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ORIGINAL PAPER

Bäcklund transformation, multiple wave solutions and lump solutions to a (3 + 1)-dimensional nonlinear evolution equation

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Abstract In this paper, a $(3+1)$-dimensional nonlinear evolution equation is cast into Hirota bilinear form with a dependent variable transformation. A bilinear Bäcklund transformation is then presented, which consists of six bilinear equations and involves nine arbitrary parameters. With multiple exponential function method and symbolic computation, nonresonant-typed one-, two-, and three-wave solutions are obtained. Furthermore, two classes of lump solutions to the dimensionally reduced cases with $y = x$ and $y = z$ are both derived. Finally, some figures are given to reveal the propagation of multiple wave solutions and lump solutions.

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1 Introduction

Nonlinear evolution equations (NLEEs) including soliton equations play an important role in the areas of mathematical physics [1–7]. Generally speaking, it is very difficult to find exact solutions to NLEEs [8–24]. The transformed rational function method and multiple exponential function method provide two effective pathways to construct multiple wave solutions [13,14]. If we can get the Hirota bilinear form for a NLEE, then we can derive the exact solutions with multiple exponential function algorithm. Furthermore, the Bäcklund transformation (BT) can also be used in solution aspects [20]. Based on a known solution, we can obtain another solution by using BT.

In this paper, we will study the following $(3 + 1)$-dimensional NLEE [7,12] as

$$u_{yt} - u_{xxxy} - 3\,(u_x\,u_y)_x - 3\,u_{xx} + 3\,u_{zz} = 0, \qquad (1)$$

which was proposed firstly in Ref. [7,12], and the resonant behavior of multiple wave solutions has been investigated [12]. With symbolic computation, two classes of lump solutions have been derived to the dimensionally reduced equations in $(2+1)$-dimensions with $z = y$ and $z = t$, respectively, by searching for positive quadratic function solutions to associated bilinear equation [7]. It is important to study other properties for Eq. (1) such as BT, nonresonant multiple wave solutions, and lump dynamics with novel dimensional reductions.

The structure of this paper is as follows: In Sect. 2, we will construct a BT for Eq. (1) based on its bilinear form, and as an application, we will derive some exact solutions via this BT. Note that the BT consists of six bilinear equations and involves nine arbitrary parameters. Nonresonant-typed multiple wave solutions will be solved in Sect. 3 by use of multiple exponential function method. In Sect. 4, we will give two classes of lump solutions to the dimensionally reduced equations in (2+1)-dimensions with $y = x$ and $y = z$, respectively. Finally, Sect. 5 presents discussions and conclusions, and we will plot some figures to describe the characteristics of multiple wave solutions and lump solutions.

2 Bilinear BT

2.1 Construction of BT

Substitution of the dependent variable transformation $u = 2(\ln f)_x$ with $f = f(x, y, z, t)$ into Eq. (1) yields the bilinear representation for Eq. (1) as

$$\left(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2\right) f \cdot f = 0, \tag{2}$$

where $D_t D_y$, $D_x^3 D_y$, D_x^2, and D_z^2 are all the bilinear derivative operators [20] defined by

$$D_x^\alpha D_y^\beta D_t^\gamma (\rho \cdot \varrho) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^\beta \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\gamma \rho(x, y, t)\varrho(x', y', t')\Big|_{x'=x,\, y'=y,\, t'=t}. \tag{3}$$

To construct a bilinear BT by means of Eq. (2), we consider

$$2P \equiv 2f^2 \left(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2\right) g \cdot g - 2g^2 \left(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2\right) f \cdot f, \tag{4}$$

in which $g = g(x, y, z, t)$ is another solution to Eq. (2).

By using exchange formulas, symbolic computation on Eq. (4) leads to

$$\begin{aligned}
-2P &= -2\left[\left(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2\right) g \cdot g\right] f^2 \\
&\quad + 2g^2 \left[\left(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2\right) f \cdot f\right] \\
&= -2\left[\left(D_t D_y g \cdot g\right) f^2 - g^2 (D_t D_y)\right] \\
&\quad + 2\left[\left(D_x^3 D_y g \cdot g\right) f^2 - g^2 \left(D_x^3 D_y f \cdot f\right)\right] \\
&\quad + 6\left[\left(D_x^2 g \cdot g\right) f^2 - g^2 \left(D_x^2 f \cdot f\right)\right] \\
&\quad - 6\left[\left(D_z^2 g \cdot g\right) f^2 - g^2 \left(D_z^2 f \cdot f\right)\right] \\
&= D_x \left(3D_x^2 D_y g \cdot f\right) \cdot fg + D_x \left(3D_x^2 g \cdot f\right) \cdot \left(D_y f \cdot g\right) \\
&\quad + D_x \left(6 D_x D_y g \cdot f\right) \cdot (D_x f \cdot g) + D_x \left(12 D_x g \cdot f\right) \cdot fg \\
&\quad + D_y \left(D_x^3 g \cdot f\right) \cdot fg + D_y \left(3D_x^2 g \cdot f\right) \cdot (D_x f \cdot g) \\
&\quad + D_t \left(-4 D_y g \cdot f\right) \cdot fg - 12 D_z \left(D_z g \cdot f\right) \cdot fg \\
&= D_x \left[\left(3D_x^2 D_y + \lambda_1 D_y \right.\right. \\
&\quad \left.\left. + \ \lambda_2 + 12 D_x + 12\lambda_8 D_z\right) g \cdot f\right] \cdot fg \\
&\quad + D_y \left[\left(D_x^3 + \lambda_3 - \lambda_1 D_x - 4D_t - 12\lambda_9 D_z\right) g \cdot f\right] \cdot fg \\
&\quad + D_x \left[\left(3D_x^2 + \lambda_4 D_y + \lambda_6\right) g \cdot f\right] \cdot \left(D_y f \cdot g\right) \\
&\quad + D_y \left[\left(3D_x^2 + \lambda_5 D_x - \lambda_6\right) g \cdot f\right] \cdot \left(D_x f \cdot g\right) \\
&\quad + D_x \left[\left(6 D_x D_y + 6\lambda_7 D_x\right) g \cdot f\right] \cdot \left(D_x f \cdot g\right) \\
&\quad - 12 D_z \left[\left(D_z + \lambda_8 D_x + \lambda_9 D_y\right) g \cdot f\right] \cdot fg,
\end{aligned} \tag{5}$$

where we have introduced nine arbitrary coefficients of λ_i $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. To this stage, equation decoupling of Eq. (5) gives rise to an alternative BT for Eq. (2) as

$$\begin{cases}
B_1 g \cdot f = (3D_x^2 D_y + \lambda_1 D_y + \lambda_2 + 12 D_x + 12\lambda_8 D_z) g \cdot f = 0, \\
B_2 g \cdot f = (D_x^3 + \lambda_3 - \lambda_1 D_x - 4D_t - 12\lambda_9 D_z) g \cdot f = 0, \\
B_3 g \cdot f = (3D_x^2 + \lambda_4 D_y + \lambda_6) g \cdot f = 0, \\
B_4 g \cdot f = (3D_x^2 + \lambda_5 D_x - \lambda_6) g \cdot f = 0, \\
B_5 g \cdot f = (6 D_x D_y + 6\lambda_7 D_x) g \cdot f = 0, \\
B_6 g \cdot f = (D_z + \lambda_8 D_x + \lambda_9 D_y) g \cdot f = 0,
\end{cases} \tag{6}$$

which consists of six bilinear equations and involves nine arbitrary parameters.

2.2 Application of BT

We take $f = 1$ as a solution to Eq. (2), which corresponds to the solution $u = 2(\ln f)_x = 0$ to Eq. (1). Solving BT of Eq. (6), we obtain six linear partial differential equations as

$$\begin{cases}
3g_{xxy} + 12 g_x + \lambda_1 g_y + 12\lambda_8 g_z + \lambda_2 g = 0, \\
g_{xxx} - 4g_t - \lambda_1 g_x - 12\lambda_9 g_z + \lambda_3 g = 0, \\
3g_{xx} + \lambda_5 g_x - \lambda_6 g = 0, \\
g_{xy} + \lambda_7 g_x = 0, \\
g_z + \lambda_8 g_x + \lambda_9 g_z = 0, \\
3g_{xx} + \lambda_4 g_y + \lambda_6 g = 0.
\end{cases} \tag{7}$$

In the following, we will derive two classes of exact solutions to Eq. (1) by solving Eq. (7) with symbolic computation.

2.2.1 *Exponential function solution to Eq. (7)*

Firstly, we consider exponential function solutions to Eq. (7) by taking $g = 1 + \varepsilon e^{\theta}$ with $\theta = kx + ly + mz - wt$, where ε, k, l, m, and w are all constants.

Selecting $\lambda_2 = \lambda_3 = \lambda_6 = 0$, and solving Eq. (7), we have

$$\begin{cases} 3k^2 l + 12k + \lambda_1 l + 12\lambda_8 m = 0, \\ k^3 + 4w - \lambda_1 k - 12\lambda_9 m = 0, \\ 3k^2 + \lambda_5 k = 0, \\ 3k^2 + \lambda_4 l = 0, \\ kl + \lambda_7 k = 0, \\ m + \lambda_8 k + \lambda_9 l = 0. \end{cases} \tag{8}$$

One choice of solutions to Eq. (8) is as follows

$$\left\{ \lambda_1 = \frac{12\lambda_8^2 k + 12\lambda_8\lambda_9 l - 3k^2 l - 12k}{l}, \quad \lambda_4 = -\frac{3k^2}{l}, \right. \\ \lambda_5 = -3k, \quad \lambda_7 = -l, \quad m = -(\lambda_8 k + \lambda_9 l), \\ \left. \omega = \frac{3\lambda_8^2 k^2 - 3\lambda_9^2 l^2 - k^3 l - 3k^2}{l} \right\}, \tag{9}$$

then,

$$u = 2(\ln g)_x \\ = \frac{2k\varepsilon e^{kx+ly-(\lambda_8 k+\lambda_9 l)z - \frac{3\lambda_8^2 k^2 - 3\lambda_9^2 l^2 - k^3 l - 3k^2}{l}t}}{1 + \varepsilon e^{kx+ly-(\lambda_8 k+\lambda_9 l)z - \frac{3\lambda_8^2 k^2 - 3\lambda_9^2 l^2 - k^3 l - 3k^2}{l}t}}, \tag{10}$$

is a solution to Eq. (1).

2.2.2 *First-order polynomial solution to Eq. (7)*

Secondly, we take a first-order polynomial solution into consideration as

$$g = kx + ly + mz - wt, \tag{11}$$

where ε, k, l, m, and w are all constants.

Selecting $\lambda_i = 0$ $(1 \le i \le 7)$ and putting Eq. (11) into (7), we obtain the following algebraic equations

$$\begin{cases} 12k + 12\lambda_8 m = 0, \\ 4w - 12\lambda_9 m = 0, \\ m + \lambda_8 k + \lambda_9 l = 0, \end{cases} \tag{12}$$

which result in $3k^2 - lw - 3m^2 = 0$.

To this stage,

$$u = 2(\ln g)_x \\ = \frac{2k}{kx + ly + mz - wt} \quad \text{with} \quad 3k^2 - lw - 3m^2 = 0, \tag{13}$$

is a solution to Eq. (1).

3 Multiple wave solutions

3.1 One-wave solution

Substituting

$$f = 1 + \varepsilon e^{\theta}, \quad \theta = kx + ly + mz - wt,$$

with ε, k, l, m, and w as constants into Eq. (2), we can obtain

$$w = -k^3 + \frac{3m^2 - 3k^2}{l}, \tag{14}$$

which is called the dispersion relation. As a result, the one-wave solution to Eq. (1) can be written as

$$u = \frac{2\varepsilon k e^{kx+ly+mz-wt}}{1 + \varepsilon e^{kx+ly+mz-wt}}. \tag{15}$$

3.2 Two-wave solution

Substituting

$$f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} \\ + \varepsilon_1\varepsilon_2 a_{12} e^{\theta_1+\theta_2}, \quad \theta_i = k_i x + l_i y + m_i z - w_i t,$$

with $\varepsilon_1, \varepsilon_2, k_i, l_i, m_i$, and w_i $(i = 1, 2)$ as constants into Eq. (2), we get

$$w_i = -k_i^3 + \frac{3m_i^2 - 3k_i^2}{l_i}, a_{12} = \frac{b_{12}}{c_{12}}, \tag{16}$$

where

$$b_{12} = k_1^3 l_1 + k_2^3 l_2 - k_1^3 l_2 - k_2^3 l_1 - 3k_1^2 k_2 l_1 - 3k_1 k_2^2 l_2 + 3k_1^2 k_2 l_2 + 3k_1 k_2^2 l_1 + l_1 w_1 + l_2 w_2 - w_1 l_2 - w_2 l_1 + 3k_1^2 + 3k_2^2 - 6k_1 k_2 - 3m_1^2 - 3m_2^2 + 6m_1 m_2, \tag{17}$$

$$c_{12} = -(k_1 + k_2)^3 (l_1 + l_2) - (l_1 + l_2)(w_1 + w_2) + 3(m_1 + m_2)^2 - 3(k_1 + k_2)^2. \tag{18}$$

As a result, the two-wave solution to Eq. (1) can be written as

$$u = \frac{2\left[k_1 \varepsilon_1 e^{\theta_1} + k_2 \varepsilon_2 e^{\theta_2} + a_{12}(k_1 + k_2)\varepsilon_1 \varepsilon_2 e^{\theta_1 + \theta_2}\right]}{1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + a_{12} \varepsilon_1 \varepsilon_2 e^{\theta_1 + \theta_2}}, \tag{19}$$

where w_i and a_{12} are determined by Eq. (16).

3.3 Three-wave solution

Following the derivation of one- and two-wave solutions, we assume

$$f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2} + \varepsilon_1 \varepsilon_3 a_{13} e^{\theta_1 + \theta_3} + \varepsilon_2 \varepsilon_3 a_{23} e^{\theta_2 + \theta_3} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \tag{20}$$

where $\theta_i = k_i x + l_i y + m_i z - w_i t$, $(i = 1, 2, 3)$, $a_{123} = a_{12} a_{13} a_{23}$, and $\varepsilon_1, \varepsilon_2, \varepsilon_3, k_i, l_i, m_i, w_i$ are constants. With symbolic computation, we get

$$w_i = -k_i^3 + \frac{3m_i^2 - 3k_i^2}{l_i}, \quad a_{ij} = \frac{b_{ij}}{c_{ij}}, \quad (i = 1, 2, 3), \tag{21}$$

where

$$b_{ij} = k_i^3 l_i + k_j^3 l_j - k_i^3 l_j - k_j^3 l_i - 3k_i^2 k_j l_i - 3k_i k_j^2 l_j + 3k_i^2 k_j l_j + 3k_i k_j^2 l_i + l_i w_i + l_j w_j - w_i l_j - w_j l_i + 3k_i^2 + 3k_j^2 - 6k_i k_j - 3m_i^2 - 3m_j^2 + 6m_i m_j,$$

$$c_{ij} = -(k_i + k_j)^3 (l_i + l_j) - (l_i + l_j)(w_i + w_j) + 3(m_i + m_j)^2 - 3(k_i + k_j)^2.$$

Finally, the three-wave solution to Eq. (1) is

$$u = \frac{2\left[\sum_{i=1}^{3} k_i \varepsilon_i e^{\theta_i} + \sum_{1 \le i < j \le 3} (k_i + k_j)\varepsilon_i \varepsilon_j a_{ij} e^{\theta_i + \theta_j} + (k1 + k2 + k3)\varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}\right]}{1 + \sum_{i=1}^{3} \varepsilon_i e^{\theta_i} + \sum_{1 \le i < j \le 3} \varepsilon_i \varepsilon_j a_{ij} e^{\theta_i + \theta_j} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}}, \tag{22}$$

where $\varepsilon_1, \varepsilon_2, k_i, l_i, m_i$ are arbitrary constants, and w_i and a_{ij} are determined by Eq. (21).

4 Lump solutions

In this section, we will search for positive quadratic function solutions to dimensionally reduced bilinear Eq. (2) via taking $y = x$ or $y = z$, correspondingly to construct lump solutions to dimensionally reduced forms of Eq. (1). We begin with the assumption

$$f = g^2 + h^2 + a_9, \tag{23}$$

and

$$g = a_1 x + a_2 z + a_3 t + a_4,$$
$$h = a_5 x + a_6 z + a_7 t + a_8,$$

where a_i $(1 \le i \le 9)$ are all real parameters to be determined. To construct the lump solutions, we note that the conditions guaranteeing the well definedness of f, positiveness of f and localization of u in all directions in the space need to be satisfied.

4.1 Lump solutions to reduction with $y = x$

With $y = x$, the dimensionally reduced form of the bilinear Eq. (2) turns out to be

$$\left(D_t\, D_x - D_x^4 - 3D_x^2 + 3\, D_z^2\right) f \cdot f = 0, \tag{24}$$

that is,

$$(f_{xt} f - f_t f_x) - \left(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2\right) - 3\left(f_{xx} f - f_x^2\right) + 3\left(f_{zz} f - f_z^2\right) = 0, \tag{25}$$

which is transformed into

$$u_{xt} - u_{xxxx} - 6\, u_x u_{xx} - 3\, u_{xx} + 3\, u_{zz} = 0. \tag{26}$$

through the link between f and u:

$$u = 2\Big[\ln f(x,z,t)\Big]_x = 2\frac{f_x(x,z,t)}{f(x,z,t)}. \tag{27}$$

Submitting Eq. (23) into (25), we obtain the following set of constraining equations for the parameters:

$$\Big\{ a_1 = a_1, \quad a_2 = a_2, \\ a_3 = \frac{3a_1\left(a_1^2 - a_2^2 + a_5^2 + a_6^2\right) - 6a_2 a_5 a_6}{a_1^2 + a_5^2}, \\ a_4 = a_4, \quad a_5 = a_5, \quad a_6 = a_6, \\ a_7 = \frac{3a_5\left(a_1^2 + a_2^2 + a_5^2 - a_6^2\right) - 6a_1 a_2 a_6}{a_1^2 + a_5^2}, \\ a_8 = a_8, \quad a_9 = \frac{\left(a_1^2 + a_5^2\right)^3}{(a_1 a_6 - a_2 a_5)^2} \Big\}, \tag{28}$$

which needs to satisfy $|a_6| > |a_5|$ and $a_1 a_6 - a_2 a_5 \neq 0$. The positive quadratic function solution to Eq. (25) is

$$f = \left(a_1 x + a_2 z + \frac{3a_1\left(a_1^2 - a_2^2 + a_5^2 + a_6^2\right) - 6a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4\right)^2 + \left(a_5 x + a_6 z + \frac{3a_5\left(a_1^2 + a_2^2 + a_5^2 - a_6^2\right) - 6a_1 a_2 a_6}{a_1^2 + a_5^2} t + a_8\right)^2 + \frac{\left(a_1^2 + a_5^2\right)^3}{(a_1 a_6 - a_2 a_5)^2}, \tag{29}$$

which, in turn, generates a class of lump solutions to dimensionally reduced Eq. (2) through the transformation $u = 2\,(\ln f)_x$ as

$$u^{(\mathrm{I})} = \frac{4(a_1 g + a_5 h)}{f}, \tag{30}$$

where the function f is defined by Eq. (29), and the functions g and h are given as follows:

$$g = a_1 x + a_2 z + \frac{3a_1\left(a_1^2 - a_2^2 + a_5^2 + a_6^2\right) - 6a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4,$$

$$h = a_5 x + a_6 z - \frac{3a_5\left(a_1^2 + a_2^2 + a_5^2 - a_6^2\right) - 6a_1 a_2 a_6}{a_1^2 + a_5^2} t + a_8.$$

4.2 Lump solutions to reduction with $y = z$

With $y = z$, the dimensionally reduced form of the bilinear Eq. (2) turns out to be

$$\left(D_t D_z - D_x^3 D_z - 3D_x^2 + 3D_z^2\right) f \cdot f = 0, \tag{31}$$

that is,

$$(f_{tz} f - f_t f_z) - (f_{xxxz} f - f_{xxx} f_z - 3 f_{xxz} f_x + 3 f_{xx} f_{xz}) - 3\left(f_{xx} f - f_x^2\right) + 3\left(f_{zz} f - f_z^2\right) = 0, \tag{32}$$

which is transformed into

$$u_{zt} - u_{xxxz} - 3\,(u_x u_z)_x - 3\, u_{xx} + 3\, u_{zz} = 0, \tag{33}$$

through the link of Eq. (27) between f and u.

Substituting $f = g^2 + h^2 + a_9$ into Eq. (32), we obtain the following set of constraining equations for the parameters:

$$\Big\{ a_1 = a_1, \quad a_2 = a_2, \\ a_3 = \frac{3a_2\left(a_1^2 - a_2^2 - a_5^2 - a_6^2\right) + 6a_1 a_5 a_6}{a_2^2 + a_6^2}, \\ a_4 = a_4, \quad a_5 = a_5, \quad a_6 = a_6, \\ a_7 = \frac{-3a_6\left(a_1^2 + a_2^2 - a_5^2 + a_6^2\right) + 6a_1 a_2 a_5}{a_2^2 + a_6^2}, \quad a_8 = a_8, \\ a_9 = -\frac{\left(a_2^2 + a_6^2\right)\left(a_1^2 + a_5^2\right)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2} \Big\}, \tag{34}$$

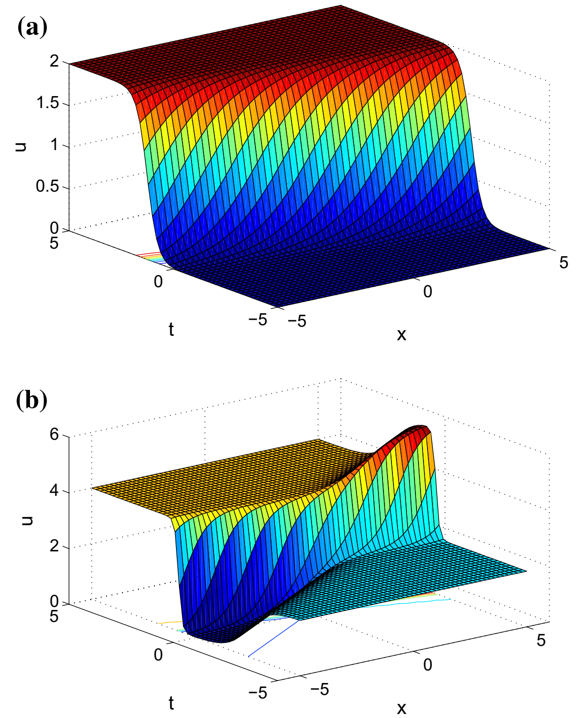

Fig. 1 **a** Characteristics of the one-wave solution via Eq. (15) with $\varepsilon = 1$, $k = 1$, $l = 1$, $m = 0$, $w = -4$, and $y = z = 0$; **b** characteristics of the two-wave solution via Eq. (19) with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $k_1 = 1$, $k_2 = 2$, $l_1 = 3$, $l_2 = 3$, $m_1 = 2$, $m_2 = 0$, $w_1 = 2$, $w_2 = -12$, and $y = z = 0$

which needs to satisfy $a_1a_6 - a_2a_5 \neq 0$ and $a_1a_2 + a_5a_6 < 0$. The positive quadratic function solution to Eq. (32) is

$$f = \left(a_1x + a_2z + \frac{3a_2\left(a_1^2 - a_2^2 - a_5^2 - a_6^2\right) + 6a_1a_5a_6}{a_2^2 + a_6^2}t + a_4\right)^2 + \left(a_5x + a_6z + \frac{-3a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) + 6a_1a_2a_5}{a_2^2 + a_6^2}t + a_8\right)^2 - \frac{\left(a_2^2 + a_6^2\right)\left(a_1^2 + a_5^2\right)(a_1a_2 + a_5a_6)}{(a_1a_6 - a_2a_5)^2}, \tag{35}$$

which, in turn, generates a class of lump solutions to dimensionally reduced Eq. (2) through the transformation $u = 2\,(\ln f)_x$ as

$$u^{(\mathrm{II})} = \frac{4(a_1g + a_5h)}{f}, \tag{36}$$

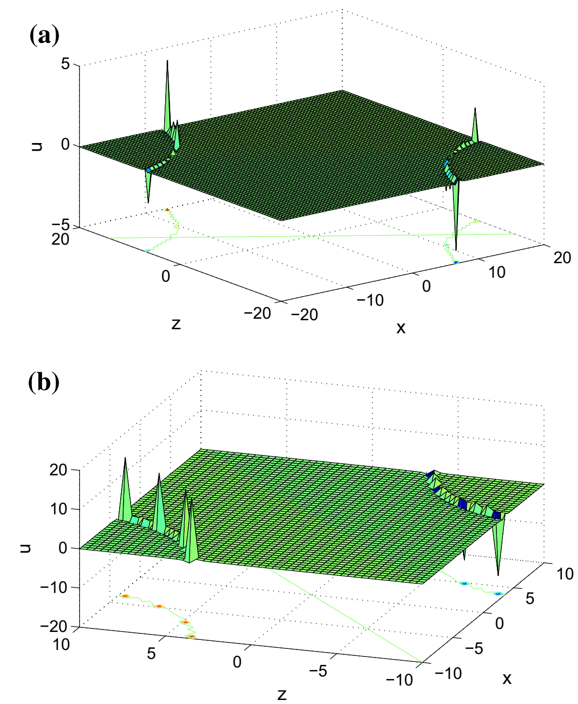

Fig. 2 **a** Characteristics of lump solution $u^{(\mathrm{I})}$ via Eq. (30) with $a_1 = 1$, $a_2 = 2$, $a_4 = 0$, $a_5 = 3$, $a_6 = 4$, $a_8 = 0$, and $t = 0$; **b** characteristics of lump solution $u^{(\mathrm{II})}$ via Eq. (36) with $a_1 = 1$, $a_2 = 2$, $a_4 = 0$, $a_5 = -3$, $a_6 = 4$, $a_8 = 0$, and $t = 0$

where the function f is defined by Eq. (35), and the functions g and h are shown as follows:

$$g = a_1x + a_2z + \frac{3a_2\left(a_1^2 - a_2^2 - a_5^2 - a_6^2\right) + 6a_1a_5a_6}{a_2^2 + a_6^2}t + a_4,$$

$$h = a_5x + a_6z + \frac{-3a_6\left(a_1^2 + a_2^2 - a_5^2 + a_6^2\right) + 6a_1a_2a_5}{a_2^2 + a_6^2}t + a_8.$$

5 Discussions and conclusion

High-dimensional problems in soliton theory attract much more attention in recent research. For example, by using multiple exp-function method and symbolic computation, one-wave, two-wave, and three-wave solutions have been presented to (3 + 1)-dimensional generalized KP and BKP equations [4,13,14]. Resonant behavior of multiple wave solutions and lump

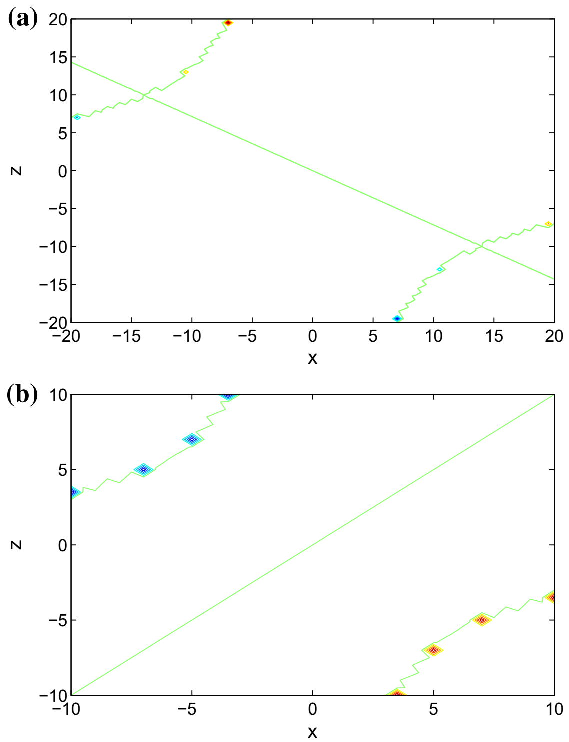

Fig. 3 **a** Contour of lump solution $u^{(\mathrm{I})}$; **b** contour of lump solution $u^{(\mathrm{II})}$

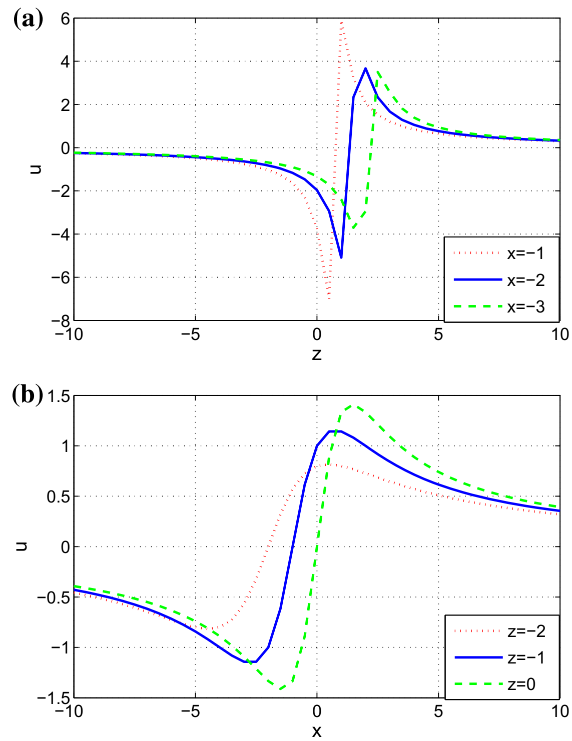

Fig. 4 **a** Plot of x curves of lump solution $u^{(\mathrm{I})}$ with $x = -1$, $x = -2$, and $x = -3$; **b** plot of z curves of lump solution $u^{(\mathrm{II})}$ with $z = 0$, $z = -1$, and $z = -2$

dynamics has been studied for a (3 + 1)-dimensional NLEE [7,19].

In this paper, we have firstly transformed the (3+1)-dimensional nonlinear partial differential equation, that is, Eq. (1) into Hirota bilinear form by a dependent transformation. Then, a bilinear BT has been constructed [see Eq. (6)], which consists of six bilinear equations and includes nine arbitrary parameters. As an application, we have derived two classes of exact solutions [see Eqs. (10) and (13)] to Eq. (1) by using this BT. Moreover, with multiple exp-function method and symbolic computation, nonresonant-typed multiple wave solutions have been given to Eq. (1) including one-wave, two-wave, and three-wave solutions [see Eqs. (15), (19), and (22)]. Characteristics of the one-wave and two-wave solutions are shown in Fig. 1.

Finally, two classes of lump solutions have been investigated to the dimensionally reduced forms of Eq. (1) with $y = x$ and $y = z$, respectively, i.e., Eqs. (26) and (33). We found no lump solution in the form of $f = g^2 + h^2 + a_9$ to reduced Eq. (1) via taking $y = t$. To reveal the lump dynamics, 3-dimensional plots, density plots and 2-dimensional curves with particular choices of the involved parameters in the potential function u are given in Figs. 2, 3, and 4, respectively.

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