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On a class of coupled Hamiltonian operators and their integrable hierarchies with two potentials

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We discuss at first in this paper the Gauge equivalence among several u-linear Hamiltonian operators and present explicitly the associated Gauge transformation of Bäcklund type among them. We then establish the sufficient and necessary conditions for the linear superposition of the discussed u-linear operators and matrix differential operators with constant coefficients of arbitrary order to be Hamiltonian, which interestingly shows that the resulting Hamiltonian operators survive only up to the third differential order. Finally, we explore a few illustrative examples of integrable hierarchies from Hamiltonian pairs embedded in the resulting Hamiltonian operators.

KEYWORDS

Gauge equivalence, Gauge transformation of Bäcklund type, Hamiltonian operator, hereditary symmetry, matrix spectral problems

1 | INTRODUCTION

A plenty of matrix spectral problems can produce Hamiltonian operators that have a component that is linear with respect to the involved potential vector $u = (u^1, \dots, u^m)$ and the partial derivatives of u (this component is said to be u-linear and can often be proven to be Hamiltonian as well), in their Hamiltonian structure. Especially when $m = 2$ and the notation $u = (p, q)^T$ is usually applied, such Hamiltonian operators are found in many soliton hierarchies that were discovered in recent decades, such as the generalized Wadati-Konno-Ichikawa hierarchy,[1] the coupled Burgers hierarchy,[2] and also a soliton hierarchy that was reported in 1992.[3] It frequently occurs that some soliton hierarchies whose original matrix problems and their deduced Hamiltonian structures might be looking rather different are indeed found to be mutually convertible through Gauge transformation of Bäcklund type[4-8] – this constitutes one of the major concerns for the mathematicians cultivating in this field. In this paper we shall present through mathematical proofs the Gauge transformations of Bäcklund type among the mathematical forms of a few matrix differential operators that are Hamiltonian, whose implications of the Gauge equivalences therein seem still not well realized.

It is also discovered that some u-linear operators are Hamiltonian, too, and they can be coupled with matrix differential operators with constant coefficients to generate new Hamiltonian operators.[9] One may take it for granted that such kind of coupling might go up to rather high (or even infinite) differential orders, and the thus, involved situation could be pretty complicated. We investigated this behavior in this paper for a special class of u-linear Hamiltonian operators and observed however that such coupling may take effect only up to a certain differential order, which for this work is determined to be three.

We begin with the terminologies and notations that will be used throughout the paper. Let $x = (x^1, \dots, x^n) \in \mathbb{R}$ so that $\partial_i = \frac{\partial}{\partial x^i}$ denote partial derivatives. Let $u = (u^1, \dots, u^m)$, where $u^i = u^i(x)$ $(1 \le i \le m)$ are sufficiently smooth functions of x. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a n-tuple multiindex that has $\alpha_i \ge 0$ for all $1 \le i \le n$. Define accordingly

$$u^i_\alpha = D^\alpha u^i, \quad \text{with} \quad D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \qquad 1 \le i \le m. \tag{1}$$

Assume that $\mathcal{A}$ denotes the space of all functions $f(x, u, \dots, u^{(k)})$ ($k \ge 0$ is an integer), where f is a smooth local function of the involved variables with $u^{(k)}$ denoting the set of all possible k-th order partial derivatives of u with respect to x. The locality here implies that the dependence of f on u, ie, $(f(u))(x)$, is completely determined by the behavior of u in a sufficiently small neighborhood of x. A typical example of $\mathcal{A}$ is the space of differential functions, ie, polynomial functions of $x, u, \dots, u^{(k)}$. Moreover, assume also that $\mathcal{B}$ denotes the space, which is defined exactly as that for $\mathcal{A}$, with the only exception that the functions in the space are allowed to be nonlocal (for instance, $\int_{-1}^{x^2} x'^2 \partial_3 u^1 dx'^2$ is admitted in $\mathcal{B}$, but not in $\mathcal{A}$–note that the superscript 2 of x' here means the second component of x' instead of power). Define also for $1 \le i \le r$ the r-th direct products of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}^r = \{(f_1, \dots, f_r)^T | f_i \in \mathcal{A}\}, \quad \mathcal{B}^r = \{(f_1, \dots, f_r)^T | f_i \in \mathcal{B}\}. \tag{2}$$

Definition 1. Let $K = K(u) = K(x, t, u)$, $S = S(u) = S(x, t, u) \in \mathcal{B}^r$. The Gateaux derivative of $K(u)$ in the direction of $S(u)$ with respect to u is defined by

$$K'[S] = K'(u)[S(u)] = \frac{\partial}{\partial \epsilon} K(u + \epsilon S)|_{\epsilon=0}. \tag{3}$$

$K(u)$ is said to be Gateaux differentiable at u, provided that the Gateaux derivative $K'[S]$ exists for all $S(u) \in \mathcal{B}^r$.

We define in $\mathcal{B}$ the following equivalence relation.

Definition 2. Let $P, Q \in \mathcal{B}$. P is said to be equivalent to Q, denoted by $P \sim Q$, if there exists $R = (R_1, \dots, R_n) \in \mathcal{B}^n$, such that $P - Q = \text{Div}R = \sum_{i=1}^n \partial_i R_i$. Let $\tilde{P} = \int P dx$ denote the equivalence class to which $P \in \mathcal{B}$ belongs. One defines also the inner product $(P, Q) = \int P^T Q dx = \int \sum_{i=1}^r P_i Q_i dx$ for $P = (P_1, \dots, P_r)^T$, $Q = (Q_1, \dots, Q_r)^T \in \mathcal{B}^r$.

Definition 3. A linear operator $\Phi : \mathcal{B}^r \to \mathcal{B}^s$ is said to have an adjoint operator $\Psi : \mathcal{B}^s \to \mathcal{B}^r$, provided that $(P, \Phi Q) = (Q, \Psi P)$ for all $P \in \mathcal{B}^s$, $Q \in \mathcal{B}^r$. The adjoint operator of Φ is often denoted as Φ^*, too. If $\Phi^* = -\Phi$, then Φ is said to be skew-symmetric.

Definition 4. A linear operator $J : \mathcal{B}^m \to \mathcal{B}^m$ is called Hamiltonian, provided that J is skew-symmetric and satisfies the so-called Jacobi identity, ie, for all $P, Q, R \in \mathcal{B}^m$,

$$\{P, Q, R\} + \{R, P, Q\} + \{Q, R, P\} = \{P, Q, R\} + \text{cycle}(P, Q, R) = 0, \tag{4}$$

where

$$\{P, Q, R\} = (P, J'[JQ]R) = \int P^T J'[JQ] R dx.$$

Definition 5. (Olver[10])
The linear operators J and M are said to constitute a Hamiltonian pair, if $\alpha_1 J + \alpha_2 M$ is always Hamiltonian for any real constants α_1 and α_2.

To be simple and brief, in this paper we assume always $n = 1$ and $m = 2$ unless otherwise stated; ie, we shall always apply $u = u(x, t) = [p, q]^T = [p(x, t), q(x, t)]^T$, $x, t \in \mathbb{R}$. Also, we define $\partial = \frac{d}{dx}$, $\partial^{-1} = \int dx$, where the constant of integral involved in the latter is always selected such that $\partial\partial^{-1} = \partial^{-1}\partial = 1$. Furthermore, from now on throughout this paper, we let $\partial^k g$ denote the k-th order total derivative of g with respect to x, except somewhere when $k = 1$ or 2, we use g_x or g_{xx} instead for convenience.

Let g be a sufficiently smooth function of x, t, and $u = [p, q]^T$. Let k and l be nonnegative integers. One can quickly conclude through a straightforward computation using integration by parts that with $\sim$ in the sense of the equivalence relation given in Definition 2,

$$x^k \partial^l g \sim \begin{cases} 0, & \text{if } k < l, \\ (-1)^l k(k-1) \ldots (k-l+1) x^{k-l} g, & \text{if } k \geq l; \end{cases}$$

and in particular, $x^k \partial^k g \sim (-1)^k k! g$. This conclusion is going to be called for multiple times below.

This article will be organized as follows. In Section 2 we discuss a class of matrix differential Hamiltonian operators H and their Gauge equivalence. The coupling of H with matrix differential operators with constant coefficients will be elucidated in Section 3, where it is particularly proven that such a coupling ceases to take effect for differential orders greater than 4. Several representative examples of matrix spectral problems are illustrated in Section 4 for the readers to perceive how the Gauge equivalence discussed in Section 2 works. Finally, a couple of concluding remarks will end the paper.

2 | A CLASS OF MATRIX DIFFERENTIAL HAMILTONIAN OPERATORS AND THEIR GAUGE EQUIVALENCE

It has been proven in Ma[9] that matrix differential operators in the form of

$$H_1 = \begin{bmatrix} p\partial + \partial p & q\partial \\ \partial q & 0 \end{bmatrix} \tag{5}$$

where $u = [p, q]^T$ is a column vector of 2 potentials, are u-linear Hamiltonian operators. Following his work, matrix differential operators in the shape of

$$H_2 = \begin{bmatrix} 0 & \partial p \\ p\partial & q\partial + \partial q \end{bmatrix} \tag{6}$$

can likewise be verified to be Hamiltonian.

Furthermore, one can also show that matrix differential operators, which are constructed from linear combinations of H_1 and H_2 through arbitrary constants β and α, ie,

$$H = \beta H_1 + \alpha H_2 = \begin{bmatrix} \beta(p\partial + \partial p) & \alpha\partial p + \beta q\partial \\ \alpha p\partial + \beta\partial q & \alpha(q\partial + \partial q) \end{bmatrix} \tag{7}$$

are Hamiltonian operators as well. Note that H_1, H_2, and H are all u-linear. To prove this claim, the conventional approach published in the same paper is certainly the most straightforward proof that one can follow. However, an alternative approach in light of the Gauge transformation of Bäcklund type for Hamiltonian operators[5] comes out to be not only even more brief, but also more insightful, in the sense that it reveals in addition that the families of matrix differential operators described by H_1, H_2, and H are indeed mutually "Gauge equivalent," which is the terminology indicating throughout this paper that 2 systems can be transformed into one another via a Gauge transformation of Bäcklund type.

Theorem 1. *Let the matrix differential operators H_1, H_2, and H be defined by Equations 5, 6, and 7, respectively. Let it be nontrivially that $\beta^2 + \alpha^2 \neq 0$. Then H_1, H_2, and H are pairwise Gauge equivalent. It follows also that H is a u-linear Hamiltonian operator.*

Proof. From the definition in (7), H it is obviously u-linear. First let

$$T = \begin{bmatrix} 1 & 0 \\ -\alpha & \beta \end{bmatrix} \quad \text{that suggests} \quad \begin{cases} \tilde{p} = p, \\ \tilde{q} = (-\alpha p + \beta q). \end{cases} \tag{8}$$

It follows immediately that

$$\begin{aligned} \tilde{H} &= THT^* \\ &= \begin{bmatrix} \beta p\partial + \beta\partial p & \beta(-\alpha p + \beta q)\partial \\ \beta\partial(-\alpha p + \beta q) & 0 \end{bmatrix} = \beta \begin{bmatrix} \tilde{p}\partial + \partial\tilde{p} & \tilde{q}\partial \\ \partial\tilde{q} & 0 \end{bmatrix}, \end{aligned} \tag{9}$$

where T^* denotes the Hermitian conjugate of T (which reduces to the transpose of T when β, α are real), is a Hamiltonian operator of H_1-type scaled by β (hence still of H_1-type). Recalling the work by Fuchssteiner and Fokas,[5]

H is thus Gauge equivalent to H_1. Alternatively, H is found to be Gauge equivalent to the H_2-type Hamiltonian operator scaled by α

$$\tilde{H}' = \alpha \begin{bmatrix} 0 & \partial \tilde{p}' \\ \tilde{p}'\partial & \tilde{q}'\partial + \partial \tilde{q}' \end{bmatrix} \tag{10}$$

through

$$T' = \begin{bmatrix} \alpha & -\beta \\ 0 & 1 \end{bmatrix} \quad \text{that suggests} \quad \begin{cases} \tilde{p}' = \alpha p - \beta q, \\ \tilde{q}' = q. \end{cases} \tag{11}$$

Furthermore, provided we start from a matrix differential operator of H_1-type, then it could be first transformed Gauge equivalently into H-type through T^{-1}, and next to H_2-type through T'. Finally, this pairwise Gauge equivalence among H_1, H_2, and H guarantees that H is a Hamiltonian operator. □

The conclusion of the above theorem suggests that the 3 families of Hamiltonian operators H_1, H_2, and H defined by Equations 5, 6, and 7 respectively, are indeed Gauge equivalent in the sense that any two of them can be transformed into one another through Gauge transformations of Bäcklund type. Also, these transformations are local since both T and T' are matrices with constant entries. This fact is yet realized by few people although its proof comes out to be rather brief.

3 | COUPLING OF *H* TO DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

It is already known that more Hamiltonian operators can be constructed by coupling matrix differential operators in forms of H_1, H_2, or H with differential operators in low orders with constant coefficients.[9,11-14] Let

$$K = \sum_{m=0}^{k} B_m \partial^m, \quad B_m = (b_{ijm})_{2\times 2} \tag{12}$$

be a matrix differential operator up to the k-th order with constant entries b_{ijm} (where $1 \le i,j \le 2$ and $0 \le m \le k$). K is apparently skew-symmetric if and only if

$$b_{ijm} = (-1)^{m+1} b_{jim} \tag{13}$$

is satisfied. This condition guarantees also the balance of the Jacobi identity; therefore, K is automatically a Hamiltonian operator. We now prove the following theorem as a generalization to the work from 1993.[9]

Theorem 2. *Let the matrix differential operators H and K be given by Equations 7 and 12, respectively, where all the coefficients b_{ijm} in K satisfy the conditions in Equation 13. Let it be nontrivially that $\beta^2 + \alpha^2 \neq 0$. Then the matrix differential operator*

$$\mathcal{J} = H + K \tag{14}$$

is a Hamiltonian operator if and only if

$$K = \begin{bmatrix} b_{111} & b_{121} \\ b_{121} & b_{221} \end{bmatrix} \partial + \begin{bmatrix} 0 & b_{122} \\ -b_{122} & 0 \end{bmatrix} \partial^2 + \begin{bmatrix} b_{113} & b_{123} \\ b_{123} & b_{223} \end{bmatrix} \partial^3, \tag{15}$$

where b_{111}, b_{121}, b_{221}, and b_{122} are arbitrary real constants; whereas, b_{113}, b_{123}, and b_{223} are real constants satisfying

$$b_{113}\alpha - b_{123}\beta = 0 \quad \textit{and} \quad b_{223}\beta - b_{123}\alpha = 0. \tag{16}$$

Proof. The required skew-symmetry is obvious. Hence, we focus on showing the Jacobi Identity.

First of all, since the H-component of $\mathcal{J}$ is Hamiltonian, the computation thus gives directly

$$\{P, Q, R\} = \sum_{k=0}^{m} \int [f_k(P,Q,R) + g_k(P,Q,R) + h_k(P,Q,R) + j_k(P,Q,R)]dx,$$

where

$$f_k(P,Q,R) = \beta(P_1 R_{1,x} - P_{1,x} R_1) b_{11k} \partial^k Q_1, \tag{17a}$$

$$g_k(P,Q,R) = \beta(P_1R_{1,x} - P_{1,x}R_1)b_{12k}\partial^k Q_2 + \alpha(P_2R_{1,x} - P_{1,x}R_2)b_{11k}\partial^k Q_1 + \beta(P_1R_{2,x} - P_{2,x}R_1)b_{21k}\partial^k Q_1, \tag{17b}$$

$$h_k(P,Q,R) = \alpha(P_2R_{1,x} - P_{1,x}R_2)b_{12k}\partial^k Q_2 + \beta(P_1R_{2,x} - P_{2,x}R_1)b_{22k}\partial^k Q_2 + \alpha(P_2R_{2,x} - P_{2,x}R_2)b_{21k}\partial^k Q_1, \tag{17c}$$

$$j_k(P,Q,R) = \alpha(P_2R_{2,x} - P_{2,x}R_2)b_{22k}\partial^k Q_2, \tag{17d}$$

for $0 \le k \le m$. Noticing the differential orders and subscripts of P_i, Q_i, R_i, it follows directly that the Jacobi identity for $H + K$ holds if and only if

$$\bar{f}_k(P,Q,R) = f_k(P,Q,R) + \text{cycle}(P,Q,R) \sim 0, \tag{18a}$$

$$\bar{g}_k(P,Q,R) = g_k(P,Q,R) + \text{cycle}(P,Q,R) \sim 0, \tag{18b}$$

$$\bar{h}_k(P,Q,R) = h_k(P,Q,R) + \text{cycle}(P,Q,R) \sim 0, \tag{18c}$$

$$\bar{j}_k(P,Q,R) = j_k(P,Q,R) + \text{cycle}(P,Q,R) \sim 0, \tag{18d}$$

where $0 \le k \le m$.

Let us first consider the case of $k = 0$. Upon noticing $b_{110} = b_{220} = 0$ and $b_{210} = -b_{120}$, a special choice of (18b) with $P_1 = r$, $Q_1 = 0$, $R_1 = x$ and $Q_2 = 1$, and a special choice of (18c) with $P_1 = 1$, $P_2 = 0$, $Q_2 = s$ and $R_2 = x$ (here one might choose r and s be arbitrary functions of x), generate

$$\bar{g}_0(P,Q,R) = \beta b_{120}(r - xr_x) \sim 2\beta b_{120} r,$$
$$\bar{h}_0(P,Q,R) = \alpha b_{120}(s - xs_x) \sim 2\alpha b_{120} s,$$

respectively. It then follows the arbitrariness of r and s that $b_{120} = 0$. Thus, when $k = 0$, (18a) to (18d) hold if and only if $B_0 = 0$.

Secondly, we consider the case of $k = 1$. We automatically have $\bar{f}_1(P,Q,R)= \bar{j}_1(P,Q,R) = 0$. Moreover, the condition $b_{211} = b_{121}$ guarantees that $\bar{g}(P,Q,R) = \bar{h}_1(P,Q,R) = 0$. So nothing is required more than $B_1^T = B_1$.

Next we investigate the case of $k = 2$. The conditions $b_{112} = b_{222} = 0$ and $b_{212} = -b_{122}$ guarantee $\bar{f}_2(P,Q,R) = \bar{j}_2(P,Q,R) = 0$ and

$$\bar{g}_2(P,Q,R) = \frac{d}{dx}[\beta b_{122}(P_1R_{1,x} - P_{1,x}R_1)Q_{2,x} + \text{cycle}(P,Q,R)] \sim 0,$$
$$\bar{h}_2(P,Q,R) = \frac{d}{dx}[\alpha b_{122}(P_2Q_{2,x} - P_{2,x}Q_2)R_{1,x} + \text{cycle}(P,Q,R)] \sim 0.$$

Hence nothing is required more than $B_2^T = -B_2$ for $k = 2$.

A little longer discussion must be carried out for the case of $k = 3$. On the one hand, we have

$$\bar{f}_3(P,Q,R) = \frac{d}{dx}[\beta b_{113}P_1(R_{1,x}Q_{1,xx} - Q_{1,x}R_{1,xx}) + \text{cycle}(P,Q,R)] \sim 0,$$
$$\bar{j}_3(P,Q,R) = \frac{d}{dx}[\alpha b_{223}P_2(R_{2,x}Q_{2,xx} - Q_{2,x}R_{2,xx}) + \text{cycle}(P,Q,R)] \sim 0.$$

Again by taking the advantages of r and s that were used above, a special choice of (18b) with $P_1 = r$, $R_1 = x$, $Q_2 = \frac{x^3}{3!}$ and $Q_1 = P_2 = R_2 = 0$ gives

$$\bar{g}_3(P,Q,R) = \beta b_{123}\left(r - xr_x + \frac{1}{2}x^3\partial^3 r\right) - \frac{1}{6}\alpha b_{113}x^3\partial^3 r \sim (-\beta b_{123} + \alpha b_{113})r,$$

based on which, (18b) would require

$$\beta b_{123} - \alpha b_{113} = 0. \tag{19}$$

Likewise, a special choice of (18c) with $P_2 = s$, $Q_1 = \frac{x^3}{3!}$, $R_2 = x$ and $P_1 = R_1 = Q_2 = 0$ leads to

$$\beta b_{223} - \alpha b_{123} = 0. \tag{20}$$

On the other hand, when (19) and (20) are satisfied, we have

$$\bar{g}_3(P,Q,R) = c_1\frac{d}{dx}[(P_1R_{1,x} - P_{1,x}R_1)Q_{2,xx} + (Q_{2,x}R_1 - Q_2R_{1,x})P_{1,xx} + (P_{1,x}Q_2 - P_1Q_{2,x})R_{1,xx} + \text{cycle}(P,Q,R)] \sim 0,$$

where $c_1 = \beta b_{123} = \alpha b_{113}$, and

$$\bar{h}_3(P,Q,R) = c_2\frac{d}{dx}[(P_1R_{2,x} - P_{1,x}R_2)Q_{2,xx} + (Q_{2,x}R_2 - Q_2R_{2,x})P_{1,xx} + (P_{1,x}Q_2 - P_1Q_{2,x})R_{2,xx} + \text{cycle}(P,Q,R)] \sim 0,$$

where $c_2 = \beta b_{223} = \alpha b_{123}$. Thus, (19) and (20) suffice to guarantee that (18a) to (18d) hold when $k = 3$.

Finally, we consider the case of $4 \leq k \leq m$. A special choice of (18a) with $P_1 = r$, $Q_1 = \frac{x^k}{k!}$ and $R_1 = x$, and a special choice of (18d) with $P_2 = s$, $Q_2 = \frac{x^k}{k!}$ and $R_2 = x$, will generate

$$\begin{aligned} \bar{f}_k &= \beta b_{11k}\left[r - xr_x + (k-1)\frac{x^k}{k!}\partial^k r\right] \sim \beta b_{11k}[2 + (-1)^k(k-1)]r, \\ \bar{j}_k &= \alpha b_{22k}\left[s - xs_x + (k-1)\frac{x^k}{k!}\partial^k s\right] \sim \alpha b_{22k}[2 + (-1)^k(k-1)]s, \end{aligned} \tag{21}$$

respectively. Since $k \geq 4$, thus

$$\beta b_{11k} = 0, \quad \alpha b_{22k} = 0, \tag{22}$$

are required to guarantee that $\bar{f}_k \sim 0$ and $\bar{j}_k \sim 0$. Then, by using (22), 2 special choices of (18b) with $P_1 = r$, $Q_1 = 0$, $R_1 = x$, and $Q_2 = \frac{x^k}{k!}$, and of (18c) with $P_1 = 0$, $Q_1 = \frac{x^k}{k!}$, $R_2 = x$, and $P_2 = s$ will give

$$\begin{aligned} \bar{g}_k(P,Q,R) &= \beta b_{12k}(r - xr_x) - \frac{1}{k!}\alpha b_{11k}x^k\partial^k r + \frac{1}{(k-1)!}\beta b_{21k}x^k\partial^k r \\ &\sim \beta b_{12k}(2-k)r - (-1)^k\alpha b_{11k}r, \end{aligned} \tag{23}$$

$$\begin{aligned} \bar{h}_k(P,Q,R) &= \alpha b_{21k}(s - xs_x) - \frac{1}{k!}\beta b_{22k}x^k\partial^k s + \frac{1}{(k-1)!}\alpha b_{12k}x^k\partial^k s \\ &\sim \alpha b_{21k}(2-k)s - (-1)^k\beta b_{22k}s. \end{aligned} \tag{24}$$

Since $\beta^2 + \alpha^2 \neq 0$, without loss of generality, assume $\beta \neq 0$. Thus, (22) implies $b_{11k} = 0$, based on which (23) gives $b_{12k} = 0$. It must be then $b_{21k} = 0$ due to (13), based on which $b_{22k} = 0$ is finally determined again from (23). Likewise we obtain again $b_{11k} = b_{12k} = b_{21k} = b_{22k} = 0$ if $\alpha \neq 0$ is assumed. In short, the conditions in (22), (23), and (24) combine to give the only possibility that $B_k = 0$ for $4 \leq k \leq m$.

Therefore, to conclude, $H + K$ is Hamiltonian if and only if K is determined by (15) and (16). This completes the proof. □

On the basis of early Hamiltonian theory,[4,5,15] Fokas[16] proved that hereditary symmetry in the form of $\Phi = MJ^{-1}$ can be constructed, provided that the differential operators M and J are known to form a Hamiltonian pair, with J being invertible. Following their conclusion, for a Hamiltonian operator $M = H + K$ with H, K given by (7), (15), respectively (and certainly all the coefficients in K satisfy the conditions given by (16), let

$$J = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}\partial + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix}\partial^3, \tag{25}$$

where $\frac{d_4}{b_{113}} = \frac{d_5}{b_{123}} = \frac{d_6}{b_{223}}$ if they are all nonzero, or $b_{113} = 0$ if and only if $d_4 = 0$, and likewise for b_{123} and d_5, b_{223} and d_6 (this condition is introduced to ensure that any linear combination of J and M is again Hamiltonian, which turns out to be a sufficient and necessary condition for J and M to constitute a Hamiltonian pair). Furthermore d_1, d_2, d_3, and a are selected properly so that J is invertible. It follows then that any linear combination of J and M is again a Hamiltonian, and J, M thus constitute a Hamiltonian. Two examples of such Hamiltonian operators J given by (25) could be, for instance, typically illustrated by (but not limited to)

$$J = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}\partial \quad \Rightarrow \quad J^{-1} = \frac{1}{\triangle}\begin{bmatrix} d_3 & -d_2 \\ -d_2 & d_1 \end{bmatrix}\partial^{-1}, \tag{26}$$

with $\triangle = d_1d_3 - d_2^2 \neq 0$; or another one which is relatively rarely used,

$$J = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix}\partial^3 \quad \Rightarrow$$
$$J^{-1} = \frac{1}{a^2}\begin{bmatrix} d_6\partial^{-1} & -d_5\partial^{-1} - a\partial^{-2} \\ -d_5\partial^{-1} + a\partial^{-2} & d_4\partial^{-1} \end{bmatrix},$$

with $a \neq 0$ and $d_4d_6 - d_5^2 = 0$. Therefore, it comes up immediately with

Theorem 3. *Let H, K and J be given by (7), (15), and (25), respectively. Let $M = H + K$. Then $\Phi = MJ^{-1}$ is a hereditary symmetry.*

The hereditary symmetry Φ given by Theorem 3, being implicitly dependent on x only, obviously comes out to be translational invariant with respect to x. Hence $\forall S \in \mathcal{B}^2$, we obtain a vanishing Lie derivative,[17] ie,

$$(L_{u_x}\Phi)S = \Phi[u_x, S] - [u_x, \Phi S] = 0. \tag{27}$$

Therefore by corollary 1 in Ref. [18] we can conclude

$$[\Phi^m u_x, \Phi^n u_x] = 0, \quad \forall m, n \geq 0,$$

and it follows naturally also that

Theorem 4. *Let H, K, and J be given by (7), (15), and (25), respectively. Let $M = H + K$. Then the class of evolution equations*

$$u_t = \Phi^m u_x, \quad m \geq 0 \tag{28}$$

has a hierarchy of infinitely many common symmetries $\{K_m = \Phi^m u_x\}$ $(m \geq 0)$ and Φ is a common hereditary strong symmetry of the hierarchy (28).

The hierarchy of evolution equations (28) is in general nonlinear and integrable in the sense that it possesses infinitely many K-symmetries. [16,18]

4 | EXAMPLES

4.1 | Example I

We present in this section a few solid examples trying to show how this kind of Gauge equivalence works. For instance, the following matrix spectral problem was computed in [3]:

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \text{with} \quad u = [p, q]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2]^T,$$

where the spectral matrix $U \in \widetilde{\mathrm{sl}}(2, \mathbb{R})$ possesses the form

$$U = (-\lambda - q)e_1 + pe_2 + \gamma e_3 = \begin{bmatrix} -\lambda - q & p \\ \gamma & \lambda + q \end{bmatrix}, \tag{29}$$

with $\gamma \neq 0$ being a constant, and

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{30}$$

constitute the basis of the Lie algebra $\mathrm{sl}(2, \mathbb{R})$. The soliton hierarchy derived from this matrix spectral problem was found to be Liouville integrable and possess the bi-Hamiltonian structure

$$u_{t_m} = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad \text{for all} \quad m \geq 1, \tag{31}$$

where $\mathcal{H}_m, \mathcal{H}_{m-1}$ are the corresponding Hamiltonian functionals, with J and M constituting the Hamiltonian pair:

$$J = -\frac{1}{\gamma}\begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \frac{1}{\gamma}(p\partial + \partial p) & \frac{1}{2\gamma}\partial^2 + \frac{1}{\gamma}q\partial \\ -\frac{1}{2\gamma}\partial^2 + \frac{1}{\gamma}\partial q & -\frac{1}{2}\partial \end{bmatrix}. \tag{32}$$

Clearly, J is invertible, and M is characteristically a case of the $\mathcal{J}$ operator presented in Theorem 2 with $\alpha = 0$ (or in another word, the H component of $\mathcal{J}$ reduces to H_1 given in (5). If by specifically letting $\gamma = 1/\beta$, and one modifies the matrix spectral problem (29) to (let $\tilde{u} = [\tilde{p}, \tilde{q}]^T$)

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(\tilde{u}, \lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} -\lambda - \alpha\tilde{p} - \beta\tilde{q} & \tilde{p} \\ \frac{1}{\beta} & \lambda + \alpha\tilde{p} + \beta\tilde{q} \end{bmatrix} \tag{33}$$

through the Gauge transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = T\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad T = \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix}, \tag{34}$$

will finally yield the Hamiltonian operator

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial + \partial\tilde{p}) & \frac{1}{2}\partial^2 - \alpha\partial\tilde{p} + \beta\tilde{q}\partial \\ -\frac{1}{2}\partial^2 - \alpha\tilde{p}\partial + \beta\partial\tilde{q} & -\frac{1}{2\beta^2}\partial - \alpha(\tilde{q}\partial + \partial\tilde{q}) \end{bmatrix} \tag{35}$$

which is obviously again characteristically a case of the $\mathcal{J}$ operator presented in Theorem 2. According to this theorem $\tilde{M}$ is associated with M in (32) via

$$\tilde{M} = T^{-1}MT^{-1^*}. \tag{36}$$

In particular, the K component in M apparently undergoes the same Gauge transformation of Bäcklund type and remains to be a (Hamiltonian) matrix differential operator with constant coefficients. This example justifies our conclusions in Theorem 2.

4.2 | Example II

In 2002, Xu generalized the Wadati-Konno-Ichikawa (WKI) hierarchy[1,19] by computing the following matrix spectral problem (for convenience of calculation, the matrix problem is presented here in a slightly different form from the original work; they are however exactly equivalent):

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \text{with} \quad u = [p,q]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2]^T,$$

where the spectral matrix $U \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ takes the form

$$U = (\lambda - \frac{\alpha}{2}p)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} \lambda - \frac{\alpha}{2}p & \lambda p \\ \lambda q & -\lambda + \frac{\alpha}{2}p \end{bmatrix}. \tag{37}$$

By solving the stationary zero-curvature equation $W_x = [U, W]$ with also $W \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ chosen to be

$$W = \begin{bmatrix} (\lambda - \frac{\alpha}{2}p)a - \frac{\alpha b_x}{2\lambda} & \lambda p a + b_x \\ \lambda q a + c_x & -(\lambda - \frac{\alpha}{2}p)a + \frac{\alpha b_x}{2\lambda} \end{bmatrix}, \tag{38}$$

where a, b, and c are functions of λ, x, and t taking the form of Laurent expansions:

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-i}, \quad c = \sum_{i\geq 0} c_i\lambda^{-i}, \tag{39}$$

it was derived that this generalized WKI soliton hierarchy possesses the bi-Hamiltonian structure $u_{t_m} = M\frac{\delta\mathcal{H}_m}{\delta u} = J\frac{\delta\mathcal{H}_{m-1}}{\delta u}$ for all $m \geq 1$, where $\mathcal{H}_m, \mathcal{H}_{m-1}$ are the corresponding Hamiltonian functionals, with J and M constituting the Hamiltonian pair. In particular M reads

$$M = \begin{bmatrix} 0 & \partial^2 + \alpha\partial p \\ -\partial^2 + \alpha p\partial & \alpha(q\partial + \partial q) \end{bmatrix}, \tag{40}$$

which is characteristically a case of the $\mathcal{J}$ operator presented in Theorem 2 with $\beta = 0$ (or in another word, the H component of $\mathcal{J}$ reduces to H_2 given in (6). This soliton hierarchy is interestingly also Liouville integrable although J in the Hamiltonian pair does not take the form of (25). The matrix spectral problem (37) of the generalized WKI hierarchy, if modified to (let $\tilde{u} = [\tilde{p}, \tilde{q}]^T$)

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(\tilde{u},\lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} \lambda - \frac{\alpha}{2}(\alpha\tilde{p} - \beta\tilde{q}) & \alpha\lambda\tilde{p} - \beta\lambda\tilde{q} \\ \lambda\tilde{q} & -\lambda + \frac{\alpha}{2}(\alpha\tilde{p} - \beta\tilde{q}) \end{bmatrix} \tag{41}$$

by admitting the Gauge transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = T\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad T = \begin{bmatrix} \alpha & -\beta \\ 0 & 1 \end{bmatrix}, \tag{42}$$

will finally yield the Hamiltonian operator

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial + \partial\tilde{p}) & \frac{1}{\alpha}\partial^2 + \alpha\partial\tilde{p} + \beta\tilde{q}\partial \\ -\frac{1}{\alpha}\partial^2 + \alpha\tilde{p}\partial + \beta\partial\tilde{q} & \alpha(\tilde{q}\partial + \partial\tilde{q}) \end{bmatrix}, \tag{43}$$

which is obviously again characteristically a case of the $\mathcal{J}$ operator presented in Theorem 2 and is associated with M in (40) through

$$\tilde{M} = T^{-1}MT^{-1^*}. \tag{44}$$

Still, the K component in M apparently undergoes the same Gauge transformation of Bäcklund type and remains to be a (Hamiltonian) matrix differential operator with constant coefficients. This example precisely illustrates our conclusions in Theorem 2.

An alternative form of the generalized WKI hierarchy was computed in Ma et al,[20] for which rather similar arguments upon Gauge transformation of Bäcklund type are also applicable.

4.3 | Example III

Zhang et al reported in 2015[2] the coupled Burgers hierarchy based on the matrix spectral problem:

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \text{with} \quad u=[p,q]^T \quad \text{and} \quad \phi=[\phi_1,\phi_2]^T,$$

where the spectral matrix $U \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ takes the form

$$U = (-\lambda+\alpha p+\beta q)e_1 + pe_2 + qe_3 = \begin{bmatrix} -\lambda+\alpha p+\beta q & p \\ q & \lambda-\alpha p-\beta q \end{bmatrix}. \tag{45}$$

To emphasize, the constants α, β are in this model subject to the constraint $\alpha\beta = -1/4$. This soliton hierarchy is also characterized by Liouville integrability and bi-Hamiltonian structure $u_{t_m} = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}$ $(m \geq 1)$, in which the Hamiltonian pair reads

$$J = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \beta(p\partial+\partial p) & -\frac{1}{2}\partial^2+\alpha\partial p+\beta q\partial \\ \frac{1}{2}\partial^2+\alpha p\partial+\beta\partial q & \alpha(q\partial+\partial q) \end{bmatrix}, \tag{46}$$

where J is clearly invertible, and M is characteristically a case of the $\mathcal{J}$ operator presented in Theorem 2 ($\alpha\beta = -1/4$). This matrix spectral problem (45), if modified to (let $\tilde{u} = [\tilde{p},\tilde{q}]^T$)

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(\tilde{u},\lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} -\lambda+2\alpha\tilde{p}+\tilde{q} & \tilde{p} \\ \frac{\alpha}{\beta}\tilde{p}+\frac{1}{\beta}\tilde{q} & \lambda-2\alpha\tilde{p}-\tilde{q} \end{bmatrix}, \tag{47}$$

by admitting the Gauge transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = T\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad T = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & \frac{1}{\beta} \end{bmatrix} \tag{48}$$

will finally yield the Hamiltonian operator

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial+\partial\tilde{p}) & -\frac{1}{2}\beta\partial^2+\beta\tilde{q}\partial \\ \frac{1}{2}\beta\partial^2+\beta\partial\tilde{q} & 0 \end{bmatrix}, \tag{49}$$

which is apparently characteristically a case of the $\mathcal{J}$ operator, with its H-component reducing to the form of H_1 in (5), presented in Theorem 2. The matrix spectral problem could also alternatively be modified into (let $\hat{u} = [\hat{p},\hat{q}]^T$)

$$\hat{\phi}_x = \hat{U}\hat{\phi} = \hat{U}(\hat{u},\lambda)\hat{\phi}, \qquad \hat{U} = \begin{bmatrix} -\lambda-\hat{p}+2\beta\hat{q} & -\frac{1}{\alpha}\hat{p}+\frac{\alpha}{\beta}\hat{q} \\ \hat{q} & \lambda+\hat{p}-2\beta\hat{q} \end{bmatrix} \tag{50}$$

by admitting the Gauge transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = T\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}, \quad \text{with} \quad T = \begin{bmatrix} -\frac{1}{\alpha} & \frac{\beta}{\alpha} \\ 0 & 1 \end{bmatrix}, \tag{51}$$

will yield the Hamiltonian operator

$$\hat{M} = \begin{bmatrix} 0 & \frac{1}{2}\alpha\partial^2+\alpha\partial\hat{p} \\ -\frac{1}{2}\alpha\partial^2+\alpha\hat{p}\partial & \alpha(\hat{q}\partial+\partial\hat{q}) \end{bmatrix}, \tag{52}$$

which is apparently also characteristically a case of the $\mathcal{J}$ operator, with its H-component reducing to the form of H_2 in (6), presented in Theorem 2. In this example, both $\tilde{M}$ given by (49) and $\hat{M}$ given by (52) can be associated with M in (46) through $\tilde{M}$(or $\hat{M}$) $= T^{-1}MT^{-1^*}$.

4.4 | Example IV

As a separate final example, we consider the Hamiltonian operator possessing the following form:

$$J = B_1\partial + B_2\partial^2 + B_3\partial^3 = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}\partial + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix}\partial^3, \tag{53}$$

where $a \neq 0$, and $\det B_3 = d_4d_6 - d_5^2 = 0$. Generally, there is no way that one can compute in an explicit way the inverse of (53), which, however, can formally be written as

$$J^{-1} = \begin{bmatrix} d_3 + d_6\partial^2 & d_2 + a\partial + d_5\partial^2 \\ d_2 - a\partial + d_5\partial^2 & d_1 + d_4\partial^2 \end{bmatrix} \triangle^{-1}\partial^{-1}, \tag{54}$$

where

$$\triangle = d_1d_3 - d_2^2 + (d_1d_6 + d_3d_4 + a^2 - 2d_2d_5)\partial^2. \tag{55}$$

Let $M = H$ (ie, M has only the H component of $\mathcal{J} = H + K$ in Theorem 2) satisfy the conditions in Theorems 3 and 4, it follows immediately that the recursion operator $\Phi = MJ^{-1}$ is a hereditary strong symmetry, and the class of evolution equations $u_t = \Phi^m u_x$ $(m \geq 0)$ hence possesses a hierarchy of infinitely many common symmetries $\{K_m = \Phi^m u_x\}$ $(m \geq 0)$. This example involves a Hamiltonian operator J in (53) that contains a differential component in third order, and the appearance of $\triangle$ in (55) attributes J with some features like the characteristic of a Camassa-Holm type hierarchy.[21]

The first nonlinear evolution equation of such a hierarchy, $u_t = (p_t, q_t)^T = \Phi u_x$ $(\Phi = MJ^{-1})$, under the introduction of another 2 implicit variables r, s (a typical treatment when computing Camassa-Holm type models) that are associated with p, q via

$$\begin{cases} p = Ar + Br_{xx}, \\ q = As + Bs_{xx}, \end{cases} \tag{56}$$

where $A = d_1d_3 - d_2^2$, $B = d_1d_6 + d_2d_5 + a^2 - 2d_3d_4$, can be written down a little lengthily as follows:

$$\begin{cases} p_t = 2\beta p(d_3r_x - d_2s_x) + \beta p_x(d_3r - d_2s) - \beta q(d_2r_x - d_1s_x) \\ \quad - \beta q(d_5r_{xxx} - d_4s_{xxx}) + 2\beta p(d_6r_{xxx} - d_5s_{xxx}) \\ \quad + \beta p_x(d_6r_{xx} - d_5s_{xx}) - \beta a(p_xs_x + 2ps_{xx} - qr_{xx}) \\ \quad + \alpha[p(d_1s - d_2r)]_x + \alpha[p(d_4s_{xx} - d_5r_{xx})]_x + \alpha a(pr_x)_x, \\ \\ q_t = \beta[q(d_3r - d_2s)]_x + \beta[q(d_6r_{xx} - d_5s_{xx})]_x - \beta a(qs_x)_x \\ \quad - 2\alpha q(d_2r_x - d_1s_x) - \alpha q_x(d_2r - d_1s) + \alpha p(d_3r_x - d_2s_x) \\ \quad + \alpha p(d_6r_{xxx} - d_5s_{xxx}) - 2\alpha q(d_5r_{xxx} - d_4s_{xxx}) \\ \quad - \alpha q_x(d_5r_{xx} - d_4s_{xx}) + \alpha a(q_xr_x + 2qr_{xx} - ps_{xx}). \end{cases} \tag{57}$$

If alternatively, one imposes in (53)$B_2 = 0$ (ie, $a = 0$) and admits but $\det B_1 < 0$, or to put it more precisely,

$$\det B_1 = d_1d_3 - d_2^2 = -\frac{d_1^2}{d_4^2}\left(d_5 - \frac{d_2d_4}{d_1}\right)^2, \tag{58}$$

which is indeed an equivalent version of the condition $d_1d_6 + d_4d_3 - 2d_2d_5 = 0$, J will possess under these assumptions an explicit inverse that reads

$$J^{-1} = \frac{1}{\det B_1}\begin{bmatrix} d_6\partial + d_3\partial^{-1} & -d_5\partial - d_2\partial^{-1} \\ -d_5\partial - d_2\partial^{-1} & d_4\partial + d_1\partial^{-1} \end{bmatrix}. \tag{59}$$

Likewise, the Hamiltonian operator J in (53) satisfying (58), together with $M = H + K$, yields a hereditary strong symmetry $\Phi = MJ^{-1}$, and the class of evolution equations $u_t = \Phi^m u_x$ $(m \geq 0)$ is associated with a hierarchy of infinitely many common symmetries $\{K_m = \Phi^m u_x\}$ $(m \geq 0)$.

5 | CONCLUSION

In this paper we showed that matrix differential operators in forms of H_1, H_2, and H given by (5), (6), and (7), respectively, are u-linear Hamiltonian operators that are also pairwise Gauge equivalent. The transformations among them are performed in terms of 2×2 matrices with constant entries (and hence the transformations can certainly be viewed as local). We think the Gauge equivalence among H_1, H_2, and H may offer some insights to the classification of some soliton hierarchies.

Furthermore, we proved also that these matrix differential operators can be coupled with differential operators with constant coefficients to generate new Hamiltonian operators. Also derived were the corresponding required conditions (15) and (16). The most interesting fact revealed by Theorem 2 is that such a coupling takes effect only up to the third order of the differential operators with constant coefficients, and this extremely succinct conclusion (which is simpler than we expected in advance) would motivate us to further explore what is hidden behind the corresponding Hamiltonian structures[11].

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