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Wronskian solution, Bäcklund transformation and Painlevé analysis to a (2 + 1)-dimensional Konopelchenko–Dubrovsky equation

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Abstract: The main work of this paper is to construct the Wronskian solution and investigate the integrability characteristics of the (2 + 1)-dimensional Konopelchenko–Dubrovsky equation. Firstly, the Wronskian technique is used to acquire a sufficient condition of the Wronskian solution. According to the Wronskian form, the soliton solution is obtained by selecting the elements in the determinant that satisfy the linear partial differential systems. Secondly, the bilinear Bäcklund transformation and Bell-polynomial-typed Bäcklund transformation are derived directly via the Hirota bilinear method and the Bell polynomial theory, respectively. Finally, Painlevé analysis proves that this equation possesses the Painlevé property, and a Painlevé-typed Bäcklund transformation is constructed to solve a family of exact solutions by selecting appropriate seed solution. It shows that the Wronskian technique, Bäcklund transformation, Bell polynomial and Painlevé analysis are applicable to obtain the exact solutions of the nonlinear evolution equations, e.g., soliton solution, single-wave solution and two-wave solution.

Keywords: Wronskian solution; bilinear Bäcklund transformation; Bell polynomial; Painlevé analysis

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1 Introduction

Nonlinear evolution equations (NLEEs) are important for describing many nonlinear physical phenomena [1]–[5]. Investigating the exact solutions and integrability of NLEEs plays an essential role in understanding the mechanisms of nonlinear waves [6]–[8]. The Hirota bilinear method is a useful and direct method to construct the N-soliton solution and bilinear Bäcklund transformation (BT) to NLEEs [9]. Based on the Hirota bilinear form, more and more kinds of exact solutions to NLEEs have been found, such as soliton solution [10]–[12], lump solution [13]–[15], M-lump solution [16], interaction solution [1], [17] and rouge wave solution [11], [15], [18]. The Wronskian technique is also a useful method to solve the exact solution of NLEEs [19]–[21], and Wronskian formulas are employed to establish N-soliton solution in terms of Wronskian-type determinan [21], [51], [52]. Moreover, the soliton solution [43], positive solution [44], [45], negative solution [46], rational solution [47] and complexion solution [48] can also be obtained by selecting the coefficient matrix corresponding to the eigenvalues.

In the study of the integrability of NLEEs, many systematic methods have been used, such as BT [35]–[38], Bell polynomial [35], [37], [52], and Painlevé analysis [10], [28], [52]. BT is a powerful method for discovering scattering problems and generating new solutions from the known ones. Bell polynomial method, firstly proposed by Lembert and Gilson, has a close connection with the Hirota bilinear method [50], and it can be used to construct Bell-polynomial-typed BT, Lax pair and conservation law for NLEEs possessing bilinear forms. In 1983, Weiss J, Tabor M and Carnevale G explored the Painlevé analysis to solve the NLEEs [49]. By determining whether an equation has the Painlevé property, one can initially determine whether the equation is integrable and has an exact solution, and then certain conditions are created for solving the equation.

Describing the nonlinear wave motion, a well-known model is the (2 + 1)-dimensional Kadomtsev–Petviashvili (KP) equation written as [22], [23]

$$(u_t + 6uu_x + u_{xxx})_x \pm u_{yy} = 0. \tag{1}$$

The equation can be applied to simulate water waves of long wavelengths with weak nonlinear restoring forces and dispersion. It is also used to simulate waves in ferromagnetic media. The coefficient ± 1 indicates weak surface tension and strong surface tension, respectively. This means that the two KP equations have different physical structures and properties for ± 1. Several analytical and numerical methods were used to solve the KP equation.

A generalized (2 + 1)-dimensional equation was firstly proposed by Konopelchenko and Dubrovsky in 1984 [42] as

$$\begin{cases} u_t - u_{xxx} - 3v_y + \frac{3}{2}a^2u^2u_x + 3au_xv - 6buu_x = 0, \\ u_y = v_x, \end{cases} \tag{2}$$

where $u = u(x, y, t)$ and $v = v(x, y, t)$ are analytic functions, and a and b are arbitrary constants. In previous research, the tanh-sech method, the Hirota bilinear method, the extended F-expansion method, the cosh-sinh method, the modified extended direct algebraic method and the exponential functions method were used to derive kinks, solitons, traveling wave solution and periodic solution in Refs. [24]–[29]. The lump solution was presented by transforming Eq. (2) into an ordinary differential equation or bilinear form [39], [40]. In addition, BT, Wronskian solution as well as Painlevé analysis were all obtained [27], [28]. Furthermore, there were some other exact solutions in Refs. [29]–[34].

When $a = 0$, Eq. (2) enjoys the following form

$$\begin{cases} u_t - u_{xxx} - 3v_y - 6buu_x = 0, \\ u_y = v_x. \end{cases} \tag{3}$$

Through the dependent variable transformation $u = \frac{2}{b}(\ln\ f)_{xx}$, Eq. (3) can be changed into the bilinear form

$$(D_xD_t - D_x^4 - 3D_y^2)f \cdot f = 0, \tag{4}$$

where the binary operator D is defined by [9]

$$\begin{aligned} &D_x^\alpha D_y^\beta D_t^\gamma[f(x, y, t) \cdot g(x, y, t)] \\ &\quad = (\partial_{x'} - \partial_x)^\alpha(\partial_{y'} - \partial_y)^\beta(\partial_{t'} - \partial_t)^\gamma f(x, y, t)g \\ &\qquad \times (x', y', t')|_{x'=x, y'=y, t'=t}. \end{aligned} \tag{5}$$

For convenience, we refer to Eq. (3) as the (2 + 1)-dimensional Konopelchenko–Dubrovsky (KD) equation in the following section.

In this paper, the main focus is to study Eq. (3). For Eq. (3), the lump solution was obtained by the positive quadratic function method in Ref. [13]. The linear superposition principle was applied to obtain resonant multiple wave solutions in real field and complex field in Ref. [41]. Besides, solitary waves and interaction phenomena have been proposed based on the Hirota bilinear form in Refs. [13], [41]. However, the Wronskian solution and the integrability of Eq. (3) have not been studied. We will derive the Wronskian solution, Bäcklund transformation, Painlevé analysis and some exact solutions of Eq. (3). The rest of this paper is structured as follows. Firstly, in Section 2, we apply the Wronskian method to obtain the Wronskian solution and derive the soliton solution based on the Wronskian form. Moreover, we take one-, two- and three-soliton solutions as examples. Secondly, in Section 3, bilinear BT and Bell-polynomial-typed BT to Eq. (3) are constructed. Then in Section 4, we use Painlevé analysis to check its integrability. Finally, we will present some conclusions in Section 5.

2 N-soliton solution in Wronskian form

2.1 Wronskian solution

In this subsection, we will construct a Wronskian solution to Eq. (3). We use the following form

$$\begin{aligned} f_N = W(\phi_1, \phi_2, \cdots, \phi_N) &= \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} \\ &= \left|\widehat{N-1}\right|, \end{aligned} \tag{6}$$

where $\phi_i^{(m)} = \frac{\partial^m \phi_i}{\partial x_i^m}$, $i = 1, 2, \ldots, N, m = 1, 2\ \ldots\ N$.

Assume that ϕ_i satisfies the following relations

$$\phi_{i,y} = \phi_{i,xx}, \quad \phi_{i,t} = 4\phi_{i,xxx}. \tag{7}$$

Based on the assumption and determinant properties, we obtain the differential equations for f_N

$$
\begin{aligned}
f_{N,x} &= \left|\widehat{N-2}, N\right|,\\
f_{N,xx} &= \left|\widehat{N-2}, N+1\right| + \left|\widehat{N-3}, N-1, N\right|,\\
f_{N,xxx} &= \left|\widehat{N-2}, N+2\right| + 2\left|\widehat{N-3}, N-1, N+1\right|\\
&\quad + \left|\widehat{N-4}, N-2, N-1, N\right|,\\
f_{N,xxxx} &= \left|\widehat{N-2}, N+3\right| + 3\left|\widehat{N-3}, N-1, N+2\right|\\
&\quad + 3\left|\widehat{N-4}, N-2, N-1, N+1\right|\\
&\quad + \left|\widehat{N-5}, N-3, N-2, N-1, N\right|,\\
f_{N,y} &= \left|\widehat{N-2}, N+1\right| - \left|\widehat{N-3}, N-1, N\right|,\\
f_{N,yy} &= \left|\widehat{N-2}, N+3\right| + 2\left|\widehat{N-3}, N, N+1\right|\\
&\quad - \left|\widehat{N-4}, N-2, N-1, N+1\right|\\
&\quad - \left|\widehat{N-3}, N-1, N+2\right|\\
&\quad + \left|\widehat{N-5}, N-3, N-2, N-1, N\right|,\\
f_{N,t} &= 4\left|\widehat{N-2}, N+2\right| - 4\left|\widehat{N-3}, N-1, N+1\right|\\
&\quad + 4\left|\widehat{N-4}, N-2, N-1, N\right|,\\
f_{N,xt} &= 4\left|\widehat{N-2}, N+3\right| - 4\left|\widehat{N-3}, N, N+1\right|\\
&\quad + 4\left|\widehat{N-5}, N-3, N-2, N-1, N\right|.
\end{aligned}
\tag{8}
$$

Substituting the derivatives of f_N into Eq. (4), we have

$$
\begin{aligned}
&(f_{xt} - f_{xxxx} - 3f_{yy})f = -12\left|\widehat{N-3}, N, N+1\right|\left|\widehat{N-1}\right|,\\
&- f_x f_t + 4f_{xxx}f_x - 3f_{xx}^2 + 3f_y^2\\
&\quad = 12\left|\widehat{N-3}, N-1, N+1\right|\left|\widehat{N-2}, N\right|\\
&\qquad - 12\left|\widehat{N-2}, N+1\right|\left|\widehat{N-3}, N-1, N\right|,
\end{aligned}
\tag{9}
$$

and further deduce that

$$
\begin{aligned}
&\left(D_xD_t - D_x^4D_y - 3D_y^2\right) f\cdot f\\
&\quad = -12\Big(|\widehat{N-1}||\widehat{N-3}, N, N+1| - |\widehat{N-2}, N\\
&\qquad \times ||\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+1\\
&\qquad \times ||\widehat{N-3}, N-1, N|\Big)\\
&\quad = -6\begin{vmatrix} \widehat{N-3} & 0 & N-2 & N-1 & N & N+1\\ 0 & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix}\\
&\quad = 0.
\end{aligned}
\tag{10}
$$

Using the Laplace expansion of the determinant, we can prove that f_N is the solution of Eq. (3).

2.2 N-soliton solution

From the calculation, we know that if ϕ_i $(i = 1 \ldots N)$ satisfies Eq. (7), the solution of Eq. (3) can be acquired. We can easily find that $\phi_i = \mathrm{e}^{\xi_i} + \mathrm{e}^{\eta_i}$ satisfies Eq. (7) with $\xi_i = l_i x + l_i^2 y + 4l_i^3 t + \xi_i^{(0)}$ and $\eta_i = k_i x + k_i^2 y + 4k_i^3 t + \eta_i^{(0)}$, where l_i, k_i, $\xi_i^{(0)}$ and $\eta_i^{(0)}$ are arbitrary constants. Therefore, we can get the N-soliton solution of Eq. (3) as

$$
u = \frac{2}{b}\ln\left[W(\phi_1, \phi_2, \cdots, \phi_N)\right]_{xx}. \tag{11}
$$

When $N = 1$, the solution of Eq. (4) is written as

$$
f = f_1 = W(\phi_i) = \phi_i = e^{\xi_1} + e^{\eta_1}. \tag{12}
$$

Substituting it into the independent variable transformation, we have

$$
u = \frac{2}{b}(\ln f_1)_{xx} = \frac{2}{b}\frac{(k_1 - l_1)^2 e^{\eta_1+\xi_1}}{(e^{\xi_1} + e^{\eta_1})^2}, \tag{13}
$$

which is the one-soliton solution of Eq. (3).

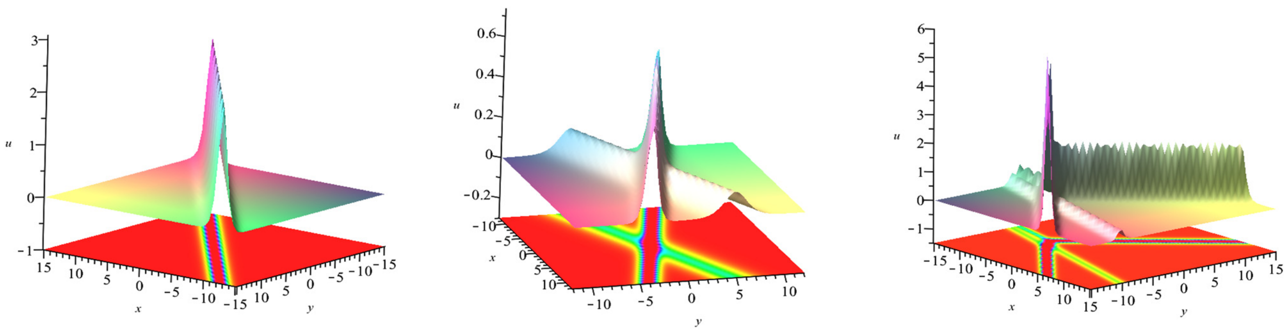
When $N = 2$, the solution of Eq. (4) is given as

$$
\begin{aligned}
f_2 = W(\phi_1, \phi_2) &= \begin{vmatrix} \phi_1 & \phi_{1,x}\\ \phi_2 & \phi_{2,x} \end{vmatrix}\\
&= (k_2 - k_1)e^{\eta_1+\eta_2}\left(1 + \frac{l_2 - k_1}{k_2 - k_1}e^{\xi_2-\eta_2} + \frac{k_2 - l_1}{k_2 - k_1}e^{\xi_1-\eta_1}\right.\\
&\quad \left. + \frac{l_2 - l_1}{k_2 - k_1}e^{\xi_1+\xi_2-\eta_1-\eta_2}\right),
\end{aligned}
\tag{14}
$$

which generates the two-soliton solution of Eq. (3) via $u = \frac{2}{b}(\ln f_2)_{xx}$.

Taking $N = 3$, we obtain

$$
\begin{aligned}
f_3 = W(\phi_1, \phi_2, \phi_3) &= \begin{vmatrix} \phi_1 & \phi_{1,x} & \phi_{1,xx}\\ \phi_2 & \phi_{2,x} & \phi_{2,xx}\\ \phi_3 & \phi_{3,x} & \phi_{3,xx} \end{vmatrix}\\
&= -(k_1 - k_2)(k_1 - k_3)(k_2 - k_3)e^{\eta_1+\eta_2+\eta_3}\\
&\quad \times \left[1 + \frac{(k_1 - l_3)(k_2 - l_3)}{(k_1 - k_3)(k_2 - k_3)}e^{\xi_3-\eta_3} - \frac{(k_1 - l_2)(k_3 - l_2)}{(k_1 - k_2)(k_2 - k_3)}\right.\\
&\quad \left. \times e^{\xi_2-\eta_2} + \frac{(k_2 - l_1)(k_3 - l_1)}{(k_1 - k_2)(k_1 - k_3)}e^{\xi_1-\eta_1}\right]\\
&\quad - (l_1 - l_2)(l_1 - l_3)(l_2 - l_3)e^{\xi_1+\xi_2+\xi_3}\\
&\quad \times \left[1 + \frac{(k_1 - l_2)(k_1 - l_3)}{(l_1 - l_2)(l_1 - l_3)}e^{\eta_1-\xi_1} - \frac{(k_2 - l_1)(k_2 - l_3)}{(l_1 - l_2)(l_2 - l_3)}\right.\\
&\quad \left. \times e^{\eta_2-\xi_2} + \frac{(k_3 - l_1)(k_3 - l_2)}{(l_1 - l_3)(l_2 - l_3)}e^{\eta_3-\xi_3}\right],
\end{aligned}
\tag{15}
$$



(a) (b) (c)

Figure 1: The soliton solution of Eq. (3) with $b = 1$. (a) One-soliton solution: $l_1 = 1.5$, $k_1 = -\frac{2}{3}$, $\xi_1^{(0)} = \eta_1^{(0)} = 0$; (b) two-soliton solution: $l_1 = -\frac{3}{2}$, $l_2 = 1$, $k_1 = -1$, $k_2 = 1$, $\xi_1^{(0)} = \eta_1^{(0)} = \xi_2^{(0)} = \eta_2^{(0)} = 0$; (c) three-soliton solution: $l_1 = -1.3$, $l_2 = 0.9$, $l_3 = 0.7$, $k_1 = 2$, $k_2 = 1.8$, $k_3 = -1.3$, $\xi_1^{(0)} = \eta_1^{(0)} = \xi_2^{(0)} = \eta_2^{(0)} = \xi_3^{(0)} = \eta_3^{(0)} = 0$.

which solves the three-soliton solution of Eq. (3) via $u = \frac{2}{b}(\ln f_3)_{xx}$.

As an example, we choose the appropriate parameters to draw one-, two- and three-soliton solutions in Figure 1.

3 Bilinear BT and Bell-polynomial-typed BT

Bell polynomial has a close association with the Hirota bilinear method. The Bell-polynomial-typed BT can be transformed into the bilinear BT through the correlation between the binary Bell polynomials and Hirota bilinear operators. In this section, we will construct the Bilinear BT and Bell-polynomial-typed BT.

3.1 Bilinear BT

In this section, we will construct a bilinear BT for Eq. (3). First of all, supposing $g = g(x, y, t)$ is another solution to Eq. (4) and introduce the following essential function

$$P = [(D_xD_t - D_x^4D_y - 3D_y^2)f \cdot f]g^2 - [(D_xD_t - D_x^4D_y - 3D_y^2)g \cdot g]f^2 = 0. \tag{16}$$

By using the exchange formulas, symbolic computation on Eq. (16) leads to

$$\begin{aligned}\frac{1}{2}P &= D_x(D_tf \cdot g) \cdot fg - D_x\big[(D_x^3f \cdot g) \cdot fg - 3(D_x^2f \cdot g) \cdot (D_xg \cdot f)\big] - 3D_y(D_yf \cdot g) \cdot fg - 3D_y(D_x^2f \cdot g) \cdot fg \\ &\quad + 3D_x(D_xD_yf \cdot g) \cdot fg + 3D_x(D_yf \cdot g) \cdot (D_xf \cdot g) \\ &= D_x[(D_t - D_x^3 + 3D_xD_y - \lambda)f \cdot g] \cdot fg \\ &\quad + 3D_x[(D_x^2 + D_y)f \cdot g] \cdot (D_xf \cdot g) \\ &\quad - 3D_y[(D_x^2 + D_y)f \cdot g] \cdot fg \\ &= 0,\end{aligned} \tag{17}$$

where λ is an arbitrary constant.

Therefore, the bilinear BT of Eq. (3) is as follows

$$\begin{cases}(D_t - D_x^3 + 3D_xD_y - \lambda)f \cdot g = 0,\\ (D_x^2 + D_y)f \cdot g = 0.\end{cases} \tag{18}$$

Let us choose a seed solution $g = 1$ to Eq. (4), then we have the following partial differential equation

$$\begin{cases}f_t - f_{xxx} + 3f_{xy} + \lambda = 0,\\ f_{xx} + f_y = 0.\end{cases} \tag{19}$$

We think about the exponential wave solution of Eq. (19)

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{20}$$

where $\theta_i = m_ix + n_iy + l_it + w_i$ $(i = 1, 2)$. With the help of symbolic computation and the selection of $\lambda = 0$, the relationship among parameters can be obtained

$$\{m_1 = m_1, m_2 = 0, n_1 = -m_1^2, n_2 = 0, l_1 = 4m_1^2, l_2 = 0, w_1 = w_1, w_2 = w_2, a_{12} = a_{12}\}. \tag{21}$$

Thus, we can acquire a class of solitary wave solutions of Eq. (3) as

$$u = \frac{2}{b}\left[\frac{m_1^2e^{\theta_1}(1 + a_{12}e^{w_2})}{1 + e^{\theta_1} + a_{12}e^{\theta_1+w_2}} - \frac{m_1^2e^{2\theta_1}(1 + a_{12}e^{w_2})^2}{(1 + e^{\theta_1} + a_{12}e^{\theta_1+w_2})^2}\right], \tag{22}$$

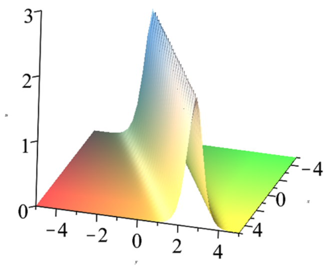


Figure 2: The single-wave solution of Eq. (3).

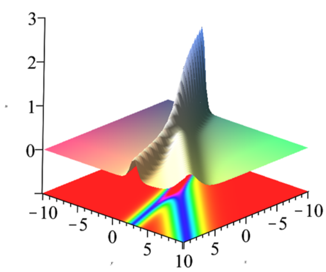


Figure 3: The two-wave solution of Eq. (3).

where $\theta_1 = m_1 x - m_1^2 y + 4m_1^3 t + w_1$. By selecting $t = 0$, $m_1 = 2, w_1 = 1, w_2 = -1, a_{12} = 1$, the single-wave solution is plotted in Figure 2.

In addition, the parameters are also interrelated in the following ways

$$\{m_1 = m_1, m_2 = m_2, n_1 = -m_1^2, n_2 = -m_2^2, l_1 = 4m_1^2, \\ l_2 = 4m_2^2, w_1 = w_1, w_2 = w_2, a_{12} = 0\}, \tag{23}$$

therefore a family of resonant two-wave solutions can be obtained

$$u = \frac{2}{b}\left[\frac{m_1^2 e^{\theta_1} + m_2^2 e^{\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}} - \frac{(m_1 e^{\theta_1} + m_2 e^{\theta_2})^2}{(1 + e^{\theta_1} + e^{\theta_2})^2}\right], \tag{24}$$

where $\theta_i = m_i x - m_i^2 y + 4m_i^3 t + w_i\,(i = 1, 2)$. Figure 3 shows the three-dimensional plot of the two-wave solution in Eq. (24) by selecting $t = 0$, $m_1 = 1$, $m_2 = 2$, $w_1 = \frac{2}{3}$, $w_2 = -1$, and $b = 1$.

3.2 Bell-polynomial-typed BT

Firstly, let us introduce Bell polynomials briefly. With the assumption f is a C^∞ function and

$$f_{r_1 x_1, \ldots, r_k x_k} = \partial_{x_1}^{r_1} \cdots \partial_{x_k}^{r_k} f, \tag{25}$$

where $r_i (i = 1 \ldots k)$ are arbitrary integers.

Then, we define the multi-dimensional Bell polynomial

$$Y_{n_1 x_1, \ldots, n_k x_k}(f) \equiv Y_{n_1, \ldots, n_k}\left(f_{r_1 x_1, \ldots, r_k x_k}\right) = \mathrm{e}^{-f} \partial_{x_1}^{n_1} \cdots \partial_{x_k}^{n_k} \mathrm{e}^{f}, \tag{26}$$

where $n_i, i = 1 \ldots k$ are arbitrary integers and $r_i, i = 1 \ldots k$.

For example, when $f = f(x, y, t)$ the Bell polynomials are as follows

$$Y_x(f) = f_x, \quad Y_{2x}(f) = f_x^2 + f_{xx}, \quad Y_{x,y}(f) = f_{x,y} + f_x f_y, \\ Y_{3x}(f) = f_x^3 + 3f_x f_{2x} + f_{3x}, \cdots \tag{27}$$

Finally, the multi-dimensional binary Bell polynomials are defined as follows

$$\mathcal{Y}_{n_1 x_1, \ldots, n_k x_k}(V, W) \\ \equiv Y_{n_1, \ldots, n_k}(f)\Big|_{f_{r_1 x_1, \ldots, r_k x_k}} \\ = \begin{cases} V_{r_1 x_1, \ldots, r_k x_k}, & \text{if } r_1 + r_2 + \cdots + r_k \text{ is odd}, \\ W_{r_1 x_1, \ldots, r_k x_k}, & \text{if } r_1 + r_2 + \cdots + r_k \text{ is even}, \end{cases} \tag{28}$$

where $V = V(x_1, x_2, \ldots, x_k)$ and $W = W(x_1, x_2, \ldots, x_k)$ are C^∞ functions.

When $V = V(x, y, t)$ and $W = W(x, y, t)$, the binary Bell polynomials are

$$\mathcal{Y}_x(V, W) = V_x, \quad \mathcal{Y}_{2x}(V, W) = W_{2x} + V_x^2, \\ \mathcal{Y}_{3x}(V, W) = V_{3x} + 3V_x W_{2x} + V_x^3, \\ \mathcal{Y}_{x,y}(V, W) = W_{x,y} + V_x W_y, \ldots \tag{29}$$

Letting $V = \ln \frac{F}{G}$ and $W = \ln FG$ in Eq. (28), we have

$$\mathcal{Y}_{n_1 x_1, \ldots, n_k x_k}\left(V = \ln \frac{F}{G}, W = \ln FG\right) \\ = (FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_k}^{n_k} F \cdot G, \tag{30}$$

where $\sum_{i=0}^{k} n_i \geq 1$, so that we can transformed the Bell-polynomial-typed BT into the bilinear BT.

Letting $V = 0$ and $q = W$ in Eq. (30), we can get the $\mathcal{P}$-polynomials

$$\mathcal{P}_{n_1 x_1, \cdots, n_k x_k}(q) = \mathcal{Y}_{n_1 x_1, \cdots, n_k x_k}(0, q). \tag{31}$$

For example, some P-polynomials are

$$\mathcal{P}_{2x}(q) = q_{2x}, \quad \mathcal{P}_{4x}(q) = q_{4x} + 3q_{2x}^2, \quad \mathcal{P}_{x,t}(q) = q_{x,t}, \cdots \tag{32}$$

where $q = q(x, y, t)$.

When $F = G$ and $q = 2\ln F$, there is a connection between the $\mathcal{P}$-polynomials and Hirota bilinear operators

$$\mathcal{P}_{n_1 x_1, \ldots, n_k x_k}(q = 2 \ln F) = F^{-2} D_{x_1}^{n_1} \ldots D_{x_k}^{n_k} F \cdot F. \tag{33}$$

Letting $u = cq_{xx}$ and $v = cq_{xy}$ in Eq. (3), we can get

$$cq_{xxt} - cq_{5x} - 3cq_{xyy} - 6c^2bq_{xx}q_{xxx} = 0. \tag{34}$$

Integral once with respect to x, Eq. (34) becomes

$$cq_{xt} - cq_{4x} - 3cq_{yy} - 3c^2bq_{xx}^2 = 0. \tag{35}$$

Letting $c = \frac{1}{b}$, we can have

$$\begin{aligned} E(q) &= q_{xt} - q_{4x} - 3q_{2y} - 3q_{xx}^2 \\ &= \mathcal{P}_{x,t}(q) - \mathcal{P}_{4x}(q) - 3\mathcal{P}_{2y}(q) = 0. \end{aligned} \tag{36}$$

In the case of $q = 2\ln f$, the variable transformation is $u = \frac{2}{b}(\ln f)_{xx}$, so Eq. (36) is equal to Eq. (4). By supposing $q' = 2\ln g$ is another solution to Eq. (36), and introducing $V = \frac{q'-q}{2} = \ln\frac{g}{f}$ and $W = \frac{q'+q}{2} = \ln fg$, the two-filed condition can be obtained

$$\begin{aligned} E(q') - E(q) &= (q'-q)_{xt} - (q'-q)_{4x} - 3(q'-q)_{2y} \\ &\quad - 3(q'-q)_{2x}(q'+q)_{2x} \\ &= 2V_{xt} - 6V_{2y} - 2V_{4x} - 12V_{2x}W_{2x} \\ &= 0. \end{aligned} \tag{37}$$

Using the binary Bell polynomials, the above formula can be expressed as follows

$$\begin{aligned} \frac{E(q') - E(q)}{2} &= \partial_x\big(\mathcal{Y}_t(V,W) - \mathcal{Y}_{3x}(V,W) \\ &\quad + 3\mathcal{Y}_{x,y}(V,W) - \lambda\big) - 3\partial_y\big(\mathcal{Y}_{2x}(V,W) \\ &\quad + \mathcal{Y}_y(V,W)\big) - 3v_{2x}\big(\mathcal{Y}_{2x}(V,W) \\ &\quad + \mathcal{Y}_y(V,W)\big) + 3v_x\partial_x\big(\mathcal{Y}_{2x}(V,W) \\ &\quad + \mathcal{Y}_y(V,W)\big) = 0, \end{aligned} \tag{38}$$

where λ is an arbitrary constant.

Therefore, the Bell-polynomial-typed BT of Eq. (3) is

$$\begin{cases} \mathcal{Y}_t(V,W) - \mathcal{Y}_{3x}(V,W) + 3\mathcal{Y}_{x,y}(V,W) - \lambda = 0, \\ \mathcal{Y}_{2x}(V,W) + \mathcal{Y}_y(V,W) = 0, \end{cases} \tag{39}$$

which indicates the bilinear BT.

4 Painlevé analysis

In this section, we will discuss whether Eq. (3) passes the Painlevé test employing the Weiss-Tabor-Carnevale method. Suppose a singular manifold $\phi(x,y,t) = 0$ and assume that

$$\begin{aligned} u &= \sum_{j=0}^{\infty} u_j(x,y,t)\phi(x,y,t)^{j+\alpha}, \\ v &= \sum_{j=0}^{\infty} v_j(x,y,t)\phi(x,y,t)^{j+\beta}, \end{aligned} \tag{40}$$

where α and β are negative integers, $u_j(x,y,t)$, $v_j(x,y,t)$ and $\phi(x,y,t)$ are analytic functions.

To balance the dominant terms, we assume

$$\begin{aligned} u &\sim u_0\phi^{\alpha}, \\ v &\sim v_0\phi^{\beta}. \end{aligned} \tag{41}$$

Substituting expressions (41) into Eq. (3) to balance the nonlinear term with the highest derivative term, we obtain

$$\begin{aligned} \alpha &= \beta = -2, \\ u_0 &= -\frac{2}{b}\phi_x^2, \\ v_0 &= -\frac{2}{b}\phi_x\phi_y. \end{aligned} \tag{42}$$

Next, we directly substitute

$$\begin{aligned} u &\sim -\frac{2}{b}\phi_x^2\phi^{-2} + u_j\phi^{j-2}, \\ v &\sim -\frac{2}{b}\phi_x\phi_y\phi^{-2} + v_j\phi^{j-2}, \end{aligned} \tag{43}$$

into Eq. (3) to get the coefficients of the lowest orders ϕ^{j-5} and ϕ^{j-2} as

$$\begin{aligned} F_1 &= (j-6)(j-4)(j+1)u_j\phi_x^3, \\ F_2 &= (j-2)(u_j\phi_y - v_j\phi_x). \end{aligned} \tag{44}$$

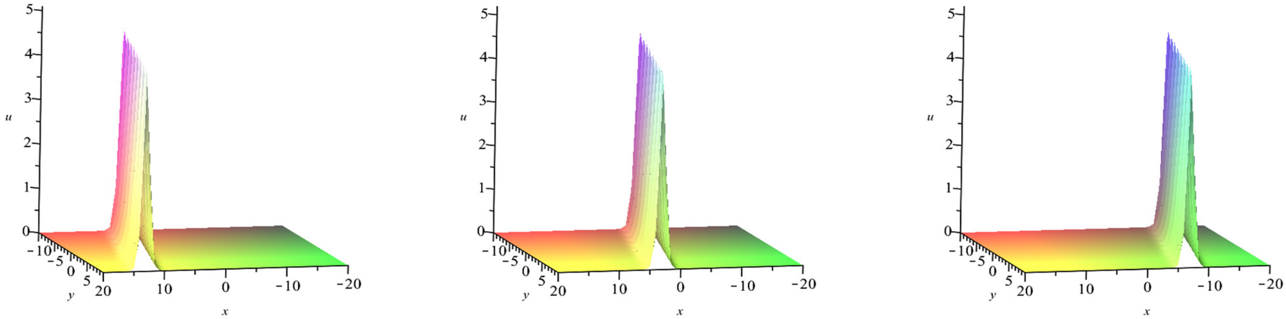
Equation (44) can be written in the following matrix form

$$\begin{aligned} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= Q(j)\begin{pmatrix} u_j \\ v_j \end{pmatrix} \\ &= \begin{pmatrix} -(j-6)(j-4)(j+1)\phi_x^3 & 0 \\ (j-2)\phi_y & -(j-2)\phi_x \end{pmatrix}\begin{pmatrix} u_j \\ v_j \end{pmatrix}, \end{aligned} \tag{45}$$

and the resonance is defined by the following formula

$$\det(Q(j)) = (j-2)(j-4)(j-6)(j+1)\phi_x^4. \tag{46}$$

Therefore, resonance occurs in $j = -1, 2, 4, 6$, and the resonance at $j = -1$ is usually related to the arbitrariness of the function $\phi(x,y,t)$, which describes singular hypersurfaces [35].



(a) (b) (c)

Figure 4: The exact solution: (a) $t = -30$; (b) $t = 0$; (c) $t = 30$ with $a_2 = -1$, $a_3 = 1$, $a_4 = \frac{1}{2}$, $b = 1$.

Finally, putting the truncated expansions

$$u = \sum_{j=0}^{6} u_j(x, y, t)\phi(x, y, t)^{j-2},$$
$$v = \sum_{j=0}^{6} v_j(x, y, t)\phi(x, y, t)^{j-2}, \tag{47}$$

into Eq. (3), and we find the compatibility conditions at $j = 2, 4$ and 6 are fully met. These facts prove that Eq. (3) is Painlevé-integrable. Moreover, we find that u_1, u_2 and u_3 satisfy the following conditions

$$u_1 = \frac{2}{b}\phi_{xx}, \tag{48}$$

$$3\phi_y^2 - \phi_x\phi_t + 6bu_2\phi_x^2 - 4\phi_{xx}^2 + 4\phi_x\phi_{xxx} = 0, \tag{49}$$

$$3\phi_x^2\phi_{yy} - 6bu_3\phi_x^4 - \phi_x^2\phi_{xt} + \phi_x\phi_{xx}\phi_t - 3\phi_{xx}\phi_y^2 + 3\phi_{xx}^3 - 4\phi_x\phi_{xx}\phi_{xxx} + \phi_x^2\phi_{xxxx} = 0. \tag{50}$$

In addition, by setting $u_i = 0$ $(i = 3, 4, \ldots)$, it is possible to get the Painlevé-typed BT as

$$u = u_0\phi^{-2} + u_1\phi^{-1} + u_2 = \frac{2}{b}(\ln\phi)_{xx} + u_2, \tag{51}$$

where u and u_2 satisfy Eq. (3), while u_1 and u_2 meet Eqs. (48) and (49).

Set a seed solution $u_2 = 0$ and assum that

$$\phi = 1 + e^{a_1x + a_2y + a_3t + a_4}, \tag{52}$$

where a_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Solving Eq. (49) gives rise to the relationship among a_i $(i = 1, 2, 3, 4)$

$$\left\{a_1 = \frac{3a_2^2}{a_3}, a_2 = a_2, a_3 = a_3, a_4 = a_4\right\}. \tag{53}$$

A family of exact solutions of Eq. (3) is obtained from the above formula

$$u = \frac{18a_2^4e^{\xi}}{ba_3^2(1 + e^{\xi})} - \frac{18a_2^4e^{2\xi}}{ba_3^2(1 + e^{\xi})^2}, \tag{54}$$

where $\xi = \frac{3a_2^2}{a_3}x + a_2y + a_3t + a_4$, which are shown in Figure 4. Observing the images we can see that as t varies the wave moves along the x-axis from the positive direction to the negative direction. The exact solution described by the above expression shows a traveling wave in (x, y)-plane. Additionally, we note that the traveling wave remains constant on the characteristic line $\frac{3a_2^2}{a_3}x + a_2y + a_3t + a_4 = \gamma$, where γ is an arbitrary real constant. The propagating speed of traveling wave in x-axis is given by $V_x = \frac{a_3^2}{3a_2^2}$, and the propagating speed in y-axis is given by $V_y = \frac{a_3}{a_2}$. Therefore, the traveling wave is affected by a_i $(i = 2, 3, 4)$.

5 Conclusions

In this paper, we have investigated the (2 + 1)-dimensional Konopelchenko–Dubrovsky equation. The main work of this paper was to obtain the Wronskian solution, Bäcklund transformation, the integrability and thus construct some exact solutions. First of all, the Wronskian solution to Eq. (3) has been constructed, and a sufficient condition for the Wronskian solution was given by Eq. (7). Selecting the appropriate type of elements in the determinant, we obtained the N-soliton solution. The three-dimensional plots of one-, two- and three-soliton solutions and the corresponding density plots have been given. It has been proved that the Wronskian technique is a powerful tool to solve the

Hirota bilinear equation, and many exact solutions can be acquired.

Besides, we have discussed the integrability aspects of Eq. (3). Bilinear BT and Bell-polynomial-typed BT are derived. Selecting the appropriate seed solution, we obtained single-wave and two-wave solutions. It has been found that Eq. (3) passes the Painlevé test. Moreover, taking the appropriate truncation, we got the Painlevé-typed BT. Furthermore, one particular solution has been obtained through choosing the appropriate seed solution.

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