin{pmatrix}\left[\frac{|\beta_2|^2}{\rho}-\beta_1^*\widetilde{\beta}_1^*-\frac{\mathrm{i}q_0}{z\rho}\beta_1^*\beta_2^*\widetilde{\beta}_2\right] & \left[\frac{\beta_2^*}{\rho}+\frac{q_0^2}{z^2\rho^2}\beta_2^*\left|\widetilde{\beta}_2\right|^2-\frac{\mathrm{i}q_0}{z\rho}\widetilde{\beta}_1^*\widetilde{\beta}_2^*\right] & \left[\frac{\mathrm{i}q_0}{z\rho}\beta_2^*\widetilde{\beta}_2+\widetilde{\beta}_1^*\right]\\ \frac{\mathrm{i}q_0}{z}\beta_1^*\widetilde{\beta}_2-\beta_2 & -\frac{q_0^2}{z^2\rho}\left|\widetilde{\beta}_2\right|^2 & -\frac{\mathrm{i}q_0}{z}\widetilde{\beta}_2\\ \beta_1^* & \frac{\mathrm{i}q_0}{z\rho}\widetilde{\beta}_2^* & 0\end{pmatrix},\tag{90}$$

with $\rho=\rho(z)$, $\beta_j=\beta_j(z)$ *and* $\widetilde{\beta}_j=\beta_j(q_0^2/z)$ *for* $j=1,2$.

To guarantee a unique solution to the aforementioned RH problem, it is imperative to establish an appropriate normalization condition. By considering the asymptotic behavior of $z\to\infty$ and $z\to 0$, we provide the following lemma.

Lemma 2. *The matrices* $\mathbf{R}^{\pm}(z;x,t)$ *defined in* (88) *have the following asymptotic behavior:*

$$\mathbf{R}^{\pm}(z;x,t)=\mathbf{R}_{\infty}+O(\frac{1}{z}),\quad z\to\infty,\quad z\in\mathbb{D}^{\pm},\tag{91a}$$

$$\mathbf{R}^{\pm}(z;x,t)=\frac{1}{z}\mathbf{R}_0+O(1),\quad z\to 0,\quad z\in\mathbb{D}^{\pm},\tag{91b}$$

where

$$\mathbf{R}_{\infty}+\frac{1}{z}\mathbf{R}_0=\mathbf{Y}_{-}(z),\quad \mathbf{R}_{\infty}=\begin{pmatrix}\mathrm{i} & 0 & 0\\ \mathbf{0}_{2\times1} & \frac{1}{q_0}\mathbf{q}_{-}^{\perp} & \frac{1}{q_0}\mathbf{q}_{-}\end{pmatrix},\quad \mathbf{R}_0=\begin{pmatrix}0 & 0 & q_0\\ \mathrm{i}\mathbf{q}_{-} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1}\end{pmatrix}.\tag{92}$$

Due to the scattering matrix breaking the symmetry between $v_{+}(z)$ and $v_{-}(z)$, the asymptotic behavior as $z\to\infty$ and $z\to 0$ is obtained using the potential value at $x\to-\infty$ (rather than $x\to\infty$). In addition to the asymptotic behavior outlined in Eq. (88), to fully formulate the RH problem presented in Eq. (89), it is also necessary to specify the residue conditions.

Lemma 3. *By using the meromorphic matrices* $\mathbf{R}^{\pm}(z;x,t)$ *as described in Lemma 1, the corresponding residue conditions are as follows:*

$$\left[\operatorname{Res}_{z=z_g}\mathbf{R}^{+}(z;x,t)\right]=C_g\left[\mathbf{r}_3^{+}(z_g),\mathbf{0},\mathbf{0}\right],\quad \left[\operatorname{Res}_{z=z_g^*}\mathbf{R}^{-}(z;x,t)\right]=\bar{C}_g\left[\mathbf{0},\mathbf{0},\mathbf{r}_1^{-}(z_g^*)\right],\tag{93a}$$

$$\left[\operatorname{Res}_{z=\theta_g}\mathbf{R}^{+}(z;x,t)\right]=-F_g\left[\mathbf{r}_2^{+}(\theta_g),\mathbf{0},\mathbf{0}\right],\quad \left[\operatorname{Res}_{z=q_0^2/\theta_g}\mathbf{R}^{-}(z;x,t)\right]=\check{F}_g\left[\mathbf{0},\mathbf{0},\mathbf{r}_2^{-}(q_0^2/\theta_g)\right],\tag{93b}$$

$$\left[\operatorname{Res}_{z=\theta_g^*}\mathbf{R}^{-}(z;x,t)\right]=\bar{F}_g\left[\mathbf{0},\mathbf{r}_1^{-}(\theta_g^*),\mathbf{0}\right],\quad \left[\operatorname{Res}_{z=q_0^2/\theta_g^*}\mathbf{R}^{+}(z;x,t)\right]=-\hat{F}_g\left[\mathbf{0},\mathbf{r}_3^{+}(q_0^2/\theta_g^*),\mathbf{0}\right],\tag{93c}$$

where $\delta_j(z)=\delta_j(z;x,t)$ *for* $j=1,2$ *and* $\mathbf{r}_j^{\pm}(z)=\mathbf{r}_j^{\pm}(z;x,t)$ *for* $j=1,2,3$, *with norming constants*

$$C_g=C_g(x,t)=\frac{c_g}{h_{11}'(z_g)}\mathrm{e}^{-2\mathrm{i}\delta_1(z_g)},\quad F_g=F_g(x,t)=\frac{f_g}{h_{11}'(\theta_g)}\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]},\quad \check{F}_g=\check{F}_g(x,t)=\frac{\check{f}_g s_{11}(q_0^2/\theta_g)}{h_{33}'(q_0^2/\theta_g)}\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]},\tag{94a}$$

$$\bar{C}_g=\bar{C}_g(x,t)=\frac{\bar{c}_g}{h_{33}'(z_g^*)}\mathrm{e}^{2\mathrm{i}\delta_1(z_g^*)},\quad \bar{F}_g=\bar{F}_g(x,t)=\frac{\bar{f}_g}{s_{11}'(\theta_g^*)}\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]},\quad \hat{F}_g=\hat{F}_g(x,t)=\frac{\hat{f}_g\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]}}{s_{33}'(q_0^2/\theta_g^*)},\tag{94b}$$

where $g=1,\ldots,G_1$ *for equations involving* z_g *and* $g=1,\ldots,G_2$ *for equations involving* θ_g.

Corollary 9. *Through Corollaries 5 and 6, it can be seen that the norming constants satisfy the following symmetry relationship:*

$$C_g^*(x,t)=\bar{C}_g(x,t)=\mathrm{e}^{-2\mathrm{i}\arg(z_g)}C_g(x,t),\quad \check{F}_g(x,t)=-\frac{\mathrm{i}q_0}{\theta_g}F_g(x,t),\quad \bar{F}_g(x,t)=-\frac{F_g^*(x,t)}{\rho(\theta_g^*)},\quad \hat{F}_g(x,t)=-\frac{\mathrm{i}q_0^3}{(\theta_g^*)^3}\frac{F_g^*(x,t)}{\rho(\theta_g^*)}.\tag{95}$$

4.2. Reconstruction formula, existence and uniqueness of the solutions of the Riemann-Hilbert problem

Regularize the RH problem by subtracting the leading asymptotics and any pole contributions associated with the discrete spectrum, then the solutions of the RH problem can be obtained with the help of the Plemelj's formula. In Appendix C, we provide a detailed proof of the following theorems.

Theorem 5. *The solutions of the RH problem defined by Lemmas 1, 2 and 3 are given by*

$$\begin{aligned}\mathbf{R}(z;x,t)=\mathbf{Y}_{-}(z)-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathbf{R}^{-}(\xi)\widetilde{\mathbf{L}}(\xi)}{\xi-z}\,\mathrm{d}\xi+\sum_{i=1}^{G_1}\left[\frac{\operatorname{Res}_{z=z_i}\mathbf{R}^{+}}{z-z_i}+\frac{\operatorname{Res}_{z=z_i^*}\mathbf{R}^{-}}{z-z_i^*}\right]\\+\sum_{j=1}^{G_2}\left[\frac{\operatorname{Res}_{z=\theta_j}\mathbf{R}^{+}}{z-\theta_j}+\frac{\operatorname{Res}_{z=\theta_j^*}\mathbf{R}^{-}}{z-\theta_j^*}\right]+\sum_{j=1}^{G_2}\left[\frac{\operatorname{Res}_{z=q_0^2/\theta_j^*}\mathbf{R}^{+}}{z-(q_0^2/\theta_j^*)}+\frac{\operatorname{Res}_{z=q_0^2/\theta_j}\mathbf{R}^{-}}{z-(q_0^2/\theta_j)}\right],\end{aligned}\tag{96}$$

where $\widetilde{\mathbf{L}}(z) = \mathbf{e}^{\mathrm{i}\boldsymbol{\Delta}(z)}\mathbf{L}(z)\mathbf{e}^{-\mathrm{i}\boldsymbol{\Delta}(z)}$*, with* $\mathbf{R}(z;x,t) = \mathbf{R}^{\pm}(z;x,t) = (\mathbf{r}_1^{\pm}, \mathbf{r}_2^{\pm}, \mathbf{r}_3^{\pm})$ *for* $z \in \mathbb{D}^{\pm}$*. Furthermore, the eigenfunctions are given by*

$$\mathbf{r}_1^-(z) = \begin{pmatrix} \mathrm{i} \\ \mathrm{i}\mathbf{q}_-/z \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_1}{\xi - z} \mathrm{d}\xi + \sum_{i=1}^{G_1} \left[\frac{C_i \mathbf{r}_3^+(z_i)}{z - z_i}\right] - \sum_{j=1}^{G_2} \left[\frac{F_j \mathbf{r}_2^+(\theta_j)}{z - \theta_j}\right], \qquad z = z_g^*, \theta_g^*, \tag{97}$$

$$\mathbf{r}_3^+(z) = \begin{pmatrix} q_0/z \\ \mathbf{q}_-/q_0 \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_3}{\xi - z} \mathrm{d}\xi + \sum_{i=1}^{G_1} \left[\frac{\bar{C}_i \mathbf{r}_1^-(z_i^*)}{z - z_i^*}\right] + \sum_{j=1}^{G_2} \left[\frac{\check{F}_j \mathbf{r}_2^-(q_0^2/\theta_j)}{z - (q_0^2/\theta_j)}\right], \qquad z = z_g, q_0^2/\theta_g^*, \tag{98}$$

$$\mathbf{r}_2^+(\theta_g) = \begin{pmatrix} 0 \\ \mathbf{q}_-^{\perp}/q_0 \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_2}{\xi - \theta_g} \mathrm{d}\xi + \sum_{j=1}^{G_2} \left[\frac{\bar{F}_j \mathbf{r}_1^-(\theta_j^*)}{\theta_g - \theta_j^*}\right] - \sum_{j=1}^{G_2} \left[\frac{\hat{F}_j \mathbf{r}_3^+(q_0^2/\theta_j^*)}{\theta_g - (q_0^2/\theta_j^*)}\right], \tag{99}$$

$$\mathbf{r}_2^-(\frac{q_0^2}{\theta_g}) = \begin{pmatrix} 0 \\ \mathbf{q}_-^{\perp}/q_0 \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_2}{\xi - (q_0^2/\theta_g)} \mathrm{d}\xi + \sum_{j=1}^{G_2} \left[\frac{\bar{F}_j \mathbf{r}_1^-(\theta_j^*)}{(q_0^2/\theta_g) - \theta_j^*}\right] - \sum_{j=1}^{G_2} \left[\frac{\hat{F}_j \mathbf{r}_3^+(q_0^2/\theta_j^*)}{(q_0^2/\theta_g) - (q_0^2/\theta_j^*)}\right], \tag{100}$$

where $g = 1, \ldots, G_1$ *for equations involving* z_g *and* $g = 1, \ldots, G_2$ *for equations involving* θ_g*.*

Usually, upon obtaining the solutions to the RH problem, the potential can be reconstructed using the norming constants and scattering coefficients. This is achieved by comparing the asymptotic behavior of the eigenfunctions with the asymptotics derived from the direct scattering process. For details on the process, see Appendix C.

Theorem 6 (*Reconstruction Formula*)**.** *Pure soliton solutions* $\mathbf{q}(x,t)$ *of the defocusing–defocusing coupled Hirota equations with NZBCs (3) are reconstructed as follows:*

$$q_k(x,t) = q_{-,k} - \frac{1}{2\pi} \int_{\mathbb{R}} \left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{(k+1)1} \mathrm{d}\xi - \sum_{i=1}^{G_1} \mathrm{i}C_i \mathbf{r}_{(k+1)3}^+(z_i) + \sum_{j=1}^{G_2} \mathrm{i}F_j \mathbf{r}_{(k+1)2}^+(\theta_j), \quad k = 1, 2. \tag{101}$$

Within the framework of the IST, we have derived the expression (96) for the solutions to the RH problem. This leads to a pertinent and significant question: can we obtain a rigorous proof regarding the existence and uniqueness of this solution? In Appendix C, we provide a rigorous proof that the existence and uniqueness of solutions to the RH problem (for simplicity, we only consider the case where there is no discrete spectrum, and the case where discrete spectra exist can also be similarly proven) are guaranteed under certain conditions.

Theorem 7 (*Uniqueness*)**.** *Under the assumption that no discrete spectrum exists, if the RH problem defined by Lemmas 1, 2 and 3 admits a solution, this solution is unique.*

Theorem 8 (*Existence*)**.** *Under the assumption that no discrete spectrum exists, if* $\mathbf{L}(\cdot) \in L^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ *and* $I - E_{\mathbf{B}}$ *has Fredholm index zero, the RH problem defined by Lemmas 1, 2 and 3 admits a unique solution.*

Remark 1. By employing methods similar to those outlined in Refs. [39–43], we can attain more robust outcomes. The defocusing–defocusing coupled Hirota equations may give rise to particularly severe spectral singularities as described in [44]. It can be shown that if $\mathbf{q}(x,t) - \mathbf{q}_{\pm}$ decays sufficiently rapidly as $x \to \pm\infty$, the scattering coefficients are infinitely differentiable functions, thereby eliminating the need for the condition $\mathbf{L}(\cdot) \in L^{\infty}(\mathbb{R})$. Furthermore, a slow decay of $\mathbf{q}(x,t) - \mathbf{q}_{\pm}$ as $x \to \pm\infty$ precludes the relevant definition of the associated Zakharov-Shabat scattering problem under zero boundary conditions considered in [44]. Similarly, it can be demonstrated that the asymptotic behavior presented in Corollary 8 implies $\mathbf{L}(\cdot) \in L^2(\mathbb{R})$. Consequently, the existence and uniqueness of the RHP solution in the presence of a discrete spectrum can be established.

Remark 2. The requirement in Theorem 8 is that the Fredholm index of operator $I - E_{\mathbf{B}}$ is zero, which proves that under the assumption in Theorem 8, the operator $I - E_{\mathbf{B}}$ is invertible on $L^2(\mathbb{R})$ [36]. Due to the zero Fredholm index of the operator $I - E_{\mathbf{B}}$, it is reversible if and only if $I - E_{\mathbf{B}}$ is injective. The methodologies presented in Refs. [39–43] substantiate that the outcome is a consequence of the scattering data's properties. Nevertheless, owing to the intricate nature of the proof, we forgo an exhaustive discussion at this juncture.

4.3. Trace formulae and pure soliton solutions

We need to reconstruct the analytical scattering coefficients $h_{11}(z)$ and $s_{11}(z)$ based on the scattering data [36]. We can define

$$\chi^+(z) = h_{11}(z) \prod_{g=1}^{G_1} \frac{z - z_g^*}{z - z_g} \prod_{g=1}^{G_2} \frac{z - \theta_g^*}{z - \theta_g}, \quad z \in \mathbb{D}^+, \qquad \chi^-(z) = s_{11}(z) \prod_{g=1}^{G_1} \frac{z - z_g}{z - z_g^*} \prod_{g=1}^{G_2} \frac{z - \theta_g}{z - \theta_g^*}, \quad z \in \mathbb{D}^-. \tag{102}$$

Using the definition of the reflection coefficients (65) and the corresponding calculation of the scattering coefficients, then we have

$$\ln \chi^+(z) + \ln \chi^-(z) = -\ln \left[1 - \left|\beta_1(z)\right|^2 - \frac{\left|\beta_2(z)\right|^2}{\rho(z)}\right], \quad z \in \mathbb{R}. \tag{103}$$

By combining (102) with (103) and using the Plemelj's formula, it can be concluded that

$$\chi^+(z) = \exp\left[-\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \ln \left[1 - \left|\beta_1(\xi)\right|^2 - \frac{\left|\beta_2(\xi)\right|^2}{\rho(\xi)}\right] \frac{\mathrm{d}\xi}{\xi - z}\right], \quad z \in \mathbb{D}^+, \tag{104a}$$

$$\chi^-(z) = \exp\left[\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \ln \left[1 - \left|\beta_1(\xi)\right|^2 - \frac{\left|\beta_2(\xi)\right|^2}{\rho(\xi)}\right] \frac{\mathrm{d}\xi}{\xi - z}\right], \quad z \in \mathbb{D}^-. \tag{104b}$$

By substituting expression (104) into definition (102), the scattering coefficient display expression can be solved

$$h_{11}(z)=\prod_{g=1}^{G_1}\frac{z-z_g}{z-z_g^*}\prod_{g=1}^{G_2}\frac{z-\theta_g}{z-\theta_g^*}\exp\left[-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^+,\tag{105a}$$

$$s_{11}(z)=\prod_{g=1}^{G_1}\frac{z-z_g^*}{z-z_g}\prod_{g=1}^{G_2}\frac{z-\theta_g^*}{z-\theta_g}\exp\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^-.\tag{105b}$$

By comparing the behavior of trace formula (105) as $z\to 0$ and the asymptotic behavior of $h_{11}(z)$ and $s_{11}(z)$ in (78), we can calculate the asymptotic phase difference:

$$\Delta\delta=\delta_+-\delta_-=2\sum_{g=1}^{G_1}\arg(z_g)+2\sum_{g=1}^{G_2}\arg(\theta_g)+\frac{1}{2\pi}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi}.\tag{106}$$

Since the discrete eigenvalues on C_o satisfy the restrictions $\arg(C_g)=\arg(z_g)$ for $g=1,\dots,G_1$, the functions $C_g(x,t)$ in Theorem 5 can be parameterized

$$C_g(x,t)\mathrm{e}^{2\mathrm{i}\delta_1(z_g)}=2\left|\lambda(z_g)\right|\mathrm{e}^{2\left|\lambda(z_g)\right|\kappa_g+\mathrm{i}\chi_g},\quad g=1,\dots,G_1,\tag{107}$$

where κ_g and χ_g are real parameters and $\chi_g=\arg(z_g)+k\pi$ for $k=0,1$.

Theorem 9. *In the reflectionless case, the pure soliton solutions (101) of the defocusing–defocusing coupled Hirota equations with NZBCs (3) may be written*

$$\mathbf{q}(x,t)=\frac{1}{\det\mathbf{K}(x,t)}\begin{pmatrix}\det\mathbf{K}_1^{\mathrm{aug}}(x,t)\\ \det\mathbf{K}_2^{\mathrm{aug}}(x,t)\end{pmatrix},\quad \mathbf{K}_n^{\mathrm{aug}}(x,t)=\begin{pmatrix}q_{-,n} & \mathbf{E}(x,t)\\ \mathbf{A}_n(x,t) & \mathbf{K}(x,t)\end{pmatrix},\quad n=1,2,\tag{108}$$

the components in vector $\mathbf{E}(x,t)=\left(E_1(x,t),\dots,E_{G_1+G_2}(x,t)\right)$ *are*

$$E_g(x,t)=\begin{cases}\mathrm{i}C_g(x,t), & g=1,\dots,G_1,\\ -\mathrm{i}F_{g-G_1}(x,t), & g=G_1+1,\dots,G_1+G_2,\end{cases}\tag{109}$$

the components in vector $\mathbf{A}_n(x,t)=\left(A_{n1}(x,t),\dots,A_{n(G_1+G_2)}(x,t)\right)^T$ *are*

$$A_{ni'}(x,t)=\begin{cases}\dfrac{q_{-,n}}{q_0}, & i'=1,\dots,G_1,\\ (-1)^{n+1}\dfrac{q_{-,\bar n}^*}{q_0}+\displaystyle\sum_{j=1}^{G_2}\frac{\mathrm{i}q_{-,n}}{\theta_j^*}f_{ji'}(x,t), & i'=G_1+1,\dots,G_1+G_2,\end{cases}\tag{110}$$

the matrix $\mathbf{K}(x,t)=\mathbf{I}+\mathbf{P}(x,t)$*, the entries of matrix* $\mathbf{P}(x,t)=(P_{jk}(x,t))$ *are defined as*

$$P_{jk}(x,t)=\begin{cases}-\dfrac{\mathrm{i}z_k}{q_0}f_k^{(2)}(z_j;x,t), & j,k=1,\dots,G_1,\\ -f_{k-G_1}^{(5)}(z_j;x,t), & j=1,\dots,G_1,\quad k=G_1+1,\dots,G_1+G_2,\\ -\displaystyle\sum_{a=1}^{G_2}f_{aj}(x,t)f_k^{(1)}(\theta_a^*;x,t), & j=G_1+1,\dots,G_1+G_2,\quad k=1,\dots,G_1,\\ \displaystyle\sum_{a=1}^{G_2}f_{aj}(x,t)f_{k-G_1}^{(3)}(\theta_a^*;x,t), & j,k=G_1+1,\dots,G_1+G_2,\end{cases}\tag{111}$$

where

$$f_{jk}(x,t)=f_j^{(4)}(\theta_{k-G_1};x,t)+\frac{\mathrm{i}\theta_j^*}{q_0}f_j^{(6)}(\theta_{k-G_1};x,t),\quad \bar n=n+(-1)^{n+1}.\tag{112}$$

4.4. Varieties of soliton solutions

Here, the different possibilities of soliton solutions (108) for the defocusing–defocusing coupled Hirota equations with NZBCs (3) are analyzed. Additionally, various schemes for these soliton solutions are studied when there is either one or two discrete eigenvalues located on or outside the circle C_o.

4.4.1. Soliton solutions for the scenario where $G_1+G_2=1$

Discuss the case where there is only one discrete eigenvalue on or outside the circle C_o, i.e., $G_1+G_2=1$. Firstly, we focus on the scenario where the eigenvalues are situated on the circle with ($G_1=1$ and $G_2=0$) and express the discrete eigenvalues and normalization constants as follows:

$$z_1=q_0\mathrm{e}^{\mathrm{i}\alpha_1},\quad c_1=\mathrm{e}^{\kappa_1+\mathrm{i}[\alpha_1+(k-\frac{1}{2})\pi]},\quad 0<\alpha_1<\pi,\quad k=0,1,\tag{113}$$

from (108) one obtains the one-soliton solution of the defocusing–defocusing coupled Hirota equations with NZBCs (3):

$$\mathbf{q}(x,t)=\mathrm{e}^{\mathrm{i}\alpha_1}\left[\cos(\alpha_1)-\mathrm{i}\sin(\alpha_1)\left[\tanh(Q_1)\right]^{(-1)^{k+1}}\right]\mathbf{q}_-,\tag{114}$$

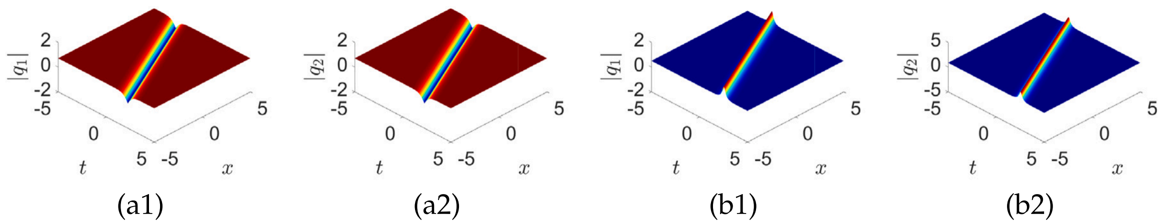

Fig. 1. (a1) and (a2): One dark–dark soliton solution by taking $\mathbf{q}_- = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^T$, $\sigma = \kappa_1 = 1$, $\alpha_1 = \frac{1}{2}\pi$, $k = 1$. (b1) and (b2): One bright–bright soliton solution by taking $\mathbf{q}_- = (\frac{1}{2}, -\frac{\sqrt{3}}{2})^T$, $\sigma = 1$, $\kappa_1 = \exp(-\frac{2}{5} - \frac{9}{5}\mathrm{i})$, $\alpha_1 = \frac{1}{2}\pi$, $k = 0$.

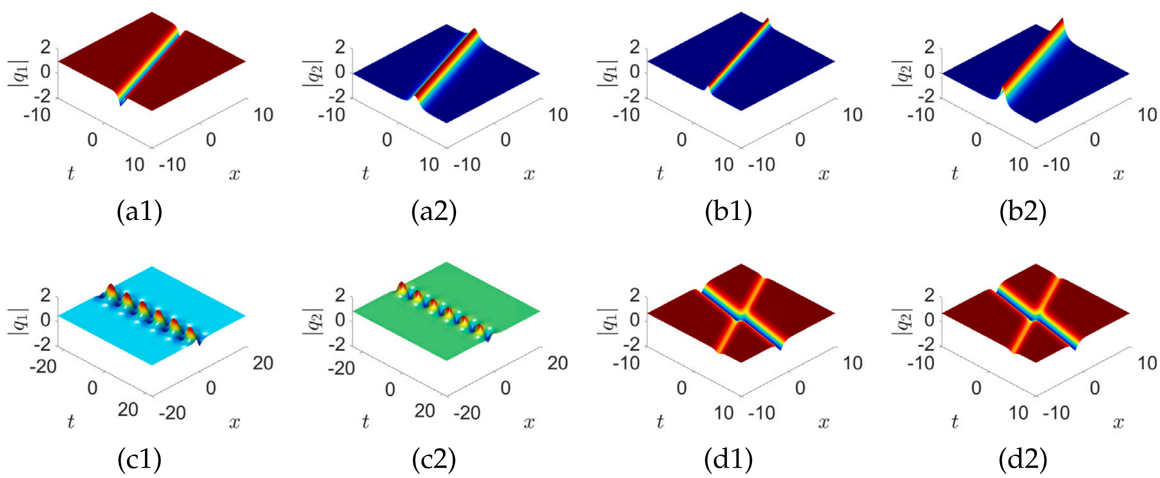

Fig. 2. (a1) and (a2): One dark-bright soliton solution by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = \kappa_2 = 1$, $\chi_2 = -1$, $K_2 = 0.5$, $\alpha_2 = \frac{1}{2}\pi$. (b1) and (b2): One bright–bright soliton solution by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = \chi_2 = 1$, $K_2 = 0.5$, $\kappa_2 = \exp(\frac{3}{5} - \frac{3}{5}\mathrm{i})$, $\alpha_2 = \frac{1}{2}\pi$. (c1) and (c2): One breather–breather soliton solution by taking $\mathbf{q}_- = (\frac{1}{2}\mathrm{e}^{-\frac{1}{10}\mathrm{i}\pi}, -\frac{\sqrt{3}}{2}\mathrm{e}^{-\frac{1}{10}\mathrm{i}\pi})^T$, $\sigma = 10^{-3}$, $\kappa_2 = 0.5$, $\chi_2 = 1$, $K_2 = 0.9$, $\alpha_2 = \frac{1}{2}\pi$. (d1) and (d2): Two dark–dark soliton solutions by taking $\mathbf{q}_- = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^T$, $\sigma = 10^{-1}$, $\kappa_3 = \kappa_4 = 1$, $\alpha_3 = \frac{1}{2}\pi$, $\alpha_4 = \frac{1}{4}\pi$, $k_1 = k_2 = 1$.

where

$$Q_1 = -q_0 \sin(\alpha_1)\left[x + [2q_0\cos(\alpha_1) + 2q_0^2\sigma\cos(2\alpha_1) + 4q_0^2\sigma]t\right] - \frac{\kappa_1}{2}. \tag{115}$$

Remark 3. In case (113), two types of soliton solutions are obtained. For $k = 1$ and $\kappa_1 \in \mathbb{R}$, one dark–dark soliton solution is given by panels (a1) and (a2) in Fig. 1. Moreover, setting $k = 0$ and $\kappa_1 \in \mathbb{C}$ generates one bright–bright soliton solution in panels (b1) and (b2) of Fig. 1.

Next, we focus on a solitary quartet of eigenvalues situated outside the circle ($G_1 = 0$ and $G_2 = 1$) and proceed to establish the relevant parameters:

$$\theta_1 = K_2\mathrm{e}^{\mathrm{i}\alpha_2}, \quad f_1 = \mathrm{e}^{\kappa_2 + \mathrm{i}\chi_2}, \quad 0 < K_2 < q_0, \quad 0 < \alpha_2 < \pi, \quad \chi_2 \in \mathbb{R}, \tag{116}$$

from (108) one generates the corresponding one-soliton solution

$$\mathbf{q}(x,t) = \frac{K_2^2(\mathrm{e}^{Q_{21}} - 1) + q_0^2}{K_2^2(\mathrm{e}^{Q_{22}} - 1) + q_0^2}\mathbf{q}_- + \frac{\mathrm{i}\mathrm{e}^{Q_{23}}K_2(K_2^2 - q_0^2)}{q_0^3 + q_0K_2^2(\mathrm{e}^{Q_{22}} - 1)}\mathbf{q}_-^{\perp}, \tag{117}$$

where

$$Q_{21} = 2\kappa_2 + 2\mathrm{i}\alpha_2 - 2\mathrm{i}K_2(x + 3q_0^2\sigma t)\sinh(\mathrm{i}\alpha_2) - 2\mathrm{i}K_2^2 t\sinh(2\mathrm{i}\alpha_2) - 2\mathrm{i}K_2^3\sigma t\sinh(3\mathrm{i}\alpha_2), \tag{118a}$$

$$Q_{23} = (1 - \mathrm{e}^{2\mathrm{i}\alpha_2})[\kappa_2 + \mathrm{i}(\chi_2 - \alpha_2) - \mathrm{i}K_2\mathrm{e}^{\mathrm{i}\alpha_2}(x + 3q_0^2\sigma t) - \mathrm{i}K_2^2\mathrm{e}^{2\mathrm{i}\alpha_2}t - \mathrm{i}K_2^3\sigma\mathrm{e}^{3\mathrm{i}\alpha_2}t], \tag{118b}$$

$$Q_{22} = 2\kappa_2 - \mathrm{i}K_2(\mathrm{e}^{-\mathrm{i}\alpha_2} - \mathrm{e}^{-3\mathrm{i}\alpha_2})[K_2 t(\mathrm{e}^{\mathrm{i}\alpha_2} + \mathrm{e}^{3\mathrm{i}\alpha_2}) + K_2^2\sigma t(1 + \mathrm{e}^{4\mathrm{i}\alpha_2}) + \mathrm{e}^{2\mathrm{i}\alpha_2}(x + (K_2^2 + 3q_0^2)\sigma t)]. \tag{118c}$$

Remark 4. In case (116), for $q_{-,1}q_{-,2} = 0$ and $\kappa_2 \in \mathbb{R}$, one dark-bright soliton solution is given by panels (a1) and (a2) in Fig. 2. Moreover, setting $q_{-,1}q_{-,2} = 0$ and $\kappa_2 \in \mathbb{C}$ yields one bright–bright soliton solution in panels (b1) and (b2) of Fig. 2. Additionally, selecting parameters such that $q_{-,1}q_{-,2} \neq 0$ results in one breather–breather soliton solution, illustrated in panels (c1) and (c2) of Fig. 2.

4.4.2. Soliton solutions for the scenario where $G_1 + G_2 = 2$

Discuss the case where there is only one discrete eigenvalue on or outside the circle C_o, i.e., $G_1 + G_2 = 2$. Firstly, taking into account a pair of eigenvalues located on the circumference ($G_1 = 2$ and $G_2 = 0$) and considering the discrete eigenvalues and normalization constants as follows:

$$z_1 = q_0\mathrm{e}^{\mathrm{i}\alpha_3}, \quad z_2 = q_0\mathrm{e}^{\mathrm{i}\alpha_4}, \quad c_1 = \mathrm{e}^{\kappa_3 + \mathrm{i}[\alpha_3 + (k_1 - \frac{1}{2})\pi]}, \quad 0 < \alpha_3, \alpha_4 < \pi, \quad k_1, k_2 = 0, 1, \quad c_2 = \mathrm{e}^{\kappa_4 + \mathrm{i}[\alpha_4 + (k_2 - \frac{1}{2})\pi]}. \tag{119}$$

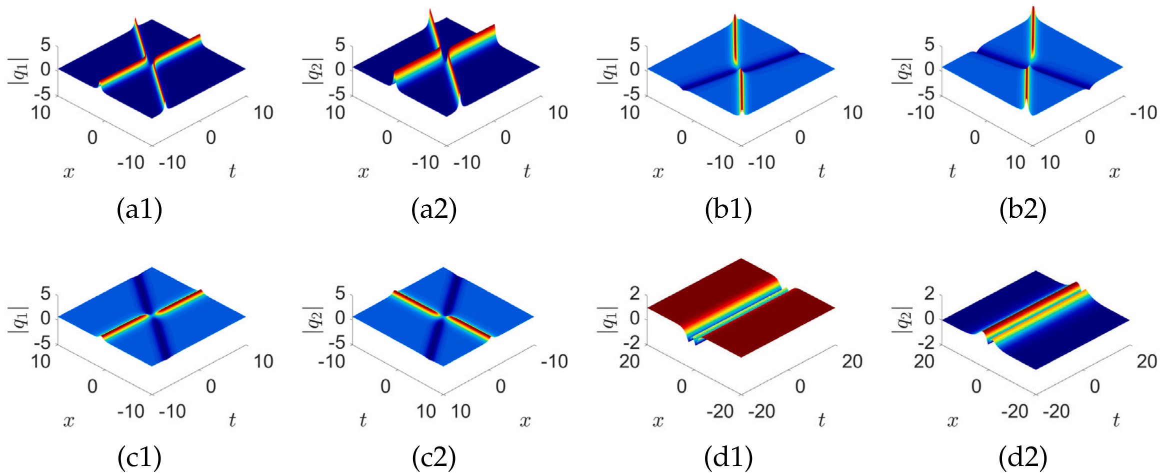

Fig. 3. (a1) and (a2): Two bright–bright soliton solutions by taking $\mathbf{q}_{-} = (\frac{1}{2}, -\frac{\sqrt{3}}{2})^T$, $\sigma = 10^{-2}$, $\kappa_3 = \mathrm{e}^{-1-\frac{3}{2}\mathrm{i}}$, $\kappa_4 = \mathrm{e}^{-1-\mathrm{i}}$, $\alpha_3 = \frac{1}{2}\pi$, $\alpha_4 = \frac{3}{4}\pi$, $k_1 = k_2 = 0$. (b1) and (b2): One dark–dark and one bright–bright soliton solutions by taking $\mathbf{q}_{-} = (\frac{1}{2}, -\frac{\sqrt{3}}{2})^T$, $\sigma = 10^{-1}$, $\kappa_3 = 1$, $\kappa_4 = \mathrm{e}^{-1-\frac{3}{2}\mathrm{i}}$, $\alpha_3 = \frac{1}{2}\pi$, $\alpha_4 = \frac{3}{4}\pi$, $k_1 = 1$, $k_2 = 0$. (c1) and (c2): One bright–bright and one dark–dark soliton solutions by taking $\mathbf{q}_{-} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^T$, $\sigma = 10^{-2}$, $\kappa_3 = \mathrm{e}^{\frac{1}{10}+\mathrm{i}}$, $\kappa_4 = 1$, $\alpha_3 = \frac{1}{2}\pi$, $\alpha_4 = \frac{3}{4}\pi$, $k_1 = 0$, $k_2 = 1$. (d1) and (d2): Two parallel dark-bright soliton solutions by taking $\mathbf{q}_{-} = (1, 0)^T$, $\sigma = 10^{-3}$, $K_6 = \kappa_5 = \kappa_6 = \chi_6 = 0.5$, $\alpha_5 = \alpha_6 = \frac{1}{2}\pi$, $k = 1$.

Remark 5. Through the expressions (119), soliton solutions and the reflectionless potentials, it can be inferred that the different structures of the two-soliton solutions are obtained. For $k_1 = k_2 = 1$ and $\kappa_3, \kappa_4 \in \mathbb{R}$, the two dark–dark soliton solutions are given by panels (d1) and (d2) in Fig. 2. Moreover, setting $k_1 = k_2 = 0$ and $\kappa_3, \kappa_4 \in \mathbb{C}$ generates two bright–bright soliton solutions in panels (a1) and (a2) of Fig. 3. In addition, one dark–dark and one bright–bright soliton solutions are obtained by selecting parameters $k_1 = 1$, $k_2 = 0$, $\kappa_3 \in \mathbb{R}$ and $\kappa_4 \in \mathbb{C}$ in panels (b1) and (b2) of Fig. 3. Finally, setting $k_1 = 0$, $k_2 = 1$, $\kappa_3 \in \mathbb{C}$ and $\kappa_4 \in \mathbb{R}$ yields one bright–bright and one dark–dark soliton solutions in (c1) and (c2) of Fig. 3.

One can discern a scenario in which one discrete eigenvalue lies on the circle while the other is situated off the circle: $G_1 = G_2 = 1$. Subsequently, we encapsulate the discrete eigenvalues and norming constants within the following expressions:

$$z_1 = q_0 \mathrm{e}^{\mathrm{i}\alpha_5}, \quad c_1 = \mathrm{e}^{\kappa_5 + \mathrm{i}[\alpha_5 + (k - \frac{1}{2})\pi]}, \quad k = 0, 1, \quad 0 < \alpha_5, \alpha_6 < \pi, \quad \theta_1 = K_6 \mathrm{e}^{\mathrm{i}\alpha_6}, \quad f_1 = \mathrm{e}^{\kappa_6 + \mathrm{i}\chi_6}, \quad 0 < K_6 < q_0, \quad \chi_6 \in \mathbb{R}. \tag{120}$$

Remark 6. Based on the analysis of two types of one-soliton solutions, the combination of these two types of soliton solutions (120) leads to six distinct outcomes in the case of $G_1 = G_2 = 1$. For $k = 1$, $q_{-,1}q_{-,2} = 0$ and $\kappa_5, \kappa_6 \in \mathbb{R}$, two parallel dark-bright soliton solutions are given by panels (d1) and (d2) in Fig. 3. In Fig. 3, panels (d1) and (d2) can be regarded as W-type soliton and M-type soliton solutions. Moreover, setting $k = 1$, $q_{-,1}q_{-,2} = 0$, $\kappa_5 \in \mathbb{R}$ and $\kappa_6 \in \mathbb{C}$ generates one parallel dark-bright and one parallel bright–bright soliton solutions in panels (a1) and (a2) of Fig. 4. Furthermore, two parallel bright–bright soliton solutions are obtained by selecting parameters $k = 0$, $q_{-,1}q_{-,2} = 0$ and $\kappa_5, \kappa_6 \in \mathbb{C}$ in panels (b1) and (b2) of Fig. 4. Then, setting $k = 0$, $q_{-,1}q_{-,2} = 0$, $\kappa_5 \in \mathbb{C}$ and $\kappa_6 \in \mathbb{R}$ yields one parallel bright–bright and one parallel dark-bright soliton solutions in panels (c1) and (c2) of Fig. 4. Additionally, setting $k = 1$, $q_{-,1}q_{-,2} \neq 0$ and $\kappa_5 \in \mathbb{R}$ generates one dark–dark and one breather–breather soliton solutions in panels (d1) and (d2) of Fig. 4. In Fig. 4, panels (b1) and (b2) can be regarded as M-type soliton and M-type soliton solutions. In addition, one bright–bright and one breather–breather soliton solutions are obtained by selecting parameters $k = 0$, $q_{1-} \times q_{2-} = 0$ and $\kappa_5 \in \mathbb{C}$ in panels (a1) and (a2) of Fig. 5.

Following that, we examine the case where both eigenvalues are located outside the circle ($G_1 = 0$ and $G_2 = 2$) and define additional parameters accordingly.

$$\theta_1 = K_7 \mathrm{e}^{\mathrm{i}\alpha_7}, \quad \theta_2 = K_8 \mathrm{e}^{\mathrm{i}\alpha_8}, \quad f_1 = \mathrm{e}^{\kappa_7 + \mathrm{i}\chi_7}, \quad f_2 = \mathrm{e}^{\kappa_8 + \mathrm{i}\chi_8}, \quad 0 < K_7, K_8 < q_0, \quad \chi_7, \chi_8 \in \mathbb{R}. \tag{121}$$

Remark 7. Through the expressions (121), soliton solutions and the reflectionless potentials, it can be inferred that the different structures of the two-soliton solutions are obtained. For $q_{-,1}q_{-,2} = 0$ and $\kappa_7, \kappa_8 \in \mathbb{R}$, two dark-bright soliton solutions are given by panels (b1) and (b2) in Fig. 5. Moreover, setting $q_{-,1}q_{-,2} = 0$ and $\kappa_7, \kappa_8 \in \mathbb{C}$ generates two bright–bright soliton solutions in panels (c1) and (c2) of Fig. 5. In addition, two breather–breather soliton solutions are obtained by selecting parameters $q_{-,1}q_{-,2} \neq 0$ in panels (d1) and (d2) of Fig. 5.

5. Multiple double-pole solutions

The situation of the defocusing–defocusing coupled Hirota equations with NZBCs (3) when the analytical scattering coefficient has double zeros was obtained in [36]. We shall denote the pertinent solutions as the "multiple double-pole" solutions associated with the Eqs. (3).

Given that $h_{11}(\theta_g) = h'_{11}(\theta_g) = 0$ and $h''_{11}(\theta_g) \neq 0$ with $\left|\theta_g\right| < q_0$, we proceed to regularize the RH problem (89) by accounting for the residue contributions, as previously discussed. However, we observe that the principal part of the Laurent series expansion of the meromorphic matrices introduces additional terms that require subtraction. Consequently, this leads to the appearance of derivatives of the eigenfunctions with respect to z as new unknowns in the RH problem. Therefore, there are additional norming constants and corresponding symmetries.

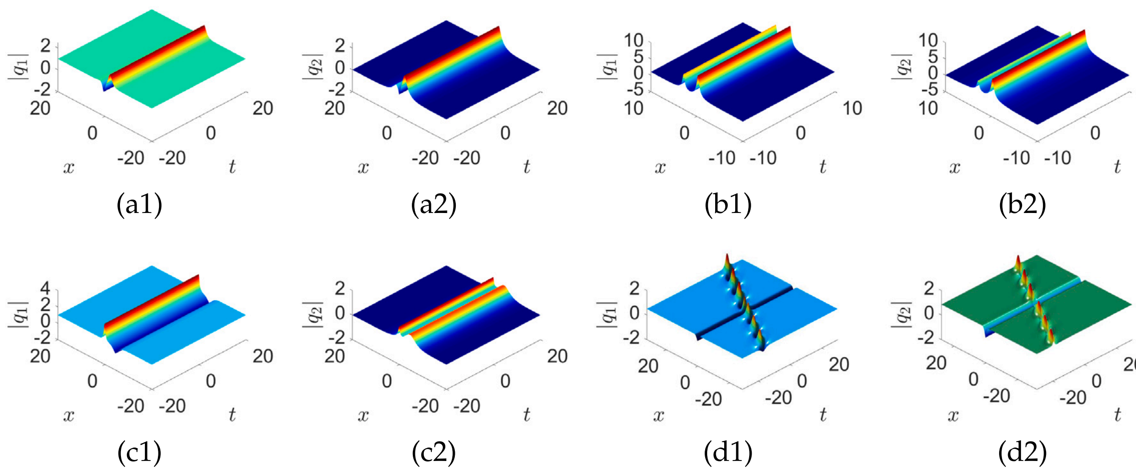

Fig. 4. (a1) and (a2): One parallel dark-bright and one parallel bright–bright soliton solutions by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-3}$, $K_6 = 0.5$, $\kappa_5 = \chi_6 = 1$, $\kappa_6 = e^{\frac{7}{10}+\frac{3}{5}i}$, $\alpha_5 = \alpha_6 = \frac{1}{2}\pi$, $k = 1$. (b1) and (b2): Two parallel bright–bright soliton solutions by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-3}$, $K_6 = 0.5$, $\kappa_5 = e^{2-2i}$, $\kappa_6 = e^{\frac{3}{5}-\frac{6}{5}i}$, $\chi_6 = 1.5$, $\alpha_5 = \alpha_6 = \frac{1}{2}\pi$, $k = 0$. (c1) and (c2): One parallel bright–bright and one parallel dark-bright soliton solutions by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-3}$, $K_6 = 0.5$, $\kappa_5 = e^{\frac{7}{10}+\frac{3}{10}i}$, $\kappa_6 = 2.8$, $\chi_6 = -0.2$, $\alpha_5 = \alpha_6 = \frac{1}{2}\pi$, $k = 0$. (d1) and (d2): One dark–dark and one breather–breather soliton solutions by taking $\mathbf{q}_- = (\frac{1}{2}e^{-\frac{1}{10}i\pi}, -\frac{\sqrt{3}}{2}e^{-\frac{1}{10}i\pi})^T$, $\sigma = 10^{-3}$, $K_6 = 0.98$, $\kappa_5 = \chi_6 = 1$, $\kappa_6 = 0.5$, $\alpha_5 = \frac{1}{2}\pi$, $\alpha_6 = \frac{4}{5}\pi$, $k = 1$.

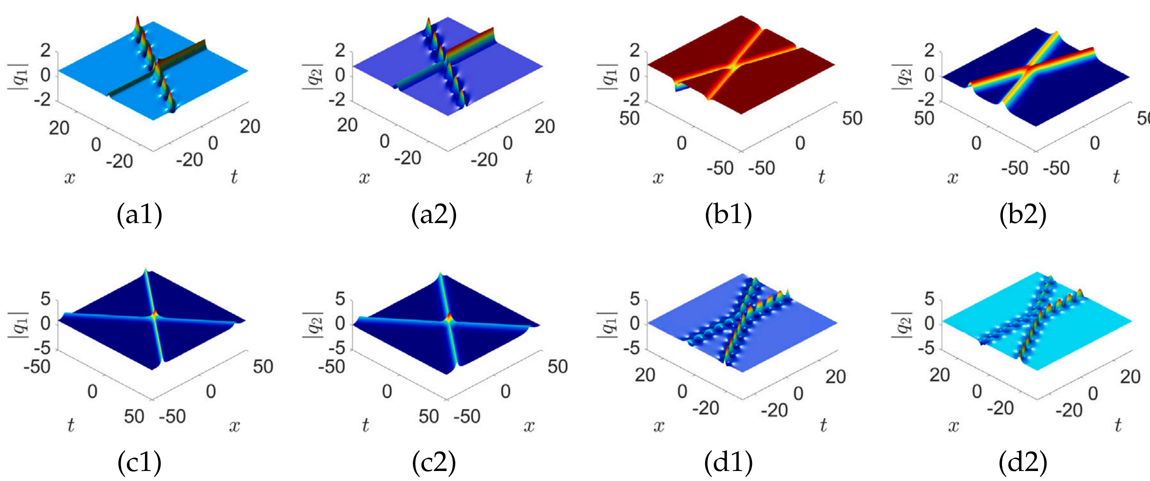

Fig. 5. (a1) and (a2): One bright–bright and one breather–breather soliton solutions by taking $\mathbf{q}_- = (\frac{1}{2}e^{-\frac{1}{10}i\pi}, -\frac{\sqrt{3}}{2}e^{-\frac{1}{10}i\pi})^T$, $\sigma = 10^{-3}$, $K_6 = 0.98$, $\kappa_5 = e^{\frac{1}{5}-i}$, $\chi_6 = 1$, $\kappa_6 = 0.2$, $\alpha_5 = \frac{1}{2}\pi$, $\alpha_6 = \frac{4}{5}\pi$, $k = 0$. (b1) and (b2): Two dark-bright soliton solutions by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-1}$, $K_7 = K_8 = 0.5$, $\chi_7 = 1$, $\kappa_7 = \kappa_8 = \chi_8 = -1$, $\alpha_7 = \frac{1}{2}\pi$, $\alpha_8 = \frac{3}{4}\pi$. (c1) and (c2): Two bright–bright soliton solutions by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-3}$, $K_7 = K_8 = 0.5$, $\chi_7 = 1$, $\chi_8 = 0.8$, $\kappa_7 = e^{\frac{1}{5}-\frac{3}{2}i}$, $\kappa_8 = e^{\frac{1}{2}+\frac{7}{10}i}$, $\alpha_7 = \frac{1}{5}\pi$, $\alpha_8 = \frac{4}{5}\pi$. (d1) and (d2): Two breather–breather soliton solutions by taking $\mathbf{q}_- = (\frac{1}{2}e^{\frac{1}{5}i\pi}, -\frac{\sqrt{3}}{2}e^{\frac{1}{5}i\pi})^T$, $\sigma = 10^{-3}$, $K_7 = K_8 = 0.98$, $\chi_7 = 0.5$, $\chi_8 = 1.5$, $\kappa_7 = -1.5$, $\kappa_8 = 2$, $\alpha_7 = \frac{1}{2}\pi$, $\alpha_8 = \frac{3}{5}\pi$.

5.1. Behavior of the eigenfunctions at multiple double poles

This section explores the behavior of eigenfunctions at multiple double poles, contrasting with the focus in [36], which is on studying the behavior at a single double pole. For the sake of brevity, we will omit the variables (x,t) from the right-hand side of the eigenfunction expressions, as they are not essential to the discussion.

Proposition 8. *Suppose that* $h_{11}(\theta_g) = h'_{11}(\theta_g) = 0$ *and* $h''_{11}(\theta_g) \neq 0$ *with* $\left|\theta_g\right| < q_0$, *then there exist constants* b_g, $\hat{b}_g$, $\check{b}_g$, $\bar{b}_g$, f_g, $\hat{f}_g$, $\check{f}_g$, $\bar{f}_g$, e_g, $\hat{e}_g$, $\check{e}_g$ *and* $\bar{e}_g$ *such that*

$$\psi'_{+,1}(\theta_g;x,t) = \frac{f_g}{s_{33}(\theta_g)}\widetilde{\gamma}'(\theta_g) + b_g\widetilde{\gamma}(\theta_g) + e_g\psi_{-,3}(\theta_g), \tag{122a}$$

$$\widetilde{\gamma}'(\frac{q_0^2}{\theta_g^*};x,t) = \hat{f}_g\psi'_{-,3}(\frac{q_0^2}{\theta_g^*}) + \hat{b}_g\psi_{-,3}(\frac{q_0^2}{\theta_g^*}) + \hat{e}_g\psi_{+,1}(\frac{q_0^2}{\theta_g^*}), \tag{122b}$$

$$\psi'_{+,3}(\frac{q_0^2}{\theta_g};x,t) = \check{f}_g\gamma'(\frac{q_0^2}{\theta_g}) + \check{b}_g\gamma(\frac{q_0^2}{\theta_g}) + \check{e}_g\psi_{-,1}(\frac{q_0^2}{\theta_g}), \tag{122c}$$

$$\gamma'(\theta_g^*;x,t)=\bar{f}_g\psi'_{-,1}(\theta_g^*)+\bar{b}_g\psi_{-,1}(\theta_g^*)+\bar{e}_g\psi_{+,3}(\theta_g^*), \tag{122d}$$

and corresponding modified eigenfunctions are

$$\nu'_{+,1}(\theta_g;x,t)=-\mathrm{i}\delta_1'(\theta_g)\nu_{+,1}(\theta_g)+e_g\nu_{-,3}(\theta_g)\mathrm{e}^{-2\mathrm{i}\delta_1(\theta_g)}+\left[\left(\frac{\mathrm{i}f_g\delta_2'(\theta_g)}{s_{33}(\theta_g)}+b_g\right)\widetilde{d}(\theta_g)+\frac{f_g}{s_{33}(\theta_g)}\widetilde{d}'(\theta_g)\right]\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]}, \tag{123}$$

$$\widetilde{d}'(\frac{q_0^2}{\theta_g^*};x,t)=-\mathrm{i}\delta_2'(\frac{q_0^2}{\theta_g^*})\widetilde{d}(\frac{q_0^2}{\theta_g^*})+\hat{e}_g\nu_{+,1}(\frac{q_0^2}{\theta_g^*})\mathrm{e}^{-\mathrm{i}[\delta_1(\theta_g^*)+\delta_2(\theta_g^*)]}+\left[\left(\hat{b}_g-\mathrm{i}\hat{f}_g\delta_1'(\frac{q_0^2}{\theta_g^*})\right)\nu_{-,3}(\frac{q_0^2}{\theta_g^*})+\hat{f}_g\nu'_{-,3}(\frac{q_0^2}{\theta_g^*})\right]\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]}, \tag{124}$$

$$d'(\theta_g^*;x,t)=-\mathrm{i}\delta_2'(\theta_g^*)d(\theta_g^*)+\bar{e}_g\nu_{+,3}(\theta_g^*)\mathrm{e}^{-\mathrm{i}[\delta_1(\theta_g^*)+\delta_2(\theta_g^*)]}+\left[\left(\mathrm{i}\bar{f}_g\delta_1'(\theta_g^*)+\bar{b}_g\right)\nu_{-,1}(\theta_g^*)+\bar{f}_g\nu'_{-,1}(\theta_g^*)\right]\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]}, \tag{125}$$

$$\nu'_{+,3}(\frac{q_0^2}{\theta_g};x,t)=\mathrm{i}\delta_1'(\frac{q_0^2}{\theta_g})\nu_{+,3}(\frac{q_0^2}{\theta_g})+\check{e}_g\nu_{-,1}(\frac{q_0^2}{\theta_g})\mathrm{e}^{-2\mathrm{i}\delta_1(\theta_g)}+\left[\left(\check{b}_g+\mathrm{i}\check{f}_g\delta_2'(\frac{q_0^2}{\theta_g})\right)d(\frac{q_0^2}{\theta_g})+\check{f}_g d'(\frac{q_0^2}{\theta_g})\right]\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]}, \tag{126}$$

where f_g, $\hat{f}_g$, $\check{f}_g$ and $\bar{f}_g$ are the same norming constants in the symmetry relationship (72), whereas b_g, $\hat{b}_g$, $\check{b}_g$, $\bar{b}_g$, e_g, $\hat{e}_g$, $\check{e}_g$ and $\bar{e}_g$ appear as a result of the double multiplicity.

Proposition 9. *Assuming that $y(z)$ and $h(z)$ are analytic in $\mathbb{D}^+$ and θ_g are double zeros of $h(z)$, then expanding $y(z)$ and $h(z)$ as Taylor expansions at $z=\theta_g$*

$$\frac{y(z)}{h(z)}=\left[\frac{2y'(\theta_g)}{h''(\theta_g)}-\frac{2y(\theta_g)h'''(\theta_g)}{3[h''(\theta_g)]^2}\right]\frac{1}{(z-\theta_g)}+\frac{2y(\theta_g)}{h''(\theta_g)}\frac{1}{(z-\theta_g)^2}+\cdots. \tag{127}$$

Therefore, it can be seen that the coefficients of $(z-\theta_g)^{-1}$ and $(z-\theta_g)^{-2}$ in the series expansion of $y(z)/h(z)$ near $z=\theta_g$ are as follows:

$$\operatorname*{Res}_{z=\theta_g}\left[\frac{y(z)}{h(z)}\right]=\frac{2y'(\theta_g)}{h''(\theta_g)}-\frac{2y(\theta_g)h'''(\theta_g)}{3[h''(\theta_g)]^2},\quad \operatorname*{Y_{-2}}_{z=\theta_g}\left[\frac{y(z)}{h(z)}\right]=\frac{2y(\theta_g)}{h''(\theta_g)}. \tag{128}$$

Corollary 10. *The generalization of the negative second power coefficients and negative first power coefficients will be obtained*

$$\operatorname*{Y_{-2}}_{z=\theta_g}\left[\frac{\nu_{+,1}(z)}{h_{11}(z)}\right]=W_g\widetilde{d}(\theta_g),\quad \operatorname*{Y_{-2}}_{z=\theta_g^*}\left[\frac{d(z)}{s_{11}(z)}\right]=\bar{W}_g\nu_{-,1}(\theta_g^*),\quad \operatorname*{Y_{-2}}_{z=q_0^2/\theta_g}\left[\frac{\nu_{+,3}(z)}{h_{33}(z)}\right]=\check{W}_g d(\frac{q_0^2}{\theta_g}),\quad \operatorname*{Y_{-2}}_{z=q_0^2/\theta_g^*}\left[-\frac{\widetilde{d}(z)}{s_{33}(z)}\right]=-\hat{W}_g\nu_{-,3}(\frac{q_0^2}{\theta_g^*}), \tag{129}$$

and

$$\operatorname*{Res}_{z=\theta_g}\left[\frac{\nu_{+,1}(z)}{h_{11}(z)}\right]=W_g\widetilde{d}'(\theta_g)+D_g\nu_{-,3}(\theta_g)+\left[B_g-\mathrm{i}x-\mathrm{i}t\left(2\theta_g+3\sigma(q_0^2+\theta_g^2)\right)\right]W_g\widetilde{d}(\theta_g), \tag{130}$$

$$\operatorname*{Res}_{z=\theta_g^*}\left[\frac{d(z)}{s_{11}(z)}\right]=\bar{W}_g\nu'_{-,1}(\theta_g^*)+\bar{D}_g\nu_{+,3}(\theta_g^*)+\left[\bar{B}_g+\mathrm{i}x+\mathrm{i}t\left(2\theta_g^*+3\sigma(q_0^2+(\theta_g^*)^2)\right)\right]\bar{W}_g\nu_{-,1}(\theta_g^*), \tag{131}$$

$$\operatorname*{Res}_{z=q_0^2/\theta_g}\left[\frac{\nu_{+,3}(z)}{h_{33}(z)}\right]=\check{W}_g d'(\frac{q_0^2}{\theta_g})+\check{D}_g\nu_{-,1}(\frac{q_0^2}{\theta_g})+\left[\check{B}_g+\frac{\mathrm{i}\theta_g^2}{q_0^2}\left(x+t(2\theta_g+3\sigma(q_0^2+\theta_g^2))\right)\right]\check{W}_g d(\frac{q_0^2}{\theta_g}), \tag{132}$$

$$\operatorname*{Res}_{z=q_0^2/\theta_g^*}\left[-\frac{\widetilde{d}(z)}{s_{33}(z)}\right]=\left[\frac{\mathrm{i}(\theta_g^*)^2}{q_0^2}\left(x+t(2\theta_g^*+3\sigma(q_0^2+(\theta_g^*)^2))\right)-\hat{B}_g\right]\hat{W}_g\nu_{-,3}(\frac{q_0^2}{\theta_g^*})-\hat{W}_g\nu'_{-,3}(\frac{q_0^2}{\theta_g^*})-\hat{D}_g\nu_{+,1}(\frac{q_0^2}{\theta_g^*}), \tag{133}$$

where

$$W_g(x,t)=\frac{2f_g\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]}}{s_{33}(\theta_g)h''_{11}(\theta_g)},\quad B_g=\frac{b_g}{f_g}s_{33}(\theta_g)-\frac{h'''_{11}(\theta_g)}{3h''_{11}(\theta_g)},\quad \bar{W}_g(x,t)=\frac{2\bar{f}_g\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]}}{s''_{11}(\theta_g^*)},\quad \bar{D}_g(x,t)=\frac{2\bar{e}_g\mathrm{e}^{-\mathrm{i}[\delta_1(\theta_g^*)+\delta_2(\theta_g^*)]}}{s''_{11}(\theta_g^*)}, \tag{134a}$$

$$\check{W}_g(x,t)=\frac{2\check{f}_g\mathrm{e}^{\mathrm{i}[\delta_2(\theta_g)-\delta_1(\theta_g)]}}{h''_{33}(q_0^2/\theta_g)},\quad \check{D}_g(x,t)=\frac{2\check{e}_g\mathrm{e}^{-2\mathrm{i}\delta_1(\theta_g)}}{h''_{33}(q_0^2/\theta_g)},\quad \bar{B}_g=\frac{\bar{b}_g}{\bar{f}_g}-\frac{s'''_{11}(\theta_g^*)}{3s''_{11}(\theta_g^*)},\quad \hat{W}_g(x,t)=\frac{2\hat{f}_g\mathrm{e}^{\mathrm{i}[\delta_1(\theta_g^*)-\delta_2(\theta_g^*)]}}{s''_{33}(q_0^2/\theta_g^*)}, \tag{134b}$$

$$\hat{D}_g(x,t)=\frac{2\hat{e}_g\mathrm{e}^{-\mathrm{i}[\delta_1(\theta_g^*)+\delta_2(\theta_g^*)]}}{s''_{33}(q_0^2/\theta_g^*)},\quad D_g(x,t)=\frac{2e_g\mathrm{e}^{-2\mathrm{i}\delta_1(\theta_g)}}{h''_{11}(\theta_g)},\quad \hat{B}_g=\frac{\hat{b}_g}{\hat{f}_g}-\frac{s'''_{33}(q_0^2/\theta_g^*)}{3s''_{33}(q_0^2/\theta_g^*)},\quad \check{B}_g=\frac{\check{b}_g}{\check{f}_g}-\frac{h'''_{33}(q_0^2/\theta_g)}{3h''_{33}(q_0^2/\theta_g)}. \tag{134c}$$

5.2. Symmetries with multiple double poles

The symmetries associated with the eigenfunctions and scattering coefficients exhibit greater complexity compared to scenarios involving only simple zeros.

Proposition 10. *Suppose that $h_{11}(\theta_g)=h'_{11}(\theta_g)=0$ and $h''_{11}(\theta_g)\neq 0$ with $\left|\theta_g\right|<q_0$, then analytic scattering coefficients has the following symmetry relationship:*

$$h''_{11}(\theta_g)=\left[s''_{11}(z)\right]^*\Big|_{z=\theta_g^*},\quad s''_{33}(\theta_g)=\left[h''_{33}(z)\right]^*\Big|_{z=\theta_g^*},\quad s''_{33}(\frac{q_0^2}{\theta_g^*})=\frac{(\theta_g^*)^4}{q_0^4}s''_{11}(z)\Bigg|_{z=\theta_g^*}, \tag{135a}$$

$$h_{11}'''(\theta_g) = \left[s_{11}'''(z)\right]^*\Big|_{z=\theta_g^*}, \quad s_{33}'''(\theta_g) = \left[h_{33}'''(z)\right]^*\Big|_{z=\theta_g^*}, \quad h_{11}''(\theta_g) = \frac{q_0^4}{\theta_g^4} h_{33}''(z)\Big|_{z=q_0^2/\theta_g}, \tag{135b}$$

$$s_{33}'''(\frac{q_0^2}{\theta_g^*}) = -\frac{(\theta_g^*)^5}{q_0^6}\left[6s_{11}''(z) + \theta_g^* s_{11}'''(z)\right]\Big|_{z=\theta_g^*}, \quad h_{11}'''(\theta_g) = -\frac{q_0^4}{\theta_g^5}\left[6h_{33}''(z) + \frac{q_0^2}{\theta_g} h_{33}'''(z)\right]\Big|_{z=q_0^2/\theta_g}. \tag{135c}$$

The eigenfunctions has the following symmetry relationship:

$$\psi_{-,1}'(\theta_g^*) = -\frac{\mathrm{i}q_0}{(\theta_g^*)^2}\left[\psi_{-,3}(\frac{q_0^2}{\theta_g^*}) + \frac{q_0^2}{\theta_g^*}\psi_{-,3}'(\frac{q_0^2}{\theta_g^*})\right], \quad \psi_{+,1}'(\theta_g) = -\frac{\mathrm{i}q_0}{\theta_g^2}\left[\psi_{+,3}(\frac{q_0^2}{\theta_g}) + \frac{q_0^2}{\theta_g}\psi_{+,3}'(\frac{q_0^2}{\theta_g})\right], \tag{136a}$$

$$\psi_{-,3}'(\theta_g) = \frac{\mathrm{i}q_0}{\theta_g^2}\left[\psi_{-,1}(\frac{q_0^2}{\theta_g}) + \frac{q_0^2}{\theta_g}\psi_{-,1}'(\frac{q_0^2}{\theta_g})\right], \quad \psi_{+,3}'(\theta_g^*) = \frac{\mathrm{i}q_0}{(\theta_g^*)^2}\left[\psi_{+,1}(\frac{q_0^2}{\theta_g^*}) + \frac{q_0^2}{\theta_g^*}\psi_{+,1}'(\frac{q_0^2}{\theta_g^*})\right], \tag{136b}$$

$$\widetilde{\gamma}'(\theta_g) = \frac{q_0^2}{\theta_g^2}\gamma'(\frac{q_0^2}{\theta_g}), \quad \gamma'(\theta_g^*) = \frac{q_0^2}{(\theta_g^*)^2}\widetilde{\gamma}'(\frac{q_0^2}{\theta_g^*}). \tag{136c}$$

Corollary 11. *The norming constants follows the following symmetry relationship:*

$$\bar{f}_g = -\frac{f_g^*}{\rho(\theta_g^*)}, \quad \hat{f}_g = \frac{\mathrm{i}q_0}{\theta_g^*\rho(\theta_g^*)} f_g^*, \quad \check{f}_g = \frac{\mathrm{i}\theta_g}{q_0 s_{33}(\theta_g)} f_g, \quad e_g = \check{e}_g = \bar{e}_g = \hat{e}_g = 0, \tag{137a}$$

$$b_g = \frac{\mathrm{i}q_0}{\theta_g}\left[\frac{\check{b}_g}{\theta_g} + \frac{q_0^2}{\theta_g^2}\check{b}_g\right], \quad b_g = -\frac{\bar{b}_g^*\rho(\theta_g)}{s_{33}(\theta_g)} - \bar{f}_g^*\left[\frac{\rho(z)}{s_{33}(z)}\right]'\Big|_{z=\theta_g}, \tag{137b}$$

$$\bar{b}_g = \frac{\mathrm{i}\theta_g^*}{q_0}\left[\frac{\hat{b}_g}{\theta_g^*} - \frac{q_0^2}{(\theta_g^*)^2}\hat{b}_g\right], \quad \bar{b}_g^* = -\frac{b_g s_{33}(\theta_g)}{\rho(\theta_g)} + \bar{f}_g^* \ln\left[\frac{s_{33}(z)}{\rho(z)}\right]'\Big|_{z=\theta_g}, \tag{137c}$$

and

$$\hat{W}_g(x,t) = \frac{\mathrm{i}q_0^5\left[s_{33}(\theta_g)\right]^*}{(\theta_g^*)^5\rho(\theta_g^*)} W_g^*(x,t), \quad \check{B}_g = \frac{\theta_g}{q_0^2} - \frac{\theta_g^2}{q_0^2}\bar{B}_g^* - \frac{\theta_g^2 s_{33}(\theta_g)}{q_0^2\rho(\theta_g)}\left[\frac{\rho(z)}{s_{33}(z)}\right]'\Big|_{z=\theta_g}, \tag{138a}$$

$$\bar{W}_g(x,t) = -\frac{\left[s_{33}(\theta_g)\right]^*}{\rho(\theta_g^*)} W_g^*(x,t), \quad \check{W}_g(x,t) = \frac{\mathrm{i}q_0^3}{\theta_g^3} W_g(x,t), \quad \hat{B}_g = \frac{3\theta_g^*}{q_0^2} - \frac{(\theta_g^*)^2}{q_0^2}\bar{W}_g, \tag{138b}$$

$$D_g(x,t) = \bar{D}_g(x,t) = \hat{D}_g(x,t) = \check{D}_g(x,t) = 0, \quad B_g = \bar{B}_g^* + \frac{s_{33}(\theta_g)}{\rho(\theta_g)}\left[\frac{\rho(z)}{s_{33}(z)}\right]'\Big|_{z=\theta_g}. \tag{138c}$$

Therefore, the residue conditions are obtained through Corollary 10:

$$\mathbf{R}^+_{-1,\theta_g}(x,t) = \left[\operatorname*{Res}_{z=\theta_g}\left[\frac{v_{+,1}(z)}{h_{11}(z)}\right], \mathbf{0}, \mathbf{0}\right], \quad \mathbf{R}^+_{-1,q_0^2/\theta_g^*}(x,t) = \left[\mathbf{0}, \operatorname*{Res}_{z=q_0^2/\theta_g^*}\left[-\frac{\widetilde{d}(z)}{s_{33}(z)}\right], \mathbf{0}\right], \tag{139a}$$

$$\mathbf{R}^+_{-2,\theta_g}(x,t) = \left[\operatorname*{Y_{-2}}_{z=\theta_g}\left[\frac{v_{+,1}(z)}{h_{11}(z)}\right], \mathbf{0}, \mathbf{0}\right], \quad \mathbf{R}^+_{-2,q_0^2/\theta_g^*}(x,t) = \left[\mathbf{0}, \operatorname*{Y_{-2}}_{z=q_0^2/\theta_g^*}\left[-\frac{\widetilde{d}(z)}{s_{33}(z)}\right], \mathbf{0}\right], \tag{139b}$$

$$\mathbf{R}^-_{-1,\theta_g^*}(x,t) = \left[\mathbf{0}, \operatorname*{Res}_{z=\theta_g^*}\left[\frac{d(z)}{s_{11}(z)}\right], \mathbf{0}\right], \quad \mathbf{R}^-_{-1,q_0^2/\theta_g}(x,t) = \left[\mathbf{0}, \mathbf{0}, \operatorname*{Res}_{z=q_0^2/\theta_g}\left[\frac{v_{+,3}(z)}{h_{33}(z)}\right]\right], \tag{139c}$$

$$\mathbf{R}^-_{-2,\theta_g^*}(x,t) = \left[\mathbf{0}, \operatorname*{Y_{-2}}_{z=\theta_g^*}\left[\frac{d(z)}{s_{11}(z)}\right], \mathbf{0}\right], \quad \mathbf{R}^-_{-2,q_0^2/\theta_g}(x,t) = \left[\mathbf{0}, \mathbf{0}, \operatorname*{Y_{-2}}_{z=q_0^2/\theta_g}\left[\frac{v_{+,3}(z)}{h_{33}(z)}\right]\right]. \tag{139d}$$

5.3. Reflectionless solutions with multiple double poles

Regularize the RH problem by subtracting the asymptotic behavior at infinity and any pole contributions associated with the discrete spectrum, then the solutions of the RH problem can be obtained with the help of Cauchy projectors. In Appendix D, we provide a detailed proof of the following theorems.

Theorem 10. *Suppose that $h_{11}(\theta_g) = h_{11}'(\theta_g) = 0$ and $h_{11}''(\theta_g) \neq 0$ with $\left|\theta_g\right| < q_0$, the multiple double-pole solutions of the RH problem defined by Lemmas 1 and 2 with residue conditions (139) are shown below:*

$$\begin{aligned}\mathbf{R}(z;x,t) = &\sum_{j=1}^{G}\left[\frac{\mathbf{R}^+_{-1,\theta_j}}{z-\theta_j} + \frac{\mathbf{R}^-_{-1,\theta_j^*}}{z-\theta_j^*} + \frac{\mathbf{R}^+_{-2,\theta_j}}{(z-\theta_j)^2} + \frac{\mathbf{R}^-_{-2,\theta_j^*}}{(z-\theta_j^*)^2}\right] + \sum_{j=1}^{G}\left[\frac{\mathbf{R}^+_{-1,q_0^2/\theta_j^*}}{z-(q_0^2/\theta_j^*)} + \frac{\mathbf{R}^-_{-1,q_0^2/\theta_j}}{z-(q_0^2/\theta_j)} + \frac{\mathbf{R}^+_{-2,q_0^2/\theta_j^*}}{[z-(q_0^2/\theta_j^*)]^2} + \frac{\mathbf{R}^-_{-2,q_0^2/\theta_j}}{[z-(q_0^2/\theta_j)]^2}\right] \\ &+ \mathbf{Y}_-(z) - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)}{\xi - z}\,\mathrm{d}\xi,\end{aligned} \tag{140}$$

where $\widetilde{\mathbf{L}}(z) = \mathrm{e}^{\mathrm{i}\mathbf{\Delta}(z)}\mathbf{L}(z)\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}(z)}$ and $\mathbf{R}(z;x,t) = \mathbf{R}^\pm(z;x,t)$ for $\Im\mathrm{m}\, z \gtrless 0$. Moreover, the eigenfunctions are given by

$$v_{-,1}(\theta_g^*;x,t)=\begin{pmatrix}\mathrm{i}\\ \mathrm{i}\mathbf{q}_-/\theta_g^*\end{pmatrix}-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_1}{\xi-\theta_g^*}\,\mathrm{d}\xi+\sum_{j=1}^{G}\left[\frac{\left[B_j-\mathrm{i}x-\mathrm{i}t\left(2\theta_j+3\sigma(q_0^2+\theta_j^2)\right)\right]W_j\widetilde{d}(\theta_j)}{\theta_g^*-\theta_j}+\frac{W_j\widetilde{d}'(\theta_j)}{\theta_g^*-\theta_j}+\frac{W_j\widetilde{d}(\theta_j)}{[\theta_g^*-\theta_j]^2}\right], \tag{141}$$

$$\begin{aligned}-\frac{\widetilde{d}(\theta_g;x,t)}{s_{33}(\theta_g)}=&\begin{pmatrix}0\\ \mathbf{q}_-^{\perp}/q_0\end{pmatrix}-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_2}{\xi-\theta_g}\,\mathrm{d}\xi+\sum_{j=1}^{G}\left\{\frac{\bar{W}_j}{(\theta_g-\theta_j^*)^2}\left[1+(\theta_g-\theta_j^*)\left[\bar{B}_j+\mathrm{i}x+\mathrm{i}t\left(2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2)\right)\right]\right]\right.\\&+\left.\frac{\mathrm{i}\theta_j^*\hat{W}_j}{q_0[\theta_g-(q_0^2/\theta_j^*)]^2}\left[1-\left(\theta_g-\frac{q_0^2}{\theta_j^*}\right)\left[\frac{\mathrm{i}(\theta_j^*)^2}{q_0^2}t\left(2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2)\right)-\hat{B}_j+\frac{\mathrm{i}(\theta_j^*)^2}{q_0^2}x+\frac{\theta_j^*}{q_0^2}\right]\right]\right\}v_{-,1}(\theta_j^*)\\&+\sum_{j=1}^{G}\left[\frac{\bar{W}_j}{\theta_g-\theta_j^*}-\frac{\mathrm{i}(\theta_j^*)^3\hat{W}_j}{q_0^3[\theta_g-(q_0^2/\theta_j^*)]}\right]v'_{-,1}(\theta_j^*),\end{aligned} \tag{142}$$

$$v'_{-,1}(\theta_g^*;x,t)=\begin{pmatrix}\mathrm{i}\\ -\mathrm{i}\mathbf{q}_-/(\theta_g^*)^2\end{pmatrix}-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_1}{[\xi-\theta_g^*]^2}\,\mathrm{d}\xi-\sum_{j=1}^{G}\left[\frac{\left[B_j-\mathrm{i}x-\mathrm{i}t\left(2\theta_j+3\sigma(q_0^2+\theta_j^2)\right)\right]W_j\widetilde{d}(\theta_j)}{[\theta_g^*-\theta_j]^2}+\frac{W_j\widetilde{d}'(\theta_j)}{[\theta_g^*-\theta_j]^2}+\frac{2W_j\widetilde{d}(\theta_j)}{[\theta_g^*-\theta_j]^3}\right], \tag{143}$$

$$\begin{aligned}-\frac{\widetilde{d}'(\theta_g;x,t)}{s_{33}(\theta_g)}=&-\frac{\widetilde{d}(\theta_g)s'_{33}(\theta_g)}{[s_{33}(\theta_g)]^2}-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_2}{(\xi-\theta_g)^2}\,\mathrm{d}\xi-\sum_{j=1}^{G}\left\{\frac{\bar{W}_j}{(\theta_g-\theta_j^*)^3}\left[2+(\theta_g-\theta_j^*)\left[\bar{B}_j+\mathrm{i}x+\mathrm{i}t\left(2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2)\right)\right]\right]\right.\\&+\left.\frac{\mathrm{i}\theta_j^*\hat{W}_j}{q_0[\theta_g-(q_0^2/\theta_j^*)]^3}\left[2-\left(\theta_g-\frac{q_0^2}{\theta_j^*}\right)\left[\frac{\mathrm{i}(\theta_j^*)^2}{q_0^2}t\left(2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2)\right)-\hat{B}_j+\frac{\mathrm{i}(\theta_j^*)^2}{q_0^2}x+\frac{\theta_j^*}{q_0^2}\right]\right]\right\}v_{-,1}(\theta_j^*)\\&+\sum_{j=1}^{G}\left[\frac{\mathrm{i}(\theta_j^*)^3\hat{W}_j}{q_0^3[\theta_g-(q_0^2/\theta_j^*)]^2}-\frac{\bar{W}_j}{(\theta_g-\theta_j^*)^2}\right]v'_{-,1}(\theta_j^*).\end{aligned} \tag{144}$$

Theorem 11 (*Reconstruction Formula*). *Multiple double-pole solutions $\hat{\mathbf{q}}(x,t)$ of the defocusing–defocusing coupled Hirota equations with NZBCs (3) are reconstructed as*

$$\hat{q}_k(x,t)=q_{-,k}-\frac{1}{2\pi}\int_{\mathbb{R}}\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{(k+1)1}\mathrm{d}\xi-\sum_{j=1}^{G}\mathrm{i}W_j\widetilde{d}'_{(k+1)}(\theta_j)-\sum_{j=1}^{G}\mathrm{i}W_j\left[B_j-\mathrm{i}x-\mathrm{i}t\left(2\theta_j+3\sigma(q_0^2+\theta_j^2)\right)\right]\widetilde{d}_{(k+1)}(\theta_j),\quad k=1,2. \tag{145}$$

5.4. Trace formulae and the multiple double-pole solutions

The construction method for the trace formula of the multiple double-pole solutions differs from that for a single pole, therefore it is assumed that

$$\chi_a^+(z)=h_{11}(z)\prod_{g=1}^{G}\frac{(z-\theta_g^*)^2}{(z-\theta_g)^2},\quad z\in\mathbb{D}^+,\quad \chi_a^-(z)=s_{11}(z)\prod_{g=1}^{G}\frac{(z-\theta_g)^2}{(z-\theta_g^*)^2},\quad z\in\mathbb{D}^-. \tag{146}$$

Using Eq. (65) and the scattering coefficients, we can derive the following result:

$$\ln\chi_a^+(z)+\ln\chi_a^-(z)=-\ln\left[1-|\beta_1(z)|^2-\frac{|\beta_2(z)|^2}{\rho(z)}\right],\quad z\in\mathbb{R}. \tag{147}$$

By combining (146) with (147) and using the Plemelj's formula, it can be concluded that

$$\chi_a^+(z)=\exp\left[-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^+, \tag{148a}$$

$$\chi_a^-(z)=\exp\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^-. \tag{148b}$$

By substituting expression (148) into definition (146), the scattering coefficient display expression can be solved

$$h_{11}(z)=\prod_{g=1}^{G}\frac{(z-\theta_g)^2}{(z-\theta_g^*)^2}\exp\left[-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^+, \tag{149a}$$

$$s_{11}(z)=\prod_{g=1}^{G}\frac{(z-\theta_g^*)^2}{(z-\theta_g)^2}\exp\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi-z}\right],\quad z\in\mathbb{D}^-. \tag{149b}$$

Subsequently, the asymptotic phase difference between the respective double poles is considered

$$\Delta\delta=\delta_+-\delta_-=4\sum_{g=1}^{G}\arg(\theta_g)+\frac{1}{2\pi}\int_{\mathbb{R}}\ln\left[1-|\beta_1(\xi)|^2-\frac{|\beta_2(\xi)|^2}{\rho(\xi)}\right]\frac{\mathrm{d}\xi}{\xi}. \tag{150}$$

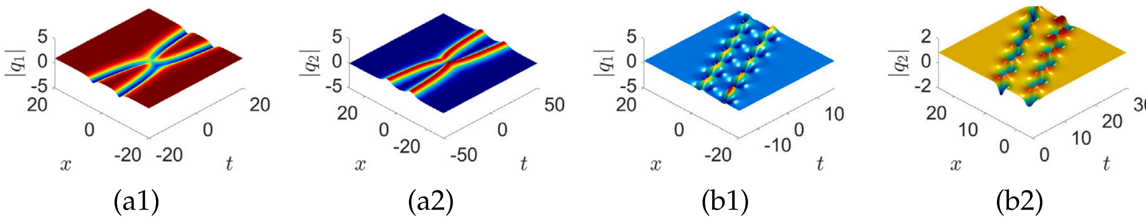

Fig. 6. (a1) and (a2): One dark-bright double-pole solution by taking $\mathbf{q}_- = (1,0)^T$, $\sigma = 10^{-3}$, $\hat{K}_1 = 0.5$, $\hat{K}_2 = \hat{K}_3 = 2$, $\hat{\kappa}_1 = \hat{\chi}_2 = 0.5$, $\hat{\chi}_1 = -2$, $\hat{\kappa}_2 = -0.5$, $\hat{\alpha}_1 = \frac{1}{2}\pi$. (b1) and (b2): One bright-breather–dark-breather double-pole solution by taking $\mathbf{q}_- = (\frac{1}{2}e^{-\frac{1}{5}i\pi}, -\frac{\sqrt{3}}{2}e^{-\frac{1}{5}i\pi})^T$, $\sigma = 10^{-2}$, $\hat{K}_1 = 0.98$, $\hat{\kappa}_2 = \hat{K}_2 = \hat{K}_3 = 2$, $\hat{\kappa}_1 = -1.5$, $\hat{\chi}_2 = -0.2$, $\hat{\chi}_1 = 1$, $\hat{\alpha}_1 = \frac{3}{5}\pi$.

Theorem 12. *In the reflectionless case, the multiple double-pole solutions (145) of the defocusing–defocusing coupled Hirota equation with NZBCs (3) may be written*

$$\hat{\mathbf{q}}(x,t) = \frac{1}{\det \hat{\mathbf{K}}(x,t)} \begin{pmatrix} \det \hat{\mathbf{K}}_1^{\text{aug}}(x,t) \\ \det \hat{\mathbf{K}}_2^{\text{aug}}(x,t) \end{pmatrix}, \quad \hat{\mathbf{K}}_n^{\text{aug}}(x,t) = \begin{pmatrix} q_{-,n} & \hat{\mathbf{E}}(x,t) \\ \hat{\mathbf{A}}_n(x,t) & \hat{\mathbf{K}}(x,t) \end{pmatrix}, \quad n = 1,2, \tag{151}$$

where the matrix $\hat{\mathbf{K}}(x,t) = \mathbf{I} + \hat{\mathbf{P}}(x,t)$*, the components in vector* $\hat{\mathbf{E}}(x,t) = \left(\hat{E}_1(x,t), \ldots, \hat{E}_{2G}(x,t)\right)$ *are*

$$\hat{E}_g(x,t) = \begin{cases} iW_g(x,t)B_g^{(1)}(x,t), & g = 1, \ldots, G, \\ iW_{g-G}(x,t), & g = G+1, \ldots, 2G, \end{cases} \tag{152}$$

the components in vector $\hat{\mathbf{A}}_n(x,t) = \left(\hat{A}_{n1}(x,t), \ldots, \hat{A}_{n(2G)}(x,t)\right)^T$ *are*

$$\hat{A}_{ni'}(x,t) = \begin{cases} (-1)^n s_{33}(\theta_{i'})\dfrac{q_{-,\bar{n}}^*}{q_0} + \displaystyle\sum_{a=1}^{G} iq_{-,n}W_a^{(7)}(\theta_{i'}), & i' = 1, \ldots, G, \\ (-1)^n s_{33}'(\theta_{i'-G})\dfrac{q_{-,\bar{n}}^*}{q_0} + \displaystyle\sum_{a=1}^{G} iq_{-,n}W_a^{(8)}(\theta_{i'-G}), & i' = G+1, \ldots, 2G, \end{cases} \tag{153}$$

where $\bar{n} = n + (-1)^{n+1}$*. For* $j = 1, \ldots, G$ *and* $k = 1, \ldots, G$*, the entries of matrix* $\hat{\mathbf{P}}(x,t) = (\hat{P}_{jk}(x,t))$ *are*

$$\hat{P}_{jk}(x,t) = s_{33}(\theta_j)\sum_{a=1}^{G}\left[W_a^{(3)}(\theta_j)W_k^{(1)}(\theta_a^*) - W_a^{(6)}(\theta_j)W_k^{(2)}(\theta_a^*)\right]. \tag{154}$$

For $j = 1, \ldots, G$ *and* $k = G+1, \ldots, 2G$*, the entries of matrix* $\hat{\mathbf{P}}(x,t) = (\hat{P}_{jk}(x,t))$ *are*

$$\hat{P}_{jk}(x,t) = s_{33}(\theta_j)\sum_{a=1}^{G}\left[W_a^{(3)}(\theta_j)b_{k-G}^{(1)}(\theta_a^*) - \frac{W_a^{(6)}(\theta_j)b_{k-G}^{(1)}(\theta_a^*)}{\theta_a^* - \theta_{k-G}}\right]. \tag{155}$$

For $j = G+1, \ldots, 2G$*, and* $k = 1, \ldots, G$*, the entries of matrix* $\hat{\mathbf{P}}(x,t) = (\hat{P}_{jk}(x,t))$ *are*

$$\hat{P}_{jk}(x,t) = -\sum_{a=1}^{G}\left[W_a^{(9)}(\theta_{j-G})W_k^{(1)}(\theta_a^*) + W_a^{(10)}(\theta_{j-G})W_k^{(2)}(\theta_a^*)\right]. \tag{156}$$

For $j = G+1, \ldots, 2G$*, and* $k = G+1, \ldots, 2G$*, the entries of matrix* $\hat{\mathbf{P}}(x,t) = (\hat{P}_{jk}(x,t))$ *are*

$$\hat{P}_{jk}(x,t) = -\sum_{a=1}^{G}\left[W_a^{(9)}(\theta_{j-G})b_{k-G}^{(1)}(\theta_a^*) + \frac{W_a^{(10)}(\theta_{j-G})b_{k-G}^{(1)}(\theta_a^*)}{\theta_a^* - \theta_{k-G}}\right]. \tag{157}$$

We have now derived the multiple double-pole solutions of the defocusing–defocusing coupled Hirota equations with NZBCs (3). Considering the one double-pole and parametrizing the discrete eigenvalues and normalization constants as follows:

$$\theta_1 = \hat{K}_1 e^{i\hat{\alpha}_1}, \quad f_1 = \hat{K}_2 e^{\hat{\kappa}_1 + i\hat{\chi}_1}, \quad \bar{b}_1 = \hat{K}_3 e^{\hat{\kappa}_2 + i\hat{\chi}_2}, \quad 0 < \hat{K}_1 < q_0, \tag{158}$$

from (151) one obtains the one double-pole solution $\hat{\mathbf{q}}_{\text{one}}(x,t)$ (223) (See Appendix D) of the defocusing–defocusing coupled Hirota equations with NZBCs (3).

Remark 8. Through the expression (158) and the reflectionless potentials, it can be inferred that the different structures of one double-pole solution are obtained. For $q_{-,1}q_{-,2} = 0$, one dark-bright double-pole solution is given by panels (a1) and (a2) of Fig. 6. Moreover, setting $q_{-,1}q_{-,2} \neq 0$ generates one bright-breather–dark-breather double-pole solution in panels (b1) and (b2) of Fig. 6.

6. Discussion and final remarks

We apply the IST tool to the defocusing–defocusing coupled Hirota equations with NZBCs (3) and derive some interesting results by constructing the matrix RH problem. We delve into the analytic properties of Jost eigenfunctions and scattering coefficients, by examining particular potential conditions that guarantee such analyticity. Innovative the analytical eigenfunctions for the defocusing–defocusing coupled Hirota equations with

NZBCs (3) adhere to two symmetry conditions. These relations are subsequently utilized to provide a rigorous characterization of the discrete spectrum. The discrete spectrum yields discrete eigenvalues in two distinct scenarios, each linked to a diverse array of soliton solution types. The characteristics of their soliton interactions are depicted through graphical illustrations. By discussing the different eigenvalues on the circle and off the circle, corresponding combinations of dark solitons, bright solitons, and breather solitons are obtained. We derived the multiple double-pole solutions of the defocusing–defocusing coupled Hirota equations with NZBCs (3) and proved the rationality of its algebraic closed system. For the defocusing–defocusing coupled Hirota equations, we have found a novel bright-breather–dark-breather double-pole solution for the first time, which may help in explaining and predicting certain characteristics in optical solitons and fluid dynamics phenomena.

In this paper, the pure soliton solutions and multiple double-pole solutions are obtained under reflectionless potential conditions. Though simple, this method inevitably has limitations. When faced with reflective potentials, even if we can derive the corresponding solutions, they often contain implicit integral terms, posing challenges for analysis and application. In future research, we plan to explore how to use the IST technique to eliminate these integral terms, thereby constructing explicit soliton solutions. For the numerical inverse scattering [33], the computational efficiency and practicability of the solution are improved while maintaining the accuracy of the analytical solutions. By combining numerical methods, we hope to understand and solve the soliton problem in the presence of reflection potential more comprehensively, and further expand the application range of IST technology in nonlinear physical phenomena.

It is worth noting that the IST technique also plays an important role in exploring the soliton solutions of nonlinear integrable flows involving coordinate reflection points [45]. This technique provides a new perspective for the analysis of soliton solutions with its unique advantages [46,47]. Recently, with the in-depth study of the 4×4 matrix spectral problem, the IST has not only been successfully applied to the generation of coupled and combined integrable models, but also has been analyzed in detail from the perspective of double Hamiltonian systems [48]. These models have significant integrable properties and show great potential and value in the fields of physics, mechanical engineering, and materials science [49,50].

At present, the IST technique has been successfully applied to deal with the parallel boundary conditions of the defocusing–defocusing coupled Hirota equations with NZBCs (3) at infinity. Similarly, this technique can also be extended to the study of non-parallel boundary conditions at infinity [51], further enriching its application in the field of mathematical physics. At the same time, the application of the robust IST method [52] in engineering and applied science will also be further developed to provide more effective solutions for practical problems. The RH representation of the high-order Darboux dressing matrix has recently attracted considerable attention in the study of the asymptotic behavior of solutions. This research investigated the far-field asymptotic behavior of multiple-pole solitons at the large-order limit of the focusing NLS equation [53]. It also examined the near-field asymptotic behavior of these solitons under the same large-order limit [54]. Moreover, this study explored the asymptotic behavior and dynamic characteristics of both large-order and infinite-order solitons within the coupled NLS equation framework [55]. The multiple higher-order poles solitons for the NLS equation [56], as well as the Nth order soliton solutions for the Wadati–Konno–Ichikawa equation [57] have been successfully derived using the IST.

In this context, future research endeavors on the defocusing–defocusing coupled Hirota equations are likely to concentrate on the following key areas:

- Non-parallel boundary problems pertain to the behavior of equations under specific boundary conditions, and they explore how these boundaries influence the stability and asymptotic properties of the solutions.
- The application of robust IST, which can effectively handle singularities in the original RH problem, offering a new perspective for solutions.
- The far-field and near-field asymptotic behavior of multiple double-pole solutions under the large-order limit, which may involve changes in the structure and dynamics of solutions at different scales.
- The asymptotic behavior of large-order and infinite-order solitons, which pertains to the stability of solitons and their applications in physical systems under extreme conditions.
- The study of multiple high-order pole solutions of coupled equations in inverse scattering analysis faces theoretical and technical challenges, such as computational complexity, uniqueness, and stability of solutions.

Future research could focus on addressing these challenges and exploring new application areas. As the fields of mathematics and physics continue to advance, we anticipate the development of more accurate algorithms and more in-depth theoretical analyses. These advancements are expected to aid in solving more complex physical problems, such as boundary issues in quantum field theory, nonlinear dynamics, and fluid mechanics.

CRediT authorship contribution statement

Peng-Fei Han: Writing – review & editing, Writing – original draft, Software, Investigation, Formal analysis, Data curation, Conceptualization, Supervision, Validation. **Wen-Xiu Ma:** Project administration, Methodology, Funding acquisition. **Ru-Suo Ye:** Software, Formal analysis. **Yi Zhang:** Visualization, Supervision, Funding acquisition, Project administration.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yi Zhang reports financial support was provided by National Natural Science Foundation of China. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Direct scattering problem

Proof of Theorem 1. We begin by recasting the Volterra integral equation as presented in Eq. (25a):

$$v_-(z;x,t)=\mathbf{Y}_-(z)+\left[\mathbf{I}+\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}(x-y)\boldsymbol{\Lambda}_1}[\mathbf{Y}_-^{-1}(z)(\mathbf{X}(z;y,t)-\mathbf{X}_-(z))v_-(z;y,t)]\mathrm{e}^{-\mathrm{i}(x-y)\boldsymbol{\Lambda}_1}\,\mathrm{d}y\right]. \tag{159}$$

The constraints of the integration boundaries indicate that the difference $x-y$ is perpetually positive for v_- (perpetually negative for v_+). Specifically, by defining $\mathbf{G}(z;x,t)=\mathbf{Y}_-(z)v_-(z;x,t)$, we can deduce that for the first column $\mathbf{G}_1(z;x,t)$ of $\mathbf{G}(z;x,t)$, the following holds:

$$\mathbf{G}_1(z;x,t)=(1,0,0)^T+\int_{-\infty}^{x}\mathbf{F}(z;x-y)[\mathbf{Q}(y,t)-\mathbf{Q}_-]\mathbf{Y}_-(z)\mathbf{G}_1(z;y,t)\,\mathrm{d}y, \tag{160}$$

where

$$\mathbf{F}(z;x-y)=\operatorname{diag}\left(1,\mathrm{e}^{-\mathrm{i}[k(z)+\lambda(z)](x-y)},\mathrm{e}^{2\mathrm{i}\lambda(z)(x-y)}\right)\mathrm{i}\mathbf{Y}_-^{-1}(z). \tag{161}$$

Next, we present a Neumann series expansion to represent $\mathbf{G}_1(z;x,t)$:

$$\mathbf{G}_1(z;x,t)=\sum_{g=0}^{\infty}\mathbf{G}_1^{(g)}(z;x,t), \tag{162}$$

with

$$\mathbf{G}_1^{(0)}=(1,0,0)^T,\quad \mathbf{G}_1^{(g+1)}(z;x,t)=\int_{-\infty}^{x}\mathbb{P}(z;x,y,t)\mathbf{G}_1^{(g)}(z;y,t)\,\mathrm{d}y, \tag{163}$$

where $\mathbb{P}(z;x,y,t)=\mathbf{F}(z;x-y)[\mathbf{Q}(y,t)-\mathbf{Q}_-]\mathbf{Y}_-(z)$. By defining the L^1 vector norm $\|\mathbf{G}_1(z;x,t)\|=|G_{11}(z;x,t)|+|G_{21}(z;x,t)|+|G_{31}(z;x,t)|$ and its associated matrix norm $\|\mathbb{P}(z;x,y,t)\|$, we subsequently obtain:

$$\left\|\mathbf{G}_1^{(g+1)}(z;x,t)\right\|\le\int_{-\infty}^{x}\|\mathbb{P}(z;x,y,t)\|\left\|\mathbf{G}_1^{(g)}(z;y,t)\right\|\,\mathrm{d}y. \tag{164}$$

Note that $\|\mathbf{Y}_\pm(z)\|\le 1+q_0/|z|$ and $\left\|\mathbf{Y}_\pm^{-1}(z)\right\|\le\left(1+q_0/|z|\right)/|\rho(z)|$, the characteristics of the matrix norm suggest that:

$$\begin{aligned}\|\mathbb{P}(z;x,y,t)\|&\le\left\|\operatorname{diag}\left(1,\mathrm{e}^{-\mathrm{i}[k(z)+\lambda(z)](x-y)},\mathrm{e}^{2\mathrm{i}\lambda(z)(x-y)}\right)\right\|\left\|\mathbf{Y}_-^{-1}(z)\right\|\|\mathbf{Q}(y,t)-\mathbf{Q}_-\|\|\mathbf{Y}_-(z)\|\\&\le p(z)\left(1+\mathrm{e}^{[k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z)](x-y)}+\mathrm{e}^{2\lambda_{\mathrm{im}}(z)(x-y)}\right)\|\mathbf{q}(y,t)-\mathbf{q}_-\|,\end{aligned} \tag{165}$$

where $\lambda_{\mathrm{im}}(z)=\operatorname{Im}\lambda(z)$, $k_{\mathrm{im}}(z)=\operatorname{Im}k(z)$ and $p(z)=\left(1+q_0/|z|\right)^2/|\rho(z)|$. Define $\mathbb{D}_I$ as the region where $\operatorname{Im}\lambda(z)>0$ and $\mathbb{D}_{II}$ as the region where $\operatorname{Im}\lambda(z)<0$. As $z\to\pm q_0$, $p(z)$ tends towards infinity. Hence, for any $\varepsilon>0$, we focus our analysis within the domain $\left(\mathbb{D}_{II}\right)_\varepsilon=\mathbb{D}_{II}\backslash\left(J_\varepsilon\left(q_0\right)\cup J_\varepsilon\left(-q_0\right)\right)$, where $J_\varepsilon\left(\pm q_0\right)=\left\{z\in\mathbb{C}:\left|z\mp q_0\right|<\varepsilon q_0\right\}$. It can be directly proven that $p_\varepsilon=\max_{z\in(\mathbb{D}_{II})_\varepsilon}p(z)=2+2/\varepsilon$. Subsequently, by employing mathematical induction, we demonstrate that for every $z\in\left(\mathbb{D}_{II}\right)_\varepsilon$ and any $n\in\mathbb{N}$,

$$\left\|\mathbf{G}_1^{(g)}(z;x,t)\right\|\le\frac{R^g(x,t)}{g!},\quad R(x,t)=2p_\varepsilon\int_{-\infty}^{x}\|\mathbf{q}(y,t)-\mathbf{q}_-\|\,\mathrm{d}y. \tag{166}$$

When $g=0$, Eq. (166) clearly holds true. Additionally, observe that for any $z\in\overline{\mathbb{D}_{II}}$ and for all $y\le x$, the inequality $1+\mathrm{e}^{[k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z)](x-y)}+\mathrm{e}^{2\lambda_{\mathrm{im}}(x-y)}\le 3$ is satisfied. Consequently, if condition (166) is valid for $g=n$, then Eq. (164) implies that:

$$\left\|\mathbf{G}_1^{(n+1)}(z;x,t)\right\|\le\frac{3p_\varepsilon}{n!}\int_{-\infty}^{x}\|\mathbf{q}(y,t)-\mathbf{q}_-\|\,R^n(y,t)\,\mathrm{d}y=\frac{1}{n!(n+1)}R^{n+1}(x,t), \tag{167}$$

the condition (166) being satisfied for $g=n$ implies its validity for $g=n+1$ as well. Therefore, if $\mathbf{q}(y,t)-\mathbf{q}_-\in L^1(-\infty,\sigma_3]$ for all finite $\sigma_3\in\mathbb{R}$ and for all $\varepsilon>0$, the Neumann series converges absolutely and uniformly with respect to $x\in(-\infty,\sigma_3)$ and $z\in\left(\mathbb{D}_{II}\right)_\varepsilon$. Analogous outcomes are observed for $v_+(z;x,t)$. □

To elucidate the analytic properties of the scattering coefficients, we deemed it essential to establish an alternative integral representation for the Jost solutions. We employ a methodology akin to that of the Manakov system as detailed in Ref. [36]. Given that the scattering matrix is independent of time, the variable t is not considered from the subsequent proof. We initially observe that the scattering problem given by Eq. (22) is analogous to another problem:

$$\psi_x(z;x)=\widehat{\mathbf{X}}(z;x)\psi(z;x)+\left[\mathbf{X}(z;x)-\widehat{\mathbf{X}}(z;x)\right]\psi(z;x), \tag{168}$$

where

$$\widehat{\mathbf{X}}(z;x)=H(x)\mathbf{X}_+(z)+H(-x)\mathbf{X}_-(z), \tag{169}$$

with $H(x)$ represents the Heaviside function, which equals 1 for $x\ge 0$ and 0 for $x<0$. For $z\in\mathbb{R}$, we define the fundamental eigenfunctions $\widehat{\psi}_\pm(z;x)$ as square matrix solutions to Eq. (168) that meet the following conditions:

$$\widehat{\psi}_\pm(z;x)=\mathrm{e}^{x\mathbf{X}_\pm(z)}[\mathbf{I}+o(1)],\quad x\to\pm\infty. \tag{170}$$

Solving Eq. (168) yields:

$$\widehat{\psi}_-(z;x)=\mathbf{F}_m(z;x,0)+\int_{-\infty}^{x}\mathbf{F}_m(z;x,y)\left[\mathbf{X}(z;y)-\widehat{\mathbf{X}}(z;y)\right]\widehat{\psi}_-(z;y)\,\mathrm{d}y, \tag{171a}$$

$$\widehat{\psi}_+(z;x)=\mathbf{F}_m(z;x,0)-\int_{x}^{\infty}\mathbf{F}_m(z;x,y)\left[\mathbf{X}(z;y)-\widehat{\mathbf{X}}(z;y)\right]\widehat{\psi}_+(z;y)\,\mathrm{d}y, \tag{171b}$$

where $\mathbf{F}_m(z;x,y)$ represents the particular solution to the homogeneous problem, i.e., $\mathbf{F}_x(z;x,y)=\widehat{\mathbf{X}}(z;x)\mathbf{F}(z;x,y)$, fulfilling the "initial conditions" $\mathbf{F}(z;x,x)=\mathbf{I}$. That is to say,

$$\mathbf{F}_m(z;x,y)=\begin{cases} \mathrm{e}^{(x-y)\mathbf{X}_+(z)}, & x\geq 0, y\geq 0. \\ \mathrm{e}^{(x-y)\mathbf{X}_-(z)}, & x\leq 0, y\leq 0. \\ \mathrm{e}^{x\mathbf{X}_+(z)}\mathrm{e}^{-y\mathbf{X}_-(z)}, & x\geq 0, y\leq 0. \\ \mathrm{e}^{x\mathbf{X}_-(z)}\mathrm{e}^{-y\mathbf{X}_+(z)}, & x\leq 0, y\geq 0. \end{cases} \tag{172}$$

By applying Eqs. (171), we deduce that

$$\widehat{\psi}_\pm(z;x)=\mathbf{F}_m(z;x,0)\left[\mathbf{S}_\mp(z)+o(1)\right],\quad x\to\mp\infty,\quad z\in\mathbb{R}, \tag{173}$$

where

$$\mathbf{S}_\mp(z)=\mathbf{I}\mp\int_{\mathbb{R}}\mathbf{F}_m(z;0,y)\left[\mathbf{X}(z;y)-\widehat{\mathbf{X}}(z;y)\right]\widehat{\psi}_+(z;y)\,\mathrm{d}y. \tag{174}$$

Given that $\mathrm{e}^{x\mathbf{X}_\pm(z)}$ remains bounded for $x\in\mathbb{R}$ when $z\in\mathbb{R}$ and assuming $\mathbf{X}(z;x)-\widehat{\mathbf{X}}(z;x)\in L^1(\mathbb{R})$, the application of Gronwall's inequality ensures that $\widehat{\psi}_\pm(z;x)$ remains bounded as $x\to\mp\infty$. Furthermore, by comparing Eq. (172) with the solutions of the asymptotic scattering problem given by Eq. (5), we find that $\bar{\phi}_\pm(x,z)\mathbf{E}_\pm(z)=\phi_\pm(x,z)$. Consequently, Eqs. (171) imply that:

$$\psi_-(z;x)=\mathbf{F}_m(z;x,0)\mathbf{Y}_-(z)+\int_{-\infty}^{x}\mathbf{F}_m(z;x,y)\left[\mathbf{X}(z;y)-\widehat{\mathbf{X}}(z;y)\right]\psi_-(z;y)\,\mathrm{d}y, \tag{175a}$$

$$\psi_+(z;x)=\mathbf{F}_m(z;x,0)\mathbf{Y}_+(z)-\int_{x}^{\infty}\mathbf{F}_m(z;x,y)\left[\mathbf{X}(z;y)-\widehat{\mathbf{X}}(z;y)\right]\psi_+(z;y)\,\mathrm{d}y. \tag{175b}$$

Observe that Eq. (175a) aligns with Eq. (25a) for all $x\leq 0$, and Eq. (175b) aligns with Eq. (25b) for all $x\geq 0$. Furthermore, the assumption that $\mathbf{q}(x)-\mathbf{q}_+\in L^1(0,\infty)$ and $\mathbf{q}(x)-\mathbf{q}_-\in L^1(-\infty,0)$ leads to the conclusion that $\mathbf{X}(z;x)-\widehat{\mathbf{X}}(z;x)\in L^1(\mathbb{R})$. Utilizing this insight along with Eqs. (175), we can validate Theorem 1 and confirm that $v_\pm(z;x)=\psi_\pm(z;x)\mathrm{e}^{-\mathrm{i}x\boldsymbol{\Lambda}_1(z)}$ are bounded as $x\to\mp\infty$. This will facilitate the demonstration of the analytical properties of the scattering coefficient.

Proof of Theorem 2. By comparing the asymptotic behaviors of $\psi_-(z;x)$ from Eq. (173) as $x\to\infty$ with those of $\psi_+(z;x)\mathbf{S}(z)$ from Eq. (19), we derive:

$$\mathbf{S}(z)=\mathbf{Y}_+^{-1}(z)\mathbf{S}_+(z)\mathbf{Y}_-(z). \tag{176}$$

Eq. (176) reduces to the following integral representation for the scattering matrix:

$$\begin{aligned}\mathbf{S}(z)=\mathbf{Y}_+^{-1}(z)\mathbf{Y}_-(z)&+\int_0^\infty \mathrm{e}^{-\mathrm{i}y\boldsymbol{\Lambda}_1(z)}\mathbf{Y}_+^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_+\right]\psi_-(z;y)\,\mathrm{d}y\\ &+\mathbf{Y}_+^{-1}(z)\mathbf{Y}_-(z)\int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}y\boldsymbol{\Lambda}_1(z)}\mathbf{Y}_-^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_-\right]\psi_-(z;y)\,\mathrm{d}y,\end{aligned} \tag{177}$$

An analogous expression can be derived for $\mathbf{H}(z)$. Specifically, the 1,1 element of (177) provides an integral representation for $s_{11}(z)$, the integral from 0 to ∞ is

$$\frac{\mathrm{e}^{-\mathrm{i}y\lambda(z)}}{\mathrm{i}\rho(z)}\left[\frac{\mathbf{q}_+^\dagger[\mathbf{q}_+-\mathbf{q}(y)]}{z}\psi_{-,11}(z;y)+(q_{+,1}^*-q_1^*(y))\psi_{-,21}(z;y)+(q_{+,2}^*-q_2^*(y))\psi_{-,31}(z;y)\right], \tag{178}$$

and the integral from $-\infty$ to 0 is

$$\sum_{j=1}^{3}\left[d_{11}(z)D_{1j}(z;y)+d_{12}(z)D_{2j}(z;y)\mathrm{e}^{\mathrm{i}y[k(z)+\lambda(z)]}+d_{13}(z)D_{3j}(z;y)\mathrm{e}^{2\mathrm{i}y\lambda(z)}\right]\psi_{-,j1}(z;y)\mathrm{e}^{-\mathrm{i}y\lambda(z)}, \tag{179}$$

where $\mathbf{Y}_+^{-1}(z)\mathbf{Y}_-(z)=\left(d_{ij}(z)\right)$ and $\mathbf{Y}_-^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_-\right]=\left(D_{ij}(z;y)\right)$. Note that $\psi_{-,1}(z;y)\mathrm{e}^{-\mathrm{i}y\lambda(z)}$ is analytic for $\operatorname{Im}z<0$ and bounded over $y\in\mathbb{R}$. Therefore, each term in (178) is analytic for $\operatorname{Im}z<0$ and bounded when $y>0$. Hence, the integral expression (178) of $s_{11}(z)$ is an analytic function for $\operatorname{Im}z<0$. Additionally, considering that the imaginary parts of $-\lambda(z)$ and $-[k(z)+\lambda(z)]$ share the same sign, it can be inferred that every term in Eq. (179) is analytic when $\operatorname{Im}z<0$. Moreover, these terms are bounded when $y<0$, the integral expression (179) of $s_{11}(z)$ is an analytic function for $\operatorname{Im}z<0$. Consequently, the integral expression (177) for $s_{11}(z)$ can be extended analytically beyond the real axis of z into the lower half-plane. The remaining parts of Theorem 2 can be substantiated using a similar approach. □

Appendix B. Discrete spectrum and asymptotic behavior

Proof of Corollary 5. By applying the symmetries defined by Eqs. (60a) and (60b), we can derive a new relationship from the second part of Eq. (69), specifically $\psi_{+,3}(z_g^*)=\bar{c}_g\psi_{-,1}(z_g^*)$, leading to the equation $\psi_{+,1}(z_g)=-\bar{c}_g\psi_{-,3}(z_g)$. Upon comparison, the first of Eq. (71) can be obtained. During the differentiation of z in Eq. (51b), apply Eq. (60c) and set $z\to z_g^*$ to achieve the desired result:

$$\begin{aligned}&\mathrm{i}s_{11}'(z_g^*)\psi_{-,3}^*(z_g)+\mathrm{i}s_{11}(z_g^*)[\psi_{-,3}^*(z_g)]'+\mathrm{i}\delta_2'(z_g^*)\mathbf{J}[\gamma(z_g^*)\times\psi_{-,1}(z_g^*)]\\ &=\mathbf{J}\left([\gamma'(z_g^*)\times\psi_{-,1}(z_g^*)]+[\gamma(z_g^*)\times\psi_{-,1}'(z_g^*)]\right)\mathrm{e}^{-\mathrm{i}\delta_2(z_g^*)}.\end{aligned} \tag{180}$$

During the differentiation of z in Eq. (51b), apply Eq. (69) and set $z \to z_g^*$ to achieve the desired result:

$$\begin{aligned}&\mathrm{i}c_g^* h_{33}'(z_g^*)\psi_{-,3}^*(z_g) + \mathrm{i}c_g^* h_{33}(z_g^*)[\psi_{-,3}^*(z_g)]' + \mathrm{i}\bar{c}_g\delta_2'(z_g^*)\mathbf{J}[\gamma(z_g^*)\times\psi_{-,1}(z_g^*)]\\&= \bar{c}_g\mathbf{J}\left([\gamma'(z_g^*)\times\psi_{-,1}(z_g^*)] + [\gamma(z_g^*)\times\psi_{-,1}'(z_g^*)]\right)\mathrm{e}^{-\mathrm{i}\delta_2(z_g^*)}.\end{aligned} \tag{181}$$

By comparing Eqs. (180) and (181), the second condition of Eq. (71) can be deduced. □

Proof of Corollary 6. Using the symmetries defined by Eqs. (60a), (60b) and (64), we can derive a new relationship from the first part of Eq. (70a), which results in the following equation:

$$\psi_{+,3}(\frac{q_0^2}{\theta_g}) = \frac{\mathrm{i}\theta_g f_g}{q_0 s_{33}(\theta_g)}\gamma(\frac{q_0^2}{\theta_g}). \tag{182}$$

Comparing Eq. (182) with the second part of Eq. (70a), the first of Eq. (72) can be obtained. By substituting $z \to \theta_g^*$ into Eq. (51b) and considering $\gamma(\theta_g^*) = \bar{f}_g\psi_{-,1}(\theta_g^*)$, and then taking the conjugate, the desired result can be obtained.

$$\psi_{+,1}(\theta_g) = -\frac{\bar{f}_g^*\rho(\theta_g)}{h_{33}^*(\theta_g^*)}\widetilde{\gamma}(\theta_g). \tag{183}$$

Comparing Eq. (183) with the first part of Eq. (70a) and considering $s_{33}(\theta_g) = h_{33}^*(\theta_g^*)$, the second of Eq. (72) can be obtained. Using the symmetry defined by Eq. (64), we can derive a new relationship from the second part of Eq. (70b), which results in the equation $\gamma(\theta_g^*) = \mathrm{i}\theta_g^*\hat{f}_g/q_0$. Comparing the first part of Eq. (70b), we obtain $\bar{f}_g = \mathrm{i}\theta_g^*\hat{f}_g/q_0$. Comparing the second part of Eq. (72) and $\bar{f}_g = \mathrm{i}\theta_g^*\hat{f}_g/q_0$, the third of Eq. (72) can be deduced. □

Appendix C. Inverse problem

Proof of Theorem 5. For the sake of simplicity, the variables x and t are excluded here. To solve Eq. (89), we perform a subtraction from both sides involving the terms specified in Eq. (91) and account for the residue contributions originating from the poles both within and on the boundary of the circle with radius q_0. Essentially, the subtraction includes the following:

$$\begin{aligned}&\mathbf{R}_\infty + \frac{1}{z}\mathbf{R}_0 + \sum_{i=1}^{G_1}\left[\frac{\mathrm{Res}_{z=z_i}\mathbf{R}^+}{z-z_i} + \frac{\mathrm{Res}_{z=z_i^*}\mathbf{R}^-}{z-z_i^*}\right]\\&+\sum_{j=1}^{G_2}\left[\frac{\mathrm{Res}_{z=\theta_j}\mathbf{R}^+}{z-\theta_j} + \frac{\mathrm{Res}_{z=\theta_j^*}\mathbf{R}^-}{z-\theta_j^*}\right] + \sum_{j=1}^{G_2}\left[\frac{\mathrm{Res}_{z=q_0^2/\theta_j^*}\mathbf{R}^+}{z-(q_0^2/\theta_j^*)} + \frac{\mathrm{Res}_{z=q_0^2/\theta_j}\mathbf{R}^-}{z-(q_0^2/\theta_j)}\right].\end{aligned} \tag{184}$$

Upon regularization, the RH problem ensures that both sides exhibit a behavior of $O(1/z)$ as $z \to \infty$, with the left-hand side being analytic in the upper half z-plane and the right-hand side analytic in the lower half z-plane. We designate the identity operator on $L^2(\mathbb{R})$ as I and proceed to define the Cauchy projection operators as follows:

$$\left(E^\pm m\right)(z) = \frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\frac{m(\xi)}{\xi-(z\pm\mathrm{i}\varepsilon)}\,\mathrm{d}\xi, \tag{185}$$

which are also properly defined within the space $L^2(\mathbb{R})$. Additionally, it is important to remember that $\left(E^\pm m\right)(z) = \lim_{s\to z}(Em)(s)$, where E represents the Cauchy-type integral

$$(Em)(s) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{m(\xi)}{\xi-s}\,\mathrm{d}\xi, \quad s\notin\mathbb{R}, \tag{186}$$

and the limit is taken from the upper or lower half plane, respectively. If $m^\pm$ is analytic in the upper (resp., lower) half of the z-plane and $m^\pm = O(1/z)$ as $z\to\infty$ in the appropriate half plane, then $E^\pm m^\pm = \pm m^\pm$ and $E^+m^- = E^-m^+ = 0$. By applying Eq. (185) and the Plemelj's formula to the regularized RH problem, we obtain Eq. (96). By considering the corresponding column vectors in $\mathbf{R}^\pm(z;x,t)$ (96) and assigning related variables, we can obtain (97), (98), (99) and (100). □

Proof of Theorem 6. Through the examination of solutions to the regularized RH problem, one can derive certain conditions by juxtaposing the first column of the matrix $\mathbf{R}^-(z;x,t)$ (96) with the asymptotic properties of the modified Jost eigenfunctions as depicted in Eq. (73).

$$q_k(x,t) = -\mathrm{i}\lim_{z\to\infty}[z\nu_{-,(k+1)1}(z;x,t)], \quad k=1,2. \tag{187}$$

Carry out a Laurent series expansion of Eq. (96) around $z\to\infty$

$$\begin{aligned}\mathbf{R}(z;x,t) = {}&\mathbf{R}_\infty + \frac{1}{z}\mathbf{R}_0 + \frac{1}{2\pi\mathrm{i}z}\int_{\mathbb{R}}\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\,\mathrm{d}\xi + \sum_{i=1}^{G_1}\frac{1}{z}\left[\mathop{\mathrm{Res}}_{z=z_i}\mathbf{R}^+ + \mathop{\mathrm{Res}}_{z=z_i^*}\mathbf{R}^-\right]\\&+\sum_{j=1}^{G_2}\frac{1}{z}\left[\mathop{\mathrm{Res}}_{z=\theta_j}\mathbf{R}^+ + \mathop{\mathrm{Res}}_{z=\theta_j^*}\mathbf{R}^- + \mathop{\mathrm{Res}}_{z=q_0^2/\theta_j^*}\mathbf{R}^+ + \mathop{\mathrm{Res}}_{z=q_0^2/\theta_j}\mathbf{R}^-\right] + O(\frac{1}{z^2}).\end{aligned} \tag{188}$$

We take $\mathbf{R}(z;x,t) = \mathbf{R}^-(z;x,t)$ in (188) and compare it with the 2,1 element of Eq. (187)

$$\mathrm{i}q_1(x,t) = \mathrm{i}q_{-,1} + \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{21}\mathrm{d}\xi + \sum_{i=1}^{G_1}C_i r_{23}^+(z_i) - \sum_{j=1}^{G_2}F_j r_{22}^+(\theta_j). \tag{189}$$

Then, compare the result with the 3,1 element of Eq. (187)

$$\mathrm{i}q_2(x,t) = \mathrm{i}q_{-,2} + \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\left[\mathbf{R}^-(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{31}\mathrm{d}\xi + \sum_{i=1}^{G_1} C_i r_{33}^+(z_i) - \sum_{j=1}^{G_2} F_j r_{32}^+(\theta_j). \tag{190}$$

We have proven the reconstruction formula (101) by combining (189) and (190). □

Proof of Theorem 7. For the sake of simplicity, the variables x and t are excluded here (as they do not affect other values). The proof employs a conventional reasoning approach (refer to [39] for details). In scenarios where there is no discrete spectrum, $\mathbf{R}(z)$ serves as a piecewise analytic function over $\mathbb{C}\backslash\mathbb{R}$, fulfilling the jump condition (89) and displaying the asymptotic behavior described in Lemma 2. Define $r(z) = \det\mathbf{R}(z)$ and compute the determinant of the jump condition (89) to obtain $r^+(z) = r^-(z)$ for $z \in \mathbb{R}$. Furthermore, $r^+(z) = r^-(z)$ suggests that $r(z)$ is an entire function, given the absence of a singularity at $z = 0$, and it is also bounded at infinity. By Liouville's theorem, since $r(z)$ is an entire function that is bounded at infinity, it must be constant. Therefore, $r(z) = 1$ for all $z \in \mathbb{C}$. This implies that $\mathbf{R}(z)$ is invertible, and its inverse $\mathbf{R}^{-1}(z)$ is analytic on $\mathbb{C}\backslash\mathbb{R}$.

Currently, consider $\widehat{\mathbf{R}}(z)$ as an additional function that is analytic in sections, fulfilling the jump condition (89) and exhibiting the asymptotic behavior as outlined in Lemma 2. Let us define the matrix $\mathbf{D}(z) = \widehat{\mathbf{R}}(z)\mathbf{R}^{-1}(z)$. By applying the jump condition (89) once more, we obtain $\mathbf{D}^+(z) = \mathbf{D}^-(z)$ for $z \in \mathbb{R}$. Lemma 2 implies $\mathbf{D}(z) = \mathbf{I} + O(1/z)$ as $z \to \infty$ and $\mathbf{D}(z) = \mathbf{I} + O(z)$ as $z \to 0$. Consequently, $\mathbf{D}(z)$ emerges as an entire function that remains bounded at infinity. By invoking Liouville's theorem, we deduce that $\mathbf{D}(z) = \mathbf{I}$ for every $z \in \mathbb{C}$, which in turn implies that $\widehat{\mathbf{R}}(z) = \mathbf{R}(z)$. □

Proof of Theorem 8. For the sake of simplicity, the variables x and t are excluded here (as they do not affect other values). In cases where the discrete spectrum is absent, as we approach $z \in \mathbb{R}$ from the appropriate directions in the complex plane, the limiting values of the Cauchy projectors $E^{\pm}$ from Eq. (185) are bounded operators on $L^2(\mathbb{R})$ [44]. For any $m \in L^2(\mathbb{R})$, elementary algebraic steps lead to:

$$\int_{\mathbb{R}}\frac{m(\xi)}{\xi-(z\pm\mathrm{i}\epsilon)}\mathrm{d}\xi = \int_{\mathbb{R}}\frac{(\xi-z)m(\xi)}{(\xi-z)^2+\epsilon^2}\mathrm{d}\xi \pm \int_{\mathbb{R}}\frac{\mathrm{i}\epsilon m(\xi)}{(\xi-z)^2+\epsilon^2}\mathrm{d}\xi. \tag{191}$$

As $\epsilon \to 0^+$, the first integral tends to $-\pi(Hm)(z)$, with H representing the Hilbert transform:

$$(Hm)(z) = \lim_{\delta\to 0^+}\frac{1}{\pi}\int_{|\xi-z|\geq\delta}\frac{m(\xi)}{z-\xi}\mathrm{d}\xi. \tag{192}$$

Additionally, the second integral approaches $\pm\mathrm{i}\pi m(z)$, given that the integrand includes a form of the Dirac delta function. Consequently,

$$\left(E^{\pm}m\right)(z) = \pm\frac{1}{2}m(z) + \frac{\mathrm{i}}{2}(Hm)(z), \quad z \in \mathbb{R}. \tag{193}$$

Since it is known that H is a bounded operator on $L^2(\mathbb{R})$ [44], and $E^{\pm}$ is also a bounded operator on $L^2(\mathbb{R})$ as the limit when $z \to \mathbb{R}$, we can leverage the properties of the Hilbert transform H once more to deduce that $E^+ - E^- = I$.

Next, we apply the techniques from [39–43] to establish the existence of the solution to the RH problem. We initiate by recasting the jump condition (89) as $\mathbf{R}^+(z) = \mathbf{R}^-(z)\mathbf{O}(z)$ for $z \in \mathbb{R}$, where the jump matrix is $\mathbf{O}(z) = \mathbf{I} - \mathrm{e}^{\mathrm{i}\boldsymbol{\Delta}(z)}\mathbf{L}(z)\mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}(z)}$. For simplicity, we can assume without loss of generality that the jump matrix can be decomposed as $\mathbf{O}(z) = \mathbf{O}_+^{-1}(z)\mathbf{O}_-(z)$ for $z \in \mathbb{R}$, where $\mathbf{O}_{\pm}(z)$ denote the upper and lower triangular matrices, respectively. Next we define

$$\mathbf{B}_{\pm}(z) = \pm\left[\mathbf{I} - \mathbf{O}_{\pm}(z)\right], \quad \mathbf{B}(z) = \mathbf{B}_+(z) + \mathbf{B}_-(z). \tag{194}$$

Ultimately, we employ these quantities to introduce a novel operator $E_{\mathbf{B}}$ within $L^2(\mathbb{R})$:

$$(E_{\mathbf{B}}m)(z) = [E^+(m\mathbf{B}_+)](z) + [E^-(m\mathbf{B}_-)](z). \tag{195}$$

Due to the boundedness and invertibility of $\mathbf{O}(z)$, it follows that $\mathbf{O}_+^{-1}(z)$ and $\mathbf{O}_-(z)$ are both bounded in $L^\infty(\mathbb{R})$ and invertible. Since $\mathbf{L}(z) \in L^\infty(\mathbb{R})$, it implies that both $\mathbf{O}_+(\cdot)$ and $\mathbf{O}_-(\cdot)$ belong to $L^\infty(\mathbb{R})$. Based on Eq. (194) and definition (195) of $E_{\mathbf{B}}$, we can conclude from the preceding discussion that $E_{\mathbf{B}}$ is a bounded operator in $L^2(\mathbb{R})$. Assuming the absence of a discrete spectrum, if $\mathbf{L}(\cdot) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $I - E_{\mathbf{B}}$ has Fredholm index zero, it follows that the operator $I - E_{\mathbf{B}}$ is invertible in $L^2(\mathbb{R})$ [36]. Let $\mathbf{W}(z)$ denote the unique solution to the ensuing integral equation:

$$\left[\left(I - E_{\mathbf{B}}\right)\mathbf{W}\right](z) = \mathbf{Y}_-(z), \quad z \in \mathbb{R}, \tag{196}$$

and $\mathbf{I} - \mathbf{W} \in L^2(\mathbb{R})$. Subsequently, we define the matrix function as:

$$\mathbf{R}_{\#}(z) = \mathbf{Y}_-(z) + [E(\mathbf{W}\mathbf{B})](z), \quad z \notin \mathbb{R}. \tag{197}$$

Next, we demonstrate that $\mathbf{R}_{\#}(z)$ satisfies the RH problem as outlined in Lemmas 1, 2 and 3. Initially, observe that $\mathbf{R}_{\#}(z)$ is analytic for all $z \in \mathbb{C}\backslash\mathbb{R}$. Next we prove that $\mathbf{R}_{\#}(z)$ satisfies the jump condition. Based on the above properties for all $z \in \mathbb{R}$, we obtain that

$$\begin{aligned}\mathbf{R}_{\#}^+(z) &= \mathbf{Y}_-(z) + [E^+(\mathbf{W}\mathbf{B})](z) = \mathbf{Y}_-(z) + [E^+(\mathbf{W}\mathbf{B}_+)](z) + [E^+(\mathbf{W}\mathbf{B}_-)](z)\\ &= \mathbf{Y}_-(z) + [E^+(\mathbf{W}\mathbf{B}_+)](z) + [E^-(\mathbf{W}\mathbf{B}_-)](z) + (\mathbf{W}\mathbf{B}_-)(z)\\ &= \mathbf{Y}_-(z) + (E_{\mathbf{B}}\mathbf{W})(z) + (\mathbf{W}\mathbf{B}_-)(z) = [\mathbf{W}(\mathbf{I}+\mathbf{B}_-)](z) = (\mathbf{W}\mathbf{O}_-)(z).\end{aligned} \tag{198}$$

In a similar manner, we deduce that $\mathbf{R}_{\#}^-(z) = (\mathbf{W}\mathbf{O}_+)(z)$. Consequently, $\mathbf{R}_{\#}^+(z) = \mathbf{R}_{\#}^-(z)\mathbf{O}(z)$, which corresponds to the jump condition (89). Ultimately, it is evident upon examination of the definition that $\mathbf{R}_{\#}(z)$ exhibits the asymptotic properties outlined in Lemma 2 as $z \to \infty$ and $z \to 0$. Consequently, $\mathbf{R}_{\#}(z)$ resolves the RH problem as delineated by Lemmas 1, 2 and 3. □

Proof of Theorem 9. For $i = 1, \ldots, G_1$ and $j = 1, \ldots, G_2$, define

$$f_i^{(1)}(z;x,t) = \frac{C_i(x,t)}{z-z_i}, \quad f_j^{(3)}(z;x,t) = \frac{F_j(x,t)}{z-\theta_j}, \quad f_j^{(5)}(z;x,t) = \frac{\check{F}_j(x,t)}{z-(q_0^2/\theta_j)}, \tag{199a}$$

$$f_i^{(2)}(z;x,t)=\frac{\bar{C}_i(x,t)}{z-z_i^*},\quad f_j^{(4)}(z;x,t)=\frac{\bar{F}_j(x,t)}{z-\theta_j^*},\quad f_j^{(6)}(z;x,t)=\frac{\hat{F}_j(x,t)}{z-(q_0^2/\theta_j^*)}. \tag{199b}$$

Under the condition of the reflectionless, there are equations that hold true

$$r_{21}^-(z_{i'}^*)=\frac{\mathrm{i}q_{-,1}}{z_{i'}^*}+\sum_{i=1}^{G_1}f_i^{(1)}(z_{i'}^*)r_{23}^+(z_i)-\sum_{j=1}^{G_2}f_j^{(3)}(z_{i'}^*)r_{22}^+(\theta_j), \tag{200a}$$

$$r_{21}^-(\theta_{j'}^*)=\frac{\mathrm{i}q_{-,1}}{\theta_{j'}^*}+\sum_{i=1}^{G_1}f_i^{(1)}(\theta_{j'}^*)r_{23}^+(z_i)-\sum_{j=1}^{G_2}f_j^{(3)}(\theta_{j'}^*)r_{22}^+(\theta_j), \tag{200b}$$

$$r_{23}^+(\frac{q_0^2}{\theta_{j'}^*})=\frac{q_{-,1}}{q_0}+\sum_{i=1}^{G_1}f_i^{(2)}(\frac{q_0^2}{\theta_{j'}^*})r_{21}^-(z_i^*)+\sum_{j=1}^{G_2}f_j^{(5)}(\frac{q_0^2}{\theta_{j'}^*})r_{22}^-(\frac{q_0^2}{\theta_j}), \tag{200c}$$

$$r_{22}^-(\frac{q_0^2}{\theta_{j'}})=\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G_2}f_j^{(4)}(\frac{q_0^2}{\theta_{j'}})r_{21}^-(\theta_j^*)-\sum_{j=1}^{G_2}f_j^{(6)}(\frac{q_0^2}{\theta_{j'}})r_{23}^+(\frac{q_0^2}{\theta_j^*}), \tag{200d}$$

$$r_{23}^+(z_{i'})=\frac{q_{-,1}}{q_0}+\sum_{i=1}^{G_1}f_i^{(2)}(z_{i'})r_{21}^-(z_i^*)+\sum_{j=1}^{G_2}f_j^{(5)}(z_{i'})r_{22}^-(\frac{q_0^2}{\theta_j}), \tag{200e}$$

$$r_{22}^+(\theta_{j'})=\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G_2}f_j^{(4)}(\theta_{j'})r_{21}^-(\theta_j^*)-\sum_{j=1}^{G_2}f_j^{(6)}(\theta_{j'})r_{23}^+(\frac{q_0^2}{\theta_j^*}), \tag{200f}$$

where $i'=1,\dots,G_1$ and $j'=1,\dots,G_2$. With the help of the analytical properties (60) and (64), the elements in piecewise meromorphic functions have the following properties:

$$r_{21}^-(z_i^*)=\frac{\mathrm{i}z_i}{q_0}r_{23}^+(z_i),\quad r_{22}^-(\frac{q_0^2}{\theta_j};x,t)=r_{22}^+(\theta_j),\quad r_{23}^+(\frac{q_0^2}{\theta_j^*})=\frac{\theta_j^*}{\mathrm{i}q_0}r_{21}^-(\theta_j^*). \tag{201}$$

Substituting (201) into Eqs. (200) yields

$$r_{23}^+(z_{i'})=\frac{q_{-,1}}{q_0}+\sum_{i=1}^{G_1}\frac{\mathrm{i}z_i}{q_0}f_i^{(2)}(z_{i'})r_{23}^+(z_i)+\sum_{j=1}^{G_2}f_j^{(5)}(z_{i'})r_{22}^+(\theta_j), \tag{202a}$$

$$r_{22}^+(\theta_{j'})=\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G_2}\left[f_j^{(4)}(\theta_{j'})+\frac{\mathrm{i}\theta_j^*}{q_0}f_j^{(6)}(\theta_{j'})\right]r_{21}^-(\theta_j^*), \tag{202b}$$

and

$$\begin{aligned}r_{22}^+(\theta_{j'})=&\sum_{j=1}^{G_2}\frac{\mathrm{i}q_{-,1}}{\theta_j^*}\left[f_j^{(4)}(\theta_{j'})+\frac{\mathrm{i}\theta_j^*}{q_0}f_j^{(6)}(\theta_{j'})\right]+\sum_{j=1}^{G_2}\sum_{i=1}^{G_1}\left[f_j^{(4)}(\theta_{j'})+\frac{\mathrm{i}\theta_j^*}{q_0}f_j^{(6)}(\theta_{j'})\right]f_i^{(1)}(\theta_j^*)r_{23}^+(z_i)\\&+\frac{q_{-,2}^*}{q_0}-\sum_{j=1}^{G_2}\sum_{j''=1}^{G_2}\left[f_j^{(4)}(\theta_{j'})+\frac{\mathrm{i}\theta_j^*}{q_0}f_j^{(6)}(\theta_{j'})\right]f_{j''}^{(3)}(\theta_j^*)r_{22}^+(\theta_{j''}).\end{aligned} \tag{203}$$

The equations pertaining to $r_{23}^+(z_i')$ and $r_{22}^+(\theta_{j'})$ constitute a self-contained set comprising G_1+G_2 equations, each with G_1+G_2 unknowns. In the same way, a closed system containing $r_{33}^+(z_i')$ and $r_{32}^+(\theta_{j'})$ can be found. These two systems can be written as $\mathbf{K}(x,t)\mathbf{X}_n(x,t)=\mathbf{A}_n(x,t)$ for $n=1,2$, while $\mathbf{X}_n(x,t)=\left(X_{n1}(x,t),\dots,X_{n(G_1+G_2)}(x,t)\right)^T$ and

$$X_{ni'}(x,t)=\begin{cases}r_{(n+1)3}^+(z_{i'};x,t), & i'=1,\dots,G_1,\\ r_{(n+1)2}^+(\theta_{i'-G_1};x,t), & i'=G_1+1,\dots,G_1+G_2.\end{cases} \tag{204}$$

Using Cramer's rule, we have

$$X_{ni}(x,t)=\frac{\det\mathbf{K}_{ni}^{\mathrm{aug}}(x,t)}{\det\mathbf{K}(x,t)},\quad i=1,\dots,G_1+G_2,\quad n=1,2, \tag{205}$$

where $\mathbf{K}_{ni}^{\mathrm{aug}}(x,t)=\left(\mathbf{K}_1(x,t),\dots,\mathbf{K}_{i-1}(x,t),\mathbf{A}_n(x,t),\mathbf{K}_{i+1}(x,t),\dots,\mathbf{K}_{G_1+G_2}(x,t)\right)$. By inserting the determinant representation of the solutions from Eq. (205) into Eq. (101), one obtains (108). □

Appendix D. Multiple double-pole solutions

Proof of Theorem 10. To solve Eq. (89), we subtract the term given in Eq. (91) and the principal parts of the Laurent series associated with the $\mathbf{R}^{\pm}(z;x,t)$ terms from both sides. Essentially, the subtraction includes the following:

$$\begin{aligned}&\mathbf{R}_{\infty}+\frac{1}{z}\mathbf{R}_0+\sum_{j=1}^{G}\left[\frac{\mathbf{R}^{+}_{-1,\theta_j}}{z-\theta_j}+\frac{\mathbf{R}^{-}_{-1,\theta_j^*}}{z-\theta_j^*}+\frac{\mathbf{R}^{+}_{-2,\theta_j}}{(z-\theta_j)^2}+\frac{\mathbf{R}^{-}_{-2,\theta_j^*}}{(z-\theta_j^*)^2}\right]\\&+\sum_{j=1}^{G}\left[\frac{\mathbf{R}^{+}_{-1,q_0^2/\theta_j^*}}{z-(q_0^2/\theta_j^*)}+\frac{\mathbf{R}^{-}_{-1,q_0^2/\theta_j}}{z-(q_0^2/\theta_j)}+\frac{\mathbf{R}^{+}_{-2,q_0^2/\theta_j^*}}{[z-(q_0^2/\theta_j^*)]^2}+\frac{\mathbf{R}^{-}_{-2,q_0^2/\theta_j}}{[z-(q_0^2/\theta_j)]^2}\right].\end{aligned}\tag{206}$$

By applying Eq. (185) and the Plemelj's formula to the regularized RH problem, we obtain Eq. (140). By considering the corresponding column vectors in $\mathbf{R}^{\pm}(z;x,t)$ (140) and assigning related variables, we can obtain (141) and (142). To derive Eqs. (143) and (144), we perform a differentiation of Eq. (140) with respect to z.

$$\begin{aligned}\mathbf{R}'(z;x,t)=&-\frac{1}{z^2}\mathbf{R}_0-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathbf{R}^{-}(\xi)\widetilde{\mathbf{L}}(\xi)}{(\xi-z)^2}\,\mathrm{d}\xi-\sum_{j=1}^{G}\left[\frac{\mathbf{R}^{+}_{-1,\theta_j}}{(z-\theta_j)^2}+\frac{\mathbf{R}^{-}_{-1,\theta_j^*}}{(z-\theta_j^*)^2}+\frac{2\mathbf{R}^{+}_{-2,\theta_j}}{(z-\theta_j)^3}+\frac{2\mathbf{R}^{-}_{-2,\theta_j^*}}{(z-\theta_j^*)^3}\right]\\&-\sum_{j=1}^{G}\left[\frac{\mathbf{R}^{+}_{-1,q_0^2/\theta_j^*}}{[z-(q_0^2/\theta_j^*)]^2}+\frac{\mathbf{R}^{-}_{-1,q_0^2/\theta_j}}{[z-(q_0^2/\theta_j)]^2}+\frac{2\mathbf{R}^{+}_{-2,q_0^2/\theta_j^*}}{[z-(q_0^2/\theta_j^*)]^3}+\frac{2\mathbf{R}^{-}_{-2,q_0^2/\theta_j}}{[z-(q_0^2/\theta_j)]^3}\right],\end{aligned}\tag{207}$$

By considering the corresponding column vectors in $[\mathbf{R}^{\pm}(z;x,t)]'$ (207) and assigning related variables, we can obtain (143) and (144). □

Proof of Theorem 11. The result can be deduced by scrutinizing the first column of $\mathbf{R}^{-}(z;x,t)$ in Eq. (140) and comparing it with equation:

$$\hat{q}_k(x,t)=-\mathrm{i}\lim_{z\to\infty}[z\nu_{-,(k+1)1}(z;x,t)],\quad k=1,2.\tag{208}$$

Carry out a Laurent series expansion of Eq. (140) around $z\to\infty$

$$\begin{aligned}\mathbf{R}(z;x,t)=&\,\mathbf{R}_{\infty}+\frac{1}{z}\mathbf{R}_0+\frac{1}{2\pi\mathrm{i}z}\int_{\mathbb{R}}\mathbf{R}^{-}(\xi)\widetilde{\mathbf{L}}(\xi)\,\mathrm{d}\xi\\&+\sum_{j=1}^{G}\frac{1}{z}\left[\mathbf{R}^{+}_{-1,\theta_j}+\mathbf{R}^{-}_{-1,\theta_j^*}+\mathbf{R}^{+}_{-1,q_0^2/\theta_j^*}+\mathbf{R}^{-}_{-1,q_0^2/\theta_j}\right]+O(\frac{1}{z^2}).\end{aligned}\tag{209}$$

We take $\mathbf{R}(z;x,t)=\mathbf{R}^{-}(z;x,t)$ in (209) and compare it with the 2,1 element of Eq. (208)

$$\begin{aligned}\mathrm{i}\hat{q}_1(x,t)=&\,\mathrm{i}q_{-,1}+\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\left[\mathbf{R}^{-}(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{21}\mathrm{d}\xi+\sum_{j=1}^{G}W_j\tilde{d}_2'(\theta_j)\\&+\sum_{j=1}^{G}W_j\left[B_j-\mathrm{i}x-\mathrm{i}t\left(2\theta_j+3\sigma(q_0^2+\theta_j^2)\right)\right]\tilde{d}_2(\theta_j).\end{aligned}\tag{210}$$

Then, compare the result with the 3,1 element of Eq. (208)

$$\begin{aligned}\mathrm{i}\hat{q}_2(x,t)=&\,\mathrm{i}q_{-,2}+\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\left[\mathbf{R}^{-}(\xi)\widetilde{\mathbf{L}}(\xi)\right]_{31}\mathrm{d}\xi+\sum_{j=1}^{G}W_j\tilde{d}_3'(\theta_j)\\&+\sum_{j=1}^{G}W_j\left[B_j-\mathrm{i}x-\mathrm{i}t\left(2\theta_j+3\sigma(q_0^2+\theta_j^2)\right)\right]\tilde{d}_3(\theta_j).\end{aligned}\tag{211}$$

We have proven the reconstruction formula (145) by combining (210) and (211). □

Proof of Theorem 12. For $j=1,\dots,G$, define

$$b_j^{(1)}(z)=\frac{W_j}{z-\theta_j},\quad b_j^{(2)}(z)=\frac{\bar{W}_j}{z-\theta_j^*},\quad b_j^{(3)}(z)=\frac{\hat{W}_j}{z-(q_0^2/\theta_j^*)},\tag{212a}$$

$$B_j^{(1)}(x,t)=B_j-\mathrm{i}x-\mathrm{i}t[2\theta_j+3\sigma(q_0^2+\theta_j^2)],\quad B_j^{(2)}(x,t)=\bar{B}_j+\mathrm{i}x+\mathrm{i}t[2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2)],\tag{212b}$$

$$B_j^{(3)}(x,t)=\frac{\theta_j^*}{q_0^2}-\hat{B}_j+\frac{\mathrm{i}(\theta_j^*)^2}{q_0^2}[x+t(2\theta_j^*+3\sigma(q_0^2+(\theta_j^*)^2))],\tag{212c}$$

and

$$W_j^{(1)}(z)=b_j^{(1)}(z)\left[B_j^{(1)}+\frac{1}{z-\theta_j}\right],\quad W_j^{(2)}(z)=\frac{b_j^{(1)}(z)}{z-\theta_j}\left[B_j^{(1)}+\frac{2}{z-\theta_j}\right],\tag{213a}$$

$$W_j^{(3)}(z)=\frac{b_j^{(2)}(z)}{z-\theta_j^*}\left[1+(z-\theta_j^*)B_j^{(2)}\right]+\frac{\mathrm{i}\theta_j^*b_j^{(3)}(z)}{q_0[z-(q_0^2/\theta_j^*)]}\left[1-\left(z-\frac{q_0^2}{\theta_j^*}\right)B_j^{(3)}\right],\tag{213b}$$

$$W_j^{(4)}(z)=\frac{b_j^{(2)}(z)}{(z-\theta_j^*)^2}\left[2+(z-\theta_j^*)B_j^{(2)}\right]+\frac{\mathrm{i}\theta_j^*b_j^{(3)}(z)}{q_0[z-(q_0^2/\theta_j^*)]^2}\left[2-\left(z-\frac{q_0^2}{\theta_j^*}\right)B_j^{(3)}\right],\tag{213c}$$

$$W_j^{(5)}(z)=\frac{\mathrm{i}(\theta_j^*)^3 b_j^{(3)}(z)}{q_0^3[z-(q_0^2/\theta_j^*)]}-\frac{b_j^{(2)}(z)}{z-\theta_j^*},\quad W_j^{(6)}(z)=b_j^{(2)}(z)-\frac{\mathrm{i}(\theta_j^*)^3 b_j^{(3)}(z)}{q_0^3}, \tag{213d}$$

with

$$W_j^{(7)}(z)=s_{33}(z)\left[W_j^{(6)}(z)/(\theta_j^*)^2-W_j^{(3)}(z)/\theta_j^*\right], \tag{214a}$$

$$W_j^{(8)}(z)=s_{33}(z)\left[\frac{W_j^{(5)}(z)}{(\theta_j^*)^2}+\frac{W_j^{(4)}(z)}{\theta_j^*}\right]+s'_{33}(z)\left[\frac{W_j^{(6)}(z)}{(\theta_j^*)^2}-\frac{W_j^{(3)}(z)}{\theta_j^*}\right], \tag{214b}$$

$$W_j^{(9)}(z)=s_{33}(z)W_j^{(4)}(z)-s'_{33}(z)W_j^{(3)}(z),\quad W_j^{(10)}(z)=s_{33}(z)W_j^{(5)}(z)+s'_{33}(z)W_j^{(6)}(z). \tag{214c}$$

Under the condition of the reflectionless, there are equations that hold true

$$v_{-,21}(\theta_j^*;x,t)=\frac{\mathrm{i}q_{-,1}}{\theta_j^*}+\sum_{j'=1}^{G}\left[\left(B_{j'}^{(1)}+\frac{1}{\theta_j^*-\theta_{j'}}\right)b_{j'}^{(1)}(\theta_j^*)\widetilde{d}_2(\theta_{j'})+b_{j'}^{(1)}(\theta_j^*)\widetilde{d}'_2(\theta_{j'})\right], \tag{215}$$

$$v'_{-,21}(\theta_j^*;x,t)=-\frac{\mathrm{i}q_{-,1}}{(\theta_j^*)^2}-\sum_{j'=1}^{G}\left[\left(B_{j'}^{(1)}+\frac{2}{\theta_j^*-\theta_{j'}}\right)\frac{b_{j'}^{(1)}(\theta_j^*)\widetilde{d}_2(\theta_{j'})}{\theta_j^*-\theta_{j'}}+\frac{b_{j'}^{(1)}(\theta_j^*)\widetilde{d}'_2(\theta_{j'})}{\theta_j^*-\theta_{j'}}\right], \tag{216}$$

$$\widetilde{d}_2(\theta_{i'};x,t)=-s_{33}(\theta_{i'})\left[\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G}W_j^{(6)}(\theta_{i'})v'_{-,21}(\theta_j^*)+W_j^{(3)}(\theta_{i'})v_{-,21}(\theta_j^*)\right], \tag{217}$$

$$\widetilde{d}'_2(\theta_{i'};x,t)=-s'_{33}(\theta_{i'})\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G}\left[s_{33}(\theta_{i'})W_j^{(4)}(\theta_{i'})-s'_{33}(\theta_{i'})W_j^{(3)}(\theta_{i'})\right]v_{-,21}(\theta_j^*)-\sum_{j=1}^{G}\left[s_{33}(\theta_{i'})W_j^{(5)}(\theta_{i'})+s'_{33}(\theta_{i'})W_j^{(6)}(\theta_{i'})\right]v'_{-,21}(\theta_j^*), \tag{218}$$

where $i'=1,\dots,G$. Substituting (215) and (216) into Eqs. (217) and (218) respectively yields

$$\begin{aligned}\widetilde{d}_2(\theta_{i'};x,t)=&-s_{33}(\theta_{i'})\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G}\sum_{j'=1}^{G}s_{33}(\theta_{i'})\left[W_j^{(6)}(\theta_{i'})W_{j'}^{(2)}(\theta_j^*)-W_j^{(3)}(\theta_{i'})W_{j'}^{(1)}(\theta_j^*)\right]\widetilde{d}_2(\theta_{j'})\\&+\sum_{j=1}^{G}\mathrm{i}q_{-,1}W_j^{(7)}(\theta_{i'})+\sum_{j=1}^{G}\sum_{j'=1}^{G}s_{33}(\theta_{i'})b_{j'}^{(1)}(\theta_j^*)\left[\frac{W_j^{(6)}(\theta_{i'})}{\theta_j^*-\theta_{j'}}-W_j^{(3)}(\theta_{i'})\right]\widetilde{d}'_2(\theta_{j'}),\end{aligned} \tag{219}$$

and

$$\begin{aligned}\widetilde{d}'_2(\theta_{i'};x,t)=&-s'_{33}(\theta_{i'})\frac{q_{-,2}^*}{q_0}+\sum_{j=1}^{G}\sum_{j'=1}^{G}\left[W_j^{(9)}(\theta_{i'})W_{j'}^{(1)}(\theta_j^*)+W_j^{(10)}(\theta_{i'})W_{j'}^{(2)}(\theta_j^*)\right]\widetilde{d}_2(\theta_{j'})\\&+\sum_{j=1}^{G}\mathrm{i}q_{-,1}W_j^{(8)}(\theta_{i'})+\sum_{j=1}^{G}\sum_{j'=1}^{G}b_{j'}^{(1)}(\theta_j^*)\left[\frac{W_j^{(10)}(\theta_{i'})}{\theta_j^*-\theta_{j'}}+W_j^{(9)}(\theta_{i'})\right]\widetilde{d}'_2(\theta_{j'}).\end{aligned} \tag{220}$$

The equations for $\widetilde{d}_2(\theta_{i'};x,t)$ and $\widetilde{d}'_2(\theta_{i'};x,t)$ form a closed system of $2G$ equations with $2G$ unknowns. In the same way, a closed system containing $\widetilde{d}_3(\theta_{i'};x,t)$ and $\widetilde{d}'_3(\theta_{i'};x,t)$ can be found. These two systems can be written as $\widehat{\mathbf{K}}(x,t)\widehat{\mathbf{X}}_n(x,t)=\widehat{\mathbf{A}}_n(x,t)$ for $n=1,2$, while $\widehat{\mathbf{X}}_n(x,t)=\left(\widehat{X}_{n1}(x,t),\dots,\widehat{X}_{n(2G)}(x,t)\right)^T$ and

$$\widehat{X}_{ni'}(x,t)=\begin{cases}\widetilde{d}_{(n+1)}(\theta_{i'};x,t), & i'=1,\dots,G,\\ \widetilde{d}'_{(n+1)}(\theta_{i'-G};x,t), & i'=G+1,\dots,2G.\end{cases} \tag{221}$$

Using Cramer's rule, we have

$$\widehat{X}_{ni}(x,t)=\frac{\det\widehat{\mathbf{K}}_{ni}^{\mathrm{aug}}(x,t)}{\det\widehat{\mathbf{K}}(x,t)},\quad i=1,\dots,2G,\quad n=1,2, \tag{222}$$

where $\widehat{\mathbf{K}}_{ni}^{\mathrm{aug}}(x,t)=\left(\widehat{\mathbf{K}}_1(x,t),\dots,\widehat{\mathbf{K}}_{i-1}(x,t),\widehat{\mathbf{A}}_n(x,t),\widehat{\mathbf{K}}_{i+1}(x,t),\dots,\widehat{\mathbf{K}}_{2G}(x,t)\right)$. By inserting the determinant representation of the solutions from Eq. (222) into Eq. (145), one obtains (151). □

One double-pole solution is obtained when $G=1$ is considered in Eq. (151).

$$\widehat{\mathbf{q}}_{\mathrm{one}}(x,t)=\frac{1}{\det\widehat{\mathbf{K}}_{\mathrm{one}}(x,t)}\begin{pmatrix}\det\widehat{\mathbf{K}}_{\mathrm{one},1}^{\mathrm{aug}}(x,t)\\ \det\widehat{\mathbf{K}}_{\mathrm{one},2}^{\mathrm{aug}}(x,t)\end{pmatrix},\quad \widehat{\mathbf{K}}_{\mathrm{one}}(x,t)=\begin{pmatrix}1+\widehat{P}_{11} & \widehat{P}_{12}\\ \widehat{P}_{21} & 1+\widehat{P}_{22}\end{pmatrix}, \tag{223}$$

where

$$\widehat{\mathbf{K}}_{\mathrm{one},1}^{\mathrm{aug}}(x,t)=\begin{pmatrix}q_{-,1} & \mathrm{i}W_1B_1^{(1)} & \mathrm{i}W_1\\ \widehat{A}_{11} & 1+\widehat{P}_{11} & \widehat{P}_{12}\\ \widehat{A}_{12} & \widehat{P}_{21} & 1+\widehat{P}_{22}\end{pmatrix},\quad \widehat{\mathbf{K}}_{\mathrm{one},2}^{\mathrm{aug}}(x,t)=\begin{pmatrix}q_{-,2} & \mathrm{i}W_1B_1^{(1)} & \mathrm{i}W_1\\ \widehat{A}_{21} & 1+\widehat{P}_{11} & \widehat{P}_{12}\\ \widehat{A}_{22} & \widehat{P}_{21} & 1+\widehat{P}_{22}\end{pmatrix}, \tag{224}$$

and

$$\hat{A}_{11} = \frac{(q_0^2-\theta_1\theta_1^*)^2}{-(q_0^2-\theta_1^2)^2}\frac{q_{-,2}^*}{q_0} + \mathrm{i}q_{-,1}W_1^{(7)}(\theta_1), \quad \hat{A}_{12} = \frac{2q_0^2(\theta_1^*-\theta_1)(q_0^2-\theta_1\theta_1^*)}{(q_0^2-\theta_1^2)^3}\frac{q_{-,2}^*}{q_0} + \mathrm{i}q_{-,1}W_1^{(8)}(\theta_1), \tag{225a}$$

$$\hat{A}_{21} = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}\frac{q_{-,1}^*}{q_0} + \mathrm{i}q_{-,2}W_1^{(7)}(\theta_1), \quad \hat{A}_{22} = \frac{2q_0^2(\theta_1-\theta_1^*)(q_0^2-\theta_1\theta_1^*)}{(q_0^2-\theta_1^2)^3}\frac{q_{-,1}^*}{q_0} + \mathrm{i}q_{-,2}W_1^{(8)}(\theta_1), \tag{225b}$$

with

$$\hat{P}_{11} = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}\left[W_1^{(3)}(\theta_1)W_1^{(1)}(\theta_1^*) - W_1^{(6)}(\theta_1)W_1^{(2)}(\theta_1^*)\right], \tag{226a}$$

$$\hat{P}_{12} = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}\left[W_1^{(3)}(\theta_1)b_1^{(1)}(\theta_1^*) - \frac{W_1^{(6)}(\theta_1)b_1^{(1)}(\theta_1^*)}{\theta_1^*-\theta_1}\right], \tag{226b}$$

$$\hat{P}_{21} = -W_1^{(9)}(\theta_1)W_1^{(1)}(\theta_1^*) - W_1^{(10)}(\theta_1)W_1^{(2)}(\theta_1^*), \tag{226c}$$

$$\hat{P}_{22} = \frac{W_1^{(10)}(\theta_1)b_1^{(1)}(\theta_1^*)}{\theta_1-\theta_1^*} - W_1^{(9)}(\theta_1)b_1^{(1)}(\theta_1^*), \tag{226d}$$

where

$$W_1^{(1)}(\theta_1^*) = b_1^{(1)}(\theta_1^*)\left[B_1^{(1)} + \frac{1}{\theta_1^*-\theta_1}\right], \quad W_1^{(2)}(\theta_1^*) = \frac{b_1^{(1)}(\theta_1^*)}{\theta_1^*-\theta_1}\left[B_1^{(1)} + \frac{2}{\theta_1^*-\theta_1}\right], \tag{227a}$$

$$W_1^{(3)}(\theta_1) = \frac{b_1^{(2)}(\theta_1)}{\theta_1-\theta_1^*}\left[1+(\theta_1-\theta_1^*)B_1^{(2)}\right] + \frac{\mathrm{i}\theta_1^* b_1^{(3)}(\theta_1)}{q_0[\theta_1-(q_0^2/\theta_1^*)]}\left[1-\left(\theta_1-\frac{q_0^2}{\theta_1^*}\right)B_1^{(3)}\right], \tag{227b}$$

$$W_1^{(4)}(\theta_1) = \frac{b_1^{(2)}(\theta_1)}{(\theta_1-\theta_1^*)^2}\left[2+(\theta_1-\theta_1^*)B_1^{(2)}\right] + \frac{\mathrm{i}\theta_1^* b_1^{(3)}(\theta_1)}{q_0[\theta_1-(q_0^2/\theta_1^*)]^2}\left[2-\left(\theta_1-\frac{q_0^2}{\theta_1^*}\right)B_1^{(3)}\right], \tag{227c}$$

$$W_1^{(5)}(\theta_1) = \frac{\mathrm{i}(\theta_1^*)^3 b_1^{(3)}(\theta_1)}{q_0^3[\theta_1-(q_0^2/\theta_1^*)]} - \frac{b_1^{(2)}(\theta_1)}{\theta_1-\theta_1^*}, \quad W_1^{(6)}(\theta_1) = b_1^{(2)}(\theta_1) - \frac{\mathrm{i}(\theta_1^*)^3 b_1^{(3)}(\theta_1)}{q_0^3}, \tag{227d}$$

$$W_1^{(7)}(\theta_1) = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}\left[\frac{W_1^{(6)}(\theta_1)}{(\theta_1^*)^2} - \frac{W_1^{(3)}(\theta_1)}{\theta_1^*}\right], \tag{227e}$$

$$W_1^{(8)}(\theta_1) = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}\left[\frac{W_1^{(5)}(\theta_1)}{(\theta_1^*)^2} + \frac{W_1^{(4)}(\theta_1)}{\theta_1^*}\right] + \frac{2q_0^2(\theta_1-\theta_1^*)(q_0^2-\theta_1\theta_1^*)}{(q_0^2-\theta_1^2)^3}\left[\frac{W_1^{(6)}(\theta_1)}{(\theta_1^*)^2} - \frac{W_1^{(3)}(\theta_1)}{\theta_1^*}\right], \tag{227f}$$

$$W_1^{(9)}(\theta_1) = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}W_1^{(4)}(\theta_1) - \frac{2q_0^2(\theta_1-\theta_1^*)(q_0^2-\theta_1\theta_1^*)}{(q_0^2-\theta_1^2)^3}W_1^{(3)}(\theta_1), \tag{227g}$$

$$W_1^{(10)}(\theta_1) = \frac{(q_0^2-\theta_1\theta_1^*)^2}{(q_0^2-\theta_1^2)^2}W_1^{(5)}(\theta_1) + \frac{2q_0^2(\theta_1-\theta_1^*)(q_0^2-\theta_1\theta_1^*)}{(q_0^2-\theta_1^2)^3}W_1^{(6)}(\theta_1), \tag{227h}$$

and

$$b_1^{(1)}(\theta_1^*) = \frac{f_1(\theta_1^*-\theta_1)(q_0^2-\theta_1^2)^2}{(q_0^2-\theta_1\theta_1^*)^2}\mathrm{e}^{[-\mathrm{i}\theta_1[x+(\theta_1+\sigma\theta_1^2+3\sigma q_0^2)t]]}, \tag{228a}$$

$$b_1^{(2)}(\theta_1) = \frac{f_1^*(\theta_1^*)^2(\theta_1^*-\theta_1)}{(\theta_1^*)^2-q_0^2}\mathrm{e}^{[\mathrm{i}\theta_1^*[x+[\theta_1^*+\sigma(\theta_1^*)^2+3\sigma q_0^2]t]]}, \tag{228b}$$

$$b_1^{(3)}(\theta_1) = \frac{\mathrm{i}f_1^* q_0^5(\theta_1-\theta_1^*)^2\mathrm{e}^{[\mathrm{i}\theta_1^*[x+[\theta_1^*+\sigma(\theta_1^*)^2+3\sigma q_0^2]t]]}}{(\theta_1^*)^2(q_0^2-\theta_1\theta_1^*)[q_0^2-(\theta_1^*)^2]}, \tag{228c}$$

$$W_1 = \frac{f_1(\theta_1-\theta_1^*)^2(q_0^2-\theta_1^2)^2}{(q_0^2-\theta_1\theta_1^*)^2}\mathrm{e}^{[-\mathrm{i}\theta_1[x+(\theta_1+\sigma\theta_1^2+3\sigma q_0^2)t]]}, \tag{228d}$$

$$B_1^{(1)} = \frac{2}{\theta_1-\theta_1^*} - \frac{2q_0^2(q_0^2+\theta_1^2-2\theta_1\theta_1^*)}{\theta_1(q_0^2-\theta_1^2)(q_0^2-\theta_1\theta_1^*)} + \frac{\bar{b}_1^*}{f_1}\left(\frac{q_0^2}{\theta_1^2}-1\right) - \mathrm{i}x - \mathrm{i}t[2\theta_1+3\sigma(q_0^2+\theta_1^2)], \tag{228e}$$

$$B_1^{(2)} = \frac{2}{\theta_1^*-\theta_1} + \frac{\bar{b}_1}{f_1^*}\left(\frac{q_0^2}{(\theta_1^*)^2}-1\right) + \mathrm{i}x + \mathrm{i}t[2\theta_1^*+3\sigma(q_0^2+(\theta_1^*)^2)], \tag{228f}$$

$$B_1^{(3)} = \frac{\theta_1^*}{q_0^2}\left[\frac{2\theta_1}{\theta_1^*-\theta_1} + \frac{\bar{b}_1\theta_1^*}{f_1^*}\left(\frac{q_0^2}{(\theta_1^*)^2}-1\right) + \mathrm{i}\theta_1^*[x+t[2\theta_1^*+3\sigma(q_0^2+(\theta_1^*)^2)]]\right]. \tag{228g}$$

Data availability

Data will be made available on request.

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