Two integrable Hamiltonian hierarchies in sl$(2, \mathbb{R})$ and so$(3, \mathbb{R})$ with three potentials

Xiang Gu, Wen-Xiu Ma, and Wen-Ying Zhang

Citation: Journal of Mathematical Physics **58**, 053512 (2017);
View online: https://doi.org/10.1063/1.4983750
View Table of Contents: http://aip.scitation.org/toc/jmp/58/5
Published by the American Institute of Physics

Articles you may be interested in

Curvature in Hamiltonian mechanics and the Einstein-Maxwell-dilaton action
Journal of Mathematical Physics **58**, 052901 (2017); 10.1063/1.4983665

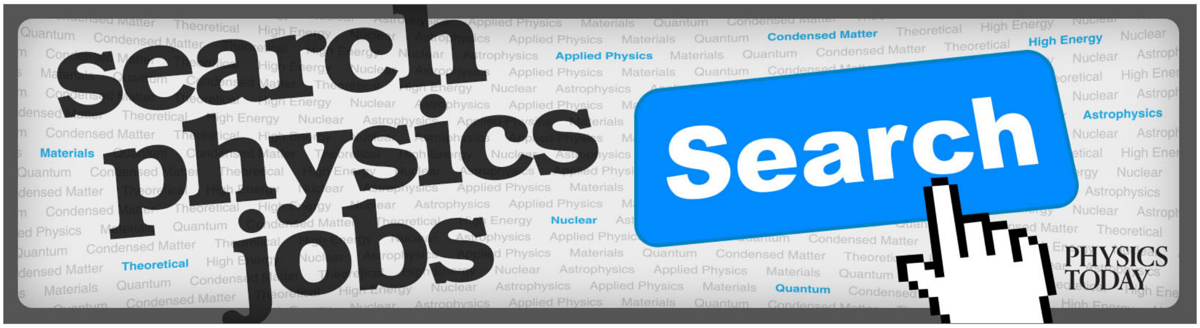
Random Lie-point symmetries of stochastic differential equations
Journal of Mathematical Physics **58**, 053503 (2017); 10.1063/1.4982639

Phase space distribution of Riemann zeros
Journal of Mathematical Physics **58**, 053504 (2017); 10.1063/1.4982737

A fundamental solution to the Schrödinger equation with Doss potentials and its smoothness
Journal of Mathematical Physics **58**, 053506 (2017); 10.1063/1.4983132

The non-autonomous YdKN equation and generalized symmetries of Boll equations
Journal of Mathematical Physics **58**, 053507 (2017); 10.1063/1.4982747

Construction of a Sturm-Liouville vessel using Gelfand-Levitan theory. On solution of the Korteweg-de Vries equation in the first quadrant
Journal of Mathematical Physics **58**, 051501 (2017); 10.1063/1.4980015



JOURNAL OF MATHEMATICAL PHYSICS **58**, 053512 (2017)

Two integrable Hamiltonian hierarchies in sl(2, $\mathbb{R}$) and so(3, $\mathbb{R}$) with three potentials

Xiang Gu,[1,a)] Wen-Xiu Ma,[1,2,3,b)] and Wen-Ying Zhang[4]
[1]*Department of Mathematics and Statistics, University of South Florida, Tampa, Florida 33620-5700, USA*
[2]*College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, People's Republic of China*
[3]*Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa*
[4]*School of Sciences, Shanghai Institute of Technology, Shanghai 201418, People's Republic of China*

(Received 6 December 2015; accepted 7 May 2017; published online 25 May 2017)

By introducing two specific matrix spectral problems associated with sl(2, $\mathbb{R}$) and so(3, $\mathbb{R}$) matrix Lie algebras, we generate two integrable Hamiltonian hierarchies with three potentials. The computation and analysis on their Hamiltonian structures by means of the trace identity show that the resulting hierarchies are Liouville integrable, namely, that each hierarchy consists of commuting Hamiltonian soliton equations. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4983750]

I. INTRODUCTION

The Korteweg-de Vries hierarchy,[1] the Ablowitz-Kaup-Newell-Segur hierarchy,[2] the Kaup-Newell hierarchy,[3] the Dirac hierarchy,[4] and the Wadati-Konno-Ichikawa hierarchy[5] are particular examples of soliton hierarchies which are generated from matrix spectral problems (or Lax pairs) based on specified matrix loop algebras. The analysis by means of the trace identity,[6,7] or in general the variational identity,[8,9] shows that those soliton hierarchies possess Hamiltonian or bi-Hamiltonian structures, and they are Liouville integrable.[10–12]

We would like to present in this paper two new integrable hierarchies which possess Hamiltonian structures and commutative conserved densities under the Poisson brackets associated with the corresponding Hamiltonian operator. There are many different approaches or theories which deal with relationships between Lie algebras and integrable equations.[2,10,12–15] Once a matrix spectral problem from a specified matrix loop algebra is properly selected as a starting point, the zero curvature formulation will be our fundamental tool.[6,7] Without using a recursion operator, the commutativity of symmetries and conservation laws and the functional independence of conserved functionals—-thus in turn the Liouville integrability— can also be guaranteed by a Virasoro algebraic structure behind zero curvature representations[16–19] and the differential recursive structure of the resulting hierarchy,[20] respectively. Moreover, there exist Lie algebraic structures of zero curvature equations generating time-dependent symmetry algebras.[21]

We shall first of all give a brief review of the procedure for constructing soliton hierarchies through zero curvature equations.[6,7] To start with, we take a matrix loop algebra $\tilde{\mathfrak{g}}$, associated with a matrix Lie algebra $\mathfrak{g}$, that is often semisimple, with the commutator given by

$$[A, B] = AB - BA, \quad \forall A, B \in \mathfrak{g}, \tag{1}$$

and consider, based on $\tilde{\mathfrak{g}}$, a spatial matrix spectral problem

$$\phi_x = U\phi, \quad U = U(u, \lambda) \in \tilde{\mathfrak{g}}, \tag{2}$$

a)Author to whom correspondence should be addressed: xianggu@mail.usf.edu.
b)Electronic mail: mawx@cas.usf.edu.

where U stands for the column vector whose components are potential functions depending on x and t, and λ is the spectral parameter. We target at searching for a solution in the shape of

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} \in \mathfrak{g}, \quad i \geq 0 \tag{3}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{4}$$

This solution W will exhibit its significance when we furnish Hamiltonian structures through the trace identity. Next, we try to formulate the temporal matrix spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi, \quad m \geq 0 \tag{5}$$

by employing the Lax matrices

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \triangle_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \tag{6}$$

where $(\lambda^m W)_+$ stands for the polynomial part of $\lambda^m W$ in λ, and $\triangle_m$ denotes the modification term to be amended to the Lax matrices whenever necessary so that the sum of $(\lambda^m W)_+$ and $\triangle_m$ would ensure that the compatibility conditions of (2) and (5), i.e., the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0 \tag{7}$$

will engender a hierarchy of soliton equations

$$u_{t_m} = K_m(u), \quad m \geq 0. \tag{8}$$

Such a soliton hierarchy generally possesses Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{9}$$

where J is a Hamiltonian operator, and the Hamiltonian functionals $\mathcal{H}_m$'s can often be computed through the trace identity[6,7] when the associated matrix Lie algebras are semisimple

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)| \tag{10}$$

or through the variational identity[8,9] when the associated matrix Lie algebras are non-semisimple

$$\frac{\delta}{\delta u}\int \langle \frac{\partial U}{\partial \lambda}, W\rangle\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle \frac{\partial U}{\partial u}, W\rangle, \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|. \tag{11}$$

Here, $\langle .,.\rangle$ stands for the non-degenerate, ad-invariant, and symmetric bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$. The Hamiltonian structures of the entire hierarchy are well furnished once J and $\{\mathcal{H}_m\}_{m=0}^{\infty}$ are determined. Usually, the recursion structure exhibited during such a computation will result in Liouville integrability or sometimes together with bi-Hamiltonian structures.

The real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and the real special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$ are the only two 3-dimensional real Lie algebras whose derived Lie algebras are exactly themselves and therefore are 3-dimensional, too. Especially in recent years, constructions of soliton hierarchies based on $\mathrm{so}(3,\mathbb{R})$, which forms a distinct basis and hence gives us a different starting point for the exploration of soliton hierarchies, were frequently reported.[22–26] This Lie algebra can be realized simply by 3×3 skew-symmetric matrices and thus it has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

with cyclic commutators

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{13}$$

Moreover, $\mathrm{sl}(2,\mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{14}$$

and the corresponding commutators read

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1. \tag{15}$$

$\mathrm{sl}(2,\mathbb{R})$ has already been used in constructing soliton hierarchies from spectral problems for many years.[1–6]

Throughout this paper, we will be applying the following matrix loop algebras:

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{ \sum_{i\geq 0} M_i \lambda^{n-i} | M_i \in \mathrm{sl}(2,\mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\} \tag{16}$$

and

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{ \sum_{i\geq 0} M_i \lambda^{n-i} | M_i \in \mathrm{so}(3,\mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\}, \tag{17}$$

i.e., the spaces of all Laurent series in λ with a finite regular part and coefficients in $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$, respectively. In particular, these matrix loop algebras include such linear combinations in the shape of

$$d_1 \lambda^k e_1 + d_2 \lambda^m e_2 + d_3 \lambda^n e_3, \tag{18}$$

where k, m, and $n \in \mathbb{Z}$ and d_1, d_2, and $d_3 \in \mathbb{R}$ are all constants, e_1, e_2, and e_3 are given by either (12) or (14).

This article will be organized as follows: in Sec. II, starting from $\widetilde{\mathrm{sl}}(2,\mathbb{R})$, inspired by the Dirac hierarchy, we shall introduce a new spectral problem and compute the corresponding soliton hierarchy through the zero curvature formulation; then in Sec. III, computations are carried out for a similar spectral problem, in which the loop algebra is replaced by $\widetilde{\mathrm{so}}(3,\mathbb{R})$. In both sections, the analysis of Hamiltonian structures will be performed by using the trace identity to show that the resulting soliton hierarchies, which provide examples of soliton hierarchies associated with the loop algebras $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ and $\widetilde{\mathrm{so}}(3,\mathbb{R})$, are Liouville integrable. (The particular conceptual details of Liouville integrability we use will be given, as we feel being more proper, in Sec. II B where it is the very topic.) The paper ends with a couple of concluding remarks in Sec. IV.

II. A NEW SOLITON HIERARCHY ASSOCIATED WITH sl(2, $\mathbb{R}$)

A. Spectral problem and the associated soliton equations

We first introduce a matrix spectral problem associated with the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$. Namely, we define in this section a new 2×2 matrix spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \text{with} \quad u = [p, q, r]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2]^T, \tag{19}$$

where the spectral matrix $U \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ takes the form

$$U = U(u,\lambda) = pe_1 + (\lambda + q)e_2 + (\lambda + r)e_3 = \begin{bmatrix} p & \lambda + q \\ \lambda + r & -p \end{bmatrix} \tag{20}$$

with the basis e_1, e_2, and e_3 given by (14). This spectral matrix is similar in mathematical form to the spectral matrix

$$U = U(u,\lambda) = pe_1 + (\lambda + q)e_2 + (-\lambda + q)e_3 = \begin{bmatrix} p & \lambda + q \\ -\lambda + q & -p \end{bmatrix} \tag{21}$$

of the Dirac hierarchy,[4] for which there has been a generalization associated with $\mathrm{so}(3,\mathbb{R})$ recently.[24] However, there are two fundamental differences between (20) and (21): First, in (20), we have identical signs in front of the spectral parameter λ, whereas in (21), the corresponding signs are distinct. Second, (20) possesses three dependent variables p, q, and r, whereas (21) has only p and q.

According to the procedure given in Sec. I, we need to solve the stationary zero curvature equation

$$W_x = [U, W], \quad W \in \widetilde{\mathrm{sl}}(2,\mathbb{R}). \tag{22}$$

We suppose that W takes the form

$$W = ae_1 + (b+c)e_2 + (b-c)e_3 = \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}, \tag{23}$$

so the stationary zero curvature Eq. (22) gives

$$\begin{cases} a_x = -2\lambda c + (q-r)b - (q+r)c, \\ b_x = -(q-r)a + 2pc, \\ c_x = -2a\lambda + 2pb - a(q+r). \end{cases} \tag{24}$$

Next letting a, b, and c take the form of Laurent expansions

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-i}, \quad c = \sum_{i\geq 0} c_i\lambda^{-i} \tag{25}$$

and applying the initial values

$$a_0 = c_0 = 0, \quad b_0 = 1, \tag{26}$$

we thus obtain, by making balances of coefficients of all powers of λ in (24),

$$\begin{cases} a_{i+1} = -\frac{1}{2}c_{i,x} + pb_i - \frac{1}{2}(q+r)a_i, \\ c_{i+1} = -\frac{1}{2}a_{i,x} + \frac{1}{2}(q-r)b_i - \frac{1}{2}(q+r)c_i, \\ b_{i+1,x} = -(q-r)a_{i+1} + 2pc_{i+1}, \end{cases} \quad i \geq 0. \tag{27}$$

It must be emphasized that while running the above recursive computation, we apply the following condition:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1 \tag{28}$$

so that the entire sequence $\{a_i, b_i, c_i | i \geq 1\}$ can be determined uniquely. Upon following (27) recursively, the first three sets of $\{a_i, b_i, c_i | i \geq 1\}$ can be worked out as follows:

$$\begin{aligned} & a_1 = p, \quad b_1 = 0, \quad c_1 = \frac{1}{2}(q-r), \\ & a_2 = \frac{1}{4}(r_x - q_x) - \frac{1}{2}p(q+r), \quad b_2 = \frac{1}{8}(q-r)^2 - \frac{1}{2}p^2, \quad c_2 = -\frac{1}{2}p_x - \frac{1}{4}(q^2 - r^2), \\ & a_3 = \frac{1}{4}p_{xx} + \frac{1}{8}p(3q^2 + 2qr + 3r^2) - \frac{1}{2}p^3 + \frac{3}{8}(qq_x - rr_x) - \frac{1}{8}(qr_x - rq_x), \\ & b_3 = \frac{1}{2}p^2(q+r) + \frac{1}{4}p(q_x - r_x) - \frac{1}{4}p_x(q-r) - \frac{1}{8}(q+r)(q-r)^2, \\ & c_3 = \frac{1}{8}(q_{xx} - r_{xx}) + \frac{1}{4}p(q_x + r_x) + \frac{1}{2}p_x(q+r) - \frac{1}{4}p^2(q-r) \\ & \quad + \frac{1}{16}(q-r)(3q^2 + 2qr + 3r^2). \end{aligned}$$

We see the localness of the first three sets of $\{a_i, b_i, c_i | i \geq 1\}$ above since they are all differential functions. Indeed, the functions a_i, b_i, and c_i ($i \geq 1$) are all local, and this reality can be justified by means of mathematical induction as follows. First, by following (22), and noting that both U and W belong to $\widetilde{\mathrm{sl}}(2, \mathbb{R})$, we have

$$\frac{d}{dx}\mathrm{tr}(W^2) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U, W]) \equiv 0.$$

This means $\mathrm{tr}(W^2)$ is a constant. Observing that $\mathrm{tr}(W^2) = 2(a^2 + b^2 - c^2)$, and recalling (26), we have

$$a^2 + b^2 - c^2 \equiv a^2 + b^2 - c^2|_{u=0} = 1. \tag{29}$$

Then we replace a, b, c by their Laurent expansions and once again apply the initial values (26) by equating the coefficients of λ^{-i} in (29) to 0, we get

$$b_i = -\frac{1}{2}\sum_{m+n=i, m,n\geq 1}(a_m a_n + b_m b_n - c_m c_n), \quad i \geq 1. \tag{30}$$

By keeping (30) in mind and considering the first two recursion relations in (27), it follows immediately via a straightforward application of mathematical induction that for all $i \geq 1$, a_i, b_i, and c_i are differential polynomials in u, that is, they are all local. The mathematical inductions conclude also that for every $i \geq 2$, the differential orders of a_i, b_i, and c_i are $i-1$, $i-2$, and $i-1$, respectively. With the recursively obtained sequence of $\{a_i, b_i, c_i | i \geq 1\}$ in mind, we can compute

$$\begin{aligned}&(\lambda^m W)_{+,x} - [U, (\lambda^m W)_+]\\ &\quad = -2c_{m+1}e_1 - 2a_{m+1}e_2 + 2a_{m+1}e_3\\ &\quad = \begin{bmatrix} -2c_{m+1} & -2a_{m+1} \\ 2a_{m+1} & 2c_{m+1} \end{bmatrix}, \quad m \geq 0,\end{aligned} \tag{31}$$

where $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ in λ. This expression does not match the shape of $U_{t_m} = p_{t_m}e_1 + q_{t_m}e_2 + r_{t_m}e_3$, so the modification terms for the Lax operators must be introduced to compensate the imbalance. By observing that

$$[U, e_2] = 2pe_2 - (\lambda + r)e_1, \qquad [U, e_3] = -2pe_3 + (\lambda + q)e_1,$$

we take a sequence of Lax operators with modification terms

$$V^{[m]} = (\lambda^m W)_+ + \triangle_m, \qquad \text{with } \triangle_m = \beta b_{m+1}e_2 + \beta b_{m+1}e_3, \quad m \geq 0,$$

where β is an arbitrarily given constant. Then we achieve

$$\begin{aligned}&V_x^{[m]} - [U, V^{[m]}]\\ &= \begin{bmatrix} -2c_{m+1} - \beta(q-r)b_{m+1} & -2a_{m+1} + \beta b_{m+1,x} - 2\beta p b_{m+1} \\ 2a_{m+1} + \beta b_{m+1,x} + 2\beta p b_{m+1} & 2c_{m+1} + \beta(q-r)b_{m+1} \end{bmatrix},\end{aligned}$$

and further, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0$$

engender a hierarchy of soliton equations

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2c_{m+1} - \beta(q-r)b_{m+1} \\ -2a_{m+1} + \beta b_{m+1,x} - 2\beta p b_{m+1} \\ 2a_{m+1} + \beta b_{m+1,x} + 2\beta p b_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{32}$$

Since the functions a_i, b_i, and c_i ($i \geq 1$) are all local, the localness of this entire hierarchy of soliton equations follows immediately. Due to the lengthy forms of the soliton equations of the hierarchy (32), we only present below explicitly the first two soliton equations

$$u_{t_1} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} K_{1,1} \\ K_{1,2} \\ K_{1,3} \end{bmatrix},$$

where

$$\begin{aligned}K_{1,1} &= p_x + \frac{1}{2}(q^2 - r^2) - \frac{1}{8}\beta(q-r)^3 + \frac{1}{2}\beta(q-r)p^2,\\ K_{1,2} &= \frac{1}{2}(q_x - r_x) + p(q+r) + \frac{1}{4}\beta(q-r)(q_x - r_x) - \beta p p_x - \frac{1}{4}\beta p(q-r)^2 + \beta p^3,\\ K_{1,3} &= -\frac{1}{2}(q_x - r_x) - p(q+r) + \frac{1}{4}\beta(q-r)(q_x - r_x) - \beta p p_x + \frac{1}{4}\beta p(q-r)^2 - \beta p^3,\end{aligned}$$

and

$$u_{t_2} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_2} = K_2 = \begin{bmatrix} K_{2,1} \\ K_{2,2} \\ K_{2,3} \end{bmatrix},$$

where

$$\begin{aligned}
K_{2,1} = &-\frac{1}{4}(q_{xx}-r_{xx}) - \frac{1}{2}p(q_x+r_x) - p_x(q+r) + \frac{1}{2}p^2(q-r) \\
&- \frac{1}{8}(q-r)(3q^2+2qr+3r^2) - \frac{1}{2}\beta p^2(q^2-r^2) - \frac{1}{4}\beta p(q-r)(q_x-r_x) \\
&+ \frac{1}{4}\beta p_x(q-r)^2 + \frac{1}{8}\beta(q+r)(q-r)^3,
\end{aligned}$$

$$\begin{aligned}
K_{2,2} = &-\frac{1}{2}p_{xx} - \frac{1}{4}p(3q^2+2qr+3r^2) + p^3 - \frac{3}{4}(qq_x-rr_x) + \frac{1}{4}(qr_x-rq_x) \\
&- \beta p^3(q+r) - \frac{1}{2}\beta p^2(q_x-r_x) + \frac{1}{2}\beta pp_x(q-r) + \frac{1}{4}\beta p(q+r)(q-r)^2 \\
&+ \frac{1}{2}\beta(q_x+r_x)p^2 + (q+r)\beta pp_x + \frac{1}{4}\beta p(q_{xx}-r_{xx}) - \frac{1}{4}\beta p_{xx}(q-r) \\
&- \frac{1}{8}\beta(q-r)(3qq_x - qr_x + q_xr - 3rr_x),
\end{aligned}$$

$$\begin{aligned}
K_{2,3} = &\frac{1}{2}p_{xx} + \frac{1}{4}p(3q^2+2qr+3r^2) - p^3 + \frac{3}{4}(qq_x-rr_x) - \frac{1}{4}(qr_x-rq_x) \\
&+ \beta p^3(q+r) + \frac{1}{2}\beta p^2(q_x-r_x) - \frac{1}{2}\beta pp_x(q-r) - \frac{1}{4}\beta p(q+r)(q-r)^2 \\
&+ \frac{1}{2}\beta(q_x+r_x)p^2 + (q+r)\beta pp_x + \frac{1}{4}\beta p(q_{xx}-r_{xx}) - \frac{1}{4}\beta p_{xx}(q-r) \\
&- \frac{1}{8}\beta(q-r)(3qq_x - qr_x + q_xr - 3rr_x).
\end{aligned}$$

B. Hamiltonian structures and Liouville integrability

A hierarchy of soliton equations very often possesses the Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{33}$$

where the Hamiltonian functionals $\mathcal{H}_m$'s can, in principle, be computed by means of the trace identity (10) or more generally the variational identity (11).

In what follows, we shall use the trace identity to generate Hamiltonian structures for the soliton hierarchy (32) due to the semisimplicity of the Lie algebra $\mathrm{sl}(2,\mathbb{R})$. First of all, it can be directly computed that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial r} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = 2b, \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial p}\Big) = 2a,$$
$$\mathrm{tr}\Big(W\frac{\partial U}{\partial q}\Big) = b-c, \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial r}\Big) = b+c.$$

By the trace identity, these indeed lead to

$$\frac{\delta}{\delta u}\int 2b\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} 2a \\ b-c \\ b+c \end{bmatrix}.$$

Making balances of coefficients of all powers of λ, we obtain

$$\frac{\delta}{\delta u}\int 2b_{m+1}dx = (\gamma - m)\begin{bmatrix} 2a_m \\ b_m - c_m \\ b_m + c_m \end{bmatrix}, \quad m \geq 0,$$

where γ, being a constant, can be addressed by computing a particular case, for instance, $m = 1$, and this gives $\gamma = 0$, thereby the above variational identity becomes

$$\frac{\delta}{\delta u}\int\Big(-\frac{2b_{m+1}}{m}\Big)dx = \begin{bmatrix} 2a_m \\ b_m - c_m \\ b_m + c_m \end{bmatrix}, \quad m \geq 1.$$

It follows immediately that the Hamiltonian functionals could be taken as

$$\mathcal{H}_m = \int\Big(-\frac{2b_{m+1}}{m}\Big)dx, \quad m \geq 1. \tag{34}$$

Considering (33) and in particular that $b_{m+1,x} = -(q-r)a_{m+1} + 2pc_{m+1}$, we obtain

$$K_m = J\begin{bmatrix} 2a_{m+1} \\ b_{m+1} - c_{m+1} \\ b_{m+1} + c_{m+1} \end{bmatrix} = \begin{bmatrix} -2c_{m+1} - \beta(q-r)b_{m+1} \\ -2a_{m+1} + \beta b_{m+1,x} - 2\beta p b_{m+1} \\ 2a_{m+1} + \beta b_{m+1,x} + 2\beta p b_{m+1} \end{bmatrix}, \tag{35}$$

where

$$J = \begin{bmatrix} 0 & 1 - \frac{1}{2}\beta(q-r) & -1 - \frac{1}{2}\beta(q-r) \\ -1 + \frac{1}{2}\beta(q-r) & \beta\partial & -2\beta p + \beta\partial \\ 1 + \frac{1}{2}\beta(q-r) & 2\beta p + \beta\partial & \beta\partial \end{bmatrix}. \tag{36}$$

A straightforward computation shows that the Poisson bracket of J satisfies the skew-symmetric condition as well as the Jacobi identity, and hence J is a Hamiltonian operator. Finally, the Hamiltonian structures of the soliton hierarchy (32) are characterized by

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \geq 0 \tag{37}$$

with the Hamiltonian functionals $\mathcal{H}_m$'s and the Hamiltonian operator J being given by (34) and (36), respectively.

In order to elucidate the Liouville integrability of the new hierarchy, it might be here the right place to recall a couple of related definitions which were stated by taking advantage of the characteristics of the associated Hamiltonian vector fields, especially when they are all local.[20]

Let $x = (x^1, \ldots, x^p)$ be the p-tuple vector of independent spatial variables and $u = (u^1, \ldots, u^q)^T$ be the q-tuple vector of dependent variables. Let J_∞ be the associated infinite jet space, whose coordinates $(x^i, u^\alpha, u^\alpha_I)$ (see Definition II.1 below) represent the union of independent variables, the dependent variables, and the derivatives of the dependent variables possibly up to an order of infinity. Consider a Hamiltonian system of evolution equations

$$u_t = J\frac{\delta \mathcal{H}}{\delta u}, \quad \text{with } u = u(x,t), \tag{38}$$

where $J = J(x, u)$ is the Hamiltonian operator and $\frac{\delta}{\delta u}$ stands for the variational derivative with respect to u. A conserved functional of a Hamiltonian system (38) is a functional $\mathcal{H} = \int H dx$ which yields a conservation law of (38), in the shape of $D_t H + \mathrm{Div} X = 0$, where Div stands for the spatial divergence.

Definition II.1 (Ref. 20). The one-form of a given differential function F is defined to be

$$dF := \sum_{k=1}^{p}\frac{\partial F}{\partial x^k}dx^k + \frac{\partial F}{\partial t}dt + \sum_{\alpha=1}^{q}\sum_{\#I\geq 0}\frac{\partial F}{\partial u^\alpha_I}du^\alpha_I, \tag{39}$$

where $u^\alpha_I = u^\alpha$ if $\#I = 0$, and if $\#I = m \geq 1$, then $u^\alpha_I = \frac{\partial^m u^\alpha}{\partial x^{i_1}\ldots\partial x^{i_m}}$ for $I = (i_1, \ldots, i_m)$, $1 \leq i_l \leq p$, and $1 \leq l \leq m$.

Definition II.2 (Ref. 20). Let K be a set of integers and $r \in \mathbb{N}$. A set of r-tuples of differential functions

$$\{F_k = (F_k^1, \ldots, F_k^r)^T \,|\, k \in K\}$$

is said to be independent if all the r-tuples of one-forms

$$\{dF_k = (dF_k^1, \ldots, dF_k^r)^T \,|\, k \in K\},$$

are linearly independent at all points in J_∞. A set of conserved functionals $\{\mathcal{H}_k | k \in K\}$ of a Hamiltonian system (38) is said to be independent if all characteristics $\{J\frac{\delta \mathcal{H}_k}{\delta u} | k \in K\}$ of the associated Hamiltonian vector fields are independent.

Definition II.3 (Ref. 20). A Hamiltonian system (38) is said to be Liouville integrable provided there exists a sequence of conserved functionals $\{\mathcal{H}_m\}_{m=0}^{\infty}$ that are in involution under the action of Poisson bracket associated with the Hamiltonian operator J,

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J = \int \left(\frac{\delta \mathcal{H}_m}{\delta u}\right)^T J \frac{\delta \mathcal{H}_n}{\delta u} = 0, \quad m, n \geq 0, \tag{40}$$

for which the characteristics of the associated Hamiltonian vector fields

$$K_m := J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0 \tag{41}$$

are independent.

According to Definition II.3, we need to show two properties to justify the Liouville integrability of a soliton hierarchy: the commutativity of the infinitely many conserved functionals and the functional independence of the characteristics of these conserved functionals. For our soliton hierarchy (32), the commutativity of conserved functionals is guaranteed by their zero curvature representations that form a commuting Virasoro algebra (see more details in related literatures[16–19]). Note that all these resulting Hamiltonian functionals correspond to common conservation laws for every soliton equation in the soliton hierarchy (32). All being local (which is rather favorable), these conservation laws can also be conveniently computed by computer algebra codes[27] or via some Riccati equations related to the underlying matrix spectral problem.[14,28,29]

Considering also the Hamiltonian structures given by (37) and the distinct differential orders of the sequence $\{a_i, b_i, c_i | i \geq 1\}$, the functional independence of the corresponding characteristics is just a consequence of the differential recursion structure of the hierarchy. (Since obviously, a set of r-tuple of differential functions is independent, if all its members have pairwise distinct differential orders.) To conclude, the soliton hierarchy (32) is Liouville integrable by Definition II.3. To be more precise, every member in the soliton hierarchy (32) has infinitely many independent conserved functionals being commutative in the sense of (40) and infinitely many generalized symmetries being commutative in the following sense:

$$[K_m, K_n] := K_m'(u)[K_n] - K_n'(u)[K_m] = J\frac{\delta\{\mathcal{H}_m, \mathcal{H}_n\}_J}{\delta u} = 0, \quad m, n \geq 0, \tag{42}$$

where K' stands for the Gateaux derivative of K.

III. THE so(3, ℝ)-COUNTERPART OF THE sl(2, ℝ) SOLITON HIERARCHY

A. Spectral problem and the associated soliton equations

Now we continue to present a so(3, ℝ)-counterpart of the spectral problem in Sec. II. Likewise, we shall start from a matrix spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \text{with} \quad u = [p, q, r]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2, \phi_3]^T, \tag{43}$$

(note that the wave function ϕ has accordingly three components) where the spectral matrix $U \in \widetilde{\text{so}}(3,\mathbb{R})$ takes the same form

$$U = U(u,\lambda) = pe_1 + (\lambda+q)e_2 + (\lambda+r)e_3 = \begin{bmatrix} 0 & -\lambda-r & -p \\ \lambda+r & 0 & -\lambda-q \\ p & \lambda+q & 0 \end{bmatrix}, \tag{44}$$

but it differs from that in (20) since the basis above is given by (12) instead. This spectral matrix has also three dependent variables.

Like what we did in Sec. II, we need to solve the stationary zero curvature equation

$$W_x = [U, W], \quad W \in \widetilde{\text{so}}(3,\mathbb{R}). \tag{45}$$

Likewise upon assuming W to possess the form

$$W = ae_1 + (b+c)e_2 + (b-c)e_3 = \begin{bmatrix} 0 & -b+c & -a \\ b-c & 0 & -b-c \\ a & b+c & 0 \end{bmatrix}, \tag{46}$$

the stationary zero curvature equation gives

$$\begin{cases} a_x = (q-r)b - (q+r+2\lambda)c, \\ b_x = pc - \frac{1}{2}(q-r)a, \\ c_x = -pb + \frac{1}{2}(q+r+2\lambda)a. \end{cases} \tag{47}$$

Next by letting a, b, and c take the form of Laurent expansions

$$a = \sum_{i\ge 0} a_i\lambda^{-i}, \quad b = \sum_{i\ge 0} b_i\lambda^{-i}, \quad c = \sum_{i\ge 0} c_i\lambda^{-i} \tag{48}$$

and admitting the initial values

$$a_0 = c_0 = 0, \quad b_0 = 1, \tag{49}$$

we thus obtain, by making balances of coefficients of all powers of λ,

$$\begin{cases} a_{i+1} = c_{i,x} + pb_i - \frac{1}{2}(q+r)a_i, \\ c_{i+1} = -\frac{1}{2}a_{i,x} + \frac{1}{2}(q-r)b_i - \frac{1}{2}(q+r)c_i, \quad i \ge 0. \\ b_{i+1,x} = -\frac{1}{2}(q-r)a_{i+1} + pc_{i+1}, \end{cases} \tag{50}$$

While running the above recursive computation, we again apply the condition

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \ge 1 \tag{51}$$

so that the entire sequence $\{a_i, b_i, c_i | i \ge 1\}$ is determined in a unique way. Upon following (50) recursively, the first three sets of $\{a_i, b_i, c_i | i \ge 1\}$ can be worked out as follows:

$$\begin{aligned}
&a_1 = p, \quad b_1 = 0, \quad c_1 = \frac{1}{2}(q-r),\\
&a_2 = \frac{1}{2}(q_x - r_x) - \frac{1}{2}p(q+r), \quad b_2 = -\frac{1}{8}(q-r)^2 - \frac{1}{4}p^2, \quad c_2 = -\frac{1}{2}p_x - \frac{1}{4}(q^2 - r^2),\\
&a_3 = -\frac{1}{2}p_{xx} + \frac{1}{8}p(q^2 + 6qr + r^2) - \frac{1}{4}p^3 - \frac{3}{4}(qq_x - rr_x) + \frac{1}{4}(qr_x - rq_x),\\
&b_3 = \frac{1}{4}p^2(q+r) + \frac{1}{4}p_x(q-r) - \frac{1}{4}p(q_x - r_x) + \frac{1}{8}(q+r)(q-r)^2,\\
&c_3 = -\frac{1}{4}(q_{xx} - r_{xx}) + \frac{1}{4}p(q_x + r_x) + \frac{1}{2}p_x(q+r) - \frac{1}{8}p^2(q-r)\\
&\qquad + \frac{1}{16}(q-r)(q^2 + 6qr + r^2).
\end{aligned}$$

Again by following the same mathematical induction recipe as that presented in Sec. II, it can be manifested that the sequence $\{a_i, b_i, c_i | i \geq 1\}$ is local as well, and the differential orders of a_i, b_i, and c_i are $i-1$, $i-2$, and $i-1$, respectively (for every $i \geq 2$).

With the recursively obtained sequence of $\{a_i, b_i, c_i | i \geq 1\}$ in mind, we can compute

$$\begin{aligned}
&(\lambda^m W)_{+,x} - [U, (\lambda^m W)_+] \\
&\quad = -2c_{m+1}e_1 + a_{m+1}e_2 - a_{m+1}e_3 \\
&\quad = \begin{bmatrix} 0 & a_{m+1} & 2c_{m+1} \\ -a_{m+1} & 0 & -a_{m+1} \\ -2c_{m+1} & a_{m+1} & 0 \end{bmatrix}, \quad m \geq 0,
\end{aligned} \tag{52}$$

where $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ in λ. This does not match the shape of $U_{t_m} = p_{t_m}e_1 + q_{t_m}e_2 + r_{t_m}e_3$, and likewise, the modification terms for the Lax operators must be introduced. Again by observing that

$$[U, e_2] = pe_3 - (\lambda + r)e_1, \qquad [U, e_3] = -pe_2 + (\lambda + q)e_1,$$

we take a sequence of Lax operators with modification terms

$$V^{[m]} = (\lambda^m W)_+ + \triangle_m, \qquad \text{with } \triangle_m = \beta b_{m+1}e_2 + \beta b_{m+1}e_3, \quad m \geq 0,$$

where β is an arbitrarily given constant. Then we achieve

$$V_x^{[m]} - [U, V^{[m]}] = \begin{bmatrix} 0 & -Z_3 & -Z_1 \\ Z_3 & 0 & -Z_2 \\ Z_1 & Z_2 & 0 \end{bmatrix},$$

where

$$\begin{aligned}
Z_1 &= -2c_{m+1} - \beta(q-r)b_{m+1}, \\
Z_2 &= a_{m+1} + \beta b_{m+1,x} + \beta p b_{m+1}, \\
Z_3 &= -a_{m+1} + \beta b_{m+1,x} - \beta p b_{m+1},
\end{aligned}$$

and further, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0$$

engender a hierarchy of soliton equations

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2c_{m+1} - \beta(q-r)b_{m+1} \\ a_{m+1} + \beta b_{m+1,x} + \beta p b_{m+1} \\ -a_{m+1} + \beta b_{m+1,x} - \beta p b_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{53}$$

Since the functions a_i, b_i, and c_i ($i \geq 1$) are all local, the localness of this entire hierarchy of soliton equations follows as an immediate consequence. Again we present here explicitly the first two soliton equations of this hierarchy

$$u_{t_1} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} K_{1,1} \\ K_{1,2} \\ K_{1,3} \end{bmatrix},$$

where

$$
\begin{aligned}
K_{1,1} &= p_x + \frac{1}{2}(q^2 - r^2) + \frac{1}{8}\beta(q-r)^3 + \frac{1}{4}\beta(q-r)p^2,\\
K_{1,2} &= \frac{1}{2}(q_x - r_x) - \frac{1}{2}p(q+r) - \frac{1}{4}\beta(q-r)(q_x - r_x) - \frac{1}{2}\beta p p_x\\
&\quad - \frac{1}{8}\beta p(q-r)^2 - \frac{1}{4}\beta p^3,\\
K_{1,3} &= -\frac{1}{2}(q_x - r_x) + \frac{1}{2}p(q+r) - \frac{1}{4}\beta(q-r)(q_x - r_x) - \frac{1}{2}\beta p p_x\\
&\quad + \frac{1}{8}\beta p(q-r)^2 + \frac{1}{4}\beta p^3,
\end{aligned}
$$

and

$$
u_{t_2} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_2} = K_2 = \begin{bmatrix} K_{2,1} \\ K_{2,2} \\ K_{2,3} \end{bmatrix},
$$

where

$$
\begin{aligned}
K_{2,1} &= \frac{1}{2}(q_{xx} - r_{xx}) - \frac{1}{2}p(q_x + r_x) - p_x(q+r) + \frac{1}{4}p^2(q-r)\\
&\quad - \frac{1}{8}(q-r)(q^2 + 6qr + r^2) - \frac{1}{4}\beta p^2(q^2 - r^2) + \frac{1}{4}\beta p(q-r)(q_x - r_x)\\
&\quad - \frac{1}{4}\beta p_x(q-r)^2 - \frac{1}{8}\beta(q-r)^3(q+r),
\end{aligned}
$$

$$
\begin{aligned}
K_{2,2} &= -\frac{1}{2}p_{xx} + \frac{1}{8}p(q^2 + 6qr + r^2) - \frac{1}{4}p^3 - \frac{3}{4}(qq_x - rr_x) + \frac{1}{4}(qr_x - rq_x)\\
&\quad + \frac{1}{4}\beta p^3(q+r) + \frac{1}{4}\beta p p_x(q-r) - \frac{1}{4}\beta p^2(q_x - r_x) + \frac{1}{8}\beta p(q+r)(q-r)^2\\
&\quad + \frac{1}{4}\beta(q_x + r_x)p^2 + \frac{1}{2}\beta p p_x(q+r) - \frac{1}{4}\beta p(q_{xx} - r_{xx}) + \frac{1}{4}\beta p_{xx}(q-r)\\
&\quad + \frac{1}{8}\beta(q-r)(3qq_x - qr_x + q_x r - 3rr_x),
\end{aligned}
$$

$$
\begin{aligned}
K_{2,3} &= \frac{1}{2}p_{xx} - \frac{1}{8}p(q^2 + 6qr + r^2) + \frac{1}{4}p^3 + \frac{3}{4}(qq_x - rr_x) - \frac{1}{4}(qr_x - rq_x)\\
&\quad - \frac{1}{4}\beta p^3(q+r) - \frac{1}{4}\beta p p_x(q-r) + \frac{1}{4}\beta p^2(q_x - r_x) - \frac{1}{8}\beta p(q+r)(q-r)^2\\
&\quad + \frac{1}{4}\beta(q_x + r_x)p^2 + \frac{1}{2}\beta p p_x(q+r) - \frac{1}{4}\beta p(q_{xx} - r_{xx}) + \frac{1}{4}\beta p_{xx}(q-r)\\
&\quad + \frac{1}{8}\beta(q-r)(3qq_x - qr_x + q_x r - 3rr_x).
\end{aligned}
$$

B. Hamiltonian structures and Liouville integrability

In this subsection, we shall again use the trace identity to present Hamiltonian structures for the soliton hierarchy (53). First of all, it can be directly computed that

$$
\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
$$

$$
\frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial r} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

and

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = -4b, \quad \operatorname{tr}\Big(W\frac{\partial U}{\partial p}\Big) = -2a,$$
$$\operatorname{tr}\Big(W\frac{\partial U}{\partial q}\Big) = -2(b+c), \quad \operatorname{tr}\Big(W\frac{\partial U}{\partial r}\Big) = -2(b-c).$$

By the trace identity, these will give

$$\frac{\delta}{\delta u}\int 2b\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix} a \\ b+c \\ b-c \end{bmatrix}.$$

Making balances of coefficients of all powers of λ, we obtain

$$\frac{\delta}{\delta u}\int 2b_{m+1}dx = (\gamma - m)\begin{bmatrix} a_m \\ b_m + c_m \\ b_m - c_m \end{bmatrix}, \quad m \geq 0,$$

where γ, like in Sec. II B, can be addressed by computing a particular case, for instance, $m = 1$, and this gives $\gamma = 0$, thereby the above variational identity becomes

$$\frac{\delta}{\delta u}\int \Big(-\frac{2b_{m+1}}{m}\Big)dx = \begin{bmatrix} a_m \\ b_m + c_m \\ b_m - c_m \end{bmatrix}, \quad m \geq 1.$$

It comes out immediately that the Hamiltonian functionals could be taken as

$$\mathcal{H}_m = \int -\frac{2b_{m+1}}{m}dx, \quad m \geq 1. \tag{54}$$

To present the Hamiltonian structures for (53), we use $b_{m+1,x} = -\frac{q-r}{2}a_{m+1} + pc_{m+1}$ to obtain

$$K_m = J\begin{bmatrix} a_{m+1} \\ b_{m+1} + c_{m+1} \\ b_{m+1} - c_{m+1} \end{bmatrix} = \begin{bmatrix} -2c_{m+1} + \beta(r-q)b_{m+1} \\ a_{m+1} + \beta b_{m+1,x} + \beta p b_{m+1} \\ -a_{m+1} + \beta b_{m+1,x} - \beta p b_{m+1} \end{bmatrix}, \tag{55}$$

where the operator

$$J = \begin{bmatrix} 0 & -1 + \frac{1}{2}\beta(r-q) & 1 + \frac{1}{2}\beta(r-q) \\ 1 - \frac{1}{2}\beta(r-q) & \beta\partial & \beta p + \beta\partial \\ -1 - \frac{1}{2}\beta(r-q) & -\beta p + \beta\partial & \beta\partial \end{bmatrix} \tag{56}$$

is also Hamiltonian since it satisfies the skew-symmetric condition and the Jacobi identity. Finally, the Hamiltonian structures of the soliton hierarchy (53) are characterized by

$$u_{t_m} = K_m = J\frac{\delta\mathcal{H}_{m+1}}{\delta u}, \quad m \geq 0 \tag{57}$$

with the Hamiltonian functionals $\mathcal{H}_m$'s and the Hamiltonian operator J being given by (54) and (56), respectively. By following the spirits of the arguments presented at the end of Sec. II B, we can also conclude that the soliton hierarchy (53) is Liouville integrable, and the Hamiltonian functionals given by (54) correspond to infinitely many independent common conservation laws.

IV. CONCLUSIONS

In this paper, we studied two matrix spectral problems based on the matrix loop algebras $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ and $\widetilde{\mathrm{so}}(3,\mathbb{R})$ and generated two integrable soliton hierarchies. By furnishing their Hamiltonian structures with trace identity, all the recursively created members in the soliton hierarchies have been proved to be Hamiltonian and Liouville integrable.

The systems presented here constitute two new soliton hierarchies with three dependent variables. We expect more concrete examples of such soliton hierarchies with three dependent variables as well as with even four or five can be discovered, this certainly require deeper understandings and more intelligent applications of such mathematical tools as trace identities, variational identities, and computer algebras.

Many other higher-order matrix spectral problems were also reported to engender soliton hierarchies.[30–35] At the very end, one thing we would like to stress is that the introduction of integrable couplings associated with enlarged matrix loop algebras[36–39] can potentially be used to enrich specific examples of such soliton hierarchies.

ACKNOWLEDGMENTS

The work was supported in part by NNSFC under the Grant Nos. 11371326, 11271008, 61070233, 10831003, and 61072147, Chunhui Plan of the Ministry of Education of China, Zhejiang Innovation Project of China (Grant No. T200905), the First-class Discipline of Universities in Shanghai, the Shanghai Univ. Leading Academic Discipline Project (No. A.13-0101-12-004) and the Distinguished Professorship by Shanghai University of Electric Power. Moreover, the authors are very grateful also to Y. Q. Yao, S. F. Shen, and S. Manukure for their constructive suggestions when carrying out this work.

[1] P. Lax, "Integrals of nonlinear equations of evolution and solitary waves," Commun. Pure Appl. Math. **21**, 467 (1968).
[2] M. Ablowitz, D. Kaup, A. Newell, and H. Segur, "The inverse scattering transform—Fourier analysis for nonlinear problems," Stud. Appl. Math. **53**, 249 (1974).
[3] D. Kaup and A. Newell, "An exact solution for a derivative nonlinear Schrödinger equation," J. Math. Phys. **19**, 798 (1978).
[4] H. Grosse, "New solitons connected to the Dirac equation," Phys. Rep. **134**, 297 (1986).
[5] M. Wadati, K. Konno, and Y. Ichikawa, "New integrable nonlinear evolution equations," J. Phys. Soc. Jpn. **47**, 1698 (1979).
[6] G. Tu, "On Liouville integrability of zero-curvature equations and the Yang hierarchy," J. Phys. A: Math. Gen. **22**, 2375 (1989).
[7] W. Ma, "A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction," Chin. J. Contemp. Math. **13**, 79 (1992).
[8] W. Ma and M. Chen, "Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras," J. Phys. A: Math. Gen. **39**, 10787 (2006).
[9] W. Ma, "Integrable couplings and matrix loop algebra," AIP Conf. Proc. **1562**, 105–122 (2013).
[10] M. Ablowitz and P. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
[11] S. Novikov, S. Manakov, L. Pitaevskii, and V. Zakharov, *Theory of Solitons—The Inverse Scattering Method* (Consultants Bureau/A Division of Plenum Publishing Corporation, New York, 1984).
[12] V. Drinfeld and V. Sokolov, "Equations of Korteweg-de Vries type, and simple Lie algebras" (Russian), Dokl. Akad. Nauk SSSR **258**, 11 (1981).
[13] F. Magri, "A simple model of the integrable Hamiltonian equation," J. Math. Phys. **19**, 1156 (1978).
[14] P. Casati, A. D. Vedova, and G. Ortenzi, "The soliton equations associated with the affine Kac-Moody Lie algebra $G_2^{(1)}$," J. Geom. Phys. **58**, 377 (2008).
[15] V. Gerdjikov, G. Vilasi, and A. Yanovski, *Integrable Hamiltonian Hierarchies, Spectral and Geometric Method* (Springer, Berlin-Heidelberg, 2008).
[16] W. Ma, "The algebraic structures of isospectral Lax operators and applications to integrable equations," J. Phys. A: Math. Gen. **25**, 5329 (1992).
[17] W. Ma, "The algebraic structure of zero curvature representations and application to coupled *k*dv systems," J. Phys. A: Math. Gen. **26**, 2573 (1993).
[18] W. Ma, "Lax representations and Lax operator algebras of isospectral and nonisospectral hierarchies of evolution equations," J. Math. Phys. **33**, 2464 (1992).
[19] W. Ma, "Lie algebra structures associated with zero curvature equations and generalized zero curvature equations," Br. J. Appl. Sci. Technol. **3**, 1336 (2013).
[20] W. Ma, "An integrable counterpart of the D-AKNS soliton hierarchy from $\mathrm{so}(3,\mathbb{R})$," Phys. Lett. A **378**, 1717 (2014).
[21] B. Fuchssteiner, "Mastersymmetries: Higher order time-dependent symmetries and conserved densities of nonlinear evolution equations," Prog. Theor. Phys. **70**, 1508 (1983).
[22] W. Ma, "A soliton hierarchy associated with $\mathrm{so}(3,\mathbb{R})$," Appl. Math. Comput. **220**, 117 (2013).
[23] W. Ma, "A spectral problem based on $\mathrm{so}(3,\mathbb{R})$ and its associated commuting soliton equations," J. Math. Phys. **54**, 103509 (2013).
[24] W. Zhang and W. Ma, "An $\mathrm{so}(3,\mathbb{R})$ counterpart of the Dirac soliton hierarchy and its bi–integrable couplings," Int. J. Theor. Phys. **53**, 4211 (2014).
[25] W. Ma, S. Shen, S. Yu, H. Zhang, and W. Zhang, "An integrable $\mathrm{so}(3,\mathbb{R})$ counterpart of the Heisenberg soliton hierarchy," Rep. Math. Phys. **74**, 283 (2014).

[26] S. Shen, L. Jiang, Y. Jin, and W. Ma, “New soliton hierarchies associated with the Lie algebra so(3, $\mathbb{R}$) and their bi-Hamiltonian structures,” Rep. Math. Phys. **75**, 113 (2015).
[27] W. Hereman, P. Adams, H. Eklund, M. Hickman, and B. Herbst, “Direct methods and symbolic software for conservation laws of nonlinear equations,” in *Advances in Nonlinear Waves and Symbolic Computation, Loose Errata* (Nova Science Publishing, New York, 2009), pp. 19–78.
[28] R. Miura, C. Gardner, and M. Kruskal, “Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion,” J. Math. Phys. **9**, 1204 (1968).
[29] G. Falqui, F. Magri, and M. Pedroni, “Bi-Hamiltonian geometry, darboux coverings, and linearization of the KP hierarchy,” Commun. Math. Phys. **197**, 303 (1998).
[30] A. Fordy, “Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces,” J. Phys. A: Math. Gen. **17**, 1235 (1984).
[31] X. Hu, “A powerful approach to generate new integrable systems,” J. Phys. A: Math. Gen. **27**, 2497 (1994).
[32] W. Ma, “Binary Bargmann symmetry constraints of soliton equations,” Nonlinear Anal.: Theory, Methods Appl. **47**, 5199 (2001).
[33] W. Ma and R. Zhou, “Adjoint symmetry constraints of multicomponent AKNS equations,” Chin. Ann. Math. **23**, 373 (2002).
[34] X. Geng and D. Du, “Two hierarchies of new nonlinear evolution equations associated with 3×3 matrix spectral problems,” Chaos, Solitons Fractals **29**, 1165 (2006).
[35] X. Zhu and D. Zhang, “Lie algebras and Hamiltonian structures of multi-component Ablowitz-Kaup-Newell-Segur hierarchy,” J. Math. Phys. **54**, 053508 (2013).
[36] W. Ma and B. Fuchssteiner, “Integrable theory of the perturbation equations,” Chaos, Solitons Fractals **7**, 1227 (1996).
[37] W. Ma, X. Xu, and Y. Zhang, “Semi-direct sums of Lie algebras and continuous integrable couplings,” Phys. Lett. A **351**, 125 (2006).
[38] W. Ma and L. Gao, “Coupling integrable couplings,” Mod. Phys. Lett. B **23**, 1847 (2009).
[39] X. Xu, “An integrable coupling hierarchy of the MKdV-integrable systems, its Hamiltonian structure and corresponding nonisospectral integrable hierarchy,” Appl. Math. Comput. **216**, 344 (2010).