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Frobenius integrable decompositions for ninth-order partial differential equations of specific polynomial type

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ABSTRACT

Frobenius integrable decompositions are presented for a kind of ninth-order partial differential equations of specific polynomial type. Two classes of such partial differential equations possessing Frobenius integrable decompositions are connected with two rational Bäcklund transformations of dependent variables. The presented partial differential equations are of constant coefficients, and the corresponding Frobenius integrable ordinary differential equations possess higher-order nonlinearity. The proposed method can be also easily extended to the study of partial differential equations with variable coefficients.

1. Introduction

While exploring soliton phenomena, one found a large number of partial differential equations (PDEs) exhibiting soliton phenomena in many science subjects such as fluid physics, solid physics, elementary particle physics, biological physics, superconductor physics, etc. It is a quite fascinating research topic how to solve such PDEs, particularly obtain interesting solutions including solitons, and the topic also attracts much attention of mathematicians, physicists and dynamicists.

Over the past several decades, there have been numerous efficient methods for constructing exact solutions to nonlinear PDEs, one after the other [1–19]. Though the existing solution methods are diverse and different, appropriate reductions (e.g., similarity reductions, symmetry constraints, travelling wave reductions) are often employed to reduce given PDEs to simpler PDEs (normally linear) and/or integrable ordinary differential equations (ODEs). By employing linear differential equations, a novel kind of exact solutions – complexitons – was presented successfully, indeed [3,4,6,7].

Recently, Ma et al. [20] presented Frobenius integrable decompositions (FIDs) for two classes of nonlinear evolution equations (NEEs) with logarithmic derivative Bäcklund transformations in soliton theory. The discussed NEEs are transformed into systems of Frobenius integrable ODEs with cubic nonlinearity. You et al. [21] obtained two classes of PDEs with variable coefficients possessing FIDs, including the KdV and the potential KdV equation, the Boussinesq equation, and the generalized BBM equation.

If a scalar PDE or a system of PDEs

$$P(u, u_t, u_x, u_{xt}, \cdots) = 0 \tag{1}$$

has solutions in the form of

$$u = v(\Phi) = v(\Phi(x,t)),$$

where v is a function of Φ, and Φ satisfies the following two compatible scalar ODEs or systems of ODEs

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$$\Phi_x = \mathrm{A}(\Phi), \quad \Phi_t = \mathrm{B}(\Phi),$$

with A and B being functions of Φ, then we say that Eq. (1) possesses a FID. FIDs generalize compatible time–space decompositions requiring Hamiltonian structures which aim to guarantee the Liouville integrability [22,23]. Through FIDs, a PDE problem can be transformed into two associated ODE problems. Thus, the existence of solutions can be guaranteed easily by the existing theories of ODEs.

In this paper, we would like to present two classes of ninth-order PDEs of specific polynomial type possessing FIDs by introducing some general ansatzes on FIDs, motivated by the works about FIDs by Ma and You et al. Two kinds of rational functions for Bäcklund transformations in the form of logarithmic derivatives are taken in our constructive computation algorithm, and the associated Frobenius integrable ODEs possess higher-order nonlinearity. Our results provide supplements to the previous ones by Ma and You et al. [20,21].

2. Specific polynomial type PDEs possessing FIDs

Let us consider a general kind of ninth-order PDEs of the following form

$$R(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{5x}, u_{7x}, u_{9x}, u_t, u_{tt}, u_{xxt}, \ldots) = 0, \tag{2}$$

where R is a differential polynomial in the dependent variable u. Many interesting wave equations belong to this set of PDEs, e.g., the KdV, potential KdV, generalized seventh-order KdV and Burgers equations, spatially periodic third-order dispersive PDEs, and the b-equation. Based on the symmetry constraint theory [24–27], we consider the following case:

$$\Phi = (\phi, \psi)^T = (\phi(x,t), \psi(x,t))^T,$$

which satisfies two Frobenius integrable systems:

$$\phi_x = \psi, \quad \psi_x = \lambda\phi, \tag{3}$$

$$\phi_t = \theta_1(\phi, \psi), \quad \psi_t = \theta_2(\phi, \psi), \tag{4}$$

where λ is a real parameter, and θ_1 and θ_2 are undetermined polynomials in ϕ and ψ. It is easy to see that (3) can be generated from the Schrödinger spectral problem with zero potential. Based on (3) and (4), we have the following mixed derivatives for the considered FIDs:

$$\begin{cases} \phi_{xt} = \psi_t = \theta_2(\phi, \psi), \\ \phi_{tx} = \theta_{1,\phi}\psi + \lambda\theta_{1,\psi}\phi, \end{cases}$$

and

$$\begin{cases} \psi_{xt} = \lambda\phi_t = \lambda\theta_1(\phi, \psi), \\ \psi_{tx} = \theta_{2,\phi}\psi + \lambda\theta_{2,\psi}\phi. \end{cases}$$

Thus, we obtain

$$\theta_2(\phi, \psi) = \theta_{1,\phi}\psi + \lambda\theta_{1,\psi}\phi, \tag{5}$$

and accordingly,

$$\lambda\theta_1(\phi, \psi) = \theta_{1,\phi\phi}\psi^2 + 2\lambda\theta_{1,\phi\psi}\phi\psi + \lambda^2\theta_{1,\psi\psi}\phi^2 + \lambda(\theta_{1,\phi}\phi + \theta_{1,\psi}\psi). \tag{6}$$

Eqs. (5) and (6) follow equivalently from the compatibility conditions $\phi_{xt} = \phi_{tx}$ and $\psi_{xt} = \psi_{tx}$. Therefore, Eq. (6) is the only condition on θ_1, while θ_2 is defined through θ_1 by Eq. (5).

To search for θ_1 which satisfies (6), we take the following ansatze:

$$\theta_1 = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} b_{i,j}\phi^i\psi^j, \tag{7}$$

where m_1 and m_2 are natural numbers, and $b_{i,j}$ ($i = 0, \ldots, m_1$; $j = 0, \ldots, m_2$) are arbitrary constants. In what follows, we set $m_1 = m_2 = 7$. Substituting (7) into (6) leads to

$$\begin{aligned} \theta_1 = {} & -\lambda^3 b_{1,6}\phi^7 - \lambda^3 b_{0,7}\phi^6\psi + 3\lambda^2 b_{1,6}\phi^5\psi^2 + 3\lambda^2 b_{0,7}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^3\psi^4 - 3\lambda b_{0,7}\phi^2\psi^5 + b_{1,6}\phi\psi^6 + b_{0,7}\psi^7 + \lambda^2 b_{1,4}\phi^5 \\ & + \lambda^2 b_{0,5}\phi^4\psi - 2\lambda b_{1,4}\phi^3\psi^2 - 2\lambda b_{0,5}\phi^2\psi^3 + b_{1,4}\phi\psi^4 + b_{0,5}\psi^5 - \lambda b_{1,2}\phi^3 - \lambda b_{0,3}\phi^2\psi + b_{1,2}\phi\psi^2 + b_{0,3}\psi^3 + b_{1,0}\phi \\ & + b_{0,1}\psi. \end{aligned} \tag{8}$$

It is now easy to obtain that

$$\theta_2 = \lambda\big(-\lambda^3 b_{0,7}\phi^6 + 6\lambda^2 b_{1,6}\phi^5\psi + 9\lambda^2 b_{0,7}\phi^4\psi^2 - 12\lambda b_{1,6}\phi^3\psi^3 - 15\lambda b_{0,7}\phi^2\psi^4 + 6b_{1,6}\phi\psi^5 + 7b_{0,7}\psi^6 + \lambda^2 b_{0,5}\phi^4 - 4\lambda b_{1,4}\phi^3\psi - 6\lambda b_{0,5}\phi^2\psi^2 + 4b_{1,4}\phi\psi^3 + 5b_{0,5}\psi^4 - \lambda b_{0,3}\phi^2 + 2b_{1,2}\phi\psi + 3b_{0,3}\psi^2 + b_{0,1}\big)\phi + \big(-7\lambda^3 b_{1,6}\phi^6 - 6\lambda^3 b_{0,7}\phi^5\psi + 15\lambda^2 b_{1,6}\phi^4\psi^2 + 12\lambda^2 b_{0,7}\phi^3\psi^3 - 9\lambda b_{1,6}\phi^2\psi^4 - 6\lambda b_{0,7}\phi\psi^5 + b_{1,6}\psi^6 + 5\lambda^2 b_{1,4}\phi^4 + 4\lambda^2 b_{0,5}\phi^3\psi - 6\lambda b_{1,4}\phi^2\psi^2 - 4\lambda b_{0,5}\phi\psi^3 + b_{1,4}\psi^4 - 3\lambda b_{1,2}\phi^2 - 2\lambda b_{0,3}\phi\psi + b_{1,2}\psi^2 + b_{1,0}\big)\psi, \tag{9}$$

where $b_{i,j}$'s are arbitrary constants. On the other hand, we consider the following specific type of the differential polynomial R:

$$R(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{5x}, u_{7x}, u_{9x}, u_t, u_{tt}, u_{xxt}, \ldots) = d_{0,0}u + d_{0,1}u_x + d_{0,2}u_{xx} + d_{0,3}u_{xxx} + d_{0,4}u_{xxxx} + d_{1,0}u^2 + d_{1,1}uu_x + d_{1,2}uu_{xx} + d_{1,3}uu_{xxx} + d_{1,4}uu_{xxxx} + d_{2,0}u_x^2 + d_{2,1}u_xu_{xx} + d_{2,2}u_xu_{xxx} + d_{2,3}u_xu_{xxxx} + d_{3,0}u_{xx}^2 + d_{3,1}u_{xx}u_{xxx} + d_{3,2}u_{xx}u_{xxxx} + d_{4,0}u_{xxx}^2 + d_{4,1}u_{xxx}u_{xxxx} + d_{5,0}u_{xxxx}^2 + e_0u_{5x} + e_1u_{7x} + e_2u_{9x} + e_3uu_{5x} + e_4uu_{7x} + e_5uu_{9x} + f_0u^2u_x + f_1u^2u_{xxx} + f_2u^2u_{5x} + f_3u_x^3 + f_4u^3u_x + f_5uu_xu_{xx} + g_0u_t + g_1u_{tt} + g_2u_{xxt} + g_3u_{xxt}^2, \tag{10}$$

where $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants. Now, a direct computation can yield the following consequence.

Theorem

(a) *If we take a Bäcklund transformation*

$$u = (\ln\phi)_x = \frac{\psi}{\phi}, \tag{11}$$

from the Frobenius integrable systems (3) and (4) to the PDE (1), then θ_1, θ_2 and R defined by (8)–(10) must be of the form

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi,$$

$$\theta_2 = \lambda\big(6\lambda^2 b_{1,6}\phi^5\psi - 12\lambda b_{1,6}\phi^3\psi^3 + 6b_{1,6}\phi\psi^5 - 4\lambda b_{1,4}\phi^3\psi + 4b_{1,4}\phi\psi^3 + 2b_{1,2}\phi\psi + b_{0,1}\big)\phi + \big(15\lambda^2 b_{1,6}\phi^4\psi^2 - 7\lambda^3 b_{1,6}\phi^6 - 9\lambda b_{1,6}\phi^2\psi^4 + b_{1,6}\psi^6 + 5\lambda^2 b_{1,4}\phi^4 - 6\lambda b_{1,4}\phi^2\psi^2 + b_{1,4}\psi^4 - 3\lambda b_{1,2}\phi^2 + b_{1,2}\psi^2 + b_{1,0}\big)\psi,$$

$$R = (224\lambda^2 e_4 + 8\lambda d_{2,3} - 40\lambda e_3 + 24d_{0,4} - 2d_{2,1} - 6d_{1,3})u^3u_x + f_1u^2u_{xxx} + f_2u^2u_{5x} + f_3u_x^3 + d_{1,4}uu_{xxxx} + d_{1,3}uu_{xxx} + (3d_{2,2} - 40\lambda f_2 - 60e_0 + 2d_{3,0} - 3f_1 - 1/2f_3 + 12d_{1,4})uu_xu_{xx} + d_{1,2}uu_{xx} + e_3uu_{5x} + e_4uu_{7x} + 630e_2u_{xxxx}^2 - 35e_4u_{xxx}u_{xxxx} + d_{2,3}u_xu_{xxxx} + d_{4,0}u_{xxx}^2 + d_{2,2}u_xu_{xxx} + d_{3,0}u_{xx}^2 + d_{2,1}u_xu_{xx} - (256\lambda^4 e_2 + 64\lambda^3 e_1 + 16\lambda^3 f_2 + 16\lambda^2 e_0 + 4\lambda^2 f_1 - 8\lambda^2 d_{1,4} + 4\lambda b_{0,1}g_2 + 4\lambda d_{0,3} - 2\lambda d_{1,2} + b_{0,1}g_0 + d_{0,1})u^2u_x/\lambda + \Big(24\lambda^2 e_3 - 288\lambda^3 e_4 - 8\lambda^2 d_{2,3} + 2b_{0,1}^2 g_1 + 2\lambda d_{2,1} - 16\lambda d_{0,4} + 2\lambda d_{1,3} + 2d_{0,2}\Big)uu_x + \Big(105e_1 - 3/4d_{4,0} + 5/2f_2 - 3/4b_{0,1}^2 g_3\Big)u_{xx}u_{xxxx} + d_{0,4}u_{xxxx} - (140\lambda e_4 + 2d_{2,3} + 10e_3)u_{xx}u_{xxx} + d_{0,3}u_{xxx} + d_{0,2}u_{xx} + e_2u_{9x} + e_1u_{7x} + e_0u_{5x} + d_{0,1}u_x - \Big(4\lambda^3 c_{0,1}^2 g_3 + 7936\lambda^4 e_2 + 4\lambda^3 d_{4,0} - 272\lambda^3 e_1 + 16\lambda^2 e_0 - 2\lambda^2 d_{2,2} + \lambda^2 f_3 - 2\lambda b_{0,1}g_2 - 2\lambda d_{0,3} + b_{0,1}g_0 + d_{0,1}\Big)u_x^2/\lambda + g_0u_t + g_1u_{tt} + g_2u_{xxt} + g_3u_{xxt}^2,$$

where $b_{i,j}$'s, $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants.

(b) *If we take a Bäcklund transformation*

$$u = (\ln\phi)_{xx} = \frac{\lambda\phi^2 - \psi^2}{\phi^2}, \tag{12}$$

from the Frobenius integrable systems (3) and (4) to the PDE (1), then θ_1, θ_2 and R defined by (8)–(10) must be of the form

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi,$$

$$\theta_2 = -\lambda^3 b_{1,6}\phi^6\psi + 3\lambda^2 b_{1,6}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^2\psi^5 + \lambda^2 b_{1,4}\phi^4\psi - 2\lambda b_{1,4}\phi^2\psi^3 - \lambda b_{1,2}\phi^2\psi + \lambda b_{0,1}\phi + b_{1,6}\psi^7 + b_{1,4}\psi^5 + b_{1,2}\psi^3 + b_{0,1}\psi,$$

$$\begin{aligned}R =& f_4u^3u_x + f_1u^2u_{xxx} + f_5uu_xu_{xx}\\&+ (443520\lambda^2e_2 - 352\lambda^2d_{4,1} - 3584\lambda^2e_4 - 120\lambda e_3 + 8\lambda f_1 - \lambda f_4 + 2\lambda f_5 - 360e_0 + 12d_{1,3} + 6d_{2,1})u^2u_x\\&+ (4d_{4,1} - 5040e_2 + 56e_4)u^2u_{5x} + e_4uu_{7x} + e_3uu_{5x} + d_{1,4}uu_{xxxx} + d_{1,3}uu_{xxx} + d_{1,2}uu_{xx}\\&+ (240\lambda d_{4,1} - 302400\lambda e_2 + 1680\lambda e_4 + 30d_{2,3} + 18d_{3,1} - 5040e_1 + 90e_3 - 3f_1 + 1/4f_4 - 3/2f_5)u_x^3\\&+ d_{2,3}u_xu_{xxxx} + d_{2,1}u_xu_{xx} + d_{3,2}u_{xx}u_{xxxx} + d_{4,1}u_{xxx}u_{xxxx} + d_{3,0}u_{xx}^2 + d_{3,1}u_{xx}u_{xxx}\\&+ \left(32\lambda^3d_{3,2} - 16\lambda^2d_{1,4} - 4\lambda^2d_{3,0} + 6b_{0,1}^2g_1 + 2\lambda d_{1,2} + 6d_{0,2}\right)u^2\\&+ (65280\lambda^3e_2 - 64\lambda^3d_{4,1} - 64\lambda^3e_4 - 16\lambda^2d_{2,3} - 16\lambda^2d_{3,1} + 4032\lambda^2e_1 - 16\lambda^2e_3 + 240\lambda e_0 - 4\lambda d_{1,3} - 4\lambda d_{2,1}\\&+12b_{0,1}g_2 + 12d_{0,3})uu_x + (10\lambda d_{1,4} - 4\lambda^2d_{3,2} + 30d_{0,4} - 3/2d_{1,2})u_x^2\\&- (5/2d_{1,4} + 3/4d_{3,0})u_xu_{xxx} + e_2u_{9x} + e_1u_{7x} + e_0u_{5x} + d_{0,4}u_{xxxx} + d_{0,3}u_{xxx} + d_{0,2}u_{xx}\\&- \left(b_{0,1}^2g_3 + 5/4d_{3,2}\right)u_{xxx}^2 - \left(4\lambda b_{0,1}^2g_1 + 16\lambda^2d_{0,4} + 4\lambda d_{0,2}\right)u\\&- (256\lambda^4e_2 + 64\lambda^3e_1 + 16\lambda^2e_0 + 4\lambda b_{0,1}g_2 + 4\lambda d_{0,3} + b_{0,1}g_0)u_x + g_0u_t + g_1u_{tt} + g_2u_{xxt} + g_3u_{xxt}^2,\end{aligned}$$

where $b_{i,j}$'s, $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants.
Let us here take a special reduction of (10) as follows:

$$\begin{aligned}R(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{5x}, u_{7x}, u_{9x}, u_t, u_{tt}, u_{xxt}, \ldots) =& d_{0,0}u + d_{0,1}u_x + d_{0,2}u_{xx} + d_{0,3}u_{xxx} + d_{0,4}u_{xxxx} + d_{1,1}uu_x\\&+ d_{1,2}uu_{xx} + d_{1,3}uu_{xxx} + d_{2,0}u_x^2 + d_{2,1}u_xu_{xx} + d_{2,2}u_xu_{xxx}\\&+ d_{2,3}u_xu_{xxxx} + d_{3,1}u_{xx}u_{xxx} + e_0u_{5x} + e_1u_{7x} + e_2u_{9x} + e_3uu_{5x}\\&+ f_0u^2u_x + f_1u^2u_{xxx} + f_2u^2u_{5x} + f_3u_x^3 + f_4u^3u_x + f_5uu_xu_{xx} + g_0u_t\\&+ g_1u_{tt} + g_2u_{xxt},\end{aligned} \tag{13}$$

where $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants. The following corollary of the previous theorem gives two classes of constant coefficient PDEs of polynomial type possessing FIDs.

Corollary

(a) *If we take a Bäcklund transformation (11) from the Frobenius integrable systems (3) and (4) to the PDE (1), then θ_1, θ_2 and R defined by (8), (9) and (13) must be of the form*

$$\begin{aligned}\theta_1 =& -\lambda^3b_{1,6}\phi^7 + 3\lambda^2b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi\\&+ b_{0,1}\psi,\end{aligned}$$

$$\begin{aligned}\theta_2 =& \lambda(6\lambda^2b_{1,6}\phi^5\psi - 12\lambda b_{1,6}\phi^3\psi^3 + 6b_{1,6}\phi\psi^5 - 4\lambda b_{1,4}\phi^3\psi + 4b_{1,4}\phi\psi^3 + 2b_{1,2}\phi\psi + b_{0,1})\phi\\&+ (-7\lambda^3b_{1,6}\phi^6 + 15\lambda^2b_{1,6}\phi^4\psi^2 - 9\lambda b_{1,6}\phi^2\psi^4 + b_{1,6}\psi^6 + 5\lambda^2b_{1,4}\phi^4 - 6\lambda b_{1,4}\phi^2\psi^2 + b_{1,4}\psi^4 - 3\lambda b_{1,2}\phi^2 + b_{1,2}\psi^2 + b_{1,0})\psi,\end{aligned}$$

$$\begin{aligned}R =& (8\lambda d_{2,3} - 40\lambda e_3 + 24d_{0,4} - 6d_{1,3} - 2d_{2,1})u^3u_x + f_1u^2u_{xxx} - 42e_1u^2u_{5x} + f_3u_x^3\\&+ (336\lambda^2e_1 - 2\lambda d_{2,2} - 4\lambda f_1 + \lambda f_3 - 6b_{0,1}g_2 - 6d_{0,3} + 2d_{1,2} + d_{2,0})u^2u_x + d_{1,2}uu_{xx}\\&+ (1680\lambda e_1 + 3d_{2,2} - 60e_0 - 3f_1 - 1/2f_3)uu_xu_{xx} + d_{1,3}uu_{xxx} + e_3uu_{5x} + d_{2,1}u_xu_{xx} + d_{2,2}u_xu_{xxx} + d_{2,3}u_xu_{xxxx} + d_{2,0}u_x^2\\&+ \left(24\lambda^2e_3 - 8\lambda^2d_{2,3} + 2b_{0,1}^2g_1 - 16\lambda d_{0,4} + 2\lambda d_{1,3} + 2\lambda d_{2,1} + 2d_{0,2}\right)uu_x - (2d_{2,3} + 10e_3)u_{xx}u_{xxx} + d_{0,2}u_{xx} + d_{0,3}u_{xxx}\\&+ d_{0,4}u_{xxxx} + e_0u_{5x} + e_1u_{7x} + (272\lambda^3e_1 + 2\lambda^2d_{2,2} - 16\lambda^2e_0 - \lambda^2f_3 + 2\lambda b_{0,1}g_2 + 2\lambda d_{0,3} - \lambda d_{2,0} - b_{0,1}g_0)u_x + g_0u_t\\&+ g_1u_{tt} + g_2u_{xxt},\end{aligned}$$

where $b_{i,j}$'s, $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants.

(b) *If we take a Bäcklund transformation (12) from the Frobenius integrable systems (3) and (4) to the PDE (1), then θ_1, θ_2 and R defined by (8), (9) and (13) must be of the form*

$$\begin{aligned}\theta_1 =& -\lambda^3b_{1,6}\phi^7 + 3\lambda^2b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi\\&+ b_{0,1}\psi,\end{aligned}$$

$$\begin{aligned}\theta_2 =& -\lambda^3b_{1,6}\phi^6\psi + 3\lambda^2b_{1,6}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^2\psi^5 + \lambda^2b_{1,4}\phi^4\psi - 2\lambda b_{1,4}\phi^2\psi^3 - \lambda b_{1,2}\phi^2\psi + \lambda b_{0,1}\phi + b_{1,6}\psi^7 + b_{1,4}\psi^5 + b_{1,2}\psi^3\\&+ \psi b_{1,0},\end{aligned}$$

$$\begin{aligned}R=&f_4u^3u_x+f_1u^2u_{xxx}+\left(443520\lambda^2e_2-120\lambda e_3+8\lambda f_1-\lambda f_4+2\lambda f_5+12d_{1,3}+6d_{2,1}-360e_0\right)u^2u_x+d_{1,2}uu_{xx}+d_{1,3}uu_{xxx}\\&+e_3uu_{5x}+d_{3,0}u_{xx}u_{xxx}+d_{2,1}u_xu_{xx}+\left(30d_{2,3}+18d_{3,0}-302400\lambda e_2-5040e_1+90e_3-3f_1+1/4f_4-3/2f_5\right)u_x^3\\&+e_2u_{9x}+d_{0,3}u_{xxx}+f_5uu_xu_{xx}+d_{0,4}u_{xxxx}+e_0u_{5x}+e_1u_{7x}+\left(30d_{0,4}-3/2d_{1,2}\right)u_x^2+\left(4/3\lambda^2d_{1,2}-16\lambda^2d_{0,4}\right)u\\&-\left(b_{0,1}^2g_1+1/3\lambda d_{1,2}\right)u_{xx}-\left(64\lambda^3e_1+256\lambda^4e_2+16\lambda^2e_0+4\lambda c_{0,1}g_2+4\lambda d_{0,3}+b_{0,1}g_0\right)u_x+d_{2,3}u_xu_{xxxx}-5040e_2u^2u_{5x}\\&+\left(65280\lambda^3e_2-16\lambda^2d_{2,3}-16\lambda^2d_{3,0}+4032\lambda^2e_1-16\lambda^2e_3-4\lambda d_{1,3}-4\lambda d_{2,1}+240\lambda e_0+12b_{0,1}g_2+12d_{0,3}\right)uu_x\\&+g_0u_t+g_1u_{tt}+g_2u_{xxt},\end{aligned}$$

where $b_{i,j}$'s, $d_{k,l}$'s, e_m's, f_n's and g_p's are arbitrary constants.
The first class of nonlinear PDEs in the corollary above includes the potential KdV equation $g_0u_t+d_{0,3}u_{xxx}+6d_{0,3}u_x^2=0$, the Burgers equation $g_0u_t+2d_{0,2}uu_x+d_{0,2}u_{xx}=0$, and the b- equation

$$g_0u_t-\frac{1}{2\lambda}u_{xxt}-2uu_x=\frac{1}{2\lambda}(-3u_xu_{xx}+uu_{xxx}),$$

which can be viewed as a general case of the famous Rod equation [28]. The second class of nonlinear PDEs in the corollary above includes the KdV equation $g_0u_t+12d_{0,3}uu_x+d_{0,3}u_{xxx}=0$, the family of spatially periodic third-order dispersive PDEs

$$b_{0,1}g_0u_t-b_{0,1}^2g_0u_x+b_{0,1}d_{0,3}u_{xxx}-d_{0,3}u_{xxt}=b_{0,1}\alpha\left(\lambda u^2+u_x^2-\frac{1}{2}uu_{xx}\right)_x,\quad \alpha=\text{const.},$$

and the generalized seventh-order KdV equation

$$\begin{aligned}&g_0u_t+(2\lambda f_5-120\lambda e_3+8\lambda f_1)u^2u_x+\left(30d_{2,3}+18d_{3,0}-\frac{3}{2}f_5+90e_3-5040e_1-3f_1\right)u_x^3+f_5uu_xu_{xx}+f_1u^2u_{xxx}\\&+d_{3,0}u_{xx}u_{xxx}+d_{2,3}u_xu_{xxxx}+e_3uu_{5x}+e_1u_{7x}=0,\end{aligned}$$

which has two well-known special cases: the Lax seventh-order equation and the Sawada–Kotera–Ito seventh-order equation [29].

3. Conclusions

Through two rational Bäcklund transformations of dependent variables, a kind of ninth-order PDEs of specific type possessing the suggested FIDs has been obtained successfully. Two special classes of such nonlinear PDEs possessing special FIDs have been presented under reduction. The obtained PDEs contain various significant nonlinear wave equations.

We point out that the approach adopted in Section 2 can be easily applied to nonlinear variable coefficient PDEs, which are quite intriguing in ocean dynamics, fluid mechanics, plasma physics, etc. In fact, in the case of variable coefficients, we just only need to take the corresponding parameters (e.g., $b_{i,j}$'s, $d_{k,l}$'s, e_m's, f_n's and g_p's) as functions of t, and the computations involved are completely similar.

It would also be very interesting but much more complicated if we take the Möbius transformation

$$u=(a\varphi+b\psi)/(c\varphi+d\psi),\quad (ad-bc\neq 0),$$

instead of the logarithmic derivative type Bäcklund transformations (11) and (12) in the process of making our computation. We will discuss this transformation in a future publication.

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References

[1] W.X. Ma, B. Fuchssteiner, Explicit and exact solutions to a Kolmogorov–Petrovskii–Piskunov equation, International Journal of Non-Linear Mechanics 31 (1996) 329–338.
[2] W.X. Ma, D.T. Zhou, Explicit exact solutions to a generalized KdV equation, Acta Mathematica Scientia 17 (1997) 168–174.
[3] W.X. Ma, Complexiton solutions to the Korteweg–de Vries equation, Physics Letters A 301 (2002) 35–44.
[4] W.X. Ma, K. Maruno, Complexiton solutions of the Toda lattice equation, Physica A 343 (2004) 219–237.
[5] W.X. Ma, Y.C. You, Rational solutions of the Toda lattice equation in Casoratian form, Chaos, Solitons and Fractals 22 (2004) 395–406.
[6] W.X. Ma, Complexiton solutions of the Korteweg–de Vries equation with self-consistent sources, Chaos, Solitons and Fractals 26 (2005) 1453–1458.
[7] W.X. Ma, Complexiton solutions to integrable equations, Nonlinear Analysis 63 (2005) e2461–e2471.

[8] Z.B. Li, S.Q. Zhang, Exact solitary wave solutions for nonlinear wave equations using symbolic computation, Acta Mathematica Scientia 17 (1997) 81–89.
[9] Y. Chen, Q. Wang, Multiple Riccati equations rational expansion method and complexiton solutions of the Whitham–Broer–Kaup equation, Physics Letters A 347 (2005) 215–227.
[10] Q. Wang, Y. Chen, A multiple Riccati equations rational expansion method and novel solutions of the Broer–Kaup–Kupershmidt system, Chaos, Solitons and Fractals 30 (2006) 197–203.
[11] H.C. Hu, B. Tong, S.Y. Lou, Nonsingular positon and complexiton solutions for the coupled KdV system, Physics Letters A 351 (2006) 403–412.
[12] E. Yomba, The modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer–Kaup–Kupershmidt equation, Chaos, Solitons and Fractals 27 (2006) 187–196.
[13] S. Zhang, T.C. Xia, A further improved extended Fan sub-equation method and its application to the (3+1)-dimensional Kadomstev–Petviashvili equation, Physics Letters A 356 (2006) 119–123.
[14] J.B. Li, Exact explicit travelling wave solutions for (n+1)-dimensional Klein–Gordon–Zakharov equations, Chaos, Solitons and Fractals 34 (2007) 867–871.
[15] H.X. Wu, T.Y. Fan, Negaton and complexiton solutions of sine-Gordon equation, Physica A 379 (2007) 471–482.
[16] L. Gao, W. Xu, Y.N. Tang, G.F. Meng, New families of travelling wave solutions for Boussinesq–Burgers equation and (3+1)-dimensional Kadomtsev–Petviashvili equation, Physics Letters A 366 (2007) 411–421.
[17] F.D. Xie, Z.Y. Yan, Exactly fractional solutions of the (2+1)-dimensional modified KP equation via some fractional transformations, Chaos, Solitons and Fractals 36 (2008) 1108–1112.
[18] L. Gao, W.X. Ma, W. Xu, Travelling wave and double-solitary-like solutions to two nonlinear evolution equations, Proceedings of the 6th International Conference on Differential Equations and Dynamical Systems (2009) 271–275.
[19] W.X. Ma, R.G. Zhou, L. Gao, Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in 2+1 dimensions, Modern Physics Letters A 24 (21) (2009) 1677–1688.
[20] W.X. Ma, H.Y. Wu, J.S. He, Partial differential equations possessing Frobenius integrable decompositions, Physics Letters A 364 (2007) 29–32.
[21] F.C. You, T.C. Xia, J. Zhang, Frobenius integrable decompositions for two classes of nonlinear evolution equations with variable coefficients, Modern Physics Letters B 23 (12) (2009) 1519–1524.
[22] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1989.
[23] W.X. Ma, in: A. Scott (Ed.), Encyclopedia of Nonlinear Science, Taylor & Francis, New York, 2005. pp. 450-453.
[24] W.X. Ma, W. Strampp, An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, Physics Letters A 185 (1994) 277–286.
[25] W.X. Ma, New finite dimensional integrable systems by symmetry constraint of the KdV equations, Journal of the Physical Society of Japan 64 (1995) 1085–1091.
[26] W.X. Ma, Z.X. Zhou, Binary symmetry constraints of N-wave interaction equations in 1+1 and 2+1 dimensions, Journal of Mathematical Physics 42 (2001) 4345–4382.
[27] Y.B. Zeng, W.X. Ma, Separation of variables for soliton equations via their binary constrained flows, Journal of Mathematical Physics 40 (1999) 6526–6557.
[28] H.H. Dai, Y. Huo, Solitary shock waves and other travelling waves in a general compressible hyperelastic rod, Proceedings of the Royal Society of London Series A 456 (2000) 331–363.
[29] M. Ito, An extension of nonlinear evolution equations of the K–dV (mK–dV) type to higher orders, Journal of the Physical Society of Japan 49 (1980) 771–779.