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# Exploring exact solitary wave solutions of Kuralay-II equation based on the truncated M-fractional derivative using the Jacobi Elliptic function expansion method

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**Abstract**
This paper focuses on the Kuralay-II equation, a prominent nonlinear evolutionary equation used in various scientific fields. Our goal was to determine exact solutions to this equation by utilizing the Jacobi Elliptic function expansion method, tailored with the truncated M-fractional derivative. This method has shown high efficiency by generating many types of solutions including periodic, solitary, and trigonometric ones, highlighting its utility in mathematics. Identifying these solutions uncovers the intricate mathematical framework of the Kuralay-II Equation and expands the scope of analytical solutions for nonlinear evolution equations. The study emphasizes the efficacy of the method in solving a range of nonlinear partial differential equations (PDEs). We represented specific replies in multiple dimensions using MATLAB to provide a clear evaluation. This study showcases the effectiveness of the Jacobi Elliptic Function Expansion Method in studying complex nonlinear phenomena in engineering and science. Our finding has significant implications, enhancing our understanding and analytical ability of nonlinear wave dynamics in different physical scenarios. This method offers a viable strategy to efficiently address a variety of nonlinear evolution equations in mathematical physics.

**Keywords** Exact solutions · Space–time fractional Kuralay-II equation · Jacobi elliptic function expansion method (JEFEM)

## 1 Introduction

Fractional differential equations play an important role in several areas of applied physics, mathematics, chemistry, economics, etc. In the field of nonlinear partial differential equations (NPDEs), the Kuralay-II Equation (K-IIE) is a challenging problem that demonstrates the complex nature of mathematical model intended to replicate the real-world phenomena. Kuralay-II Equation is well known for its wide range of applications, that extend from the complex domain of quantum mechanics to the dynamic environments of fluid dynamics. However, the nonlinearity of the equation presents

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a significant challenge, frequently making standard approaches ineffective. The quest for creative and useful alternatives has been powered by this problem. In addition to being a useful academic exercise, exploring new methodologies is a critical first step towards obtaining exact and significant insights into the nature of the equation. A solution to Kuralay-II Equation could result in revolutionary breakthroughs in science and technology. Our main goal is to make a major contribution in the field of mathematical physics. With its roots in the rich historical background of elliptic functions, the Jacobi Elliptic Function Expansion Method becomes a powerful mathematical approach for solving nonlinear partial differential equations. This method offers a flexible and effective way to solve equations that are often difficult to solve using traditional analytical methods. Fractional derivatives are a fundamental mathematical tool with several applications in engineering, physics and mathematical science, among other areas. Fractal geometry is closely related to them, offering a framework for studying genetic and memory-based systems. Many variations have been studied, including the fractional derivatives of Caputo and Hattaf. The solution and modelling of differential equation of fractional nature, both linear and nonlinear, are significantly impacted by these derivatives. Accurate estimates of fractional nature derivative and solution to the corresponding equations can be constructed through the use of numerical techniques like Lagrange polynomial interpolation.

In the field of optical fiber, in particular, the space–time fractional nonlinear Schrödinger equation is very important. Researcher frequently use techniques such as the modified simple equation method and the extended $\left(\frac{G'}{G^2}\right)$ expansion method to solve this equation and obtained soliton solutions. These methods are essential for comprehending and improving optical fiber communication networks (Akram et al. 2023a). The improved tan $(\frac{\Psi(\eta)}{2})$ expansion method is used to obtain the traveling wave solutions of the nonlinear space–time fractional form of Cahn–Allen equation with beta and M-truncated derivative (Sadaf et al. 2023a). The extended-$\left(\frac{G'}{G^2}\right)$ expansion method, the $\exp(-\phi(\Omega))$-expansion method, and the generalized Kudryshov method are some the methods to obtain the accurate closed-form solutions of the Manakov system (Akram et al. 2023b). The generalized projective Riccati equation method and the two variables $(\frac{G'}{G},\frac{1}{G})$ expansion method are used to obtain chirped optical soliton solution and chirp-free solutions for the nonlinear Schrödinger equation with the anti-cubic law of nonlinearity (Arshed et al. 2023a). The generalized exponential rational function method is used in the study of the fractional complex Ginzburg–Landau model (Akram et al. 2023c). Gardner's equation has been discussed to find the solitary wave solution by using the Kudryashov's R function method, the generalized projective Riccati equation method and $(\frac{G'}{G^2})$ expansion method (Abundant solitary wave solutions of Gardner's equation using three effective integration techniques [J]. 2023). The extended sinh-Gordon equation expansion method is used to study the solitary wave properties of the (2 + 1)-dimensional Chaffee-Infante equation (Arshed et al. 2023b). The improved tan $(\frac{\Psi(\eta)}{2})$ expansion method is used to obtain the traveling wave solutions of the Complex Ginzburg–Landau Equation (Sadaf et al. 2022).

Islam et al. (2023) did an extensive investigation on the LGH equation, providing an in-depth analysis of its stability, phase plane analysis, and the separation of soliton solutions. Islam et al. (2024) advanced the comprehension of nonlinear optical phenomena by studying optical soliton solutions and doing a bifurcation analysis of the paraxial nonlinear Schrödinger equation. Expanding upon the first study conducted by Bashar et al. (2022a) on soliton solutions and the consequences of fractional components in the time-fractional modified equal width equation, our research seeks to delve deeper into the complexities of soliton theory. Arafat et al. (2023) researched nonlinear optical solitons in the fractional Biswas-Arshed Model, which advanced our understanding of intricate wave phenomena. Bashar et al. (2022b) demonstrated the effectiveness of analytical techniques in deriving optical solutions for the multidimensional (2 + 1)-dimensional Kundu–Mukherjee–Naskar equation, highlighting the significance of these methods in analyzing intricate wave phenomena. The groundbreaking research conducted by Islam et al. (2022) on analytical soliton solutions for nonlinear evolution equations has significantly contributed to the progress of our comprehension of wave dynamics, particularly in the fields of ocean engineering and science.

In a series of comprehensive studies, Akram and colleagues advanced the field of fractional calculus applied to nonlinear dynamical systems. Through a comparative analysis, they delved into the time fractional nonlinear Drinfeld–Sokolov–Wilson system, employing the modified auxiliary equation method for equation resolution (Akram et al. 2023d). Further, they uncovered soliton solutions for the generalized time-fractional Boussinesq-like equation using three innovative approaches (Akram et al. 2023e). Batool and team contributed by examining the dynamics and soliton formation within the (2 + 1)-dimensional zoomeron equation and foam drainage equation, offering deep insights into their nonlinear mathematical phenomena (Batool et al. 2023). Akram and colleagues also presented analytical solutions for the fractional complex Ginzburg–Landau model, utilizing the Generalized Exponential Rational Function Method to explore two distinct types of nonlinearities (Akram et al. 2023c). Zainab and team's research shed light on the effects of the β-derivative on the time fractional Jaulent–Miodek system, employing both the modified auxiliary equation method and the exp(− g(Ω))-expansion method to reveal intricate details of nonlinear dynamics (Zainab and Akram 2023). Additionally, Akram et al. (2022a) explored optical soliton solutions of the fractional Sasa-Satsuma equation with beta and conformable derivatives, and the dynamics of optical solitons under the complex Ginzburg–Landau equation, highlighting the impacts of Kerr law and non-Kerr law nonlinearity on soliton behavior (Akram et al. 2022b).

Investigating fractional nature derivative sheds light on numerous of phenomena such as the control mechanisms behind explosive events and the dynamics of collective behavior in nonlinear models as well as the suppression of oscillation (Zhongkui 2022; Khalid et al. 2022). In the study of fractional calculus, M-fractional derivatives are a sophisticated extension of extension of numerous well-known fractional derivatives that have been documented in academic publications. These derivatives show basic properties such as linearity, product rule, and chain rule that are characteristic of classical derivatives. With in the category of M-fractional derivatives is a recently introduced derivative called the K-fractional truncated derivative (K-FTD) (Van Mieghem. 2022). Furthermore, in this setting of the nonlinearity space–time fractional Cahn–Allen equation (FCAE), a special form of the derivative of fractional nature known as the M-truncated derivative is employed (Lemnaouar et al. 2023). It is essential to investigate fractional differential equations, anomalous diffusion processes, and fractal geometry through the study of fractal derivatives, which is closely related to Haussdorff's fractional dimension geometry theories (Sadaf et al. 2023b.) Caputo's derivative, a continuous approximation of the fractal derivative, is extensively applied in the study that investigates fractional functions and integral inequalities (Deppman et al. 2023). There are many different methods are used to construct the exact solutions of nonlinear PDEs such as, the Lie symmetry analysis (Kumar et al. 2023), the functional variable method (Rezazadeh et al. 2020), the exp-function method (Ellahi et al. 2018), the Khater method (Bibi et al. 2017), the new modified simplest equation method

(Irshad et al. 2017), the new mapping method (Dahiya et al. 2021), the extended simplest equation method (El-Ganaini et al. 2023), the first integral method (El-Ganaini and Kumar 2023), the (ψ − φ)-expansion method (Alam 2023), the improved generalized Riccati equation mapping method (Rani et al. 2021), and many more (Kumar et al. 2020, 2022; Osman 2019; Osman et al. 2019; Alquran and Jarrah 2019; Jaradat et al. 2015, 2017; Alquran et al. 2015, 2018; Syam et al. 2017; Khan et al. 2022; Yépez-Martínez and Gómez-Aguilar 2019a, b; Yépez-Martínez et al. 2018; Akinyemi 2021; Sulaiman et al. 2019).

The lack of study on the Kuralay-II equation using the Jacobi Elliptic Function Expansion Method, especially for the integration of the truncated M-fractional derivative, prompted us to launch our investigation. The Kuralay-II equation is one of the many scientific domains where this approach has not yet been fully utilized, despite its encouraging track record in solving nonlinear PDEs. We lost a chance to learn more about nonlinear evolution equations and find novel analytical solutions because of this void in the literature. We hope that by filling this knowledge gap, our study will contribute to the ongoing discussion of nonlinear wave dynamics and its applications in engineering and mathematical physics. Using this new method, we want to learn about the subtle dynamics of the Kuralay-II equation and maybe find out how systems controlled by this model behave. To shed light on its intricate dynamics and offer precise solutions, this work explores the unexplored realm of using the M-fractional derivative with the space–time fractional Kuralay-II Equation. Doing so advances our knowledge of the mechanics of the equation and adds to the larger discussion of nonlinear mathematical physics.

The remainder of this article is structured in a structured ordered way: Some of the properties and definitions of the M-fractional derivative and applications of Kuralay-II Equation in many scientific fields is provided in Sect. 1. The methodology descriptions of the Jacobi Elliptic Function Expansion technique are outlined in Sect. 2. A methodical guidance for the progressive application of the procedure to obtain exact solutions for Kuralay-IIA Equation and Kuralay-IIA Equation are provided in Sect. 3. In Sect. 4, graphs are examined for a more thorough analysis. In Sect. 5, the conclusion of the study.

## 2 Truncated M-fractional derivative and its properties

We define some important terms and discuss some properties of the truncated M-fractional derivative in this section.

**Definition 1** Let $u(\theta) : [o, \infty) \to R$, then the truncated M-derivative of $u$ of orfer $\alpha$ is shown in Sousa and Oliveira (2018)

$$D_{M,\theta}^{\alpha,\Upsilon} u(\theta) = \lim_{\tau \to 0} \frac{u\left(\theta E_\gamma\left(\tau \theta^{1-\alpha}\right)\right) - u(\theta)}{\tau}, 0 < \alpha \leq 1, \Upsilon > 0, \tag{1}$$

Here $E_\Upsilon(.)$ Represents truncated Mittag-Lliffler of one parameter that given as (El-Ganaini et al. 2023):

$$E_\Upsilon(z) = \sum_{j=0}^{i} \frac{z^j}{\Gamma\left(\Upsilon_j + 1\right)}, \Upsilon > 0 \text{ and } z \in C \tag{2}$$

**Property 1** *Consider* $\alpha\epsilon(0,1], \gamma > 0, r, s\epsilon R$, *and* $g, f\alpha-$ *differentiable at a point* $\theta > 0$, *then by Sousa and Oliveira* (2018):

$$D^{\alpha,\Upsilon}_{M,\theta}(rg(\theta) + sf(\theta) = rD^{\alpha,\Upsilon}_{M,\theta}g(\theta) + sD^{\alpha,\Upsilon}_{M,\theta}f(\theta) \tag{3}$$

$$D^{\alpha,\Upsilon}_{M,\theta}(g(\theta).f(\theta)) = g(\theta)D^{\alpha,\Upsilon}_{M,\theta}f(\theta) + f(\theta)D^{\alpha,\Upsilon}_{M,\theta}g(\theta) \tag{4}$$

$$D^{\alpha,\Upsilon}_{M,\theta}\left(\frac{g(\theta)}{f(\theta)}\right) = \frac{f(\theta)D^{\alpha,\Upsilon}_{M,\theta}g(\theta) - g(\theta)D^{\alpha,\Upsilon}_{M,\theta}f(\theta)}{(f(\theta))^2} \tag{5}$$

$$D^{\alpha,\Upsilon}_{M,\theta}(A) = 0, \text{ where A is constant} \tag{6}$$

$$D^{\alpha,\Upsilon}_{M,\theta}g(\theta) = \frac{\theta^{1-\alpha}}{\Gamma(1+\Upsilon)}\frac{dg(\theta)}{d\theta} \tag{7}$$

## 3 Jacobi elliptic function expansion scheme

In this section, we discussed the methodology description. The following steps will be carried out while pursuing this study.

*Step 1* Let us consider the given nonlinear partial differential equation say, in two variables.

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\alpha u}{\partial^\alpha x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^{2\gamma} u}{\partial x^{2\gamma}}, \dots.\right) = 0 \tag{8}$$

*Step 2* Transforming Eq. (8) by applying the traveling wave transformation as

$$u = u(\xi), \xi = w\frac{t^\alpha}{\alpha} + c\frac{x^\gamma}{\gamma} \tag{9}$$

. The Eq. (9) is reduced to the ordinary differential equation of integral order.

$$F\left(u', u'', u''', \dots\right) = 0 \tag{10}$$

In addition to this extended indirect approach, the principal objective is to increase the probability of finding solutions an auxiliary ordinary differential equation (the first category of three parameter Jacobian equation) in order to produce a significant number of Jacobian elliptic solutions for the particular equation. The auxiliary equation is visualizable.

$$\left(F'\right)^2(\xi) = PF^4(\xi) + QF^2(\xi) + R \tag{11}$$

where $F' = \frac{\partial F}{\partial \xi}, \xi = (x, t)$ and P, Q are real constants. The Eq. (11) has solution in Table (A), Where $i^2 = -1$, the jacobian elliptic functions $sn\xi = sn(\xi, m), cn\xi = cn(\xi, m)$, and $dn\xi = dn(\xi, m,)$, here $m(0 < m < 1)$ is the modulus.

The Jacobi elliptic functions mentioned in Table 2, were reduced to trigonometric and hyperbolic functions in the limiting manner for $m \to 1$ and $m \to 0$.

As a result, we construct multiple solutions i.e., periodic, hyperbolic, and trigonometric solutions for the problem. The Jacobi elliptic function expansion method can be used represented $u(\xi)$ as a finite series of Jacobi elliptic function expansion method.

$$u(\xi) = \sum_{i=0}^{\infty} a_i F^i(\xi) \tag{12}$$

where $\mathrm{F}(\xi)$ is the solution of the nonlinear ordinary Eq. (11) and n, $\mathrm{a_i}(\mathrm{i} = 0, 1, 2, \dots, \mathrm{n})$ are constants to be determine later, the n can be determining the highest order linear term

$$O\left(\frac{\partial^p u}{\partial \xi^p}\right) = n + p, p = 1, 2, 3, \dots \tag{13}$$

And the order of highest nonlinear term is

$$O\left(u^q \frac{\partial^p u}{\partial \xi^p}\right) = (q+1)n + p, q = 0, 1, 2, 3, \dots p = 1, 2, 3, \dots \tag{14}$$

In Eq. (10). Substituting Eq. (12) and setting all the coefficients of power $F$ to be zero, then a system of nonlinear algebraic equations for $\mathrm{a_i}(\mathrm{i} = 0, 1, 2, \dots .)$ is derived, by solving this system using all the values for P, Q, R, (11) in Table 1.

After all this procedure combining the values with (12) and the auxiliary equation we choose, we can get exact solutions for the given PDE.

## 4 Model description

Consider the M-fractional Kuralay-II equation given as (Sagidullayeva et al. 2022):

$$iD_{M,t}^{\alpha,\Upsilon} u - D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon} u\right) - vu = 0 \tag{15}$$

$$D_{M,x}^{\alpha,\Upsilon} v - 2\epsilon D_{M,t}^{\alpha,\Upsilon}\left(|u|^2\right) = 0 \tag{16}$$

Here $u = u(x,t)$ is a complex valued function while $v = v(x,t)$ is a real valued function. The conjugate of complex valued function $u$ is $u^*, \epsilon = \pm 1$ where x and t are spatio-temporal real variables. There are two forms of Kuralay-II equation.

Kuralay-IIA Equation (K-IIAE)

Let us assume the following form of the Kuralay-IIA equation, given as:

$$iD_{M,t}^{\alpha,\Upsilon} u - D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon} u\right) - vu = 0, \tag{17}$$

$$iD_{M,t}^{\alpha,\Upsilon} r + D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon} r\right) + vr = 0, \tag{18}$$

$$D_{M,x}^{\alpha,\Upsilon} v + 2d^2 D_{M,t}^{\alpha,\Upsilon}(ru) = 0. \tag{19}$$

**Table 1** Possible solutions of $F(\xi)$ in Eq. (11) for the selected P, Q and R values

| | P | Q | R | F |
|---|---|---|---|---|
| 1 | $m^2$ | $-(1+m^2)$ | $1$ | $sn, cd$ |
| 2 | $-m^2$ | $2m^2-1$ | $1-m^2$ | $cn$ |
| 3 | $-1$ | $2-m^2$ | $m^2-1$ | $dn$ |
| 4 | $1$ | $-(1=m^2)$ | $m^2$ | $ns, dc$ |
| 5 | $1-m^2$ | $2m^2-1$ | $-m^2$ | $nc$ |
| 6 | $m^2-1$ | $2-m^2$ | $-1$ | $nd$ |
| 7 | $1-m^2$ | $2-m^2$ | $1$ | $sc$ |
| 8 | $-m^2(1-m^2)$ | $2m^2-1$ | $1$ | $sd$ |
| 9 | $1$ | $2-m^2$ | $1-m^2$ | $cs$ |
| 10 | $1$ | $2m^2-1$ | $m^2(1-m^2)$ | $-ds$ |
| 11 | $\frac{-1}{4}$ | $\frac{m^2+1}{2}$ | $\frac{-(1-m^2)^2}{4}$ | $mcn \mp dn$ |
| 12 | $\frac{1}{4}$ | $\frac{-2m^2+1}{2}$ | $\frac{1}{4}$ | $nc \mp cs$ |
| 13 | $\frac{1-m^2}{4}$ | $\frac{m^2+1}{2}$ | $\frac{1-m^2}{4}$ | $nc \mp sc$ |
| 14 | $\frac{1}{4}$ | $\frac{m^2-2}{2}$ | $\frac{m^4}{4}$ | $ns \mp ds$ |
| 15 | $\frac{m^2}{4}$ | $\frac{m^2-2}{2}$ | $\frac{m^4}{4}$ | $sn \mp icn, \frac{sn}{\sqrt{1-m^2 sn \mp cn}}$ |
| 16 | $\frac{1}{4}$ | $\frac{1-2m^2}{2}$ | $\frac{1}{4}$ | $mcn \mp idn, \frac{sn}{1 \mp cn}$ |
| 17 | $\frac{m^2}{4}$ | $\frac{m^2-2}{2}$ | $\frac{1}{4}$ | $\frac{sn}{1 \mp dn}$ |
| 18 | $\frac{m^2-1}{4}$ | $\frac{m^2+1}{2}$ | $\frac{m^2-1}{4}$ | $\frac{dn}{1 \mp msn}$ |
| 19 | $\frac{1-m^2}{4}$ | $\frac{m^2+1}{2}$ | $\frac{-m^2+1}{4}$ | $\frac{cn}{1 \mp sn}$ |
| 20 | $\frac{(1-m^2)^2}{4}$ | $\frac{m^2+1}{2}$ | $\frac{1}{4}$ | $\frac{sn}{dn \mp cn}$ |
| 21 | $\frac{m^4}{4}$ | $\frac{m^2-2}{2}$ | $\frac{1}{4}$ | $\frac{cn}{\sqrt{1-m^2 \mp dn}}$ |

this is called the K-IIAE. It is integrable.

Kuralay-IIB Equation (K-IIBE)

Let us assume the second form of the Kuralay-IIB equation, given as:

$$iD_{M,t}^{\alpha,\Upsilon}u + D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon}u\right) - vu = 0, \tag{20}$$

$$iD_{M,t}^{\alpha,\Upsilon}r - D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon}r\right) + vr = 0, \tag{21}$$

$$D_{M,t}^{\alpha,\Upsilon}v - 2D_{M,x}^{\alpha,\Upsilon}(ru) = 0. \tag{22}$$

This is called the K-IIAE. This is also integrable.

Mathematical Treatment of the Model

K-IIAE

Let us consider the truncated M-fractional Kuralay-IIA equation for $d = 1$ and $r = \epsilon u^*$, given as:

$$iD_{M,t}^{\alpha,\Upsilon}u - D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon}u\right) - vu = 0 \tag{23}$$

$$D_{M,x}^{\alpha,\Upsilon}v - 2\epsilon D_{M,t}^{\alpha,\Upsilon}\left(|u|^2\right) = 0 \tag{24}$$

Let us consider the following traveling wave transformation

$$u(x,t) = U(\xi)e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}, v(x,t) = v(\xi), \xi = \frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha) \tag{25}$$

where $\mu, \tau$ and $\Omega$ are parameters and $\lambda$ is defined below. By substituting the wave transformations into PDE we find the real and imaginary parts, given is the real part, we have

$$2U^3\lambda\epsilon - U\Omega(\mu(\tau-1) - c) + \lambda\Omega^2U'' = 0 \tag{26}$$

Imaginary part

$$(\lambda - \tau\lambda - \mu\Omega)U' = 0 \tag{27}$$

From the (above equation), we find the speed of the soliton gives us:

$$\lambda = \frac{\mu\Omega}{1-\tau} \tag{28}$$

By using the balancing procedure in (26), we find $N = 1$.

K-IIBE

Let us consider the truncated M-fractional Kuralay-IIB equation for $d = 1$ and $r = \epsilon u^*$, given as follows:

$$iD_{M,t}^{\alpha,\Upsilon}u + D_{M,x}^{\alpha,\Upsilon}\left(D_{M,t}^{\alpha,\Upsilon}u\right) - vu = 0 \tag{29}$$

$$D_{M,t}^{\alpha,\Upsilon}v + 2\epsilon D_{M,x}^{\alpha,\Upsilon}\left(|u|^2\right) = 0 \tag{30}$$

Let us consider the following traveling wave transformation

$$u(x,t) = U(\xi)e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}, v(x,t) = v(\xi), \xi = \frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha) \tag{31}$$

where $\mu, \lambda, \tau$ and $\Omega$ are parameters. By substituting the wave transformations into PDE we find the real and imaginary parts, given is the real part, we have.

Real part:

$$2U^3\Omega\epsilon - \lambda U(\mu(\tau+1) - c) + \lambda^2\Omega U'' = 0 \tag{32}$$

Imaginary part

$$(\lambda + \tau\lambda + \mu\Omega)U\prime = 0 \tag{33}$$

From the (above equation), we find the speed of the soliton gives us:

$$\lambda = -\frac{\mu\Omega}{1+\tau} \tag{34}$$

**Table 2** In the sense of limiting sense for $m \to 1$ and $m \to 0$, the Jacobi elliptic functions are:

| | F | $m \to 1$ | $m \to 0$ | | $F$ | $m \to 1$ | $m \to 0$ |
|---|---|---|---|---|---|---|---|
| 1 | Snu | tanhu | sinu | 7 | dcu | 1 | secu |
| 2 | cnu | sechu | cosu | 8 | ncu | Coshu | secu |
| 3 | dnu | sechu | 1 | 9 | scu | sinhu | tanu |
| 4 | Cdu | coshu | 1 | 10 | nsu | cothu | cscu |
| 5 | sdu | sinhu | sinu | 11 | dsu | cschu | cotu |
| 6 | Ndu | cosh | 1 | 12 | csu | cschu | cotu |

By using the balancing procedure in (32), we find $N = 1$

### 4.1 Analytical solutions of K-IIAE using the Jacobi Elliptic function expansion method

For $N = 1$. Equation (12) change into the following simple form.

$$U(\xi) = a_0 + a_1 F(\xi) \tag{35}$$

Inserting Eq. (35) into Eq. (26), we get the following system of equations

$$-\Omega(\mu\tau - \mu - c)a_0 + 2\lambda\epsilon a_0^3 = 0 \tag{36}$$

$$-\Omega(\mu\tau - \mu - c)a_1 + 6\lambda\epsilon a_0^2 a_1 + \lambda\Omega^2 a_1 Q = 0 \tag{37}$$

$$6\lambda\epsilon a_0 a_1^2 = 0 \tag{38}$$

$$2\lambda\epsilon a_1^3 + 2\lambda\Omega^2 a_1 P = 0 \tag{39}$$

By solving the system of equations via Maple, we find

$$a_0 = 0,\ a_1 = \pm\sqrt{-\frac{P}{\in}}\Omega,\ \mu = \frac{c + \lambda\Omega Q}{\tau - 1} \tag{40}$$

Using the above-mentioned method for solving the problem and the value of Eq. (35), in combination, we obtained the exact solution of M-fractional Kuralay-IIA equation.

$$u(x,t) = \pm\sqrt{-\frac{P}{\epsilon}}\Omega F(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

When $P = m^2, Q = -\left(1 + m^2\right), R = 1$, and $F = sn$ from Table 1, Thus

$$\mathrm{u}_{1,1}(x,t) = \pm\sqrt{-\frac{m^2}{\epsilon}}\Omega sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And considering $m \to 1$ from Table 2, the solitary wave solution can be obtained as:

$$u_{1,2}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

When $P = m^2, Q = -\left(1+m^2\right), R = 1$, and $F = cd$ can be deduced from Table 1. the solution can be evaluated as:

$$u_{1,3}(x,t) = \pm\sqrt{-\frac{m^2}{\epsilon}}\Omega cd\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

As $m \to 1$ from Table 2, the solitary wave solution as,

$$u_{1,4}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega cosh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Considering $P = -m^2, Q = 2m^2 - 1$, and $F = cn$ from Table 1, the solution can be acquired as,

$$u_{1,5}(x,t) = \pm\sqrt{\frac{m^2}{\epsilon}}\Omega cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Furthermore, if $m \to 1$ from Table 2, the solitary wave solution is as follow:

$$u_{1,6}(x,t) = \pm\sqrt{\frac{1}{\epsilon}}\Omega sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Setting $P = -1, Q = 2 - m^2$, and $F = dn$ from Table 1, so

$$u_{1,7}(x,t) = \pm\sqrt{\frac{1}{\epsilon}}\Omega dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for $m \to 1$ from Table 2, it is distinctly show that this solution is similar to the solitary wave of $u_{1,6}(x,t)$

For choices $P = 1, Q = -\left(1+m^2\right)$ and $F = ns$, from Table 1, in this way the solution is found as:

$$u_{1,8}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For choices $m \to 1$ the solitary wave solution can be evaluated as,

$$u_{1,9}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For $m \to 0$, with the help of Table 2, the periodic solution can be obtained as,

$$u_{1,10}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For selecting $P = 1, Q = -\left(1+m^2\right)$ and $F = dc$, from Table 1, in this way the solution can be determined as,

$$u_{1,11}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega dc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For $m\to 0$ the periodic solution can be acquired as,

$$u_{1,12}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

*Assigning* $P=1-m^2, Q=2m^2-1$, and $F=nc$, from Table 1, so the solution can be found as,

$$u_{1,13}(x,t) = \pm\sqrt{\frac{m^2-1}{\epsilon}}\Omega nc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for, $m\to 0$, in the light of Table 2, so the solution can be found as

$$u_{1,14}(x,t) = \pm\sqrt{\frac{-1}{\epsilon}}\Omega sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Regarding $P=1-m^2, Q=2-m^2, F=sc$ from Table 1, the solution can be expressed as,

$$u_{1,15}(x,t) = \pm\sqrt{\frac{m^2-1}{\epsilon}}\Omega sc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Also, $m\to 0$, by way of Table 2, the periodic solution can be acquired as,

$$u_{1,16}(x,t) = \pm\sqrt{\frac{-1}{\epsilon}}\Omega tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Supposing $P=1, Q=2-m^2, F=cs$, from Table 1, the solution can be obtained as,

$$u_{1,17}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega cs\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For $m\to 1$, with the help of Table 2, the solitary wave solution can be acquired as,

$$u_{1,18}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Moreover $m\to 0$, from Table 2, the periodic solution can be stated as,

$$u_{1,19}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

If we choices $P=1, Q=2m^2-1, F=ds$, from Table 1, in this way the solution can be expressed as,

$$u_{1,20}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\Omega ds\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for $m \rightarrow 1$ and $m \rightarrow 0$ with the help of Table 2, the solitary wave solution and periodic solution are similar to $u_{1,18}(x,t)$ and $u_{1,19}(x,t)$, respectively.

When $P = \frac{-1}{4}, Q = \frac{m^2+1}{2}, F = mcn \mp dn$, from Table 1, so the solution can be obtained as,

$$u_{1,21}(x,t) = \pm\sqrt{\frac{1}{4\epsilon}}\Omega\left(mcn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And as $m \rightarrow 1$, from Table 2, the solitary wave solution can be written as

$$u_{1,22}(x,t) = \pm\sqrt{\frac{1}{4\epsilon}}\Omega\left(sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Considering $P=\frac{1}{4}, Q = \frac{-2m^2+1}{2}, F = ns \mp cs$, from Table 1, the solution can be acquired as,

$$u_{1,23}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp cs\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \rightarrow 1$, in the light of Table 2, the solitary wave solution can be written as,

$$u_{1,24}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \rightarrow 0$, the periodic solution can be acquired as,

$$u_{1,25}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Also assigning $P=\frac{1-m^2}{4}, Q = \frac{m^2+1}{2}, F = nc \mp sc$, from Table 1, in this way the solution can be expressed as,

$$u_{1,26}(x,t) = \pm\sqrt{\frac{m^2-1}{\epsilon}}\Omega\left(nc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp sc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \rightarrow 0$, from Table 2, the periodic solution can be written as,

$$u_{1,27}(x,t) = \pm\sqrt{\frac{-1}{\epsilon}}\Omega\left(sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Considering $P=\frac{1}{4}, Q = \frac{m^2-2}{2}, F = ns \mp ds$, from Table 1, so the solution can be found as,

$$u_{1,28}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp ds\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \rightarrow 1$, from Table 2, the solitary wave solution is as follow,

$$u_{1,29}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Also, regarding $P=\frac{m^2}{4}, Q = \frac{m^2-2}{2}, F = sn \mp icn$ from Table 1, the solution can be stated as,

$$u_{1,30}(x,t) = \pm\sqrt{-\frac{m^2}{4\epsilon}}\Omega\left(sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp icn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

As m → 1, from Table 2, the solitary wave solution can be obtained as,

$$u_{1,31}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp isech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Also if, P=$\frac{m^2}{4}$, $Q = \frac{m^2-2}{2}$, $F = \frac{sn}{\sqrt{1-m^2 sn \mp cn}}$ from Table 1, the solution can be expressed as,

$$u_{1,32}(x,t) = \pm\sqrt{-\frac{m^2}{4\epsilon}}\Omega\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}{\sqrt{1 - m^2 sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And if m → 1, the solitary wave solution can be obtained as,

$$u_{1,33}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(\frac{tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}{\sqrt{1 - tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

While P=$\frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $F = msn \mp idn$, from Table 1, the solution can be written as,

$$u_{1,34}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(msn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp idn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

For m → 1 the solitary wave is same as that of $u_{1,29}(x,t)$.
Supposing P=$\frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $F = \frac{sn}{1 \mp dn}$, from Table 1, the solution can be determining as,

$$u_{1,35}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \to 1$, from Table 2, the solitary wave solution can be written as,

$$u_{1,36}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(\frac{tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}{\sqrt{1 \mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \to 0$, from Table 2, the periodic solution can be obtained as,

$$u_{1,37}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\Omega sin\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Supposing P=$\frac{m^2}{4}$, $Q = \frac{m^2-2}{2}$, $F = \frac{sn}{1 \mp dn}$, from Table 1, the solution can be determining as,

$$u_{1,38}(x,t) = \pm\sqrt{-\frac{m^2}{4\epsilon}}\Omega\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

For m approach to 1 the solution is same as that of $u_{1,36}(x,t)$

When P=$\frac{1-m^2}{4}$, $Q=\frac{m^2+1}{2}$, $F=\frac{cn}{1\mp sn}$, from Table 1, the solution can be written as,

$$u_{1,39}(x,t)=\pm\sqrt{\frac{m^2-1}{4\epsilon}}\Omega\left(\frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{1\mp sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

If m → 0, from Table 2, the periodic solution can be determined as,

$$\mathrm{u}_{1,40}(x,t)=\pm\sqrt{\frac{-1}{4\epsilon}}\Omega\left(\frac{cos\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{1\mp sin\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Supposing P=$\frac{(1-m^2)^2}{4}$, $Q=\frac{m^2+1}{2}$, $F=\frac{sn}{dn\mp cn}$, from Table 1, the solution can be found as,

$$u_{1,41}(x,t)=\pm\sqrt{-\frac{\left(1-m^2\right)^2}{4\epsilon}}\Omega\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Also, regarding P=$\frac{m^4}{4}$, $Q=\frac{m^2-2}{2}$, $F=\frac{cn}{\sqrt{1-m^2\mp dn}}$, from Table 1, the solution can be stated as,

$$u_{1,42}(x,t)=\pm\sqrt{-\frac{m^4}{4\epsilon}}\Omega\left(\frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{\sqrt{1-m^2\mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Setting m → 1 by using Table 2, due to this setting,

$$u_{1,43}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\Omega\left(\frac{sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{\sqrt{\mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

## 4.2 Analytical solutions of K-IIBE using the Jacobi Elliptic function expansion method

For $N=1$. Equation (12) change into the following simple form.

$$U(\xi)=b_0+b_1F(\xi) \tag{41}$$

Inserting Eq. (35) into Eq. (32), we get the following system of equations

$$-\lambda(\mu\tau-\mu-c)b_0+2\Omega\epsilon b_0^3=0 \tag{42}$$

$$-\lambda(\mu\tau + \mu - c)b_1 + 6\Omega\epsilon b_0^2 b_1 + \Omega\lambda^2 b_1 Q = 0 \tag{43}$$

$$6\Omega\epsilon b_0 b_1^2 = 0 \tag{44}$$

$$2\Omega\epsilon b_1^3 + 2\Omega\lambda^2 b_1 P = 0 \tag{45}$$

By solving the system of equations via Maple, we find

$$b_0 = 0,\ b_1 = \pm\sqrt{-\frac{P}{\in}}\lambda,\ \mu = \frac{c + \lambda\Omega Q}{\tau + 1} \tag{46}$$

Using the above-mentioned method for solving the problem and the value of Eq. (41), in combination, we obtained the exact solution of M-fractional Kuralay-IIA equation.

$$\mathrm{u}(x,t) = \pm\sqrt{-\frac{P}{\epsilon}}\lambda F\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

When $P = m^2, Q = -\left(1 + m^2\right), R = 1,$ and $F = sn$ from Table 1, Thus

$$\mathrm{u}_{2,1}(x,t) = \pm\sqrt{-\frac{m^2}{\epsilon}}\lambda sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And considering $m \to 1$ from Table 2, the solitary wave solution can be obtained as:

$$\mathrm{u}_{2,2}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

When $P = m^2, Q = -\left(1 + m^2\right), R = 1,$ and $F = cd$ can be deduced from Table 1. the solution can be evaluated as:

$$u_{2,3}(x,t) = \pm\sqrt{-\frac{m^2}{\epsilon}}\lambda cd\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

As $m \to 1$ from Table 2, the solitary wave solution as,

$$u_{2,4}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda cosh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Considering $P = -m^2, Q = 2m^2 - 1,$ and $F = cn$ from Table 1, the solution can be acquired as,

$$u_{2,5}(x,t) = \pm\sqrt{\frac{m^2}{\epsilon}}\lambda cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Furthermore, if $m \to 1$ from Table 2, the solitary wave solution of Eq. (1) is as follow:

$$u_{2,6}(x,t) = \pm\sqrt{\frac{1}{\epsilon}}\lambda sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Setting $P = -1, Q = 2 - m^2,$ and $F = dn$ from Table 1, so

$$u_{2,7}(x,t) = \pm\sqrt{\frac{1}{\epsilon}}\lambda dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for $m \to 1$ from Table 2, it is distinctly show that this solution is similar to the solitary wave of $u_{2,6}(x,t)$

For choices $P = 1, Q = -\left(1+m^2\right)$ and $F = ns$, from Table 1, in this way the solution is found as:

$$u_{2,8}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For choices m → 1 the solitary wave solution can be evaluated as,

$$u_{2,9}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For $m \to 0$, with the help of Table 2, the periodic solution can be obtained as,

$$u_{2,10}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For selecting $P = 1, Q = -\left(1+m^2\right)$ and $F = dc$, from Table 1, in this way the solution can be determined as,

$$u_{2,11}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda dc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For $m \to 0$ the periodic solution can be acquired as,

$$u_{2,12}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Assigning $P = 1-m^2, Q = 2m^2-1$, and $F = nc$, from Table 1, so the solution can be found as,

$$u_{2,13}(x,t) = \pm\sqrt{\frac{m^2-1}{\epsilon}}\lambda nc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for, $m \to 0$, in the light of Table 2, so the solution can be found as

$$u_{2,14}(x,t) = \pm\sqrt{\frac{-1}{\epsilon}}\lambda sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Regarding $P = 1-m^2, Q = 2-m^2, F = sc$ from Table 1, the solution can be expressed as,

$$u_{2,15}(x,t) = \pm\sqrt{\frac{m^2-1}{\epsilon}}\lambda sc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Also, $m \to 0$, by way of Table 2, the periodic solution can be acquired as,

$$u_{2,16}(x,t) = \pm\sqrt{\frac{-1}{\epsilon}}\lambda tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Supposing $P = 1, Q = 2 - m^2, F = cs$, from Table 1, the solution can be obtained as,

$$u_{2,17}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda cs\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

For $m \to 1$, with the help of Table 2, the solitary wave solution can be acquired as,

$$u_{2,18}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Moreover $m \to 0$, from Table 2, the periodic solution can be stated as,

$$u_{2,19}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

If we choices $P = 1, Q = 2m^2 - 1, F = ds$, from Table 1, in this way the solution can be expressed as,

$$u_{2,20}(x,t) = \pm\sqrt{-\frac{1}{\epsilon}}\lambda ds\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \to 1$ and $m \to 0$ with the help of Table 2, the solitary wave solution and periodic solution are similar to $u_{2,18}(x,t)$ and $u_{2,19}(x,t)$ respectively.

When $P = \frac{-1}{4}, Q = \frac{m^2+1}{2}, F = mcn \mp dn$, from Table 1, so the solution can be obtained as,

$$u_{2,21}(x,t) = \pm\sqrt{\frac{1}{4\epsilon}}\lambda\left(mcn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And as $m \to 1$, from Table 2, the solitary wave solution can be written as

$$u_{2,22}(x,t) = \pm\sqrt{\frac{1}{4\epsilon}}\lambda\left(sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Considering $P=\frac{1}{4}, Q = \frac{-2m^2+1}{2}, F = ns \mp cs$, from Table 1, the solution can be acquired as,

$$u_{2,23}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp cs\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \to 1$, in the light of Table 2, the solitary wave solution can be written as,

$$u_{2,24}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

And for $m \to 0$, the periodic solution can be acquired as,

$$u_{2,25}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(csc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right) \mp cot\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha + \lambda t^\alpha)\right)\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha + \mu t^\alpha)\right)}$$

Also assigning $P=\frac{1-m^2}{4}, Q = \frac{m^2+1}{2}, F = nc \mp sc$, from Table 1, in this way the solution can be expressed as,

$$u_{2,26}(x,t)=\pm\sqrt{\frac{m^2-1}{\epsilon}}\lambda\left(nc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp sc\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

And for $m\to 0$, from Table 2, the periodic solution can be written as,

$$u_{2,27}(x,t)=\pm\sqrt{\frac{-1}{\epsilon}}\lambda\left(sec\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp tan\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Considering P$=\frac{1}{4}$, $Q=\frac{m^2-2}{2}$, $F=ns\mp ds$, from Table 1, so the solution can be found as,

$$u_{2,28}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(ns\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp ds\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

And for $m\to 1$, from Table 2, the solitary wave solution is as follow,

$$u_{2,29}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp csch\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Also, regarding P$=\frac{m^2}{4}$, $Q=\frac{m^2-2}{2}$, $F=sn\mp icn$ from Table 1, the solution can be stated as,

$$u_{2,30}(x,t)=\pm\sqrt{-\frac{m^2}{4\epsilon}}\lambda\left(sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp icn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

As m$\to$ 1, from Table 2, the solitary wave solution can be obtained as,

$$u_{2,31}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp isech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

Also if, P$=\frac{m^2}{4}$, $Q=\frac{m^2-2}{2}$, $F=\frac{sn}{\sqrt{1-m^2 sn\mp cn}}$ from Table 1, the solution can be expressed as,

$$u_{2,32}(x,t)=\pm\sqrt{-\frac{m^2}{4\epsilon}}\lambda\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{\sqrt{1-m^2 sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

And if m$\to$ 1, the solitary wave solution can be obtained as,

$$u_{2,33}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(\frac{tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{\sqrt{1-tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

While P$=\frac{1}{4}$, $Q=\frac{1-2m^2}{2}$, $F=msn\mp idn$, from Table 1, the solution can be written as,

$$u_{2,34}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(msn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\mp idn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

For m$\to$ 1 the solitary wave is same as that of $u_{2,29}(x,t)$.

Supposing P$=\frac{1}{4}$, $Q=\frac{1-2m^2}{2}$, $F=\frac{sn}{1\mp dn}$, from Table 1, the solution can be determining as,

$$u_{2,35}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}{1\mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^\alpha+\lambda t^\alpha)\right)}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha+\mu t^\alpha)\right)}$$

And for $m \to 1$, from Table 2, the solitary wave solution can be written as,

$$u_{2,36}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(\frac{tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{\sqrt{1 \mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

And for $m \to 0$, from Table 2, the periodic solution can be obtained as,

$$u_{2,37}(x,t) = \pm\sqrt{-\frac{1}{4\epsilon}}\lambda sin\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Supposing $P=\frac{m^2}{4}, Q = \frac{m^2-2}{2}, F = \frac{sn}{1 \mp dn}$, from Table 1, the solution can be determining as,

$$u_{2,38}(x,t) = \pm\sqrt{-\frac{m^2}{4\epsilon}}\lambda\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{1 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

For m approach to 1 the solution is same as that of $u_{2,36}(x,t)$
When $P=\frac{1-m^2}{4}, Q = \frac{m^2+1}{2}, F = \frac{cn}{1 \mp sn}$, from Table 1, the solution can be written as,

$$u_{2,39}(x,t) = \pm\sqrt{\frac{m^2-1}{4\epsilon}}\lambda\left(\frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{1 \mp sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

If $m \to 0$, from Table 2, the periodic solution can be determined as,

$$u_{2,40}(x,t) = \pm\sqrt{\frac{-1}{4\epsilon}}\lambda\left(\frac{cos\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{1 \mp sin\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Supposing $P=\frac{(1-m^2)^2}{4}, Q = \frac{m^2+1}{2}, F = \frac{sn}{dn \mp cn}$, from Table 1, the solution can be found as,

$$u_{2,41}(x,t) = \pm\sqrt{-\frac{\left(1-m^2\right)^2}{4\epsilon}}\lambda\left(\frac{sn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right) \mp cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Also, regarding $P=\frac{m^4}{4}, Q = \frac{m^2-2}{2}, F = \frac{cn}{\sqrt{1-m^2 \mp dn}}$, from Table 1, the solution can be stated as,

$$u_{2,42}(x,t) = \pm\sqrt{-\frac{m^4}{4\epsilon}}\lambda\left(\frac{cn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}{\sqrt{1-m^2 \mp dn\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha} + \lambda t^{\alpha})\right)}}\right) \times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

Setting $m \to 1$ by using Table 2, due to this setting,

$$u_{2,43}(x,t)=\pm\sqrt{-\frac{1}{4\epsilon}}\lambda\left(\frac{sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)}{\sqrt{\mp sech\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\Omega x^{\alpha}+\lambda t^{\alpha})\right)}}\right)\times e^{i\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}+\mu t^{\alpha})\right)}$$

## 5 Graphical results

Here, using the Jacobi elliptic function expansion technique, we graphically examine the complexities of wave solutions. In particular, the truncated M-Fractional Kuralay-II Equation is subjected to this approach in order to clarify the features and actions of these wave solutions. We have deliberated over the parameter values and chosen them with care to ensure that our illustration is clear and precise.

The behavior of a wave solution under different parameters is illustrated in Figs. 1 and 2, which have a sequence of four sub-figures (a-d) each. A steady single wave keeping its shape is shown by the time-varying wave solution in Figs. 1a and 2a, which exhibits a constant peak location with amplitude fluctuations. The results of changing the parameter α (which ranges from 0.2 to 1 in Fig. 1b and 0.8 to 1 in Fig. 2b) are examined in Figs. 1b and 2b, which both reiterate the idea of stability. The figures show that when α increases, the wave's peak becomes noticeably sharper and more defined, demonstrating how the parameter greatly affects the wave's features.

3D surface plots in Figs. 1c, 2c, and 3c give a more complete picture of the wave's spatial and temporal evolution and add more clarity. One characteristic of solitary waves is their highly localized peak, which is shown by the plots' surface height and color fluctuations, which indicate the wave's size. Figures 1d, 2d, and 3d display contour plots that corroborate this viewpoint. These charts highlight the stationary peak of the wave over time, supporting the notion of a stable solitary wave solution, through color coding, where warmer hues reflect larger magnitudes.

Further investigation into the wave's characteristics and its behavior under different circumstances is presented in Figs. 3, 4, and 5, which expand upon these findings. Wave dispersion or nonlinearity may be indicated by changes in amplitude and width, as seen in Fig. 3a and b, which concentrate on different time moments and α values, respectively. The stability and evolution of the solitary wave profile over time are illustrated in Fig. 4a, which is a series of plots tracking this profile. In order to determine if the wave is dispersed and deformed or acts like a real soliton, this representation is essential.

Figure 4b shows how the solitary wave profile varies when α values change, which helps understand how the wave dynamics are affected by changes in system properties and how solid the solitary wave is under different conditions. Similar to previous pictures, Fig. 4c's three-dimensional surface plots combine spatial distribution with temporal evolution, revealing the wave's energy distribution, propagation velocity, and interactions with the medium. To better understand the wave's energy distribution and locate its core, Fig. 4d uses a contour map to highlight the wave's spatial and temporal history. Finally, for discrete time steps and varied α values, Fig. 5a and b investigate the spatial evolution of a wave function and a physical quantity, respectively. Small volumes, typical of standing waves that keep their shape as they travel, are shown in these charts.

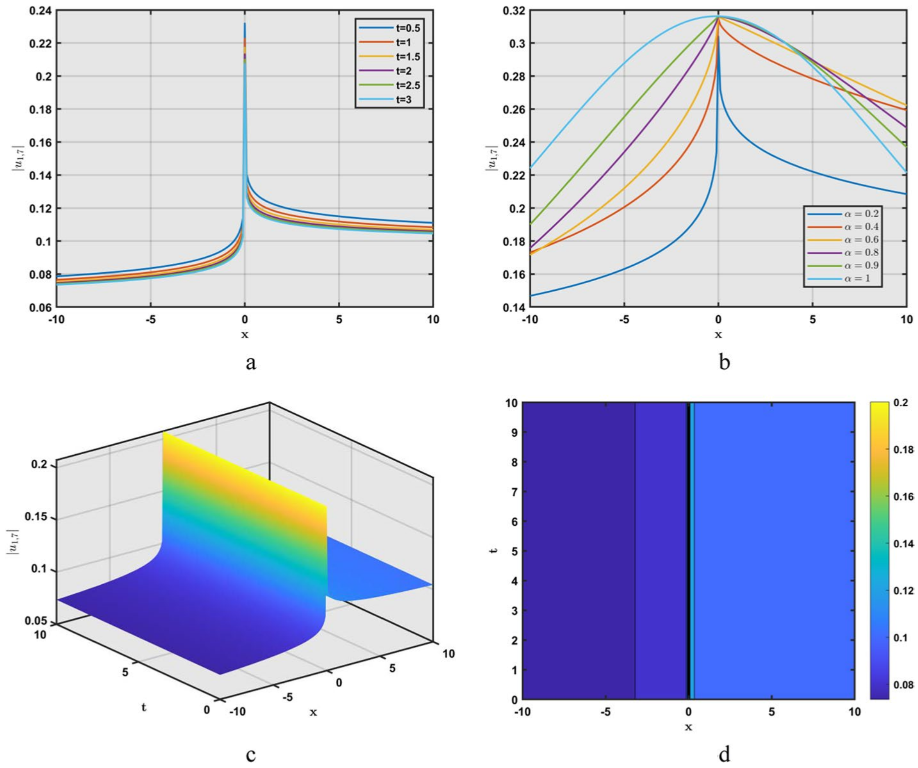


**Fig. 1** Two-Dimensional plot for the solution $u_{1,7}$ for varying $t$ and $\alpha$, surface and contour for $\in= 0.1, \alpha = 0.1, \Omega = 0.1, \Upsilon = 0.1, \tau = 0.1, \mu = 0.1$and$m = 0.9$.

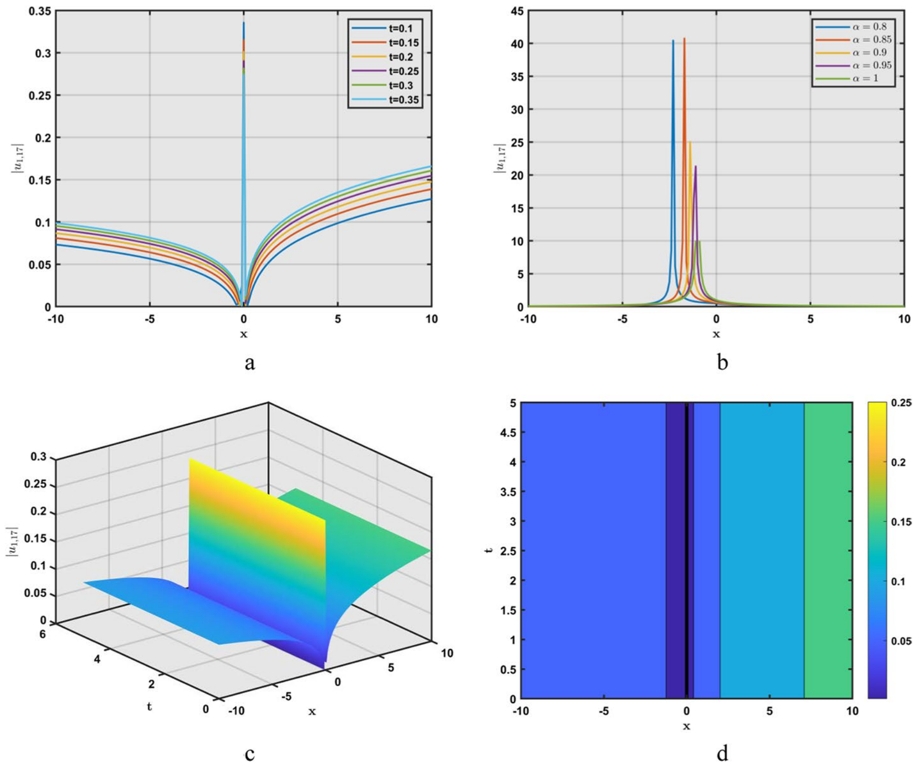


**Fig. 2** Two-Dimensional plot for the solution $u_{1,17}$ for varying $t$ and $\alpha$, surface and contour for $\in= 0.1, \alpha = 0.1, \Omega = 0.1, \Upsilon = 0.1, \tau = 0.1, \mu = 0.1$ and $m = 0.9$.

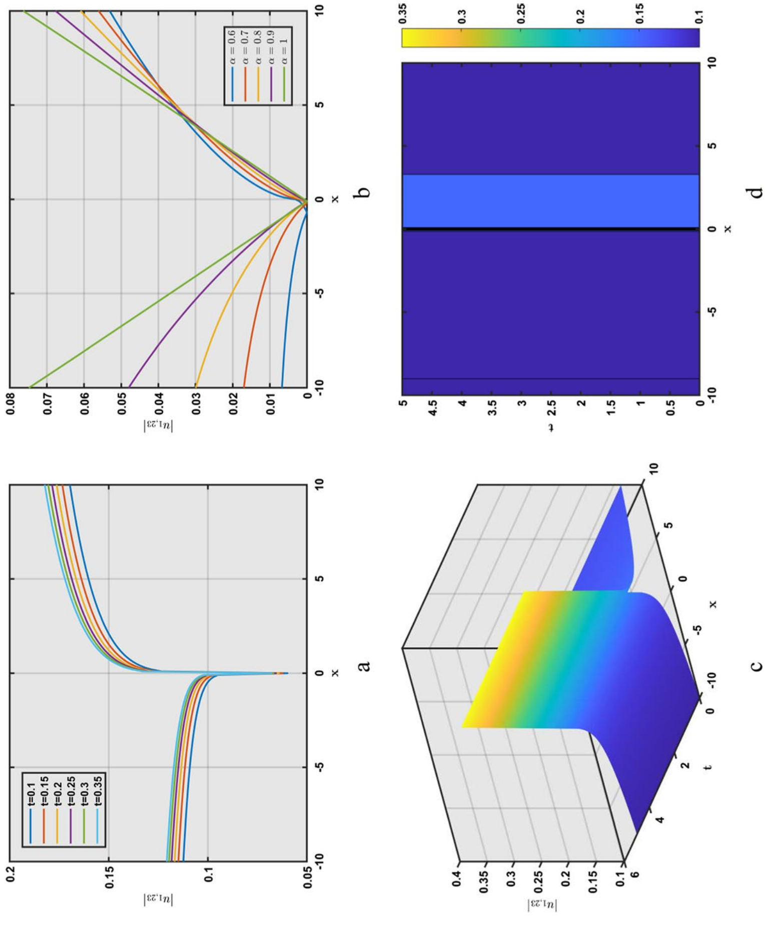


**Fig. 3** Two-Dimensional plot for the solution $u_{1,23}$ for varying $t$ and $\alpha$, surface and contour for $\in = 0.1, \alpha = 0.1, \Omega = 0.1, \Upsilon = 0.1, \tau = 0.1, \mu = 0.1$ and $m = 0.5$.

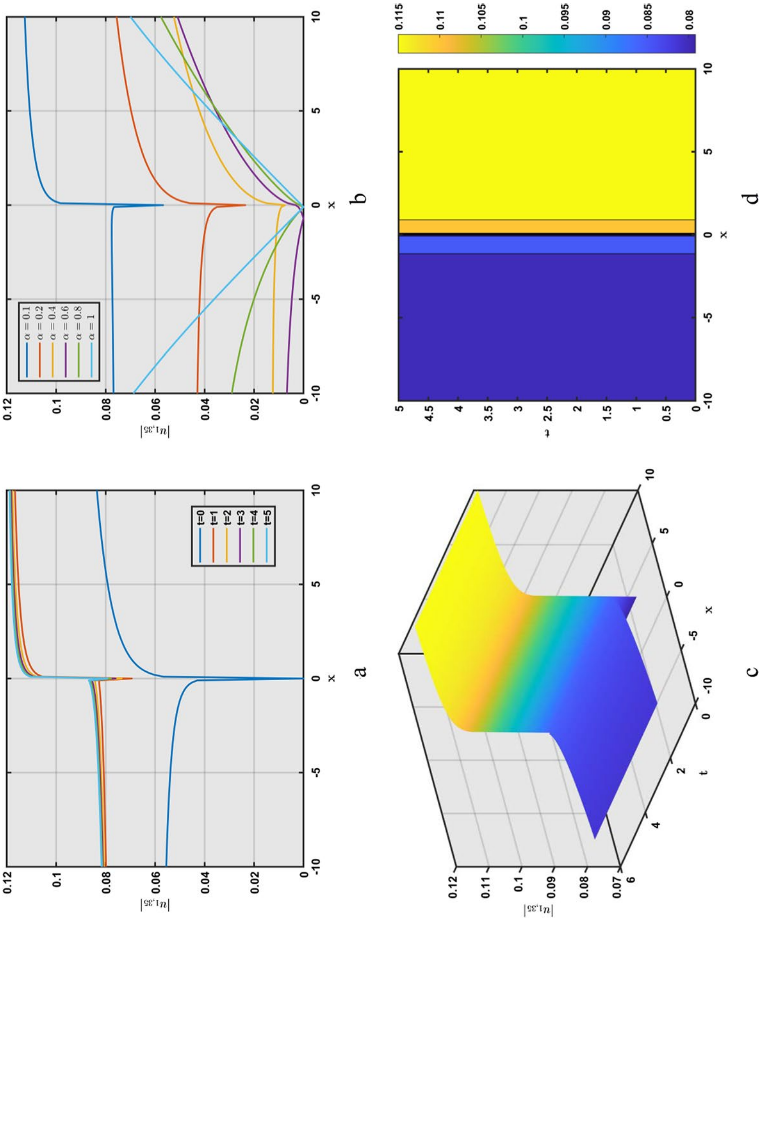


**Fig. 4** Two-Dimensional plot for the solution $u_{1,35}$ for varying $t$ and $\alpha$, surface and contour for $\in= 0.1, \alpha = 0.1, \Omega = 0.1, \Upsilon = 0.1, \tau = 0.1, \mu = 0.1 \text{and} m = 0.9$.

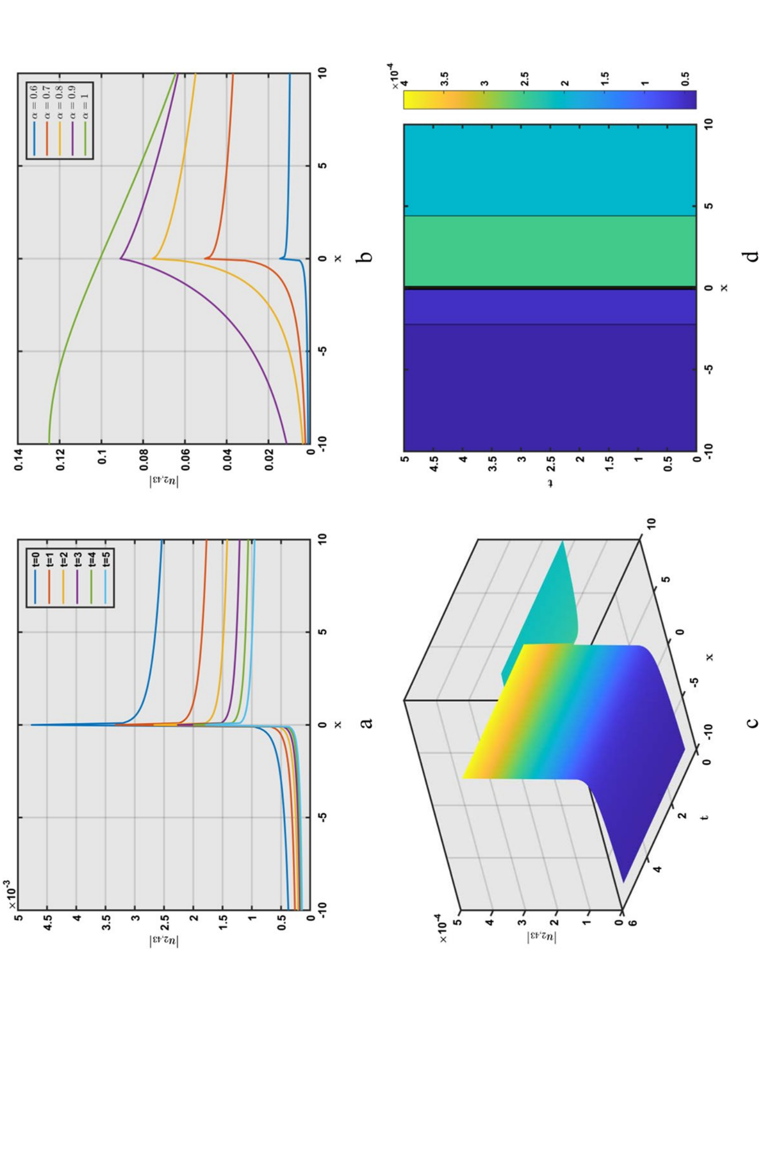


**Fig. 5** Two-Dimensional plot for the solution $u_{2,43}$ for varying $t$ and $\alpha$, surface and contour for $\in= 1, \alpha = 0.4, \Omega = 0.1, \Upsilon = 1, \tau = 0.5, \mu = 0.5$, and$m = 0.5$.

Finally, these figures show the wave's stability, parameter responsiveness, and spatial and temporal evolution, which give a multi-faceted picture of the wave's behavior. The behavior and features of the single wave under diverse conditions can be better understood by applying different graphical techniques.

## 6 Conclusion

We have used the Jacobi Elliptic Function Expansion Method (JEFEM) to obtain exact solutions to the Kuralay-II Equation via the truncated M-fractional derivative in this study. A number of novel exact solutions have been found by using this method, which has shed light on the complicated dynamics of the problem under different circumstances. Not only can these solutions, which include periodic, solitary, and trigonometric forms, help us understand mathematics better, but they also show how complicated the problem is. Insights into the model's temporal evolution are provided by the periodic solutions, which highlight its cyclical nature, while crucial information on localized events is offered by the solitary waves. The oscillatory nature of the problem can be better understood with the help of trigonometric solutions. This study opens the door to further investigations in mathematical physics and related fields by proving how flexible JEFEM is when dealing with nonlinear partial differential equations. We suggest a multi-pronged strategy for future research that takes into account possible uses in quantum mechanics, numerical simulations, fluid dynamics, and mathematical physics. Research comparing JEFEM to alternative solution methods could shed light on its advantages. We may be able to get new insights into the dynamics of complex differential equations and perhaps discover novel solutions if we apply this methodology to multi-dimensional equations and use Physics-Informed Neural Networks (PINNs). Breakthrough discoveries could be possible as a result of this combination of theoretical investigation and practical application, which aims to connect theoretical mathematical models with their corresponding real-world occurrences.

**Author contributions** M.I.K., A.F., and W.-X.M. collectively contributed to the conceptualization and development of the study. Specifically, M.I.K. (M. Ishfaq Khan) played a pivotal role in formulating the research question, designing the methodology, and conducting the primary analysis. A.F. (Aamir Farooq) substantially contributed to the data collection, data analysis, and interpretation of the results. W.-X.M. (Wen-Xiu Ma) was instrumental in providing critical feedback, refining the methodologies, and validating the findings.In terms of the manuscript preparation, M.I.K. took the lead in drafting the original manuscript. A.F. was responsible for the preparation and editing of the manuscript's supplementary materials, ensuring their alignment with the core findings. W.-X.M. conducted a thorough review and editing of the manuscript, providing substantial revisions for intellectual content.All authors, M.I.K., A.F., and W.-X.M., were actively involved in the discussion and interpretation of the study's outcomes and implications. They collaboratively worked on revising the manuscript critically for important intellectual content. Each author has read and approved the final version of the manuscript to be published. Additionally, all authors agree to be accountable for all aspects of the work, ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

## Declarations

**Competing interests** The authors declare no competing interests.

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