Nonlinear Analysis 71 (2009) e711–e724

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Travelling wave solutions to Zufiria's higher-order Boussinesq type equations

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ARTICLE INFO

Keywords:
Zufiria's higher-order Boussinesq type equations
Periodic wave
Solitary wave
Periodic-like wave
Soliton-like wave
Jacobi elliptic function periodic wave
Weierstrass elliptic function periodic
Rational solutions

ABSTRACT

Zufiria's higher-order Boussinesq type equations are studied by transforming them into solvable ordinary differential equations. Various families of their travelling wave solutions are generated, which include periodic wave, solitary wave, periodic-like wave, soliton-like wave, Jacobi elliptic function periodic wave, combined non-degenerative Jacobi elliptic function-like wave, Weierstrass elliptic function periodic and rational solutions. The presented approach can be also applied to nonlinear wave equations with variable coefficients in mathematical physics and mechanics.

1. Introduction

It is well-known that nonlinear evolution equations (NEEs) play an important role in describing many significant dynamical behaviors in mechanics, physics, optics, biology, chemistry etc., and the investigation of the exact solutions to NEEs can undoubtedly help us further analyze and understand their various dynamical properties. For constructing exact travelling wave solutions, many powerful and efficient methods have been established and developed based on the traditional solving methods [1–8], such as the homogeneous balance method [9], tanh-function expansion [10–12], Jacobi elliptic function expansion [13–16], method of bifurcation [17], F-expansion method [18–22], ADM method [23], auxiliary ordinary differential equation method [24–26], variable separation approach [27], Wronskian and Casorati techniques [28] etc. Fan et al. [29] found a series of travelling wave solutions for some nonlinear evolution equations by applying a direct approach with computerized symbolic computations. Yan et al. [30–33] improved this algebraic method such that it can be used to seek more types of solutions. More recently, Yomba [34] proposed an extended Fan's sub-equation method and obtained many general solutions of some nonlinear wave equations.

Although new solutions to the general elliptic equation were given (see [34]) through the connection between the general elliptic equation and the other existing sub-equations like the Riccati equation, the first kind elliptic equation, the auxiliary ordinary differential equation, the generalized Riccati equation etc., the solutions of the elliptic equation which Yomba built can not include all the corresponding ones in [29]. In other words, Yomba's work is complementary to Fan's. Therefore, more diverse solutions can be constructed by synthesizing the solutions of the elliptic equation in [29,34].

In this paper, we consider seeking more new explicit and exact travelling wave solutions to the following Zufiria's higher-order Boussinesq type equations:

$$h_t + (hu)_x + \frac{1}{3}u_{xxx} + \frac{2}{15}u_{xxxxx} = 0, \qquad u_t + h_x + uu_x + \alpha u_{xxxx} = 0, \tag{1}$$

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doi:10.1016/j.na.2008.11.069

where α is a constant, and $u(x,t)$ and $h(x,t)$ represent the depth-averaged horizontal velocity and the elevation of the free surface, respectively. Eqs. (1) were firstly proposed in [35] by setting $\alpha = 0$. By using numerical methods, some periodic wave solutions of Eqs. (1) were obtained in order to confirm that non-symmetric periodic waves can appear via spontaneous symmetry-breaking bifurcations from the symmetric waves [35]. Some solitary and periodic waves of Eqs. (1) with $\alpha = 0$ were already presented by applying a simple ansatze and an auxiliary equation, and it was shown that Eqs. (1) had a nice property which led to a global energy type argument for linear stability of the solitary waves when $\alpha = 0$ (see [36]).

This paper is organized as follows: In the next section, many new families of explicit and exact solutions to Zufiria's higher-order Boussinesq type Equation (1) with $\alpha = 0$ are successfully constructed via a further improved Fan's sub-equation method, and some types of solutions are also obtained in the limiting cases. In Section 3, various examples of travelling wave solutions to Eqs. (1) with $\alpha = 1$ are presented. The case with $\alpha \neq 0$ and $\alpha \neq 1$ are shortly discussed in Section 4. Finally, in Section 5, some conclusions are given.

2. New diverse wave solutions to Eqs. (1) with $\alpha = 0$

Let us firstly consider travelling wave solutions to Eqs. (1) with $\alpha = 0$. Making the following travelling wave transformation

$$u(x,t) = u(\zeta), \qquad h(x,t) = h(\zeta), \quad \zeta = x + \lambda t, \tag{2}$$

where the wave velocity λ is a constant to be determined. Substituting (2) into Eqs. (1) and integrating them once read

$$\lambda u + h + \frac{1}{2}u^2 = l_1, \tag{3a}$$

$$\lambda h + hu + \frac{1}{3}u'' + \frac{2}{15}u^{(4)} = l_2, \tag{3b}$$

where l_1 and l_2 are integration constants. Substituting (3a) into (3b)) yields

$$\frac{2}{15}u^{(4)} + \frac{1}{3}u'' + (\lambda + u)\left(l_1 - \lambda u - \frac{1}{2}u^2\right) - l_2 = 0. \tag{4}$$

Compared with the ansatze (7) in [35], a more general formal solution to Eqs. (1) is introduced as follows

$$u(x,t) = \sum_{j=0}^{2} a_j\phi^j + \sum_{j=1}^{2} \phi^{-j}[b_j + \phi'(c_j\phi^{2j-1} + d_j)], \tag{5}$$

where $a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1$, and d_2 are undetermined constants, and $\phi = \phi(\zeta)$ satisfies

$$\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \varepsilon\left(\sum_{k=0}^{4}\lambda_k\phi^i\right)^{1/2}, \tag{6}$$

where $\varepsilon = \pm 1$, and λ_k $(k = 0, 1, 2, 3, 4)$ are constants to be determined later. With the aid of *Maple*, substituting (5) into Eq. (4) along with (6) and setting the coefficients of $\phi^{w_1}\left(\sum_{k=0}^{s}\lambda_i\phi^i\right)^{w_2/2}$ $(w_1 = 0, \pm 1, \pm 2, \ldots; w_2 = 0, 1)$ to zero yield a system of nonlinear algebraic equations with respect to $a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, \lambda, \lambda_0, \lambda_1, \lambda_2, \lambda_3$ and λ_4. Then, explicit and exact wave solutions can be constructed through our ansatze (5) via the associated solutions of Eq. (6).

In the process of constructing exact solutions to NEEs, Eq. (6) is often viewed as a key auxiliary equation, and the types of its solutions determine the solutions for the original NEEs indirectly. In order to seek more new solutions to Eqs. (1), we here combine the solutions to Eq. (6) which were listed in [29,34]. Our computation results show that this combination is an efficient way to obtain more diverse families of explicit exact solutions.

The solutions of the corresponding algebraic system of equations are listed below:

Case 1

$$\lambda_1 = \lambda_3 = a_1 = b_1 = b_2 = c_1 = c_2 = 0, \qquad a_0 = \pm\frac{\sqrt{2}}{6}(1 + 8\lambda_2) - \lambda, \qquad a_2 = \pm 4\sqrt{2}\lambda_4,$$
$$l_1 = \frac{16}{5}\lambda_2^2 - \frac{1}{2}\lambda^2 \mp \frac{6\sqrt{2}}{5}a_2\lambda_0 + \frac{1}{12}, \qquad l_2 = \frac{4}{15}a_2\lambda_0(1 - 8\lambda_2) \pm \frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2 - 1\right) \pm \frac{\sqrt{2}}{108}.$$

In this case, λ_0, λ_2 and λ_4 are arbitrary constants, then ϕ is one of the 16 ϕ_l^{IV} $(l = 1, 2, \ldots, 16)$.

Case 2

$$\lambda_0=\lambda_4=a_2=b_2=c_1=c_2=0,\qquad a_0=\pm\frac{\sqrt{2}}{3}\left(\lambda_2+\frac{1}{2}\right)\mp\lambda,\qquad a_1=\pm\sqrt{2}\lambda_3,\qquad b_1=\pm\sqrt{2}\lambda_1,$$
$$l_1=\frac{6}{5}a_1b_1+\frac{2}{5}\left(\frac{1}{3}+\lambda^2\right)\mp3(a_0+\lambda)\left(\frac{\sqrt{2}}{10}+\frac{3}{5}a_0\right),$$
$$\begin{aligned}l_2 = {} & \frac{2}{5}a_1b_1\left[\pm\frac{\sqrt{2}}{3}-4(a_0+\lambda)\right]+\frac{a_0}{5}\left[\frac{2}{3}\mp3\sqrt{2}\left(\lambda+\frac{1}{2}a_0\right)+2a_0^2\right]+\frac{6}{5}\lambda a_0(\lambda+a_0)\\ & +\frac{\lambda}{5}\left[\frac{2}{3}\mp\lambda\left(\frac{3\sqrt{2}}{2}+2\lambda\right)\right],\end{aligned}$$

where λ_1, λ_2 and λ_3 are arbitrary constants.

Case 3

$$\lambda_0=\lambda_4=a_2=b_2=c_1=c_2=0,\qquad a_0=-\lambda,\qquad a_1=\pm\sqrt{2}\lambda_3,\qquad b_1=\pm\sqrt{2}\lambda_1,\qquad \lambda_2=-\frac{1}{2},$$
$$l_1=\frac{3}{2}a_1b_1\left(1\pm\frac{1}{5}\right)+\frac{2}{15}-\frac{1}{2}\lambda^2,\qquad l_2=\pm\sqrt{2}\left(\frac{1}{15}a_1b_1+\frac{1}{2}\right),$$

where λ_1 and λ_3 are arbitrary constants.

Case 4

$$\lambda_0=a_1=a_2=b_2=c_1=c_2=0,\qquad a_0=\pm\frac{\sqrt{2}}{3}\left(\lambda_2+\frac{1}{2}\right)\mp\lambda,\qquad b_1=\pm\sqrt{2}\lambda_1,$$
$$l_1=\frac{2}{15}\mp\frac{3\sqrt{2}}{10}(\lambda+a_0+b_1\lambda_3)+\frac{9}{5}a_0\left(\frac{1}{2}a_0+\lambda\right)+\frac{2}{5}\lambda^2,$$
$$\begin{aligned}l_2 = {} & \frac{2}{15}(\lambda+a_0+b_1\lambda_3)\pm\frac{\sqrt{2}}{5}b_1[b_1\lambda_4-\lambda_3(\lambda+a_0)]\mp\frac{3\sqrt{2}}{5}\left[a_0\left(\lambda+\frac{1}{2}a_0\right)+\frac{1}{2}\lambda^2\right]\\ & +\frac{6}{5}\lambda a_0(\lambda+a_0)+\frac{2}{5}(\lambda^3+a_0^3),\end{aligned}$$

where λ_1, λ_2, λ_3 and λ_4 are arbitrary constants.

Case 5

$$\lambda_4=a_2=b_1=b_2=c_1=c_2=0,\qquad a_0=\pm\frac{\sqrt{2}}{3}\left(\lambda_2+\frac{1}{2}\right)\mp\lambda,\qquad a_1=\pm\sqrt{2}\lambda_3,$$
$$l_1=\frac{2}{15}\mp\frac{3\sqrt{2}}{10}(\lambda+a_0+\lambda_1a_1)+\frac{9}{5}a_0\left(\frac{1}{2}a_0+\lambda\right)+\frac{2}{5}\lambda^2,$$
$$\begin{aligned}l_2 = {} & \frac{2}{15}(\lambda+a_0+a_1\lambda_1)\pm\frac{\sqrt{2}}{5}a_1[a_1\lambda_0-\lambda_1(\lambda+a_0)]\mp\frac{3\sqrt{2}}{5}\left[a_0\left(\lambda+\frac{1}{2}a_0\right)+\frac{1}{2}\lambda^2\right]\\ & +\frac{6}{5}\lambda a_0(\lambda+a_0)+\frac{2}{5}(\lambda^3+a_0^3),\end{aligned}$$

where λ_0, λ_1, λ_2 and λ_3 are arbitrary constants.

Case 6

$$\lambda_1=a_1=a_2=b_1=b_2=c_2=l_2=0,\qquad a_0=-\lambda,\qquad c_1=\pm4\sqrt{2\lambda_4},\qquad \lambda_2=-\frac{1}{2},\qquad \lambda_3=\pm\frac{1}{4}\mathrm{i},$$
$$l_1=\frac{9}{20}c_1^2\lambda_0+\frac{2}{15}-\frac{\lambda^2}{2},$$

where λ_0 and λ_4 are arbitrary constants.

Case 7

$$a_1=a_2=b_1=b_2=c_2=l_2=0,\qquad a_0=-\lambda,\qquad c_1=\pm4\sqrt{2\lambda_4},\qquad \lambda_1=\lambda_3(\lambda_3^2+2\lambda_4)/(16\lambda_4^2),$$
$$\lambda_2=(3\lambda_3^2+2\lambda_4)/(8\lambda_4),$$
$$l_1=[\lambda_3^2(27+109\lambda_4)+4\lambda_4^2(60\lambda^2-1728\lambda_4\lambda_0+11)]/(480\lambda_4^2),$$

where λ_0, λ_3 and λ_4 are arbitrary constants.

(1) If $\lambda_0 = r^2$, $\lambda_1 = 2rp$, $\lambda_2 = 2rq + p^2$, $\lambda_3 = 2pq$, $\lambda_4 = q^2$, then ϕ is one of the $24\phi_l^I$ $(l = 1, 2, \ldots, 24)$.
(2) If $\lambda_3 = 0$, then $\lambda_1 = 0$, $\lambda_2 = \frac{1}{4}$.

Case 8

$$\lambda_0 = \lambda_1 = b_1 = b_2 = c_2 = 0, \qquad a_0 = \pm\frac{\sqrt{2}}{6}(1 + 2\lambda_2) - \lambda, \qquad a_1 = \pm\sqrt{2}\lambda_3, \qquad a_2 = \pm 2\sqrt{2}\lambda_4,$$
$$c_1 = \pm 2\sqrt{2\lambda_4},$$
$$l_1 = -\frac{1}{2}\lambda^2 + \frac{1}{5}\lambda_2^2 + \frac{1}{12}, \qquad l_2 = \pm\frac{\sqrt{2}}{540}(1 + 2\lambda_2)(8\lambda_2^2 + 5 - 10\lambda_2).$$

In this case, λ_2, λ_3 and λ_4 are arbitrary constants, then ϕ is one of the 10 ϕ_l^{III} $(l = 1, 2, \ldots, 10)$.

Case 9

$$\lambda_1 = b_1 = b_2 = c_2 = 0, \qquad a_0 = \pm\sqrt{2}(1 + 2\lambda_2)/6 - \lambda, \qquad a_1 = \pm\sqrt{2}\lambda_3, \qquad a_2 = \pm 2\sqrt{2}\lambda_4,$$
$$c_1 = \pm 2\sqrt{2\lambda_4},$$
$$l_1 = 3c_1^2\lambda_0/10 - \lambda^2/2 + \lambda_2^2/5 + 1/12, \qquad l_2 = \mp\sqrt{2}\{18\lambda_0[c_1^2(1 + 4\lambda_2) - 6a_1^2] + 4\lambda_2^2(3 - 4\lambda_2) - 5\}/540,$$

where λ_0, λ_2, λ_3 and λ_4 are arbitrary constants. If $\lambda_3 = 0$, then ϕ is one of the 16 ϕ_l^{IV} $(l = 1, 2, \ldots, 16)$.

Case 10

$$\lambda_0 = \lambda_3 = a_1 = b_1 = b_2 = c_2 = 0, \qquad a_0 = \pm\sqrt{2}(1 + 2\lambda_2)/6 \mp \lambda, \qquad a_2 = \pm 2\sqrt{2}\lambda_4, \qquad c_1 = \pm 2\sqrt{2\lambda_4},$$
$$l_1 = \frac{3}{10}[3a_0(a_0 + 2\lambda) \mp \sqrt{2}(\lambda + a_0)] + \frac{\lambda}{5}(9 + 2\lambda) + \frac{2}{15},$$
$$l_2 = \pm\frac{\sqrt{2}}{20}c_1^2\lambda_1^2 + (a_0 + \lambda)\left\{\frac{2}{5}\left[(a_0 + \lambda)^2 + \frac{1}{3}\right] \mp \frac{3\sqrt{2}}{10}(a_0 + \lambda)\right\}^2,$$

where λ_1, λ_2 and λ_4 are arbitrary constants.

Case 11

$$\lambda_1 = \lambda_2 = \lambda_4 = a_2 = b_1 = c_1 = c_2 = 0, \qquad a_0 = \pm\frac{\sqrt{2}}{6} - \lambda, \qquad a_1 = \pm\sqrt{2}\lambda_3, \qquad b_2 = \pm 4\sqrt{2}\lambda_0,$$
$$l_1 = \frac{1}{12} - \frac{1}{2}\lambda^2, \qquad l_2 = -\frac{27}{20}a_1^2 b_2 \pm \frac{\sqrt{2}}{108},$$

where λ_0 and λ_3 are arbitrary constants.

Case 12

$$\lambda_1 = \lambda_3 = a_1 = b_1 = c_1 = c_2 = 0, \qquad a_0 = \pm\frac{4\sqrt{2}}{3}\lambda_2 \pm \frac{\sqrt{2}}{6} - \lambda, \qquad a_2 = \pm 4\sqrt{2}\lambda_4, \qquad b_2 = \pm 4\sqrt{2}\lambda_0,$$
$$l_1 = \frac{1}{12} - \frac{1}{2}\lambda^2 + \frac{6}{5}a_2 b_2 + \frac{16}{5}\lambda_2^2, \qquad l_2 = \pm\frac{\sqrt{2}}{540}[64\lambda_2^2(16\lambda_2 - 3) + 5 - 72a_2 b_2(1 + 16\lambda_2)],$$

where λ_0, λ_2 and λ_4 are arbitrary constants. In this case, ϕ is one of the 16 ϕ_l^{IV} $(l = 1, 2, \ldots, 16)$.

Case 13

$$\lambda_1 = \lambda_3 = a_1 = b_1 = c_1 = c_2 = 0, \qquad a_0 = -\lambda, \qquad a_2 = \pm 4\sqrt{2}\lambda_4, \qquad b_2 = \pm 4\sqrt{2}\lambda_0, \qquad \lambda_2 = -\frac{1}{8},$$
$$l_1 = \frac{2}{15} - \frac{1}{2}\lambda^2 + \frac{3}{10}a_2 b_2(5 \mp 1), \qquad l_2 = \pm\sqrt{2}\left(\frac{1}{15}a_2 b_2 + \frac{1}{2}\right),$$

where λ_0 and λ_4 are arbitrary constants and ϕ is one of the 16 ϕ_l^{IV} $(l = 1, 2, 3, 7, 10, 12, 13)$.

Using the results above, we can obtain the following 7 families of travelling wave solutions through careful calculations.

Family 1: Periodic wave solutions

$$u_{11} = \mp 7\sqrt{2}/6 - \lambda \pm 4\sqrt{2}\csc^2\zeta,$$
$$\lambda_1 = \lambda_3 = 0, \qquad \zeta = x + \lambda t, \qquad l_1 = \frac{16}{5}\lambda_2^2 - \frac{1}{2}\lambda^2 \mp \frac{6\sqrt{2}}{5}a_2\lambda_0 + \frac{1}{12},$$
$$l_2 = \frac{4}{15}a_2\lambda_0(1 - 8\lambda_2) \pm \frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2 - 1\right) \pm \frac{\sqrt{2}}{108};$$

$$u_{12}=\pm\sqrt{2}(\lambda_2+1/2)/3\mp\lambda\mp\sqrt{2}\lambda_2\sec^2(\sqrt{-\lambda_2}\zeta/2),\qquad \lambda_1=0,\qquad \lambda_2<0,\qquad \lambda_3\neq 0,$$
$$u_{13}=\pm\sqrt{2}(\lambda_2+1/2)/3\mp\lambda\pm 2\sqrt{2}\lambda_2/[\pm\sin(\sqrt{-\lambda_2}\zeta)-1],\qquad \lambda_1\neq 0,\qquad \lambda_2<0,\qquad \lambda_3=0,$$
$$\lambda_0=\lambda_4=0,\qquad \zeta=x+\lambda t,\qquad l_1=\frac{6}{5}a_1b_1+\frac{2}{5}\left(\frac{1}{3}+\lambda^2\right)\mp 3(a_0+\lambda)\left(\frac{\sqrt{2}}{10}+\frac{3}{5}a_0\right),$$
$$\begin{aligned}l_2 = {} & \frac{2}{5}a_1b_1\left[\pm\frac{\sqrt{2}}{3}-4(a_0+\lambda)\right]+\frac{a_0}{5}\left[\frac{2}{3}\mp 3\sqrt{2}\left(\lambda+\frac{1}{2}a_0\right)+2a_0^2\right]+\frac{6}{5}\lambda a_0(\lambda+a_0)\\ & +\frac{\lambda}{5}\left[\frac{2}{3}\mp\lambda\left(\frac{3\sqrt{2}}{2}+2\lambda\right)\right];\end{aligned}$$
$$u_{14}=-\lambda\pm[\sqrt{2}\cos^{-2}(\sqrt{2}\zeta/4)]/2,$$
$$\lambda_1=\lambda_3=l_2=0,\qquad \lambda_0=\frac{1}{64\lambda_4},\qquad \lambda_2=\frac{1}{4},\qquad \lambda_4>0,\qquad \zeta=x+\lambda t,\qquad l_1=\frac{15\lambda^2-4}{30}.$$

Family 2: Solitary wave solutions

$$u_{21}=\mp 5\sqrt{2}/2-\lambda\pm 4\sqrt{2}\coth^2\zeta,$$
$$\lambda_1=\lambda_3=0,\qquad \zeta=x+\lambda t,\qquad l_1=\frac{16}{5}\lambda_2^2-\frac{1}{2}\lambda^2\mp\frac{6\sqrt{2}}{5}a_2\lambda_0+\frac{1}{12},$$
$$l_2=\frac{4}{15}a_2\lambda_0(1-8\lambda_2)\pm\frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2-1\right)\pm\frac{\sqrt{2}}{108};$$
$$u_{22}=\pm\sqrt{2}(\lambda_2+1/2)/3\mp\lambda\mp\sqrt{2}\lambda_2\operatorname{sech}^2(\sqrt{\lambda_2}\zeta/2),\qquad \lambda_1=0,\qquad \lambda_2>0,\qquad \lambda_3\neq 0,$$
$$u_{23}=\pm\sqrt{2}(\lambda_2+1/2)/3\mp\lambda\pm 2\sqrt{2}\lambda_2/[\pm\sinh(2\sqrt{\lambda_2}\zeta)-1],\qquad \lambda_1\neq 0,\quad \lambda_2>0,\qquad \lambda_3=0,$$
$$\lambda_0=\lambda_4=0,\qquad \zeta=x+\lambda t,\qquad l_1=\frac{6}{5}a_1b_1+\frac{2}{5}\left(\frac{1}{3}+\lambda^2\right)\mp 3(a_0+\lambda)\left(\frac{\sqrt{2}}{10}+\frac{3}{5}a_0\right),$$
$$\begin{aligned}l_2 = {} & \frac{2}{5}a_1b_1\left[\pm\frac{\sqrt{2}}{3}-4(a_0+\lambda)\right]+\frac{a_0}{5}\left[\frac{2}{3}\mp 3\sqrt{2}\left(\lambda+\frac{1}{2}a_0\right)+2a_0^2\right]+\frac{6}{5}\lambda a_0(\lambda+a_0)\\ & +\frac{\lambda}{5}\left[\frac{2}{3}\mp\lambda\left(\frac{3\sqrt{2}}{2}+2\lambda\right)\right].\end{aligned}$$

Family 3: Periodic-like solutions

$$u_{31}=\pm 5\sqrt{2}/6-\lambda\pm\sqrt{2}(\sec\zeta\pm\tan\zeta)^2,$$
$$\lambda_1=\lambda_3=0,\qquad \zeta=x+\lambda t,\qquad l_1=\frac{16}{5}\lambda_2^2-\frac{1}{2}\lambda^2\mp\frac{6\sqrt{2}}{5}a_2\lambda_0+\frac{1}{12},$$
$$l_2=\frac{4}{15}a_2\lambda_0(1-8\lambda_2)\pm\frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2-1\right)\pm\frac{\sqrt{2}}{108};$$
$$u_{32}=\pm\sqrt{2}(1+2\lambda_2)/6\pm\lambda\pm 2\sqrt{2}\lambda_2\left[\sec^2(\sqrt{-\lambda_2}\zeta)+\lambda_2\left|\sec(\sqrt{-\lambda_2}\zeta)\tan(\sqrt{-\lambda_2}\zeta)\right|\right],$$
$$\lambda_0=\lambda_1=\lambda_3=0,\qquad \lambda_2<0,\qquad \lambda_4>0,\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{3}{10}[3a_0(a_0+2\lambda)\mp\sqrt{2}(\lambda+a_0)]+\frac{\lambda}{5}(9+2\lambda)+\frac{2}{15},$$
$$l_2=\pm\frac{\sqrt{2}}{20}c_1^2\lambda_1^2+(a_0+\lambda)\left\{\frac{2}{5}\left[(a_0+\lambda)^2+\frac{1}{3}\right]\mp\frac{3\sqrt{2}}{10}(a_0+\lambda)\right\}^2;$$
$$u_{33}=-\lambda\pm 4\sqrt{2\lambda_4}\cos^{-2}\left(\frac{\sqrt{2}}{4}\zeta\right)\left|\frac{8\sqrt{\lambda_4}\cos^2\left(\frac{\sqrt{2}}{4}\zeta\right)-4\sqrt{\lambda_4}-\sqrt{2}\sin\left(\frac{\sqrt{2}}{4}\zeta\right)\pm \mathrm{i}\cos\left(\frac{\sqrt{2}}{4}\zeta\right)}{32\lambda_4-32\lambda_4\cos^2\left(\frac{\sqrt{2}}{4}\zeta\right)+4\sqrt{2\lambda_4}\sin\left(\frac{\sqrt{2}}{2}\zeta\right)\pm \mathrm{i}\mp\cos^2\left(\frac{\sqrt{2}}{4}\zeta\right)}\right|,$$
$$\lambda_0=\lambda_1=l_2=0,\qquad \lambda_2=-\frac{1}{2},\qquad \lambda_3=\pm\frac{1}{4}\mathrm{i},\qquad \zeta=x+\lambda t,\qquad l_1=\frac{9}{20}c_1^2\lambda_0+\frac{2}{15}-\frac{\lambda^2}{2};$$

$$u_{34} = -\lambda \pm 8\sqrt{2}\,|q| \left| \left\{ \frac{r\cos(\sqrt{4qr-p^2}\zeta)}{p\cos(\sqrt{4qr-p^2}\zeta) + \sqrt{4qr-p^2}[\sin(\sqrt{4qr-p^2}\zeta) \pm 1]} \right\}' \right|,$$

$$l_2 = 0, \quad \lambda_0 = r^2, \quad \lambda_1 = 2rp, \quad \lambda_2 = 2rq + p^2, \quad \lambda_3 = 2pq, \quad \lambda_4 = q^2, \quad \zeta = x + \lambda t,$$

$$l_1 = \frac{\lambda_3^2(27 + 109\lambda_4) + 4\lambda_4^2(60\lambda^2 - 1728\lambda_4\lambda_0 + 11)}{480\lambda_4^2}.$$

Family 4: Soliton-like solutions

$$u_{41} = \mp\sqrt{2}/2 - \lambda \pm \sqrt{2}(\tanh\zeta \pm \operatorname{sech}\zeta)^2,$$

$$\lambda_1 = \lambda_3 = 0, \quad \zeta = x + \lambda t, \quad l_1 = \frac{16}{5}\lambda_2^2 - \frac{1}{2}\lambda^2 \mp \frac{6\sqrt{2}}{5}a_2\lambda_0 + \frac{1}{12},$$

$$l_2 = \frac{4}{15}a_2\lambda_0(1 - 8\lambda_2) \pm \frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2 - 1\right) \pm \frac{\sqrt{2}}{108};$$

$$u_{42} = -\lambda \mp \frac{\sqrt{2}\lambda_4\,|\sinh(\zeta/2)|}{\cosh^2(\zeta/2)}, \quad \lambda_4 < 0,$$

$$u_{43} = -\lambda \pm \frac{\sqrt{2}\lambda_4\,|\cosh(\zeta/2)|}{\sinh^2(\zeta/2)}, \quad \lambda_4 > 0,$$

$$\lambda_0 = \lambda_1 = \lambda_3 = l_2 = 0, \quad \lambda_2 = \frac{1}{4}, \quad \zeta = x + \lambda t, \quad l_1 = \frac{60\lambda^2 + 11}{120};$$

$$u_{44} = \pm\frac{3\sqrt{2}}{2} - \lambda \pm \frac{4\sqrt{2}(d - 2b)\operatorname{csch}^2\zeta}{b\operatorname{csch}^2\zeta + c\coth\zeta + d} \pm \frac{2\sqrt{2}(c^2 + 4b^2 - 4bd)\operatorname{csch}^4\zeta}{(b\operatorname{csch}^2\zeta + c\coth\zeta + d)^2}$$
$$\pm 2\sqrt{2(c^2 + 4b^2 - 4bd)}\left|\left(\frac{\operatorname{csch}^2\zeta}{b\operatorname{csch}^2\zeta + c\coth\zeta + d}\right)'\right|,$$

$$\lambda_0 = \lambda_1 = 0, \quad \lambda_2 = 4, \quad \lambda_3 = \frac{4(d - 2b)}{a}, \quad \lambda_4 = \frac{c^2 + 4b^2 - 4bd}{a^2}, \quad \zeta = x + \lambda t,$$

$$l_1 = \frac{16}{5} - \frac{1}{2}\lambda^2 + \frac{1}{12}, \quad l_2 = \pm\frac{31\sqrt{2}}{20};$$

$$u_{45} = \pm\frac{\sqrt{2}}{6}(1 + 2a^2) - \lambda \mp 2\sqrt{2}ac\left\{\frac{a}{c + \cosh(a\zeta) - \sinh(a\zeta)} \mp \frac{b^2ac}{b^2[c + \cosh(a\zeta) - \sinh(a\zeta)]^2}\right.$$
$$\left.\pm \left|([c + \cosh(a\zeta) - \sinh(a\zeta)]^{-1})'\right|\right\},$$

$$u_{46} = \pm\frac{\sqrt{2}}{6}(1 + 2a^2) - \lambda \mp \frac{2\sqrt{2}a^2(\cosh(a\zeta) + \sinh(a\zeta))}{[c + \cosh(a\zeta) - \sinh(a\zeta)]} \pm \frac{2\sqrt{2}a^2[\cosh(a\zeta) + \sinh(a\zeta)]^2}{[c + \cosh(a\zeta) - \sinh(a\zeta)]^2}$$
$$\pm 2\sqrt{2}\left|\left\{a[\cosh(a\zeta) + \sinh(a\zeta)][c + \cosh(a\zeta) - \sinh(a\zeta)]^{-1}\right\}'\right|,$$

$$\lambda_0 = \lambda_1 = 0, \quad \lambda_2 = a^2, \quad \lambda_3 = 2ab, \quad \lambda_4 = b^2, \quad \zeta = x + \lambda t, \quad l_1 = \frac{1}{5}a^4 - \frac{1}{2}\lambda^2 + \frac{1}{12},$$

$$l_2 = \pm\frac{\sqrt{2}}{540}(1 + 2a^2)(8a^4 + 5 - 10a^2);$$

$$u_{47} = -\lambda \pm 2\sqrt{2}\left|\cosh(\pm\sqrt{2}\mathrm{i}\zeta/2) \mp \sinh(\pm\sqrt{2}\mathrm{i}\zeta/2)\right|[\cosh(\pm\sqrt{2}\mathrm{i}\zeta/2) \pm 1 \mp \sinh(\pm\sqrt{2}\mathrm{i}\zeta/2)]^{-2},$$

$$\lambda_0 = \lambda_1 = l_2 = 0, \quad \lambda_2 = -\frac{1}{2}, \quad \lambda_3 = \pm\frac{1}{4}\mathrm{i}, \quad l_1 = \frac{2}{15} - \frac{\lambda^2}{2}, \quad \zeta = x + \lambda t;$$

$$u_{48} = -\lambda \pm 8\sqrt{2}\,|q|\left|\left\{\frac{r\sinh(\sqrt{p^2 - 4qr}\zeta)}{\sqrt{p^2 - 4qr}[\cosh(\sqrt{p^2 - 4qr}\zeta) \pm 1] - p\sinh(\sqrt{p^2 - 4qr}\zeta)}\right\}'\right|,$$

$$l_2 = 0, \quad \lambda_0 = r^2, \quad \lambda_1 = 2rp, \quad \lambda_2 = 2rq + p^2, \quad \lambda_3 = 2pq, \quad \lambda_4 = q^2, \quad \zeta = x + \lambda t,$$

$$l_1 = \frac{\lambda_3^2(27 + 109\lambda_4) + 4\lambda_4^2(60\lambda^2 - 1728\lambda_4\lambda_0 + 11)}{480\lambda_4^2}.$$

Family 5: Jacobi elliptic function periodic solutions

$$u_{51} = \pm\sqrt{2}(8m^2-7)/6 - \lambda \pm 4\sqrt{2}m^2 sn^2\zeta,$$
$$u_{52} = \pm\sqrt{2}(16m^2-7)/6 - \lambda \mp 4\sqrt{2}m^2 cn^2\zeta,$$
$$u_{53} = \pm\sqrt{2}(17-8m^2)/6 - \lambda \mp 4\sqrt{2}dn^2\zeta,$$
$$u_{54} = \pm\sqrt{2}(8m^2-7)/6 - \lambda \pm 4\sqrt{2}m^2 cd^2\zeta,$$
$$u_{55} = \pm\sqrt{2}(17-8m^2)/6 - \lambda \pm 4\sqrt{2}(1-m^2) sc^2\zeta,$$
$$u_{56} = \mp\sqrt{2}(8m^2+7)/6 - \lambda \pm 4\sqrt{2}ns^2\zeta,$$
$$u_{57} = \pm\sqrt{2}(16m^2-7)/6 - \lambda \pm 4\sqrt{2}(1-m^2) nc^2\zeta,$$
$$u_{58} = \mp\sqrt{2}(8m^2+7)/6 - \lambda \pm 4\sqrt{2}dc^2\zeta,$$
$$u_{59} = \pm\sqrt{2}(17-8m^2)/6 - \lambda \pm 4\sqrt{2}(m^2-1) nd^2\zeta,$$
$$u_{510} = \pm\sqrt{2}(16m^2-7)/6 - \lambda \mp 4\sqrt{2}m^2(1-m^2) sd^2\zeta,$$
$$u_{511} = \pm\sqrt{2}(16m^2-7)/6 - \lambda \pm 4\sqrt{2}ds^2\zeta,$$
$$u_{512} = \pm\sqrt{2}(17-8m^2)/6 - \lambda \pm 4\sqrt{2}cs^2\zeta,$$
$$\lambda_1 = \lambda_3 = 0, \qquad \zeta = x+\lambda t, \qquad l_1 = \frac{16}{5}\lambda_2^2 - \frac{1}{2}\lambda^2 \mp \frac{6\sqrt{2}}{5}a_2\lambda_0 + \frac{1}{12},$$
$$l_2 = \frac{4}{15}a_2\lambda_0(1-8\lambda_2) \pm \frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2 - 1\right) \pm \frac{\sqrt{2}}{108};$$
$$u_{513} = -\lambda \pm 4\sqrt{2}ds^2\zeta \mp \frac{63\sqrt{2}}{64}ds^{-2}\zeta,$$
$$\lambda_1 = \lambda_3 = 0, \qquad \lambda_0 = -\frac{63}{256}, \qquad \lambda_2 = -\frac{1}{8}, \qquad \lambda_4 = 1, \qquad \zeta = x+\lambda t, \qquad l_1 = \frac{2}{15} - \frac{1}{2}\lambda^2 + \frac{3}{10}a_2 b_2(5\mp 1),$$
$$l_2 = \pm\sqrt{2}\left(\frac{1}{15}a_2 b_2 + \frac{1}{2}\right);$$
$$u_{514} = -\lambda \pm 2\sqrt{\frac{2\lambda_4 m^2}{1-2m^2}}\left|cn'\left(\frac{1}{2}\sqrt{\frac{1}{2m^2-1}}\zeta\right)\right|, \qquad \lambda_0 = \frac{m^2(m^2-1)}{16\lambda_4(1-2m^2)^2}, \qquad \lambda_4 < 0,$$
$$u_{515} = -\lambda \pm 2\sqrt{\frac{2\lambda_4}{m^2-2}}\left|dn'\left(\frac{1}{2}\sqrt{\frac{1}{2-m^2}}\zeta\right)\right|, \qquad \lambda_0 = \frac{1-m^2}{16\lambda_4(2-m^2)^2}, \qquad \lambda_4 < 0,$$
$$\lambda_1 = \lambda_3 = l_2 = 0, \qquad \lambda_2 = \frac{1}{4}, \qquad \zeta = x+\lambda t, \qquad l_1 = \frac{60\lambda^2 - 1728\lambda_4\lambda_0 + 11}{120}.$$

Family 6: Combined non-degenerative Jacobi elliptic function-like solutions

$$u_{61} = \pm\sqrt{2}(5-8m^2)/6 - \lambda \pm \sqrt{2}(ns\zeta \pm cs\zeta)^2,$$
$$u_{62} = \pm\sqrt{2}(5+4m^2)/6 - \lambda \pm \sqrt{2}(1-m^2)(nc\zeta \pm sc\zeta)^2,$$
$$u_{63} = \pm\sqrt{2}(4m^2-7)/6 - \lambda \pm \sqrt{2}(ns\zeta \pm \mathrm{d}s\zeta)^2,$$
$$u_{64} = \pm\sqrt{2}(4m^2-7)/6 - \lambda \pm \sqrt{2}m^2(sn\zeta \pm \mathrm{i}cn\zeta)^2,$$
$$\lambda_1 = \lambda_3 = 0, \qquad \zeta = x+\lambda t, \qquad l_1 = \frac{16}{5}\lambda_2^2 - \frac{1}{2}\lambda^2 \mp \frac{6\sqrt{2}}{5}a_2\lambda_0 + \frac{1}{12},$$
$$l_2 = \frac{4}{15}a_2\lambda_0(1-8\lambda_2) \pm \frac{16\sqrt{2}\lambda_2^2}{45}\left(\frac{16}{3}\lambda_2 - 1\right) \pm \frac{\sqrt{2}}{108};$$
$$u_{65} = \pm\sqrt{2}(m^2-1)/6 - \lambda \pm \sqrt{2}(ns\zeta \pm ds\zeta)\left[\pm\frac{1}{2}(ns\zeta \pm \mathrm{d}s\zeta)\right] \pm \sqrt{2}\left|(ns\zeta \pm ds\zeta)'\right|,$$
$$\lambda_1 = \lambda_3 = 0, \qquad \lambda_0 = \frac{m^2}{4}, \qquad \lambda_2 = \frac{m^2-2}{2}, \qquad \lambda_4 = \frac{1}{4}, \qquad \zeta = x+\lambda t,$$
$$l_1 = \frac{3c_1^2\lambda_0}{10} - \frac{\lambda^2}{2} + \frac{\lambda_2^2}{5} + \frac{1}{12}, \qquad l_2 = \frac{\mp\sqrt{2}\left\{18\lambda_0[c_1^2(1+4\lambda_2) - 6a_1^2] + 4\lambda_2^2(3-4\lambda_2) - 5\right\}}{540};$$

$$\lambda_1=\lambda_3=0,\qquad \lambda_0=\frac{m^2}{4},\qquad \lambda_2=\frac{m^2-2}{2},\qquad \lambda_4=\frac{1}{4},\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{1}{12}-\frac{1}{2}\lambda^2+\frac{6}{5}a_2b_2+\frac{16}{5}\lambda_2^2,$$
$$l_2=\pm\frac{\sqrt{2}}{540}[64\lambda_2^2(16\lambda_2-3)+5-72a_2b_2(1+16\lambda_2)].$$

Family 7: Weierstrass elliptic function solutions

$$u_{71}=\pm\sqrt{2}/6\mp\lambda\pm\sqrt{2}\lambda_3\wp(\sqrt{\lambda_3}\zeta/2,f_2,f_3),$$
$$\lambda_2=\lambda_4=0,\qquad \lambda_3>0,\qquad f_2=-\frac{4\lambda_1}{\lambda_3},\qquad f_3=-\frac{4\lambda_0}{\lambda_3},\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{2}{15}\mp\frac{3\sqrt{2}}{10}(\lambda+a_0+\lambda_1a_1)+\frac{9}{5}a_0\left(\frac{1}{2}a_0+\lambda\right)+\frac{2}{5}\lambda^2,$$
$$l_2=\frac{2}{15}(\lambda+a_0+a_1\lambda_1)\pm\frac{\sqrt{2}}{5}a_1[a_1\lambda_0-\lambda_1(\lambda+a_0)]\mp\frac{3\sqrt{2}}{5}\left[a_0\left(\lambda+\frac{1}{2}a_0\right)+\frac{1}{2}\lambda^2\right]$$
$$+\frac{6}{5}\lambda a_0(\lambda+a_0)+\frac{2}{5}(\lambda^3+a_0^3);$$
$$u_{72}=\pm\sqrt{2}/6\mp\lambda\pm\sqrt{2}\lambda_1\wp^{-1}(\sqrt{\lambda_3}\zeta/2,f_2,0),$$
$$\lambda_0=\lambda_2=\lambda_4=0,\qquad \lambda_3>0,\qquad f_2=-\frac{4\lambda_1}{\lambda_3},\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{2}{15}\mp\frac{3\sqrt{2}}{10}(\lambda+a_0+b_1\lambda_3)+\frac{9}{5}a_0\left(\frac{1}{2}a_0+\lambda\right)+\frac{2}{5}\lambda^2,$$
$$l_2=\frac{2}{15}(\lambda+a_0+b_1\lambda_3)\pm\frac{\sqrt{2}}{5}b_1[b_1\lambda_4-\lambda_3(\lambda+a_0)]\mp\frac{3\sqrt{2}}{5}\left[a_0\left(\lambda+\frac{1}{2}a_0\right)+\frac{1}{2}\lambda^2\right]$$
$$+\frac{6}{5}\lambda a_0(\lambda+a_0)+\frac{2}{5}(\lambda^3+a_0^3);$$
$$u_{73}=\pm\frac{\sqrt{2}}{6}\mp\lambda\pm\sqrt{2}\lambda_3\wp\left(\frac{\sqrt{\lambda_3}\zeta}{2},f_2,0\right)\pm\sqrt{2}\lambda_1\wp^{-1}\left(\frac{\sqrt{\lambda_3}\zeta}{2},f_2,0\right),$$
$$\lambda_0=\lambda_2=\lambda_4=0,\qquad \lambda_3>0,\qquad f_2=-\frac{4\lambda_1}{\lambda_3},\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{6}{5}a_1b_1+\frac{2}{5}\left(\frac{1}{3}+\lambda^2\right)\mp3(a_0+\lambda)\left(\frac{\sqrt{2}}{10}+\frac{3}{5}a_0\right),$$
$$l_2=\frac{2}{5}a_1b_1\left[\pm\frac{\sqrt{2}}{3}-4(a_0+\lambda)\right]+\frac{a_0}{5}\left[\frac{2}{3}\mp3\sqrt{2}\left(\lambda+\frac{1}{2}a_0\right)+2a_0^2\right]+\frac{6}{5}\lambda a_0(\lambda+a_0)$$
$$+\frac{\lambda}{5}\left[\frac{2}{3}\mp\lambda\left(\frac{3\sqrt{2}}{2}+2\lambda\right)\right];$$
$$u_{74}=\pm\sqrt{2}/6-\lambda\pm\sqrt{2}\lambda_3\wp(\sqrt{\lambda_3}\zeta/2,f_2,f_3)\pm4\sqrt{2}\lambda_0\wp^{-2}(\sqrt{\lambda_3}\zeta/2,f_2,f_3),$$
$$\lambda_1=\lambda_2=\lambda_4=0,\qquad \lambda_3>0,\qquad f_2=0,\qquad f_3=-\frac{4\lambda_0}{\lambda_3},\qquad \zeta=x+\lambda t,$$
$$l_1=\frac{1}{12}-\frac{1}{2}\lambda^2,\quad l_2=-\frac{27}{20}a_1^2b_2\pm\frac{\sqrt{2}}{108}.$$

Remark 1. Only the expressions of u are listed and the ones of h are omitted for the limit of length. m $(0\le m\le 1)$ is the modulus of the Jacobi elliptic function. The more detailed notations for the Jacobi and Weierstrass elliptic functions can be found in [37–40].

Remark 2. It is easy to see that the solitary wave solutions and the Jacobi elliptic function periodic solutions are recovered, and u_{22} and u_{52} are correspond to (12) and (13) in [36] under proper transformation. Figs. 1–5 depict some typical graphs of the solutions above, which exhibit some singularities of the solutions.

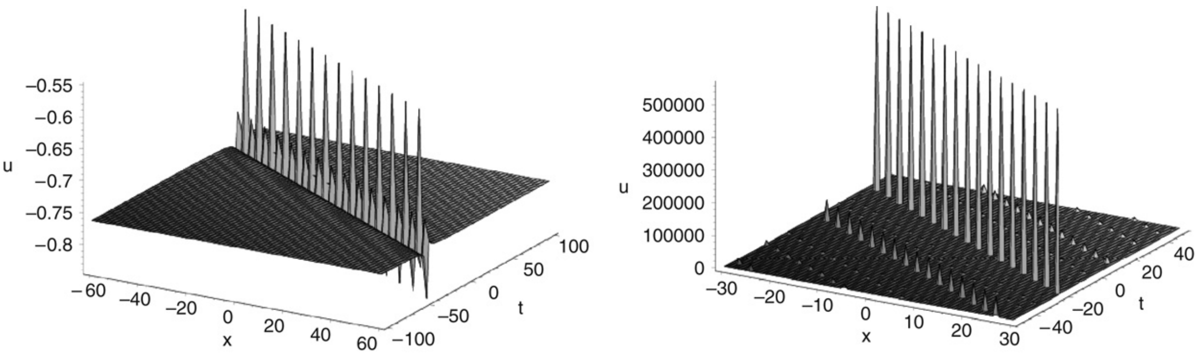

Fig. 1. Solitary wave solution $u_{23} : \lambda_2 = 0.001$, $\lambda = 1$ (left) and periodic-like solution $u_{31} : \lambda = 1$ (right).

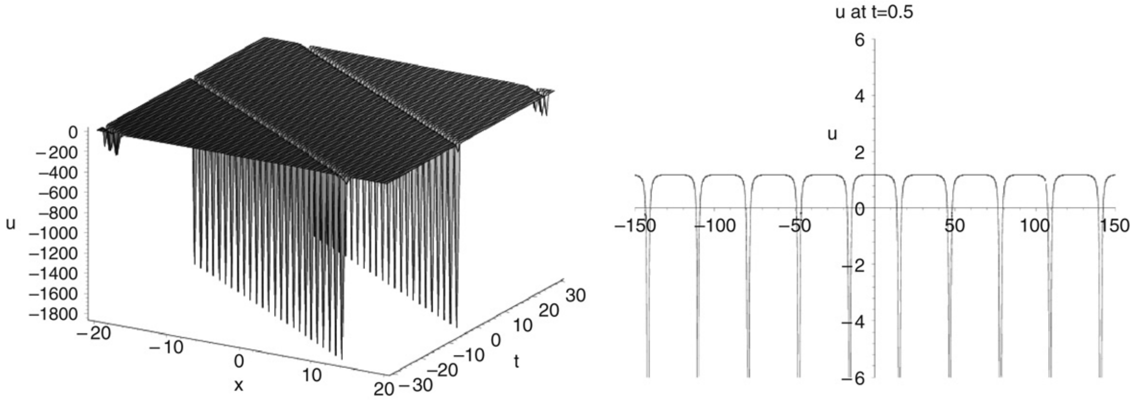

Fig. 2. Periodic-like solution u_{32} with $\lambda_2 = -0.01$, $\lambda = 1$.

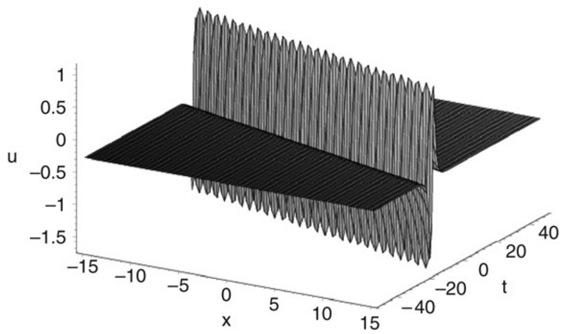

Fig. 3. u_{41} with $\lambda = 1$.

Remark 3. due to the limit of length, we only show some examples of the solutions in Family 3 and Family 4. In fact, more exact solutions can be obtained by applying the relevant results in [34]. Obviously, the solutions obtained by our further improved Fan's sub-equation method are more diverse than those in [36] and a lot of new and different solutions of Eqs. (1) are first reported here.

3. Travelling wave solutions to Eqs. (1) with $\alpha = 1$

By using the travelling wave transformation shown in Section 2, the resulting ordinary differential equations from Eqs. (1) read as (here we set integral constants to be zero so as to simplify our computation)

$$\lambda u + h + \frac{1}{2}u^2 + u''' = 0, \tag{7a}$$

$$\lambda h + hu + \frac{1}{3}u'' + \frac{2}{15}u^{(4)} = 0. \tag{7b}$$

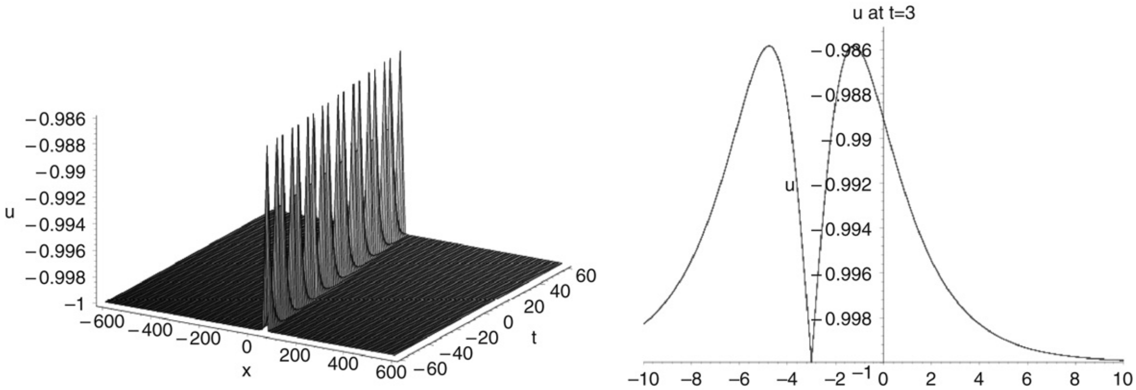

Fig. 4. An example of the soliton-like solution u_{42} with $\lambda = 1$ and $\lambda_4 = -0.02$.

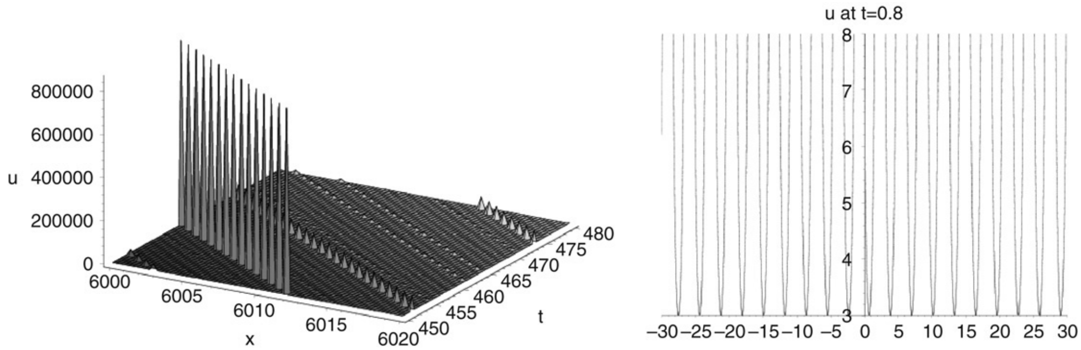

Fig. 5. u_{56} with $\lambda = 1$ and $m = 0.00001$.

The substitution of (7a) into (7b) yields

$$\frac{2}{15}u^{(4)} + \frac{1}{3}u'' + (\lambda + u)\left(-\lambda u - \frac{1}{2}u^2 - u'''\right) = 0. \tag{8}$$

The solutions of the corresponding system of algebraic equations are shown below:

Case 1

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{59}{120}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225}, \qquad a_0 = \pm\frac{2\sqrt{1770}}{225}, \qquad d_1 = \frac{8}{15}.$$

Case 2

$$a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_4 = \frac{-225}{16}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225},$$
$$a_0 = \pm\frac{2\sqrt{1770}}{225}, \qquad a_1 = \mathrm{i}, \qquad d_1 = \frac{4}{15}.$$

Case 3

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225}, \qquad a_0 = \pm\frac{2\sqrt{1770}}{225}, \qquad d_1 = \frac{4}{15}.$$

Case 4

$$a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_4 = \frac{225a_1^2}{16}, \qquad \lambda = \pm\frac{2\sqrt{1770}}{225}, \qquad a_0 = \mp\frac{2\sqrt{1770}}{225},$$
$$d_1 = \frac{4}{15}.$$

Case 5

$$a_1 = a_2 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_1 = \pm\frac{\sqrt{1770}b_1}{4}, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225},$$

$$a_0 = \pm\frac{2\sqrt{1770}}{225}, \qquad d_1 = -\frac{4}{15}.$$

Case 6

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda = \pm\frac{2\sqrt{1770}}{225}, \qquad a_0 = \mp\frac{2\sqrt{1770}}{225}, \qquad d_1 = -\frac{4}{15}.$$

Using the similar technique shown in Section 2, many travelling wave solutions to the generalized Zufiria's higher-order Boussinesq equation can be obtained. Here we just give some intriguing ones, shown as follows:

$$u_1 = \pm\frac{2\sqrt{1770}}{225}\left[1 \pm \tanh\left(\frac{\sqrt{1770}}{60}\zeta\right)\right],$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{59}{120}, \qquad \lambda_4 < 0, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225}; \quad \text{or} \quad \lambda_0 = \lambda_1 = \lambda_4 = 0, \qquad \lambda_2 = \frac{59}{30},$$

$$\lambda = \mp\frac{2\sqrt{1770}}{225}, \lambda_3$$

is an arbitrary constant;

$$u_2 = \pm\frac{2\sqrt{1770}}{225} \pm \frac{2}{225}\mathrm{i}\sqrt{1770}\,\mathrm{sech}\left(\frac{\sqrt{1770}}{30}\zeta\right) \pm \frac{2}{225}\sqrt{1770}\tanh\left(\frac{\sqrt{1770}}{30}\zeta\right),$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_4 = -\frac{225}{16}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225};$$

$$u_3 = \pm\frac{2\sqrt{1770}}{225} \pm \frac{4}{15}\sqrt{\frac{59}{30}}\tanh\left(\frac{\sqrt{1770}}{60}\zeta\right),$$

$$\lambda_0 = \lambda_1 = \lambda_4 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_3 \neq 0, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225};$$

$$u_4 = \pm\frac{2\sqrt{1770}}{225} \pm \frac{2\sqrt{1770}}{225}\sqrt{1 + \frac{\mathrm{sech}^4\left(\frac{\sqrt{1770}}{120}\zeta\right)}{4\tanh^2\left(\frac{\sqrt{1770}}{120}\zeta\right)}},$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{59}{120}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225}, \qquad \lambda_4 \text{ is an arbitrary constant;}$$

$$u_5 = \mp\frac{2\sqrt{1770}}{225} \pm \frac{\mathrm{sech}^2\left(\frac{\sqrt{1770}}{60}\zeta\right)\sqrt{1770}}{225\tanh\left(\frac{\sqrt{1770}}{60}\zeta\right)} \pm \frac{2\sqrt{1770}}{225}\sqrt{1 + \frac{\mathrm{sech}^4\left(\frac{\sqrt{1770}}{60}\zeta\right)}{4\tanh^2\left(\frac{\sqrt{1770}}{60}\zeta\right)}},$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_4 = \frac{225a_1^2}{16}, \qquad \lambda = \pm\frac{2\sqrt{1770}}{225};$$

$$\begin{aligned} u_6 = {} & \mp\frac{2\sqrt{1770}}{225} + \frac{59}{30}\frac{a_1\,\mathrm{sech}^2\left(\frac{\sqrt{1770}}{60}\zeta\right)}{\pm\frac{1}{4}\sqrt{1770}\,|a_1|\tanh\left(\frac{\sqrt{1770}}{60}\zeta\right) - \lambda_3} \\ & \pm \frac{2}{225}\sqrt{\frac{59}{30} + \frac{59}{30}\frac{\lambda_3\,\mathrm{sech}^2\left(\frac{\sqrt{1770}}{60}\zeta\right)}{\pm\frac{1}{4}\sqrt{1770}\,|a_1|\tanh\left(\frac{\sqrt{1770}}{60}\zeta\right) - \lambda_3} + \frac{3481}{64}\frac{a_1^2\,\mathrm{sech}^4\left(\frac{\sqrt{1770}}{60}\zeta\right)}{\left[\pm\frac{1}{4}\sqrt{1770}\,|a_1|\tanh\left(\frac{\sqrt{1770}}{60}\zeta\right) - \lambda_3\right]^2}}, \end{aligned}$$

$$\lambda_0 = \lambda_1 = 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda_4 = \frac{225a_1^2}{16}, \qquad \lambda = \pm\frac{2\sqrt{1770}}{225};$$

$$u_7 = \pm\frac{2\sqrt{1770}}{225} + \frac{b_1 - \frac{2}{15}\sqrt{\frac{225}{4}b_1^2 \pm \sqrt{1770}b_1\left(\mathrm{e}^{\frac{\pm\sqrt{1770}\zeta}{30}} \mp \frac{15\sqrt{1770}}{236}b_1\right) + \frac{118}{15}\left(\mathrm{e}^{\frac{\pm\sqrt{1770}\zeta}{30}} \mp \frac{15\sqrt{1770}}{236}b_1\right)^2}}{\mathrm{e}^{\frac{\pm\sqrt{1770}\zeta}{30}} \mp \frac{15\sqrt{1770}}{236}b_1},$$

$$\lambda_3 = \lambda_4 = 0, \qquad \lambda_0 = \frac{\lambda_1^2}{4\lambda_2}, \qquad \lambda_1 = \pm\frac{\sqrt{1770}}{4}b_1, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda = \mp\frac{2\sqrt{1770}}{225};$$

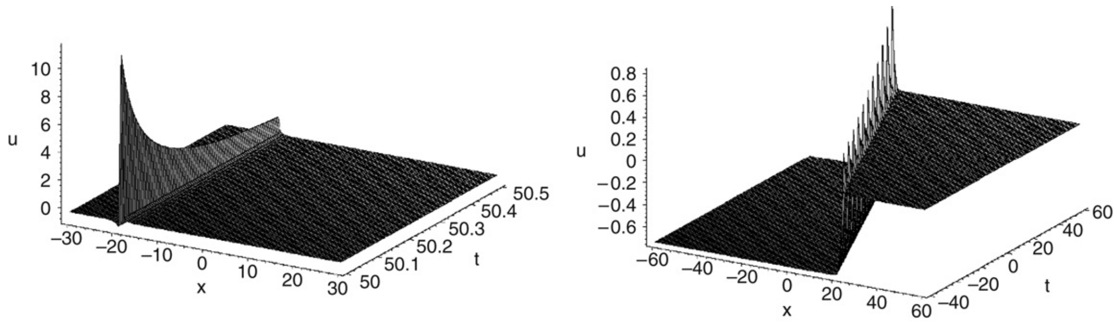

Fig. 6. $u_6 : a_1 = \lambda_3 = 1$ (left) and u_8 (right).

$$u_8 = \mp \frac{2\sqrt{1770}}{225} \pm \frac{236}{225} \frac{\sqrt{\frac{15}{59}\left[\pm \sinh\left(\frac{\sqrt{1770}}{15}\zeta\right) - 1\right] + \frac{15}{118}\left[\pm \sinh\left(\frac{\sqrt{1770}}{15}\zeta\right) - 1\right]^2}}{\pm \sinh\left(\frac{\sqrt{1770}}{15}\zeta\right) - 1},$$

$$\lambda_0 = \lambda_3 = \lambda_4 = 0, \qquad \lambda_1 \neq 0, \qquad \lambda_2 = \frac{59}{30}, \qquad \lambda = \pm \frac{2\sqrt{1770}}{225}.$$

Some graphs of the above solutions are depicted in Fig. 6, which show the distribution of some singularities of the solutions.

4. New explicit wave solutions to Eqs. (1) with $\alpha \neq 0$ and $\alpha \neq 1$

In this case, many intriguing travelling wave solutions can be constructed by using our method. We just give two of these solutions due to the limit of length.

Case 1

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{75\alpha^2 - 16}{120\alpha^2}, \qquad \lambda = \mp \frac{\sqrt{40\alpha^2 - \frac{128}{15}}}{15\alpha^2}$$

$$a_0 = \pm \frac{\sqrt{40\alpha^2 - \frac{128}{15}}}{15\alpha^2}, \qquad d_1 = \frac{8}{15\alpha}.$$

Case 2

$$a_2 = b_1 = b_2 = c_1 = c_2 = d_2 = 0, \qquad \lambda_2 = \frac{75\alpha^2 - 16}{30\alpha^2}, \qquad \lambda_4 = \frac{225\alpha^2 a_1^2}{16}, \qquad \lambda = \mp \frac{\sqrt{40\alpha^2 - \frac{128}{15}}}{15\alpha^2},$$

$$a_0 = \pm \frac{\sqrt{40\alpha^2 - \frac{128}{15}}}{15\alpha^2}, \qquad d_1 = \frac{4}{15\alpha}.$$

$$u_1 = \pm \frac{2\sqrt{10\alpha^2 - \frac{32}{15}}}{15\alpha^2} \pm \frac{4}{15\alpha^2}\sqrt{\frac{75\alpha^2 - 16}{30}} \tanh\left(\pm \frac{\sqrt{30(75\alpha^2 - 16)}\zeta}{60\alpha}\right),$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{75\alpha^2 - 16}{120\alpha^2} > 0, \qquad \lambda = \mp \frac{2\sqrt{10\alpha^2 - \frac{32}{15}}}{15\alpha^2}, \qquad \lambda_4 < 0;$$

$$u_2 = \pm \frac{2\sqrt{10\alpha^2 - \frac{32}{15}}}{15\alpha^2} \pm \frac{2}{15\alpha^2}\sqrt{\frac{2}{15}(75\alpha^2 - 16)}\, \mathrm{sech}\left(\pm \frac{\sqrt{30(75\alpha^2 - 16)}}{30\alpha}\zeta\right)$$

$$\pm \frac{2}{225\alpha^2}\sqrt{75\alpha^2 - 16} \tanh\left(\pm \frac{\sqrt{30(75\alpha^2 - 16)}}{30\alpha}\zeta\right),$$

$$\lambda_0 = \lambda_1 = \lambda_3 = 0, \qquad \lambda_2 = \frac{75\alpha^2 - 16}{30\alpha^2} > 0, \qquad \lambda = \mp \frac{2\sqrt{10\alpha^2 - \frac{32}{15}}}{15\alpha^2}, \qquad \lambda_4 = \frac{225}{16}\alpha^2 a_1^2 < 0.$$

5. Conclusions

Motivated by seeking more solutions to Zufiria's higher-order Boussinesq type equations, we extended the algebraic method on the basis of [29–34]. The main improvement of our method is to synthesize the corresponding solutions (see [29, 34]) for the first-order ordinary differential equation (6) depending on different values of λ_i ($i = 0, 1, 2, 3, 4$). New various families of explicit solutions are successfully obtained, including periodic wave, solitary wave, periodic-like, soliton-like, Jacobi elliptic function periodic, combined non-degenerative Jacobi elliptic function-like, Weierstrass elliptic function periodic etc. The properties of some solutions to Eqs. (1) are also shown by Figs. 1–6. These diverse new explicit and exact solutions will be helpful to the further research of the physical and mechanical meaning and laws of motion of the nature and realistic models, and the investigation of these diverse solutions will be quite important for the understanding and discussion of corresponding dynamical behaviors.

Moreover, the existence of smooth and non-smooth wave solutions and the dynamical properties of Zufiria's higher-order Boussinesq type equations could be studied by using the theory of bifurcations of dynamical systems. In addition, it deserves more investigation whether there exist other types of exact solutions to the generalized Zufiria's higher-order Boussinesq equations. We plan to analyze and discuss these situations in a future publication.

Acknowledgments

The work was supported in part by the National Natural Science Foundation of China (Grant No. 10332030, Grant No. 10472091 and Grant No. 10502042), the Doctorate Foundation of Northwestern Polytechnical University (Grant No. CX200616), the Established Researcher Grant of the University of South Florida, the CAS Faculty Development Grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and Wang Kuancheng Foundation. One of the authors (L. Gao) would like to express her sincere thanks to Prof. Wen-Xiu Ma for his guidance during her Ph.D. study in the University of South Florida.

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