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ORIGINAL PAPER

Lump-type solution and breather lump–kink interaction phenomena to a (3+1)-dimensional GBK equation based on trilinear form

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Abstract In this paper, the multivariate trilinear operators in the $(3 + 1)$-dimensional space are applied to a $(3 + 1)$-dimensional GBK equation. The resulting trilinear form is used to study its wave dynamics. Particularly, we generate a type of new interaction solutions between breather lump-type solitons and other multi-kink solitons, thereby formulating a kind of breather lump–kink solitons. By setting time constants, we change the coordinates of kink solitons to make them collide with the breather lump-type soliton, during which breather lump-type soliton is swallowed eventually by those kink solitons. The evolution behaviours of the breather lump–kink solitons are depicted by plotting 3-D and density graphs from the perspective of wave characteristics.

Keywords Trilinear form · $(3+1)$-dimensional GBK equation · Lump-type solution · Breather lump–kink soliton

1 Introduction

A new direct method for constructing multi-soliton solutions to integrable nonlinear evolution equations is introduced by Hirota in 1971, namely the Hirota bilinear method [1]. The idea is to make a transformation into new variables, so that in these new variables multi-soliton solutions appear in a particularly simple form. Researchers usually use the Hirota bilinear method to solve nonlinear PDEs and obtain their lump solutions [2–13], multi-soliton solutions [14–17], multi-lump solutions [18–20], interaction solutions [7,8,11,21–24] and others [25–33]. For example,

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the following (2+1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_xu_{xy}=0, \tag{1}$$

under the transformation $u=2(\ln f)_x$, turns into the Hirota bilinear form

$$\left(D_tD_y+D_yD_x^3\right)f\cdot f=0, \tag{2}$$

where D_x, D_y and D_t are Hirota's bilinear operators, and some lump solutions, soliton solutions and interaction solutions are generated, with the aid of the Hirota bilinear method [17].

Up to now, there are few studies on using trilinear forms to solve nonlinear PDEs. The nonlinear terms in trilinear equations have great influence on computation. Hence, it brings us a new research topic—the trilinear method for constructing analytic solutions to nonlinear PDEs. Recently, we find that some new analytic solutions of nonlinear PDEs can be obtained through trilinear differential equations. Some interesting examples and general properties of trilinear equations have been studied in [34].

In this work, we would like to construct lump-type solutions and breather lump–kink interaction solutions for a (3+1)-dimensional nonlinear equation, by utilizing the trilinear method. The rest of the paper is organized as follows. In Sect. 2, we present a trilinear equation and construct its lump-type solutions. In Sect. 3, we compute breather lump–kink soliton solutions. Finally, we give some discussions and concluding remarks in Sect. 4.

2 Trilinear form and lump-type solution

In this section, we present a trilinear form and its corresponding (3+1)-dimensional nonlinear equation, with the aid of trilinear differential operators. And then, we get a set of lump-type solutions for the resulting (3+1)-dimensional nonlinear equation.

2.1 Trilinear form

Consider the trilinear operators in (3+1)-dimensional space defined by the following rule [34]

$$\begin{aligned}&\left(D_{\bar{p}_1,x}^K\cdot D_{\bar{p}_2,y}^L\cdot D_{\bar{p}_3,z}^M\cdot D_{\bar{p}_4,t}^N\right)f\cdot g\cdot h\\&=\left(\alpha_{p_1}\partial_x+\alpha_{p_1'}\partial_{x'}+\alpha_{p_1''}\partial_{x''}\right)^K\\&\cdot\left(\alpha_{p_2}\partial_y+\alpha_{p_2'}\partial_{y'}+\alpha_{p_2''}\partial_{y''}\right)^L\\&\cdot\left(\alpha_{p_3}\partial_z+\alpha_{p_3'}\partial_{z'}+\alpha_{p_3''}\partial_{z''}\right)^M\\&\cdot\left(\alpha_{p_4}\partial_t+\alpha_{p_4'}\partial_{t'}+\alpha_{p_4''}\partial_{t''}\right)^N\\&\times f(x,y,z,t)g\left(x',y',z',t'\right)\\&h\left(x'',y'',z'',t''\right)|_{x=x'=x'',y=y'=y'',z=z'=z'',t=t'=t''},\end{aligned} \tag{3}$$

where $\bar{p}_i=\langle p_i,p_i',p_i''\rangle$ $(1\le i\le 4)$, K, L, M and N are arbitrary nonnegative integers. And the powers of α_s^m $(s\ge 1)$ are signs, defined by:

$$\begin{aligned}&s=2k(k\in\mathbb{N}):+,-,+,-,\cdots,\\&s=1:+,+,+,+,\cdots,\\&s=3:+,-,+,+,-,+,\cdots,\\&s=5:+,-,+,-,+,+,-,+,-,+,\cdots,\\&s=7:+,-,+,-,+,-,+,+,-,+,-,+,-,+,\cdots,\end{aligned}$$

for $m=0,1,2,\ldots$, and then, by setting $\bar{p}_i=\langle p_i,p_i',p_i''\rangle=\langle 1,2,3\rangle$, we have

$$\begin{aligned}D_xf\cdot g\cdot h&=f_xgh-fg_xh-fgh_x,\\D_x^2f\cdot g\cdot h&=f_{xx}gh-2f_xg_xh-2f_xgh_x\\&\quad+fg_{xx}h+2fg_xh_x+fgh_{xx},\\D_xD_yf\cdot g\cdot h&=f_{xy}gh-f_xg_yh-f_xgh_y-f_yg_xh\\&\quad+fg_{xy}h+fg_xh_y-f_ygh_x\\&\quad+fg_yh_x+fgh_{xy},\end{aligned} \tag{4}$$

and others. Let $f=g=h$, we also get the following cases:

$$\begin{aligned}D_x^2f\cdot f\cdot f&=3f_{xx}f^2-2{f_x}^2f,\\D_xD_yf\cdot f\cdot f&=3f_{xy}f^2-2f_xf_yf,\\D_x^3D_yf\cdot f\cdot f&=f_{xxxy}f^2-2f_{xxx}f_yf-6f_{xxy}f_xf\\&\quad-6f_{xx}f_xf_y-6f_{xy}{f_x}^2\\&\quad+18f_{xy}f_{xx}f,\\D_x^4f\cdot f\cdot f&=f_{xxxx}f^2-8f_{xxx}f_xf-12f_{xx}{f_x}^2\\&\quad+18{f_{xx}}^2f,\end{aligned} \tag{5}$$

where $f=f(x,y,z,t)$ is an unknown function. Consider a P-polynomial in (3+1)-dimensional reads [35]

$$P(x,y,z,t)=\delta x^4+\gamma x^3y+\alpha x^3z+xt+yt-zz, \tag{6}$$

and the corresponding trilinear equation is

$$\left(\delta D_x^4+\gamma D_x^3 D_y+\alpha D_x^3 D_z+D_x D_t +D_y D_t-D_z^2\right) f\cdot f\cdot f=0, \tag{7}$$

namely,

$$\begin{aligned}&\delta\left(f_{xxxx} f^2-8 f_{xxx} f_x f-12 f_{xx} {f_x}^2 +18 {f_{xx}}^2 f\right)+\gamma\left(f_{xxxy} f^2-2 f_{xxx} f_y f-6 f_{xxy} f_x f\right.\\&\left.-6 f_{xx} f_x f_y-6 f_{xy} {f_x}^2+18 f_{xx} f_{xy} f\right)+\alpha\left(f_{xxxz} f^2\right.\\&-2 f_{xxx} f_z f-6 f_{xxz} f_x f-6 f_{xx} f_x f_z\\&\left.-6 f_{xz} {f_x}^2+18 f_{xx} f_{xz} f\right)+3 f_{xt} f^2\\&-2 f_x f_t f+3 f_{yt} f^2-2 f_y f_t f-3 f_{zz} f^2+2 {f_z}^2 f=0,\end{aligned} \tag{8}$$

under the transformation $u=2(\ln f)_x$, Eq. (8) is mapped into a $(3+1)$-dimensional form

$$\delta u_{xxxx}+6\delta u_x u_{xx}+\gamma u_{xxxy}+\alpha u_{xxxz} +3\gamma(u_x u_{xy}+u_{xx} u_y)+3\alpha(u_x u_{xz}+u_{xx} u_z) +u_{xt}+u_{yt}-u_{zz}=0, \tag{9}$$

and the corresponding generalized form of Eq. (9) is

$$\delta u_{xxxx}+\lambda u_x u_{xx}+\gamma u_{xxxy} +\alpha u_{xxxz}+\mu(u_x u_{xy}+u_{xx} u_y) +\beta(u_x u_{xz}+u_{xx} u_z)+\eta_1 u_{xt} +\eta_2 u_{yt}-\eta_3 u_{zz}=0. \tag{10}$$

By setting $\alpha=\eta_2=\eta_3=0$, $\lambda=6\delta$, $\mu=3\gamma$ and $\eta_1=1$ in Eq. (10), one can be written as

$$\delta(u_{xxxx}+6 u_x u_{xx}) +\gamma(u_{xxxy}+3 u_x u_{xy}+3 u_{xx} u_y)+u_{xt}=0, \tag{11}$$

which is the $(2+1)$-dimensional Bogoyavlensky–Konopelchenko equation [36]. Thus, Eq. (10) is an extension of Eq. (11), namely, a $(3+1)$-dimensional generalized Bogoyavlensky–Konopelchenko (GBK) equation, which can describe interaction of Riemann wave along with three space directions in nonlinear media.

2.2 Lump-type solution

By taking the parameters $\delta=\gamma=\alpha=\eta_1=\eta_2=\eta_3=1$, $\lambda=6$, $\mu=\beta=3$, Eq. (10) has the following form:

$$u_{xxxx}+6 u_x u_{xx}+u_{xxxy}+u_{xxxz} +3(u_x u_{xy}+u_{xx} u_y)+3(u_x u_{xz} +u_{xx} u_z)+u_{xt}+u_{yt}-u_{zz}=0, \tag{12}$$

whose corresponding trilinear equation is

$$\left(D_x^4+D_x^3 D_y+D_x^3 D_z +D_x D_t+D_y D_t-D_z^2\right) f\cdot f\cdot f=0, \tag{13}$$

i.e.

$$\begin{aligned}&f_{xxxx} f^2-8 f_{xxx} f_x f-12 f_{xx} {f_x}^2+18 {f_{xx}}^2 f\\&+f_{xxxy} f^2-2 f_{xxx} f_y f-6 f_{xxy} f_x f\\&-6 f_{xx} f_x f_y-6 f_{xy} {f_x}^2+18 f_{xx} f_{xy} f+f_{xxxz} f^2\\&-2 f_{xxx} f_z f-6 f_{xxz} f_x f-6 f_{xx} f_x f_z\\&-6 f_{xz} {f_x}^2+18 f_{xx} f_{xz} f+3 f_{xt} f^2-2 f_x f_t f\\&+3 f_{yt} f^2-2 f_y f_t f-3 f_{zz} f^2+2 {f_z}^2 f=0.\end{aligned} \tag{14}$$

Take the function f in Eq. (14) as the following form [2–13]

$$f={\zeta_1}^2+{\zeta_2}^2+g_0,\quad \zeta_i=a_i x+b_i y+c_i z+d_i t+h_i,\ i=1,2, \tag{15}$$

where $g_0>0$, and $a_i, b_i, c_i, d_i, h_i (i=1,2)$ are real constants to be calculated. Substituting (15) into trilinear Eq. (14), we reach a set of algebraic equations, and then, we can derive $a_i, b_i, c_i, d_i, h_i (i=1,2)$ with the aid of Maple. This way, we can obtain the following solutions of coefficients:

$$\begin{aligned}&a_i=a_i (i=1,2),\ b_1=d_1-a_1,\\&c_1=-d_1,\ d_i=d_i (i=1,2),\\&b_2=d_2-a_2,\ c_2=-d_2,\ h_i=h_i (i=1,2),\ g_0=g_0,\end{aligned} \tag{16}$$

and under the transformation $u=2(\ln f)_x$, we can get the following lump-type solutions to Eq. (12):

$$u=\frac{4(a_1\zeta_1+a_2\zeta_2)}{f}, \tag{17}$$

and

$$f=(a_1 x+(d_1-a_1) y-d_1 z+d_1 t+h_1)^2 +(a_2 x+(d_2-a_2) y-d_2 z+d_2 t+h_2)^2+g_0, \tag{18}$$

where the functions ζ_1 and ζ_2 are given as follows:

$$\begin{aligned}&\zeta_1=a_1 x+(d_1-a_1) y-d_1 z+d_1 t+h_1,\\&\zeta_2=a_2 x+(d_2-a_2) y-d_2 z+d_2 t+h_2.\end{aligned} \tag{19}$$

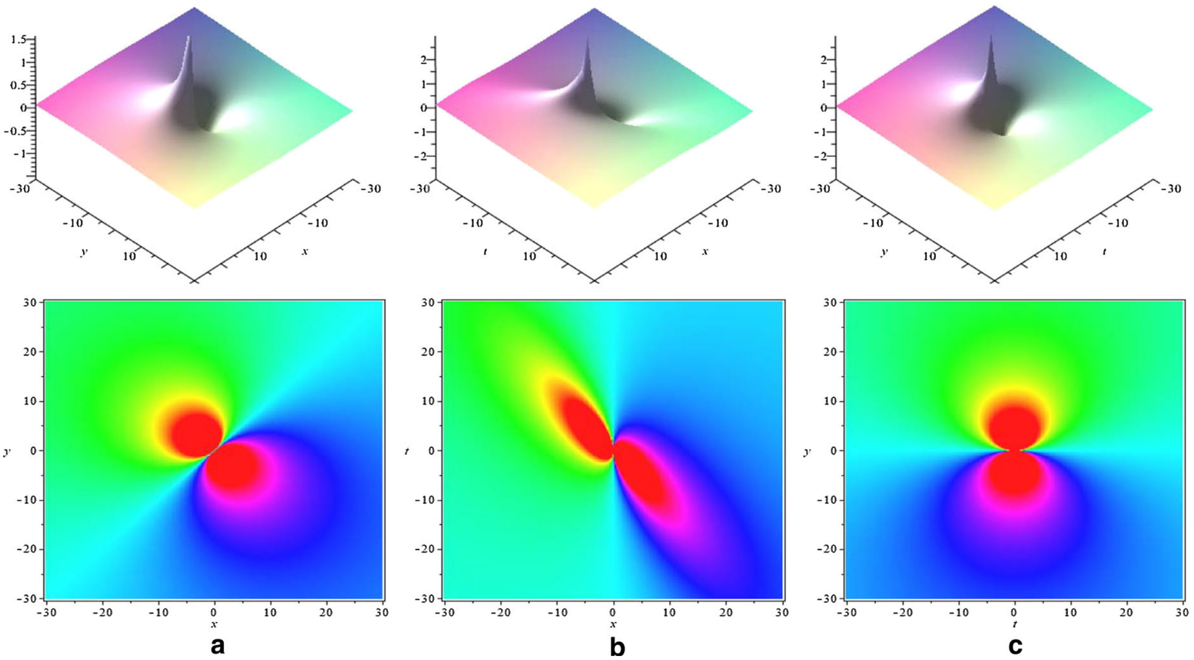

Fig. 1 3-D plots (top) and density plots (bottom) of u via Eq. (17) at three coordinates: **a** the x_y_u-coordinate by $z=-x, t=0$; **b** the x_t_u-coordinate by $z=-x, y=0$; and **c** the t_y_u-coordinate by $z=y, x=0$; with $a_1=-2, d_1=2, h_1=0, a_2=2, d_2=2, h_2=0, g_0=1$

By setting $g_0>0$, the solution f is positive, and u is localized in all directions in the space. Thus, the lump-type solution $u\to 0$ at any given time t, if the corresponding sum of squares ${\zeta_1}^2+{\zeta_2}^2\to\infty$. Figure 1 shows the 3-D and density plots of u, by choosing appropriate values of these parameters in Eq. (17).

3 Breather lump–kink solitons

In this section, we first propose a type of new interaction solutions in terms of a new combination of cosine function, hyperbolic cosine function and exponential functions, namely, breather lump–kink soliton. Then, we will investigate the interaction between the breather lump-type soliton and multi-kink solitons for Eq. (12), by taking the function f in Eq. (14) in the following form

$$f=k_1\cos(\zeta_1)+k_2\cosh(\zeta_2)+\sum_{j=3}^{N}k_je^{\zeta_j}+g_0, \quad (20)$$

where $\zeta_i=a_ix+b_iy+c_iz+d_it+h_i(i=1,2\ldots N)$, and $k_i(i=1,2,\ldots N)$, $N\geq 3$ are real constants.

Procedure 3.1 (Acquisition of interaction solutions) Substituting Eq. (20) into Eq. (14), we reach the coefficients of different polynomials and set the coefficients of these terms to zero, and we get a set of algebraic equations in the parameters. And then, we can derive $a_i, b_i, c_i, d_i, h_i(i=1,2\ldots N)$ and $k_i(i=1,2,\ldots N)$. Finally, substituting the identified values of $a_i, b_i, c_i, d_i, h_i(i=1,2\ldots N)$ and $k_i(i=1,2,\ldots N)$ into Eq. (20), and through the transformation $u=2(\ln f)_x$, we finally obtain abundant interaction solutions of Eq. (12).

3.1 Breather lump-type soliton and a kink soliton

To search for interaction solutions between the breather lump-type soliton and a kink soliton of Eq. (12), we suppose $N=3$ and Eq. (20) has the following form

$$f=k_1\cos(\zeta_1)+k_2\cosh(\zeta_2)+k_3e^{\zeta_3}+g_0, \quad (21)$$

where $\zeta_i=a_ix+b_iy+c_iz+d_it+h_i(i=1,2,3)$ and $g_0>0$. Here $a_i, b_i, c_i, d_i, h_i(i=1,2,3)$ and $k_i(i=1,2,3)$ are real constants to be calculated. According

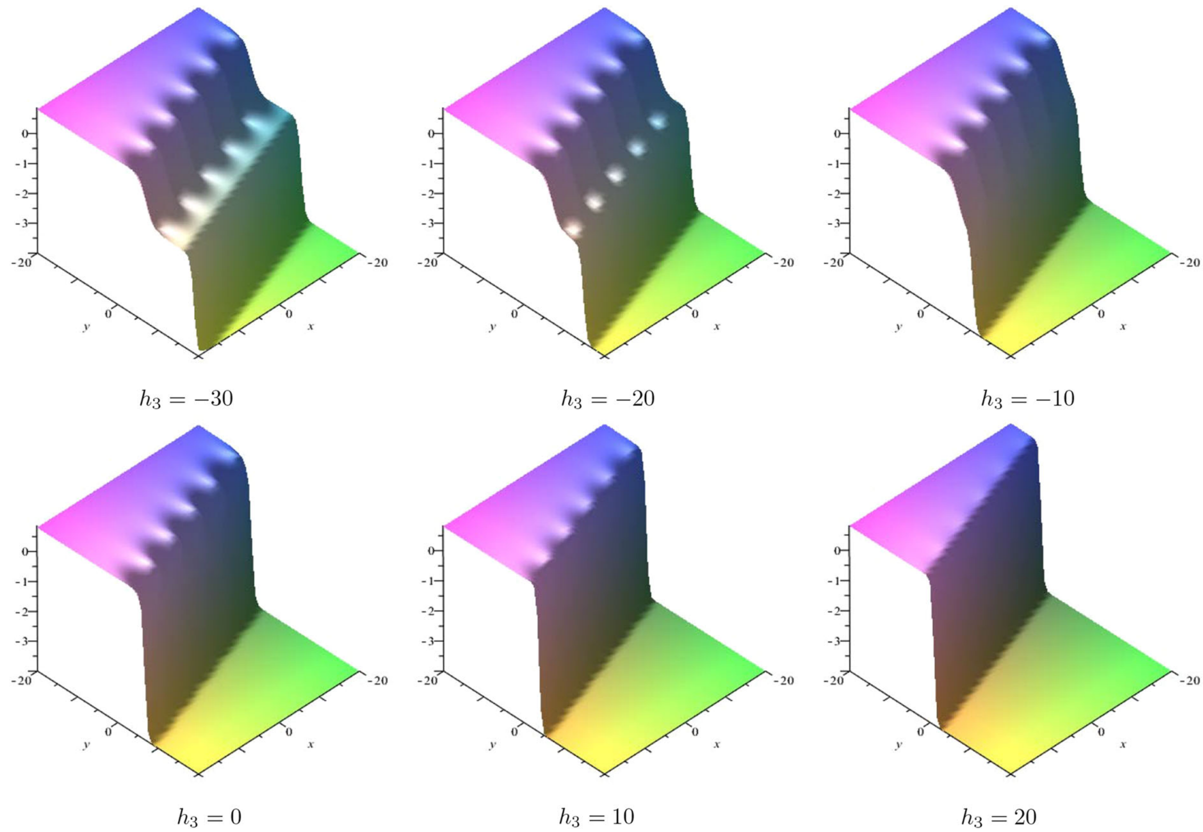

Fig. 2 The 3-D evolution plots of u via Eq. (23) at $h_3=-30$, $h_3=-20$, $h_3=-10$, $h_3=0$, $h_3=10$ and $h_3=20$

to Procedure 3.1, we can obtain the following solutions of coefficients:

$$\begin{aligned} &a_i=a_i(i=1,2,3),\ b_1=d_1-a_1,\ c_1=-d_1,\\ &b_2=d_2-a_2,\ c_2=-d_2, b_3=d_3-a_3,\\ &c_3=-d_3,\ d_i=d_i(i=1,2,3),\ h_i=h_i(i=1,2,3),\\ &g_0=g_0,\ k_i=k_i(i=1,2,3), \end{aligned} \tag{22}$$

and under the transformation $u=2(\ln f)_x$, we can get the following breather lump–kink soliton to Eq. (12):

$$u=-\frac{2(k_1a_1\sin(\zeta_1)-k_2a_2\sinh(\zeta_2)-k_3a_3e^{\zeta_3})}{f}, \tag{23}$$

and

$$\begin{aligned} f=&k_1\cos(a_1x+(d_1-a_1)y-d_1z+d_1t+h_1)\\ &+k_2\cosh(a_2x+(d_2-a_2)y-d_2z+d_2t+h_2)\\ &+k_3e^{(a_3x+(d_3-a_3)y-d_3z+d_3t+h_3)}+g_0, \end{aligned} \tag{24}$$

where the functions ζ_1, $\zeta_2$and ζ_3 are given as follows:

$$\begin{aligned} \zeta_1&=a_1x+(d_1-a_1)y-d_1z+d_1t+h_1,\\ \zeta_2&=a_2x+(d_2-a_2)y-d_2z+d_2t+h_2,\\ \zeta_3&=a_3x+(d_3-a_3)y-d_3z+d_3t+h_3. \end{aligned} \tag{25}$$

Figures 2 and 3 show 3-D and corresponding density evolution behaviours of u, by setting $t=0$, $z=-x$, $a_1=0.4$, $d_1=0.4$, $h_1=0$, $a_2=-0.4$, $d_2=0.2$, $h_2=0$, $a_3=-2$, $d_3=1$, $g_0=1$ and $k_i=1(i=1,2,3)$; and picking different values for h_3 from -30 to 20, where we change the coordinate of the kink soliton to make it collide with the breather lump-type soliton. We can see the mutual reaction of breather lump-type wave and kink solitary wave.

The breather lump-type soliton and a kink soliton begin appearing at $h_3=-30$. The kink soliton begins colliding with the breather lump-type soliton at $h_3=-20$. Finally, the breather lump-type soliton is swallowed by the kink soliton at $h_3=20$ in Figs. 2 and 3.

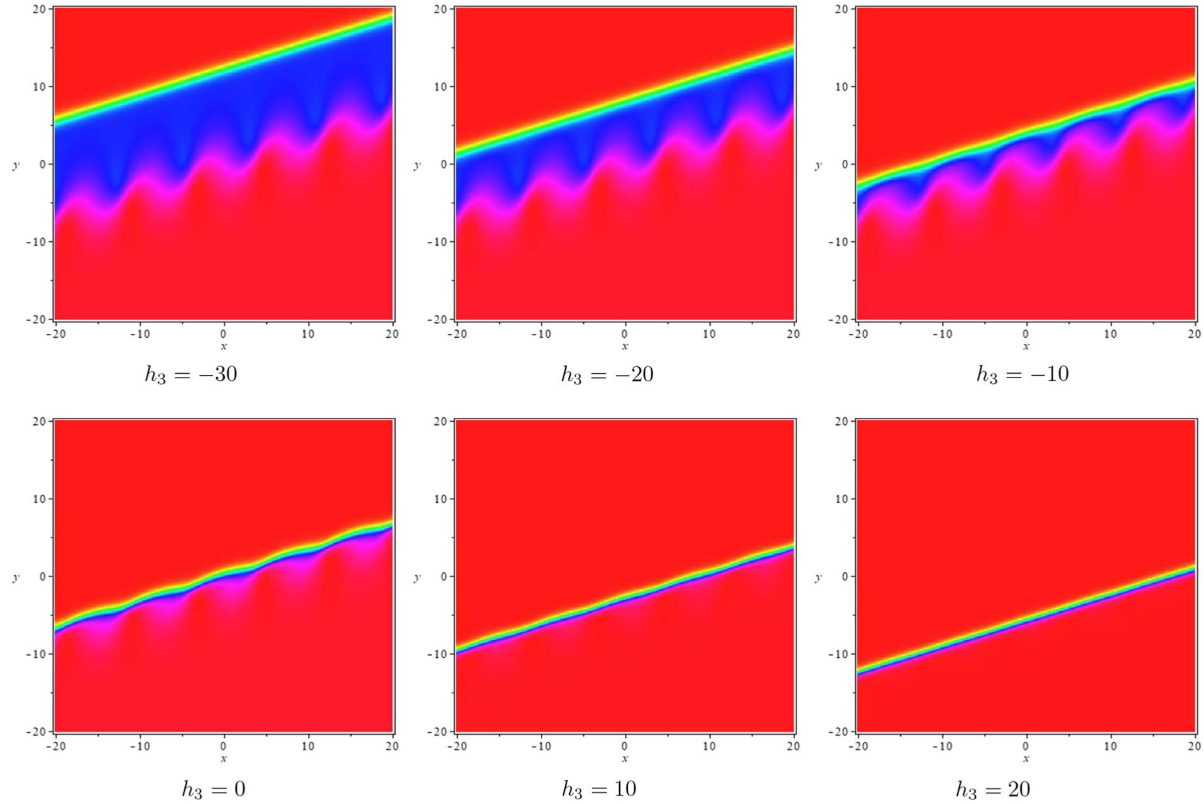

Fig. 3 The density evolution plots of u via Eq. (23) at $h_3 = -30$, $h_3 = -20$, $h_3 = -10$, $h_3 = 0$, $h_3 = 10$ and $h_3 = 20$

3.2 Breather lump-type soliton and two kink solitons

By choosing $N = 4$, Eq. (20) has the following form

$$f = k_1 \cos(\zeta_1) + k_2 \cosh(\zeta_2) + \sum_{j=3}^{4} k_j e^{\zeta_j} + g_0, \quad (26)$$

where $\zeta_i = a_i x + b_i y + c_i z + d_i t + h_i (i = 1, 2 \ldots 4)$ and $g_0 > 0$. Here, $a_i, b_i, c_i, d_i, h_i (i = 1, 2 \ldots 4)$ and $k_i (i = 1, 2 \ldots 4)$ are real constants to be calculated. According to Procedure 3.1, we can obtain the following solutions of coefficients:

$$\begin{aligned} &a_i = a_i (i = 1, 2 \ldots 4),\ b_1 = d_1 - a_1, c_1 = -d_1, \\ &d_1 = d_1, b_2 = -4{a_2}^3 - a_2,\ c_2 = 4{a_2}^3, \\ &d_2 = -4{a_2}^3,\ b_3 = d_3 - a_3,\ c_3 = -d_3, \\ &d_3 = d_3,\ h_i = h_i (i = 1, 2 \ldots 4), \\ &b_4 = b_4,\ c_4 = c_4,\ d_4 = d_4, \\ &g_0 = g_0,\ k_i = k_i (i = 1, 2 \ldots 4), \end{aligned} \quad (27)$$

and under the transformation $u = 2(\ln f)_x$, we can get the following breather lump–kink soliton to Eq. (12):

$$u = -\frac{2(k_1 a_1 \sin(\zeta_1) - k_2 a_2 \sinh(\zeta_2) - k_3 a_3 e^{\zeta_3} - k_4 a_4 e^{\zeta_4})}{f}, \quad (28)$$

and

$$\begin{aligned} f = {} & k_1 \cos(a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1) \\ & + k_2 \cosh\left(a_2 x - \left(4{a_2}^3 + a_2\right) y \right. \\ & \left. + 4{a_2}^3 z - 4{a_2}^3 t + h_2\right) \\ & + k_3 e^{(a_3 x + (d_3 - a_3) y - d_3 z + d_3 t + h_3)} \\ & + k_4 e^{(a_4 x + b_4 y + c_4 z + d_4 t + h_4)} + g_0, \end{aligned} \quad (29)$$

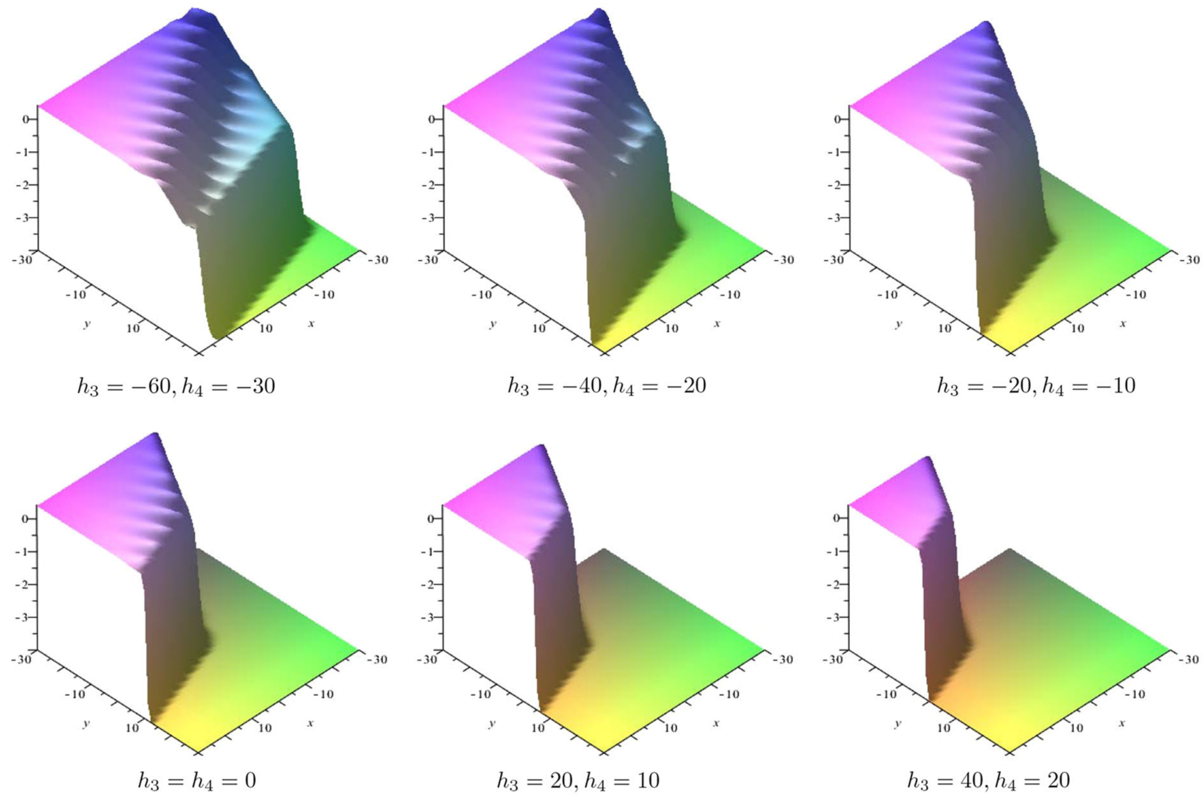

Fig. 4 3-D evolution plots of u via Eq. (28) at $h_3 = -60, h_4 = -30, h_3 = -40, h_4 = -20, h_3 = -20, h_4 = -10, h_3 = 0, h_4 = 0, h_3 = 20, h_4 = 10$ and $h_3 = 40, h_4 = 20$

where the functions $\zeta_i (i = 1, 2 \ldots 4)$ are given as follows:

$$\begin{aligned}
\zeta_1 &= a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1, \\
\zeta_2 &= a_2 x - \left(4 a_2{}^3 + a_2\right) y + 4 a_2{}^3 z - 4 a_2{}^3 t + h_2, \\
\zeta_3 &= a_3 x + (d_3 - a_3) y - d_3 z + d_3 t + h_3, \\
\zeta_4 &= a_4 x + b_4 y + c_4 z + d_4 t + h_4. \qquad (30)
\end{aligned}$$

Figures 4 and 5 show 3-D and corresponding density evolution behaviours of u, by setting $t = 0$, $z = -x$, $a_1 = 0.2$, $d_1 = 0.6$, $h_1 = 0$, $a_2 = -0.2$, $h_2 = 0$, $a_3 = -2$, $d_3 = 1$, $a_4 = -2$, $b_4 = 0.6$, $c_4 = -0.8$, $d_4 = 1$, $g_0 = 1$ and $k_i (i = 1, 2 \ldots 4) = 1$ and picking different values for h_3 from -60 to 40; and h_4 from -30 to 20, where we change the coordinates of the two kink solitons to make them collide with the breather lump-type soliton.

We can see that the breather lump-type soliton is surrounded by two kink solitons at $h_3 = -60, h_4 = -30$ in Figs. 4 and 5. Then, the breather lump-type soliton is swallowed eventually by the kink solitons when $h_3 = 40, h_4 = 20$.

3.3 Breather lump-type soliton and three kink solitons

By choosing $N = 5$, Eq. (20) has the following form

$$f = k_1 \cos(\zeta_1) + k_2 \cosh(\zeta_2) + \sum_{j=3}^{5} k_j e^{\zeta_j} + g_0, \quad (31)$$

where $\zeta_i = a_i x + b_i y + c_i z + d_i t + h_i (i = 1, 2 \ldots 5)$ and $g_0 > 0$. Here, $a_i, b_i, c_i, d_i, h_i (i = 1, 2 \ldots 5)$ and $k_i (i = 1, 2 \ldots 5)$ are real constants to be calculated. According to Procedure 3.1, we can obtain the following solutions of coefficients:

$$\begin{aligned}
&a_i = a_i (i = 1, 2 \ldots 5),\ b_1 = d_1 - a_1,\ c_1 = -d_1, \\
&d_1 = d_1,\ b_2 = -4 a_2{}^3 - a_2,\ c_2 = 4 a_2{}^3, \\
&d_2 = -4 a_2{}^3,\ b_3 = d_3 - a_3,\ c_3 = -d_3,\ d_3 = d_3,
\end{aligned}$$

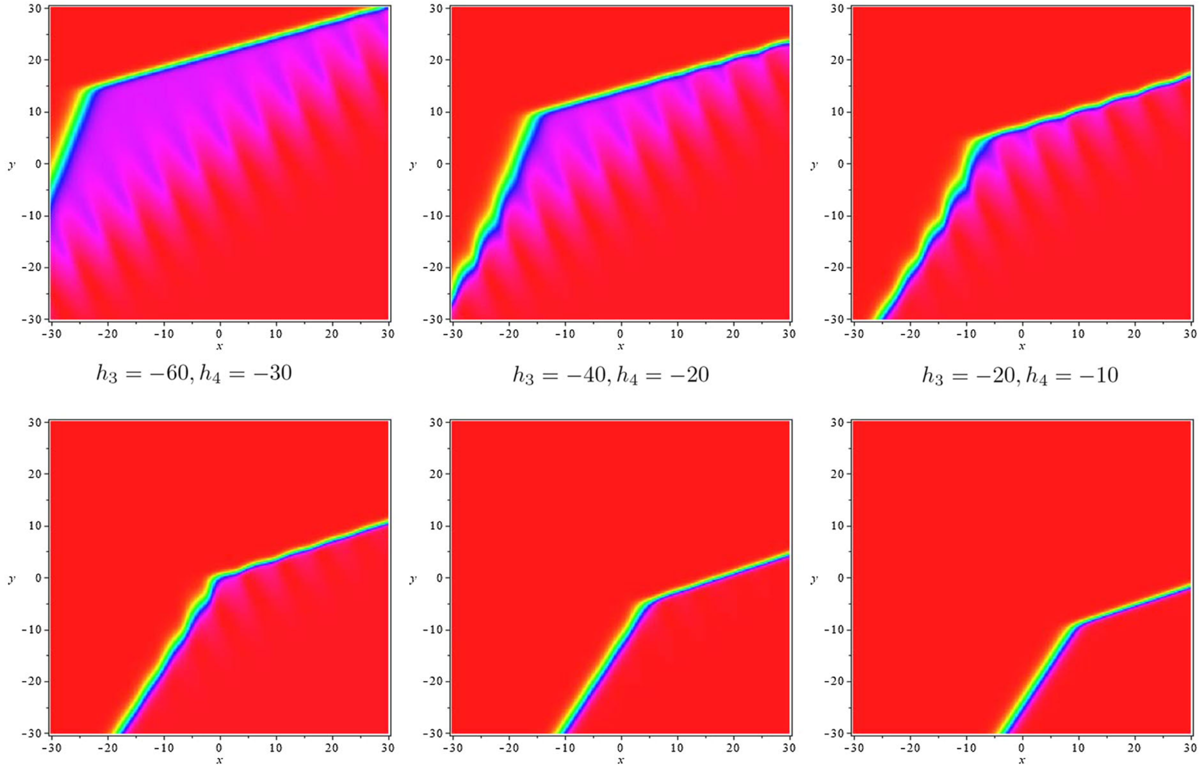

Fig. 5 Density evolution plots of u via Eq. (28) at $h_3 = -60$, $h_4 = -30$, $h_3 = -40$, $h_4 = -20$, $h_3 = -20$, $h_4 = -10$, $h_3 = 0$, $h_4 = 0$, $h_3 = 20$, $h_4 = 10$ and $h_3 = 40$, $h_4 = 20$

$$h_i = h_i (i = 1, 2 \ldots 5),\ b_4 = b_4,\ c_4 = c_4,\ d_4 = d_4,\ b_5 = d_5 - a_5,\ c_5 = -d_5,\ d_5 = d_5,\ g_0 = g_0,\ k_i = k_i (i = 1, 2 \ldots 5), \tag{32}$$

and under the transformation $u = 2(\ln f)_x$, we can get the following breather lump–kink soliton to Eq. (12):

$$u = -\frac{2(k_1 a_1 \sin(\zeta_1) + k_2 a_2 \sinh(\zeta_2) - k_3 a_3 e^{\zeta_3} - k_4 a_4 e^{\zeta_4} - k_5 a_5 e^{\zeta_5})}{f}, \tag{33}$$

and

$$\begin{aligned} f = {} & k_1 \cos(a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1) \\ & + k_2 \cosh \left(a_2 x - \left(4{a_2}^3 + a_2 \right) y + 4{a_2}^3 z - 4{a_2}^3 t + h_2 \right) \\ & + k_3 e^{(a_3 x + (d_3 - a_3) y - d_3 z + d_3 t + h_3)} \\ & + k_4 e^{(a_4 x + b_4 y + c_4 z + d_4 t + h_4)} \\ & + k_5 e^{(a_5 x + (d_5 - a_5) y - d_5 z + d_5 t + h_5)} + g_0, \end{aligned} \tag{34}$$

where the functions $\zeta_i (i = 1, 2 \ldots 5)$ are given as follows:

$$\begin{aligned} \zeta_1 &= a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1, \\ \zeta_2 &= a_2 x - \left(4{a_2}^3 + a_2 \right) y + 4{a_2}^3 z - 4{a_2}^3 t + h_2, \\ \zeta_3 &= a_3 x + (d_3 - a_3) y - d_3 z + d_3 t + h_3, \\ \zeta_4 &= a_4 x + b_4 y + c_4 z + d_4 t + h_4, \\ \zeta_5 &= a_5 x + (d_5 - a_5) y - d_5 z + d_5 t + h_5. \end{aligned} \tag{35}$$

Figures 6 and 7 show 3-D and corresponding density evolution behaviours of u, by setting $t = 0$, $z = -x$, $a_1 = 0.2$, $d_1 = 0.4$, $h_1 = 0$, $a_2 = -0.2$, $h_2 = 0$, $a_3 = -2$, $d_3 = 1$, $a_4 = -2$, $b_4 = 0.6$, $c_4 = -0.8$, $d_4 = 1$, $a_5 = 0.4$, $d_5 = -0.4$, $h_5 = -10$, $g_0 = 1$ and $k_i (i = 1, 2 \ldots 5) = 1$ and picking different values for h_3 from -60 to 40; and h_4 from -30 to 20, where we change the coordinates of the two kink solitons to make them collide with the breather lump-type soliton, and the third one does not move.

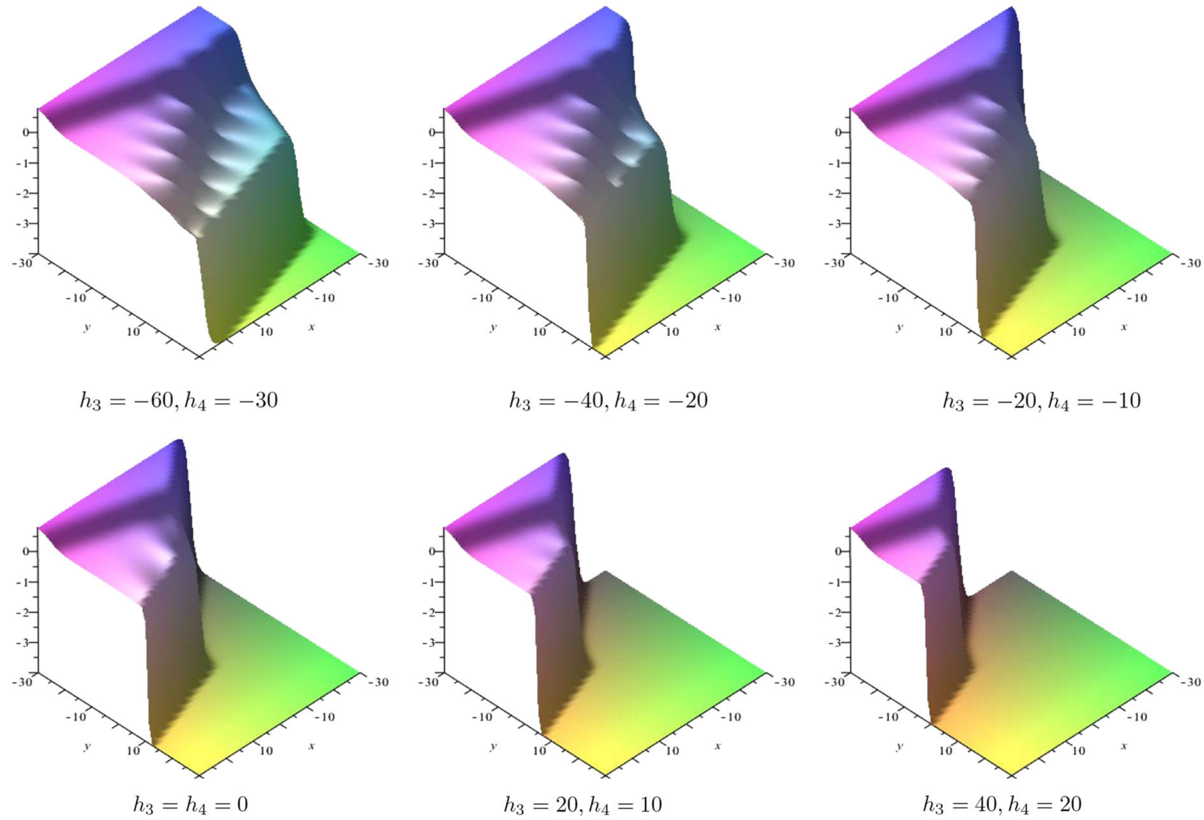

Fig. 6 3-D evolution plots of u via Eq. (33) at $h_3=-60, h_4=-30, h_3=-40, h_4=-20, h_3=-20, h_4=-10, h_3=0, h_4=0, h_3=20, h_4=10$ and $h_3=40, h_4=20$

The breather lump-type soliton is surrounded by three kink solitons at $h_3=-60, h_4=-30$ in Figs. 6 and 7. Finally, the breather lump-type soliton completely decays and is swallowed by the kink solitons when $h_3=40, h_4=20$.

3.4 Breather lump-type soliton and four kink solitons

By choosing $N=6$, Eq. (20) has the following form

$$f=k_1\cos(\zeta_1)+k_2\cosh(\zeta_2)+\sum_{j=3}^{6}k_je^{\zeta_j}+g_0, \quad (36)$$

where $\zeta_i=a_ix+b_iy+c_iz+d_it+h_i(i=1,2\ldots6)$ and $g_0>0$. Here $a_i, b_i, c_i, d_i, h_i(i=1,2\ldots6)$ and $k_i(i=1,2\ldots6)$ are real constants to be calculated. According to Procedure 3.1, we can obtain the following solutions of coefficients:

$$\begin{aligned}&a_i=a_i(i=1,2,4,5,6),\ b_1=d_1-a_1,\\&c_1=-d_1,\ d_1=d_1,\ b_2=-4{a_2}^3-a_2,\\&c_2=4{a_2}^3,\ d_2=-4{a_2}^3,\ a_3=-a_5,\\&b_3=a_5-d_5,\ c_3=d_5,\ d_3=-d_5,\\&h_i=h_i(i=1,2\ldots6),\ b_4=b_4,\\&c_4=c_4,\ d_4=d_4,\ b_5=d_5-a_5,\ c_5=-d_5,\\&d_5=d_5,\ b_6=d_6-a_6,\ c_6=-d_6,\ d_6=d_6,\ g_0=g_0,\\&k_i=k_i(i=1,2\ldots6),\end{aligned} \quad (37)$$

and under the transformation $u=2(\ln f)_x$, we can get the following breather lump–kink soliton to Eq. (12):

$$u=-\frac{2(k_1a_1\sin(\zeta_1)+k_2a_2\sinh(\zeta_2)+k_3a_5e^{\zeta_3}-k_4a_4e^{\zeta_4}-k_5a_5e^{\zeta_5}-k_6a_6e^{\zeta_6})}{f}, \quad (38)$$

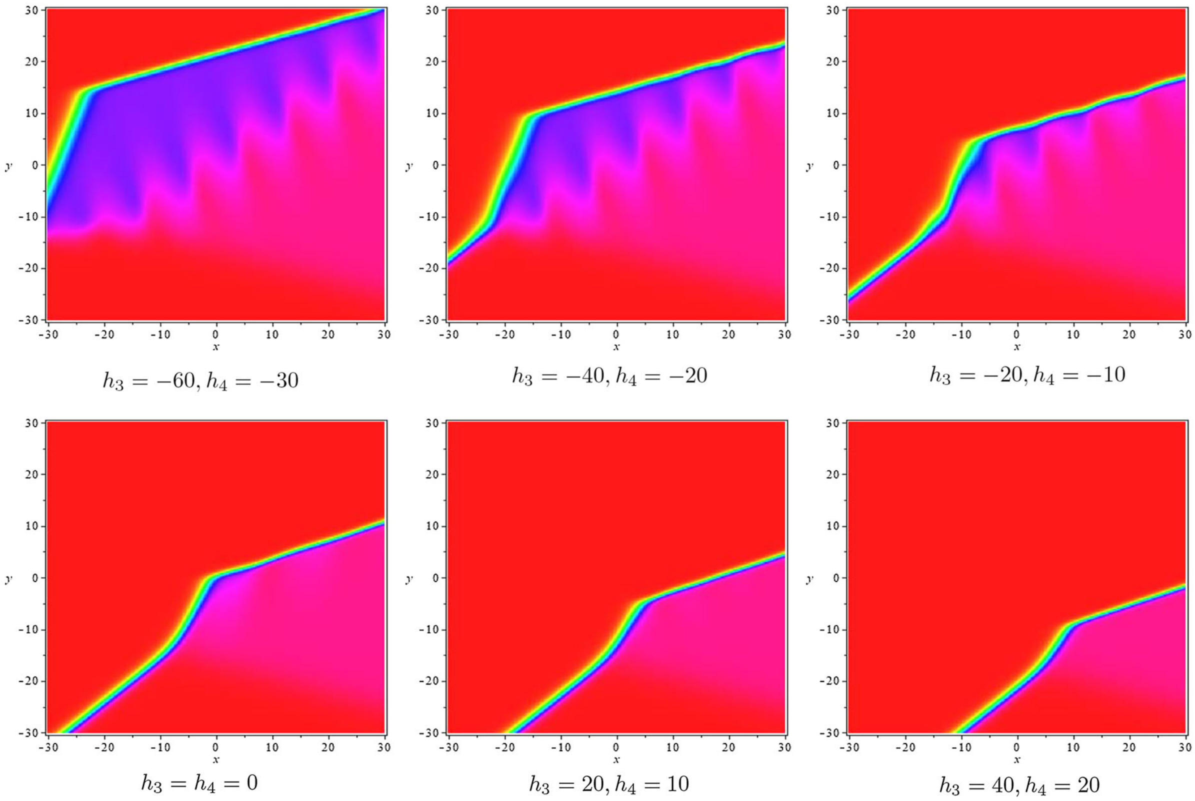

Fig. 7 Density evolution plots of u via Eq. (33) at $h_3 = -60$, $h_4 = -30$, $h_3 = -40$, $h_4 = -20$, $h_3 = -20$, $h_4 = -10$, $h_3 = 0$, $h_4 = 0$, $h_3 = 20$, $h_4 = 10$ and $h_3 = 40$, $h_4 = 20$

and

$$\begin{aligned} f &= k_1 \cos(a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1) \\ &\quad + k_2 \cosh\left(a_2 x - \left(4{a_2}^3 + a_2\right) y \right. \\ &\quad \left. + 4{a_2}^3 z - 4{a_2}^3 t + h_2\right) \\ &\quad + k_3 e^{(-a_5 x + (a_5 - d_5) y + d_5 z - d_5 t + h_3)} \\ &\quad + k_4 e^{(a_4 x + b_4 y + c_4 z + d_4 t + h_4)} \\ &\quad + k_5 e^{(a_5 x + (d_5 - a_5) y - d_5 z + d_5 t + h_5)}, \\ &\quad + k_6 e^{(a_6 x + (d_6 - a_6) y - d_6 z + d_6 t + h_6)} + g_0, \end{aligned} \tag{39}$$

where the functions $\zeta_i (i = 1, 2 \ldots 6)$ are given as follows:

$$\begin{aligned} \zeta_1 &= a_1 x + (d_1 - a_1) y - d_1 z + d_1 t + h_1, \\ \zeta_2 &= a_2 x - \left(4{a_2}^3 + a_2\right) y + 4{a_2}^3 z - 4{a_2}^3 t + h_2, \\ \zeta_3 &= -a_5 x + (a_5 - d_5) y + d_5 z - d_5 t + h_3, \\ \zeta_4 &= a_4 x + b_4 y + c_4 z + d_4 t + h_4, \\ \zeta_5 &= a_5 x + (d_5 - a_5) y - d_5 z + d_5 t + h_5, \\ \zeta_6 &= a_6 x + (d_6 - a_6) y - d_6 z + d_6 t + h_6. \end{aligned} \tag{40}$$

Figures 8 and 9 show 3-D and corresponding density evolution behaviours of u, by setting $t = 0$, $z = -x$, $a_1 = 0.2$, $d_1 = 0.4$, $h_1 = 0$, $a_2 = -0.2$, $h_2 = 0$, $a_4 = -2$, $b_4 = 0.6$, $c_4 = -0.8$, $d_4 = 1$, $h_4 = -20$, $a_5 = -1$, $d_5 = 0.4$, $h_5 = -10$, $a_6 = -0.4$, $d_6 = 0.8$, $h_6 = -20$, $g_0 = 1$ and $k_i (i = 1, 2 \ldots 6) = 1$; and picking different values for h_3 from -30 to 20, where we change the coordinate of one kink soliton to make it collides with the breather lump-type soliton, and three others do not move.

We can see that the breather lump-type soliton is swallowed by the kink solitons when $h_3 = 20$, in Figs. 8 and 9.

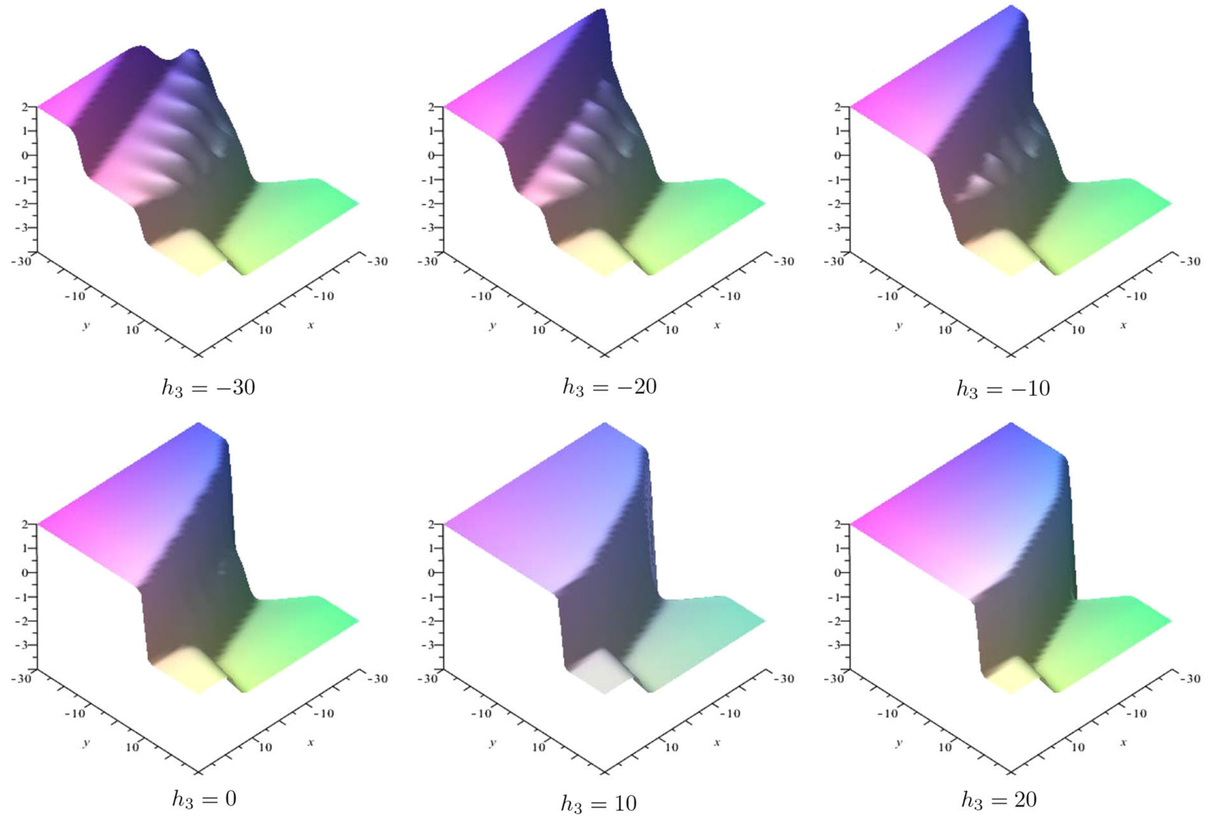

Fig. 8 3-D evolution plots of u via Eq. (38) at $h_3 = -30$, $h_3 = -20$, $h_3 = -10$, $h_3 = 0$, $h_3 = 10$ and $h_3 = 20$

Thus, we find that the breather lump–kink solitons are completely non-elastic, $u(x, y, z, t)$ is mixed solitary waves of cosine function, hyperbolic cosine function and exponential functions, and the breather lump-type soliton begins decaying as the kink solitons approach it. This interaction phenomena is similar to the fusion and fission for soliton solutions.

4 Summary and discussions

In this study, we obtained a trilinear form for a $(3+1)$-dimensional GBK equation, based on the multivariate trilinear operators in $(3+1)$ dimensions. Then, we successfully constructed a group of lump-type solutions and four types of breather lump–kink solitons, by utilizing the resulting trilinear form. Abundant interaction phenomena of the breather lump–kink solitons were depicted by plotting evolution graphs from the perspective of dynamics. It is expected that those new interesting solutions would be helpful in better understanding the $(3+1)$-dimensional GBK equation.

We presented 3-D and density graphs at three coordinates of the lump-type soliton in Fig. 1. The 3-D and density evolution behaviours of the breather lump–kink solitons were depicted in Figs. 2, 3, 4, 5, 6, 7, 8 and 9. We changed the coordinates of the kink solitons to make them collided with the breather lump-type soliton. We saw that finally, the breather lump-type soliton was swallowed by those kink solitons. It should be particularly interesting to note that, by taking $\bar{p}_i = \langle p_i, p_i', p_i'' \rangle = \langle 1, 2, 2 \rangle$, the trilinear operators $D_x^3 D_y f \cdot f \cdot f$ and $D_x^4 f \cdot f \cdot f$ can take the following forms:

$$\begin{aligned} D_x^3 D_y f \cdot f \cdot f &= 3 f_{xxxy} f^2 - 2 f_{xxx} f_y f - 6 f_{xxy} f_x f \\ &\quad -6 f_{xx} f_x f_y - 6 f_{xy} {f_x}^2 + 18 f_{xy} f_{xx} f, \\ D_x^4 f \cdot f \cdot f &= 3 f_{xxxx} f^2 - 8 f_{xxx} f_x f \\ &\quad -12 f_{xx} {f_x}^2 + 18 {f_{xx}}^2 f, \end{aligned} \tag{41}$$

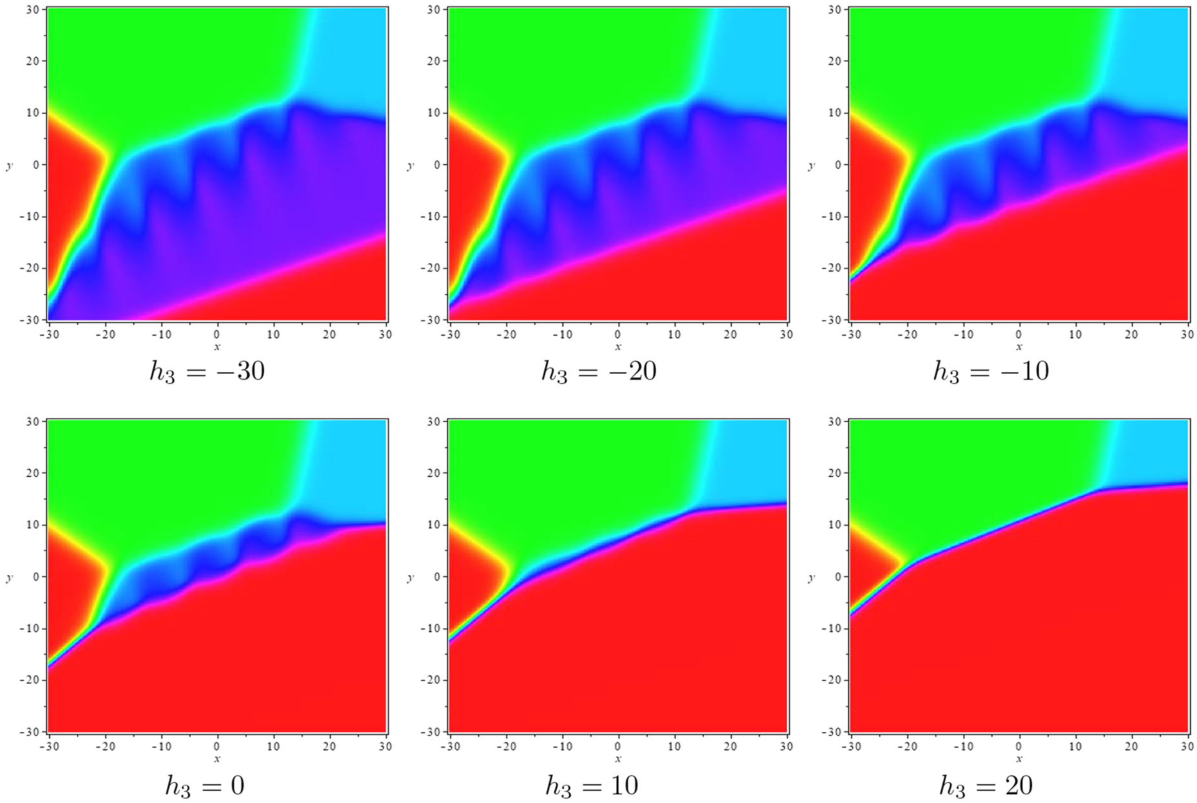

Fig. 9 Density evolution plots of u via Eq. (38) at $h_3 = -30$, $h_3 = -20$, $h_3 = -10$, $h_3 = 0$, $h_3 = 10$ and $h_3 = 20$

and thus, we can obtain another type of nonlinear equations from Eq. (13), for which the lump-type solutions and breather lump–kink solitons can be similarly generated.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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