Optical and Quantum Electronics (2024) 56:542
https://doi.org/10.1007/s11082-023-06258-7

Exact solutions for the improved mKdv equation with conformable derivative by using the Jacobi elliptic function expansion method

Aamir Farooq[1] · **Muhammad Ishfaq Khan**[2] · **Wen Xiu Ma**[1,3,4,5]

Received: 24 September 2023 / Accepted: 29 December 2023

Abstract
The goal of this paper is to find exact solutions to the improved modified Korteweg-de Vries (mKdV) equation with a conformable derivative using the Jacobi elliptic function expansion method. The improved mKdV equation is a prominent mathematical model in the realm of nonlinear partial differential equations, with widespread applicability in diverse scientific and engineering domains. This study leverages the well-known effectiveness of the Jacobi elliptic function expansion method in solving nonlinear differential equations, specifically focusing on the intricacies of the improved mKdV problem. The investigation reveals innovative and explicit solutions, providing insight into the dynamics of the related physical processes. This paper provides a comprehensive examination of these solutions, emphasizing their distinct features and depictions using Jacobi elliptic functions. These findings are especially advantageous for specialists in the fields of nonlinear science and mathematical physics, providing significant insights into the behavior and development of nonlinear waves in various physical situations. This work also contributes to our knowledge of the improved mKdV equation and shows that the Jacobi elliptic function expansion method is a useful tool for solving complex nonlinear situations. The study is enhanced with graphical illustrations of various solutions, which further enhance its analytical complexity.

Keywords Improved mKdV Equation · Conformable derivative · Jacobi elliptic function expansion method · Exact solutions

1 Introduction

A wide range of scientific and engineering fields place a significant amount of importance on nonlinear differential equations. These equations stand out due to their intricate interdependencies between variables, which do not follow a linear or proportional relationship. When it comes to accurately modeling, characterizing, and predicting the behavior of multifaceted and dynamic systems, where linear approaches fall short, their unrivalled proficiency is the source of their paramount role. These equations are especially important in fields such as modeling the climate, biological systems, and complex engineering

Extended author information available on the last page of the article

Published online: 30 January 2024

structures. They provide insights that are necessary for both theoretical and practical advancements, making them particularly important in these areas. Researchers and engineers are able to develop a more profound comprehension of their respective fields through the use of linear differential equations. These equations capture the unpredictable and frequently chaotic nature of real-world phenomena. In turn, this helps in making decisions that are more informed, driving innovation, and paving the way for breakthroughs in a variety of fields. Because of the nuanced portrayal of the environment and phenomena that they provide, the accuracy and depth of analysis are elevated, and as a result, they play a transformative role in the advancement of knowledge and application within these domains.

Obtaining accurate solutions for fractional nonlinear partial differential equations can be accomplished through the utilization of a number of various alternative approaches. A technique that is successful for obtaining both exact and approximation series solutions for a variety of differential equations is the Laplace residual power series technique, which is referred to as (Khresat et al. 2023). These solutions can be obtained by using this methodology. This approach is renowned for its simplicity and usefulness. Additionally, the extended G/G' strategy, detailed in (Bekir and Güner 2013; Gepreel and Omran 2012; Özkan et al. 2023; Zheng 2012), is another robust approach that produces soliton solutions for space-time fractional equations. Not only that, but there is also a fascinating finding concerning sub linear fractional equations. It has been demonstrated in (Řehák 2023) that they are capable of producing solutions that exhibit behavior that is asymptotically super-linear. This set of solutions is distinguished by a consistent pattern of change, and the behavior of these solutions can be correctly portrayed through the use of asymptotic equations. The generation of functions, particularly multivariable power series, is another method that is worthy of mention. According to (Hwang and Lu 2023), it has been discovered that these functions are quite effective in the process of solving partial difference equations. Moreover, various other methods, including the fractional sub equation method (Guo et al. 2012; Lu 2012; Tang et al. 2012; Wen and Zheng 2013; Zhang and Zhang 2011; Zheng and Wen 2013) the Jacobi elliptic function expansion method (Inc and Ergüt 2005; Khan et al. 2022; Zheng 2014) the variational iteration method (Inc 2008), and the sine–cosine method (Hirota and Satsuma 1981; Inç and Evans 2004) also noteworthy in this subject. The area of fractional calculus has a wide range of applications across a variety of fields, such as engineering, biology, and other social sciences, as illustrated in (Abdulhameed et al. 2019; Chang et al. 2019; Dubey et al. 2019; Goulart et al. 2019). Not only are these methodologies capable of solving partial difference equations, but they also provide vital mathematical techniques for solving fractional nonlinear partial differential equations. This was noted before in the sentence. A deeper understanding of a wide variety of different physical scenarios can be achieved through the utilization of these methodologies. Alquran et al. (2022) presented new topological and non-topological unidirectional-wave solutions for the modified-mixed KdV equation and bidirectional-wave solutions for the Benjamin Ono equation, utilizing recent techniques to advance the field. In (Alquran et al. 2023a), the authors investigated dual-wave solutions to the Kadomtsev–Petviashvili model with second-order temporal and spatial–temporal dispersion terms. In (Alquran and Alhami 2022), the authors discussed the analysis of lumps, single-stripe, breather-wave, and two-wave solutions to the generalized perturbed-KdV equation by means of Hirota's bilinear method. For better understanding also see (Alquran 2022, 2023; Alquran et al. 2023b; Eslami and Rezazadeh 2016; Ghanbari 2019; Ghanbari and Baleanu 2020; Ghanbari and Gómez-Aguilar 2019a, b; Ghanbari and Kuo 2019; Jaradat and Alquran 2020; Khater and Ghanbari 2021; Qiao et al. 2022; Rezazadeh 2018; Younas et al. 2023).

When it comes to solving nonlinear wave equations and other nonlinear phenomena, the Jacobi elliptic function expansion method is a significant innovation that represents a substantial advancement. There are a number of advantages and disadvantages associated with the Jacobi elliptic function approach, which is utilized for the resolution of nonlinear partial differential equations (PDEs). The capacity to produce solutions that are both accurate and periodic is one of its key characteristics. This ability is essential for comprehending complex nonlinear dynamics. This technique is especially useful for equations that display particular symmetries and invariants, since it enables a more in-depth study of the solutions to be performed. On the other hand, one of its key shortcomings is that it has a restricted applicability; it might not produce solutions for nonlinear PDEs that are more generic or complex and do not comply with its particular criteria. When compared to other methods, such as the inverse scattering transform, the Jacobi elliptic function method may be easier to apply and require less computational effort. However, it does not possess the generality and resilience that are found in techniques that are more adaptable, such as numerical simulations. The Jacobi technique excels in offering more explicit analytical solutions for the equations that fall within its scope, in contrast to other approaches such as the Hirota bilinear approach, which are capable of handling a wider variety of nonlinear partial differential equations. In comparison to the hyperbolic tangent function expansion strategy, which is more restricted in its scope, this method is renowned for its capacity for diversity. These answers frequently include solitary wave solutions as well as shock wave solutions (Sait 2023), which exemplifies the method's capacity to be applied in a variety of contexts. Particularly remarkable is the fact that the Jacobi elliptic function expansion approach incorporates a wide range of different methodologies. Among these are the trial function method, the general projective Riccati equations method, the first integral method, the simple equations technique, the F-expansion method, the modified simple equation method, and the trial function method. Every one of them can be considered a particular instance or variant of the Jacobi technique from a different perspective. The Jacobi method is an all-encompassing method, which emphasizes the crucial role it plays in nonlinear analysis. There is a void in the study addressing the application of this method to the improved mKdV equation with conformable derivative, which is an interesting observation to make. It has been noted that there has been no considerable examination in this field as of yet, which indicates that there is a viable avenue for future research. This unknown region gives a chance to obtain a deeper understanding of the Jacobi elliptic function expansion approach and to apply it in unexpected circumstances. This could potentially lead to new discoveries and breakthroughs in the field of nonlinear wave dynamics.

To solve complicated nonlinear PDE problems is the driving force behind this study, which focuses on the well-known and very versatile Jacobi elliptic function expansion approach. The exact solutions of the well-known improved mKdV equation, which incorporates conformable derivatives, are studied using this approach. A more comprehensive framework is introduced by conformable derivatives, which allows for a deeper investigation of the behavior of the equation. We use the well-established effectiveness of the Jacobi elliptic function expansion method to systematically find accurate solutions to difficult nonlinear PDEs. To fully understand the system under investigation, these answers are essential. Our goal is to help better understand and apply these complicated mathematical models in different scientific and engineering contexts by shedding light on the dynamic nature of their solutions and the intricate characteristics of the improved mKdV equations with these derivatives. The improved mKdV equation with the conformable derivatives can be express as:

$$D_t^{\alpha} u + a u^2 D_x^{\gamma} u + b D_t^{\alpha} D_{2x}^{\gamma} u + \beta D_{3x}^{\gamma} u = 0, \quad t > 0, \quad 0 < \alpha, \ \gamma \ \leq 1 \tag{1.1}$$

here α and γ are the parameters describing the order of the conformable derivative.

The rest of the article is organized like this: The relevance of the improved mKdV equation using conformable derivatives is highlighted in Sect. 2 by providing a brief overview of the equation. In Sect. 3, the method of Jacobi elliptic function expansion is introduced. Several graphs depicting the answers are shown in Sect. 4, allowing a direct visual examination of the different behavior's displayed by these solutions. In Sect. 5, the paper concludes by discussing the study's general conclusions. Before moving on to the next section, we will review some basic features of the conformable derivative, as described in reference (Khalil et al. 2014), to make sure that all readers understand and can follow along.

Definition Abdeljawad 2015): Let $\mathfrak{A} : [0, \infty] \to \mathbb{R}$ be a function. The $\alpha - th$ order conformable derivative of $\mathfrak{A}$ is defined by

$$D^{\alpha}(\mathfrak{A})(t) = \lim_{\varepsilon \to 0} \frac{\mathfrak{A}\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \quad \varepsilon > 0, \quad \alpha \in (0, 1).$$

If $\mathfrak{A}$ is $\alpha-$ differentiable in some $(0, \alpha)$ and $\lim_{\varepsilon \to t^{0+}} \mathfrak{A}^{(\alpha)}(t)$ if it does, we characterize $\mathfrak{A}^{(\alpha)}(0) = \lim_{\varepsilon \to t^{0+}} \mathfrak{A}^{(\alpha)}(t)$.

Theorem *Abdeljawad* (2015): *Let* $\alpha \in (0, 1]$ *and suppose f, g is* $\alpha-$ *differential at point* $t > 0$. *Then, the following are satisfied*:

1. $D^{\alpha}(\delta f + \gamma g) = \delta D^{\alpha}(f) + \gamma D^{\alpha}(g) \quad \forall \, \delta, \gamma \in \mathbb{R}$.
2. $D^{\alpha}(t^{q}) = q t^{q-\alpha} \ \ \forall \, q \in \mathbb{R}$.
3. $D^{\alpha}(\chi) = 0$, for all constant functions $f(t) = \chi$.
4. $D^{\alpha}(fg) = f D^{\alpha}(g) + D^{\alpha}(f) g$.
5. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{g D^{\alpha}(f) - f D^{\alpha}(g)}{g^2}$.
6. If, in addition, f is differential, then $D^{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Theorem *Abdeljawad* (2015): *Assume* $f, g : (0, \infty) \to \mathbb{R}$ *be* $\alpha-$ *differentiable functions, where* $0 < \alpha \leq 1$. *Let* $h(t) = f(g(t))$. *Then* $h(t)$ *is* $\alpha -$ differentiable *for all t with* $t \neq 0$ *and* $g(t) \neq 0$ *we have*

$$D^{\alpha}(h)(t) = D^{\alpha}(f)(g(t)).D^{\alpha}(g)(t).g(t)^{\alpha-1}.$$

If $t = 0$ we have

$$D^{\alpha}(h)(0) = \lim_{t \to 0} D^{\alpha}(f)(g(t)).D^{\alpha}(g)(t).g(t)^{\alpha-1}.$$

2 Methodology of the Proposed Technique

Step 1: Let us take into account the nonlinear partial differential equation that has been provided to us, say with two variables.

$$F\left(u, \frac{\partial^{\alpha}u}{\partial t^{\alpha}}, \frac{\partial^{\alpha}u}{\partial^{\alpha}x}, \frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}, \frac{\partial^{2\gamma}u}{\partial x^{2\gamma}}, \dots\right) = 0 \tag{2.1}$$

Step 2: Changing Eq. (2.1) by using the traveling wave transformation in the sense that

$$u = u(\xi), \quad \xi = w\frac{t^{\alpha}}{\alpha} + c\frac{x^{\gamma}}{\gamma} \tag{2.2}$$

where c and w, are constants. In order to simplify things, we may rewrite Eq. (2.1) as an ordinary differential equation of integral order with the help of Eq. (2.2).

$$F\left(u', u'', u''', \dots\right) = 0 \tag{2.3}$$

In addition to this extended indirect approach, the primary objective is to increase the probability of finding solutions to an auxiliary ordinary differential equation (the first category of the parameter Jacobian equation). The goal is to produce many Jacobian elliptic solutions for the specific equation. This is done to accomplish the goal of building many Jacobian elliptic solutions. The auxiliary equation can be represented pictorially.

$$\left(F'\right)^2(\xi) = PF^4(\xi) + QF^2(\xi) + R \tag{2.4}$$

here $F' = \frac{\partial F}{\partial \xi}, \xi = (x, t)$ and P, Q are real constants. The Eq. 2.4 has solution in Table 1 , here $i^2 = -1$, the jacobian elliptic functions $sn\xi = sn(\xi, m), cn\xi = cn(\xi, m)$, and $dn\xi = dn(\xi, m,)$, here $m(0 < m < 1)$ is the modulus.

The Jacobi elliptic functions, as indicated in Table 2, were simplified to trigonometric and hyperbolic functions by a limiting process as m approaches 1 and 0, respectively.

Consequently, we get many solutions, including periodic, hyperbolic, and trigonometric solutions, for the given problem. The Jacobi elliptic function expansion method may be utilized to express $u(\xi)$ as a finite series of Jacobi elliptic functions.

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi) \tag{2.5}$$

where $F(\xi)$ is the solution of the nonlinear ordinary Eq. (2.4) and n, $a_i (i = 0, 1, 2, \dots, n)$ are constants to be determine later, the n can be determining the highest order linear term

$$O\left(\frac{\partial^p u}{\partial \xi^p}\right) = n + p, \quad p = 1, 2, 3, \dots \tag{2.6}$$

And the order of highest nonlinear term is

$$O\left(u^q \frac{\partial^p u}{\partial \xi^p}\right) = (q + 1)n + p, \quad q = 0, 1, 2, 3, \dots \; p = 1, 2, 3, \dots \tag{2.7}$$

In Eq. (2.3) by substituting Eq. (2.5) and equating the coefficients of power F to zero, a system of nonlinear algebraic equations for *a_i* (where *i=0, 1, 2,…, n*) may be obtained.

Table 1 The Eq. (2.4) might have many solutions for the function $F(\xi)$ depending on the values chosen for *P, Q*, and *R*

	P	Q	R	F
1	m^2	$-(1+m^2)$	1	sn, cd
2	$-m^2$	$2m^2-1$	$1-m^2$	cn
3	-1	$2-m^2$	m^2-1	dn
4	1	$-(1=m^2)$	m^2	ns, dc
5	$1-m^2$	$2m^2-1$	$-m^2$	nc
6	m^2-1	$2-m^2$	-1	nd
7	$1-m^2$	$2-m^2$	1	sc
8	$-m^2(1-m^2)$	$2m^2-1$	1	sd
9	1	$2-m^2$	$F(\xi)$	cs
10	1	$2m^2-1$	$m^2(1-m^2)$	$-ds$
11	$\frac{-1}{4}$	$\frac{m^2+1}{2}$	$\frac{-(1-m^2)^2}{4}$	$mcn \mp dn$
12	$\frac{1}{4}$	$\frac{-2m^2+1}{2}$	$\frac{1}{4}$	$nc \mp cs$
13	$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$nc \mp sc$
14	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$ns \mp ds$
15	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$sn \mp icn, \frac{sn}{\sqrt{1-m^2 sn \mp cn}}$
16	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$mcn \mp idn, \frac{sn}{1 \mp cn}$
17	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \mp dn}$
18	$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{dn}{1 \mp msn}$
19	$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{-m^2+1}{4}$	$\frac{cn}{1 \mp sn}$
20	$\frac{(1-m^2)^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1}{4}$	$\frac{sn}{dn \mp cn}$
21	$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-m^2 \mp dn}}$

Table 2 In the sense of limiting sense for $m \to 1$ and $m \to 0$, the Jacobielliptic functions reduce to:

	F	$m \to 1$	$m \to 0$		F	$m \to 1$	$m \to 0$
1	snu	tanhu	sinu	7	dcu	1	secu
2	cnu	sechu	cosu	8	ncu	coshu	secu
3	dnu	sechu	1	9	scu	sinhu	tanu
4	cdu	coshu	1	10	nsu dsu	cothu	cscu
5	sdu	sinhu	sinu	11	dsu	cschu	cotu
6	ndu	coshu	1	12	csu	cschu	cotu

Solving this system using *P, Q,* and *R* values, as provided in Table 1, yields the desired solution. By using the technique above of combining the data with the auxiliary equation, it is possible to obtain precise answers for Eq. (2.1).

3 Exact solutions of improved mKdV equation with conformable derivatives

In this section, we construct the exact solutions of the space-time improved mKdV equation by using the Jacobi elliptic function expansion method, the space-time improved mKdV Eq. (1.1). We obtain a nonlinear ordinary differential equation of integer order by substituting the wave variable Eq. (2.2) into Eq. (1.1).

$$w\frac{du}{d\xi} + acu^2\frac{du}{d\xi} + \left(bc^2w + \beta c^3\right)\frac{d^3u}{d\xi^3} = 0 \tag{3.1}$$

By equating the highest order nonlinear component with the highest order derivative term in Eq. (3.1), we deduce that *n=1*. Consequently, the proposed solution for Eq. (3.1) takes a simplified form.

$$u(\xi) = a_0 + a_1F(\xi) \tag{3.2}$$

The differentiation to u w.r.t ξ yields

$$\frac{du}{d\xi} = a_1F'(\xi) \tag{3.3}$$

and

$$u^2\frac{du}{d\xi} = a_0^2a_1F'(\xi) + a_1^3F^2(\xi)F'(\xi) + 2a_0a_1^2F(\xi)F'(\xi) \tag{3.4}$$

$$\frac{d^3u}{d\xi^3} = aF'''(\xi) = a_1(6PF^2(\xi)F'(\xi) + QF'(\xi) \tag{3.5}$$

Putting the values of $u(\xi)$, $\frac{\partial u}{\partial \xi}$, $u^2\frac{du}{d\xi}$ and $\frac{d^3u}{d\xi^3}$ from Eq. 3.2 to 3.5 into Eq. 3.1, we obtain.

$$\begin{aligned} &w\left(a_1F'(\xi)\right) + ac\left(a_0^2a_1F'(\xi) + a_1^3F^2(\xi)F'(\xi) + 2a_0a_1^2F(\xi)F'(\xi)\right) \\ &+\left(bc^2w + \beta c^3\right)(a_1\left(6PF^2(\xi)F'(\xi) + QF'(\xi)\right) = 0 \end{aligned} \tag{3.6}$$

.By collecting different power of $F^i(\xi)$, we get the system of algebraic equations as:

$$\begin{aligned} &F^0 : wa_1 + aca_0^2a_1 + a_1\left(bc^2w + \beta c^3\right)Q = 0, \\ &F^1 : 2aca_0a_1^2 = 0, \\ &F^2 : aca_1^3 + a_16P\left(bc^2w + \beta c^3\right). \end{aligned} \tag{3.7}$$

We obtain the coefficients associated with the series shown in Eq. (3.2) by solving the system above equations.

$$a_0 = 0 \tag{3.8}$$

$$a_1 = \pm\sqrt{\frac{-6Pc(bw + \beta c)}{a}}, \tag{3.9}$$

and

$$w = \frac{-\beta c^3 Q}{1 + bc^2 Q}. \tag{3.10}$$

By employing the approach above to address the problem and utilizing the value provided in Eq. (2.3), we successfully derived the precise answer for Eq. (1.1).

$$u = \pm\sqrt{\frac{-6\beta P c^2\left(1 - \frac{bc^2 Q}{1+bc^2 Q}\right)}{a}} F\left(c\frac{x^\gamma}{\gamma} - \frac{\beta c^3 Q}{1 + bc^2 Q}\frac{t^\alpha}{\alpha}\right) \tag{3.11}$$

When $P = m^2, Q = -\left(1 + m^2\right), R = 1,$ and $F = sn$ from Table 1, Thus

$$u_{1,1} = \pm\sqrt{\frac{-6\beta m^2 c^2\left(1 + \frac{bc^2\left(1+m^2\right)}{1-bc^2\left(1+m^2\right)}\right)}{a}}\,\mathrm{sn}\left(c\frac{x^\gamma}{\gamma} + \frac{c^3\beta\left(1 + m^2\right)}{1 - bc^2\left(1 + m^2\right)}\frac{t^\alpha}{\alpha}\right) \tag{3.12}$$

Taking into account m →1 from Table 2, the solution for the solitary wave can be found by using the following:

$$u_{1,2} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{2bc^2}{1-2bc^2}\right)}{a}}\tanh\left(c\frac{x^\gamma}{\gamma} + \frac{2c^3\beta}{1 - 2bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.13}$$

When $P = m^2, Q = -\left(1 + m^2\right), R = 1,$ and $F = cd$ can be deduced from Table 1. The solution can be evaluated as:

$$u_{1,3} = \pm\sqrt{\frac{-6\beta m^2 c^2\left(1 + \frac{bc^2\left(1+m^2\right)}{1-bc^2\left(1+m^2\right)}\right)}{a}}\,cd\left(c\frac{x^\gamma}{\gamma} + \frac{c^3\beta\left(1 + m^2\right)}{1 - bc^2\left(1 + m^2\right)}\frac{t^\alpha}{\alpha}\right) \tag{3.14}$$

As the limit of m approaches 1 from Table 2, the solitary wave solution may be observed,

$$u_{1,4} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{2bc^2}{1-2bc^2}\right)}{a}}\cosh\left(c\frac{x^\gamma}{\gamma} + \frac{2c^3\beta}{1 - 2bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.15}$$

Considering $P = -m^2, Q = 2m^2 - 1,$ and $F = cn$ from Table 1, the solution can be acquired as,

$$u_{1,5} = \pm\sqrt{\frac{6\beta m^2 c^2\left(1 - \frac{bc^2\left(2m^2-1\right)}{1+bc^2\left(2m^2-1\right)}\right)}{a}}\,cn\left(c\frac{x^\gamma}{\gamma} - \frac{\beta c^3\left(2m^2 - 1\right)}{1 + bc^2\left(2m^2 - 1\right)}\frac{t^\alpha}{\alpha}\right) \tag{3.16}$$

Moreover, if the limit of m approaches 1 according to Table 2, the solution for the solitary wave of Equation (1.1) can be expressed as follows:

$$u_{1,6} = \pm\sqrt{\frac{6\beta c^2\left(1 - \frac{bc^2}{1+bc^2}\right)}{a}}\,\mathrm{sech}\left(c\frac{x^\gamma}{\gamma} - \frac{\beta c^3}{1 + bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.17}$$

Setting $P = -1, Q = 2 - m^2$, and $F = dn$ from Table 1, so

$$u_{1,7} == \pm\sqrt{\frac{6\beta c^2\left(1 - \frac{bc^2(2-m^2)}{1+bc^2(2-m^2)}\right)}{a}}dn\left(c\frac{x^\gamma}{\gamma} - \frac{\beta c^3(2-m^2)}{1+bc^2(2-m^2)}\frac{t^\alpha}{\alpha}\right) \tag{3.18}$$

When examining the data in Table B for the limit as m approaches 1, it becomes evident that this solution resembles the single wave described in Eq. 3.17.

For choices $P = 1, Q = -(1 + m^2)$ and $F = ns$, from Table 1, in this way the solution is found as:

$$u_{1,8} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{bc^2(1+m^2)}{1-bc^2(1+m^2)}\right)}{a}}ns\left(c\frac{x^\gamma}{\gamma} + \frac{\beta c^3(1+m^2)}{1-bc^2(1+m^2)}\frac{t^\alpha}{\alpha}\right) \tag{3.19}$$

For choices m → 1 the solitary wave solution can be evaluated as,

$$u_{1,9} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{2bc^2}{1-2bc^2}\right)}{a}}\coth\left(c\frac{x^\gamma}{\gamma} + \frac{2\beta c^3}{1-2bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.20}$$

As the limit of m approaches zero, the periodic solution can be derived by utilizing Table 2,

$$u_{1,10} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{bc^2}{1-bc^2}\right)}{a}}\csc\left(c\frac{x^\gamma}{\gamma} + \frac{\beta c^3}{1-bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.21}$$

For selecting $P = 1,\ Q = -(1 + m^2)$ and $F = dc$, from Table 1, in this way the solution can be determined as,

$$u_{1,11} = \pm\sqrt{\frac{-6\beta c^2\left(1 + \frac{bc^2(1+m^2)}{1-bc^2(1+m^2)}\right)}{a}}dc\left(c\frac{x^\gamma}{\gamma} + \frac{\beta c^3(1+m^2)}{1-bc^2(1+m^2)}\frac{t^\alpha}{\alpha}\right) \tag{3.22}$$

The periodic solution can be obtained as the limit of m approaches zero.,

$$u_{1,12} = \sqrt{\frac{-6\beta c^2\left(1 + \frac{bc^2}{1-bc^2}\right)}{a}}sec\left(c\frac{x^\gamma}{\gamma} + \frac{\beta c^3}{1-bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.23}$$

Assigning $P = 1 - m^2, Q = 2m^2 - 1$, and $F = nc$, from table A, so the solution can be found as,

$$u_{1,13} = \pm\sqrt{\frac{-6\beta c^2(1-m^2)\left(1 - \frac{bc^2(2m^2-1)}{1+bc^2(2m^2-1)}\right)}{a}}nc\left(c\frac{x^\gamma}{\gamma} - \frac{\beta c^3(2m^2-1)}{1+bc^2(2m^2-1)}\frac{t^\alpha}{\alpha}\right) \tag{3.24}$$

In the context of m approaching zero, concerning Table 2, the solution can be determined as in Eq. 3.23.

Regarding the information found in Table A's *P = 1 − m^2, Q = 2 − m^2*, and *F = sc*, the solution can be written as,

$$u_{1,14} = \pm\sqrt{\frac{-6\beta\left(1-m^2\right)c^2\left(1-\frac{bc^2\left(2-m^2\right)}{1+bc^2\left(2-m^2\right)}\right)}{a}}sc\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(2-m^2\right)}{1+bc^2\left(2-m^2\right)}\frac{t^\alpha}{\alpha}\right) \tag{3.25}$$

Furthermore, as *m* approaches zero, the periodic solution can be obtained according to Table 2,

$$u_{1,15} = \pm\sqrt{\frac{-6\beta c^2\left(1-\frac{2bc^2}{1+2bc^2}\right)}{a}}tan\left(c\frac{x^\gamma}{\gamma}-\frac{2\beta c^3}{1+2bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.26}$$

Assuming the given values *P = 1, Q = 2 − m^2*, and *F = cs*, the solution can be derived by referring to Table 1,

$$u_{1,16} = \pm\sqrt{\frac{-6\beta c^2\left(1-\frac{bc^2\left(2-m^2\right)}{1+bc^2\left(2-m^2\right)}\right)}{a}}cs\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(2-m^2\right)}{1+bc^2\left(2-m^2\right)}\frac{t^\alpha}{\alpha}\right) \tag{3.27}$$

When the parameter m approaches 1, the solitary wave solution can be obtained using Table 2,

$$u_{1,17} = \pm\sqrt{\frac{-6\beta c^2\left(1-\frac{bc^2}{1+bc^2}\right)}{a}}\mathrm{csch}\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3}{1+bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.28}$$

Furthermore, as the limit of m approaches zero, according to the data shown in Table 2, the periodic solution can be expressed as,

$$u_{1,18} = \pm\sqrt{\frac{-6\beta c^2\left(1-\frac{2bc^2}{1+2bc^2}\right)}{a}}cot\left(c\frac{x^\gamma}{\gamma}-\frac{2\beta c^3}{1+2bc^2}\frac{t^\alpha}{\alpha}\right) \tag{3.29}$$

By making the corresponding selections from table A, (*P = 1, Q = 2m^2−1, F = ds*), the solution becomes

$$u_{1,19} = \pm\sqrt{\frac{-6\beta c^2\left(1-\frac{bc^2\left(2m^2-1\right)}{1+bc^2\left(2m^2-1\right)}\right)}{a}}ds\left[c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(2m^2-1\right)}{1+bc^2\left(2m^2-1\right)}\frac{t^\alpha}{\alpha}\right] \tag{3.30}$$

For the limit as m approaches 1 and m approaches 0, the solitary wave solution and periodic solution exhibit similarities to equations (3.29) and (3.21), respectively, when utilizing Tables 1 and 2.

When $P = \frac{-1}{4}, Q = \frac{m^2+1}{2}, F = mcn \mp dn$, from Table 1, so the solution can be obtained as,

$$u_{1,20}=\pm\sqrt{\frac{\frac{3}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\right)}{a}}\left[mcn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp dn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.31}$$

As the limit of m approaches 1, according to the data presented in Table 22, the solitary wave solution can be expressed as,

$$u_{1,21}=\pm\sqrt{\frac{\frac{3}{2}\beta c^2\left(1-\frac{bc^2}{1+bc^2}\right)}{a}}\left[sech\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3}{1+bc^2}\frac{t^\alpha}{\alpha}\right)\mp sech\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3}{1+bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.32}$$

Considering P= $\frac{1}{4}, Q=\frac{-2m^2+1}{2}, F=ns\mp cs$, from Table 1, the solution can be acquired as,

$$u_{1,22}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{-2m^2+1}{2}\right)}{1+bc^2\left(\frac{-2m^2+1}{2}\right)}\right)}{a}}\left[ns\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{-2m^2+1}{2}\right)}{1+bc^2\left(\frac{-2m^2+1}{2}\right)}\right)\mp cs\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^2\left(\frac{-2m^2+1}{2}\right)}{1+bc^2\left(\frac{-2m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.33}$$

When m approaches 1, according to the information provided in Table 2, the solitary wave solution can be expressed as,

$$u_{1,23}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\coth\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp \mathrm{csch}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.34}$$

For the limit, as m approaches zero, the periodic solution can be obtained as,

$$u_{1,24}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\csc\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp \cot\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.35}$$

Also assigning $P=\frac{1-m^2}{4}, Q=\frac{m^2+1}{2}, F=nc\mp sc$, according to the data presented in Table 1, the solution can be represented in the following manner,

$$u_{1,25}=\pm\sqrt{\frac{-3\beta\left(\frac{1-m^2}{2}\right)c^2\left(1-\frac{bc^2\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\right)}{a}}\left[nc\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp sc\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.36}$$

As the limit of m approaches zero, according to the data provided in Table 2, the periodic solution can be expressed as,

$$u_{1,26} = \pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{\frac{1}{2}bc^2}{1+\frac{1}{2}bc^2}\right)}{a}}\left[sec\left(c\frac{x^\gamma}{\gamma}-\frac{\frac{1}{2}\beta c^3}{1+\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp tan\left(c\frac{x^\gamma}{\gamma}-\frac{\frac{1}{2}\beta c^3}{1+\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.37}$$

Considering $P=\frac{1}{4}$, $Q=\frac{m^2-2}{2}$, $F=ns\mp ds$, the solution can be determined based on the data shown in Table 1,

$$u_{1,27} = \pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\right)}{a}}\left[ns\left(c\frac{x^\gamma}{\gamma}-\frac{c^3\beta\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp ds\left(c\frac{x^\gamma}{\gamma}-\frac{c^3\beta\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.38}$$

For the case when m tends to 1, according to the data provided in table B, the solution for the solitary wave can be expressed as follows.,

$$u_{1,28} = \pm\sqrt{\frac{-\frac{3}{2}\beta bc^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\coth\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp \operatorname{csch}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.39}$$

Also, regarding P $=\frac{m^2}{4}$, $Q=\frac{m^2-2}{2}$, $F=sn\mp icn$ from table A, the solution can be stated as,

$$u_{1,29} = \pm\sqrt{\frac{-\frac{3}{2}\beta m^2 c^2\left(1-\frac{bc^2\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\right)}{a}}\left[sn\left(c\frac{x^\gamma}{\gamma}-\frac{c^3\beta\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp icn\left(c\frac{x^\gamma}{\gamma}-\frac{c^3\beta\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.40}$$

As the m limit approaches 1, the solitary wave solution can be derived from Table 2,

$$u_{1,30} = \pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\tanh\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp i\operatorname{sech}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \tag{3.41}$$

Also if, P $=\frac{m^2}{4}$, $Q=\frac{m^2-2}{2}$, $F=\frac{sn}{\sqrt{1-m^2sn\mp cn}}$ from Table 1, the solution can be expressed as,

$$u_{1,31} = \pm\sqrt{\frac{-\frac{3}{2}\beta m^2 c^2\left(1-\frac{bc^2\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\right)}{a}}\left[\frac{sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{\sqrt{1-m^2sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp icn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}}\right] \tag{3.42}$$

If the limit of m approaches 1, the solitary wave solution can be derived as,

$$u_{1,32}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\frac{\tanh\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}{\sqrt{1-\tanh\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\mp i\text{sech}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}}\right] \quad (3.43)$$

While $P=\frac{1}{4}$, $Q=\frac{1-2m^2}{2}$, $F=msn\mp idn$, the solution can be expressed as indicated in Table 1,

$$u_{1,33}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\right)}{a}}\left[msn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp idn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\frac{t^\alpha}{\alpha}\right)\right] \quad (3.44)$$

For the limit as m approaches 1, the single wave is identical to the one described in Eq. (3.44).

Supposing $P=\frac{1}{4}$, $Q=\frac{1-2m^2}{2}$, $F=\frac{sn}{1\mp dn}$, the solution can be determined using the information in table A as,

$$u_{1,34}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\right)}{a}}\left[\frac{sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{1\mp dn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{1-2m^2}{2}\right)}{1+bc^2\left(\frac{1-2m^2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}\right] \quad (3.45)$$

In addition, using Table 2, as a reference $m\to 1$, the expression for solitary waves can be expressed as follows:

$$u_{1,35}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\frac{\tanh\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}{1\mp \text{sech}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}\right] \quad (3.46)$$

The periodic solution can be derived for $m\to 0$ by consulting table B, and it is as follows:

$$u_{1,36}=\pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{\frac{1}{2}bc^2}{1+\frac{1}{2}bc^2}\right)}{a}}\left[\sin\left(c\frac{x^\gamma}{\gamma}-\frac{\frac{1}{2}\beta c^3}{1+\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)\right] \quad (3.47)$$

Supposing P $=\frac{m^2}{4}$, $Q=\frac{m^2-2}{2}$, $F=\frac{sn}{1\mp dn}$, from table A, you can figure out the solution as,

$$u_{1,37} = \pm\sqrt{\frac{-\frac{3m^2}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\right)}{a}}\left[\frac{sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{1\mp dn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}\right] \tag{3.48}$$

When $P=\frac{1-m^2}{4}, Q=\frac{m^2+1}{2}, F=\frac{cn}{1\mp sn}$, the solution can be expressed as indicated in Table 1,

$$u_{1,38} = \pm\sqrt{\frac{-3\left(\frac{1-m^2}{2}\right)\beta c^2\left(1-\frac{bc^2\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\right)}{a}\left[\frac{cn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{1\mp sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)}\right]} \tag{3.49}$$

If the limit of m approaches zero, the periodic solution can be calculated using the information provided in table B.

$$u_{1,39} = \pm\sqrt{\frac{-\frac{3}{2}\beta c^2\left(1-\frac{\frac{1}{2}bc^2}{1+\frac{1}{2}bc^2}\right)}{a}}\left[\frac{cos\left(c\frac{x^\gamma}{\gamma}-\frac{\frac{1}{2}\beta c^3}{1+\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}{1\mp sin\left(c\frac{x^\gamma}{\gamma}-\frac{\frac{1}{2}\beta c^3}{1+\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}\right] \tag{3.50}$$

Supposing P= $\frac{(1-m^2)^2}{4}$, $Q=\frac{m^2+1}{2}$, $F=\frac{sn}{dn\mp cn}$, from Table 1, the solution can be found as,

$$u_{1,40} = \pm\sqrt{\frac{-\frac{3\beta(1+m^2)^2}{2}c^2\left(1-\frac{bc^2\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\right)}{a}}\left[\frac{sn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{dn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)\mp cn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2+1}{2}\right)}{1+bc^2\left(\frac{m^2+1}{2}\right)}\frac{t^\alpha}{\alpha}\right)}\right] \tag{3.51}$$

Also, regarding P= $\frac{m^4}{4}$, $Q=\frac{m^2-2}{2}$, $F=\frac{cn}{\sqrt{1-m^2\mp dn}}$, from Table 1, the solution can be stated as,

$$u_{1,41} = \pm\sqrt{\frac{\frac{-3m^4}{2}\beta c^2\left(1-\frac{bc^2\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\right)}{a}}\left[\frac{cn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}{\sqrt{1-m^2\mp dn\left(c\frac{x^\gamma}{\gamma}-\frac{\beta c^3\left(\frac{m^2-2}{2}\right)}{1+bc^2\left(\frac{m^2-2}{2}\right)}\frac{t^\alpha}{\alpha}\right)}}\right] \tag{3.52}$$

Setting $m\to 1$ by using Table 2, due to this setting,

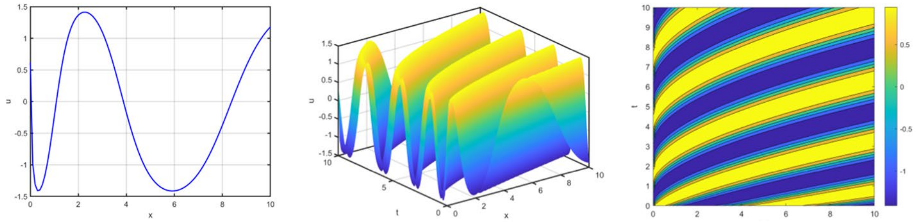

Fig. 1 Two-Dimensional, surface and contour plot for the solution $u_{1,1}$ for $t = 1$, $a = 1$, $b = 2$, $c = 2$, $B = 3$, $m = 0.5$, $\gamma = 0.5$, $\alpha = 0.8$.

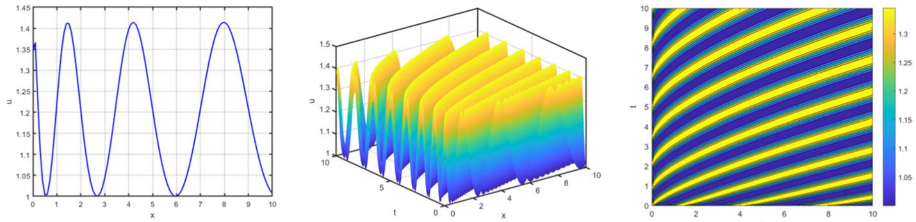

Fig. 2 Two-Dimensional, surface and contour plot for the solution $u_{1,3}$ for $t = 1$, $a = 1$, $b = 2$, $c = 2$, $B = 3$, $m = 0.5$, $\gamma = 0.6$, $\alpha = 0.8$.

$$u_{1,42} = \pm\sqrt{\frac{\frac{-3}{2}\beta c^2\left(1+\frac{\frac{1}{2}bc^2}{1-\frac{1}{2}bc^2}\right)}{a}}\left[\frac{\operatorname{sech}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}{\sqrt{\mp\operatorname{sech}\left(c\frac{x^\gamma}{\gamma}+\frac{\frac{1}{2}\beta c^3}{1-\frac{1}{2}bc^2}\frac{t^\alpha}{\alpha}\right)}}\right] \tag{3.53}$$

4 Graphical results

In this section, we explore into the complexities of wave solutions that are derived by the Jacobi elliptic function expansion method. In specifically, we apply this method to the improved mKdv equation that incorporates conformable derivatives. In order to provide a clear and concise illustration of the characteristics and actions of these wave solutions, we have carefully chosen particular values for the parameters that are involved. This option was made with the intention of displaying a wide variety of soliton wave patterns, which are depicted in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. The improved mKdV equation results in wave patterns that are diverse and complex and each figure is a witness to the diversity and complexity of these wave patterns. For the purpose of providing a more comprehensive perspective, we have utilized both two-dimensional and three-dimensional representations together. The surface profiles of the waves are the primary emphasis of the two-dimensional plots, whilst the three-dimensional plots add depth and provide a more tangible feel of the wave patterns. Additionally, contour plots are given in order to highlight the gradients and intensities that are present within the wave patterns. This provides a

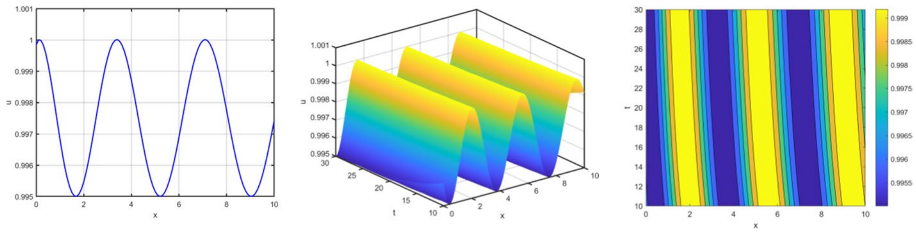

Fig. 3 Two-Dimensional, surface and contour plot for the solution $u_{1,7}$ for $t = 1$, $a = 1$, $b = 1$, $c = -1$, $B = 0.5$, $m = 0.5$, $\gamma = 0.9$, $\alpha = 0.1$.

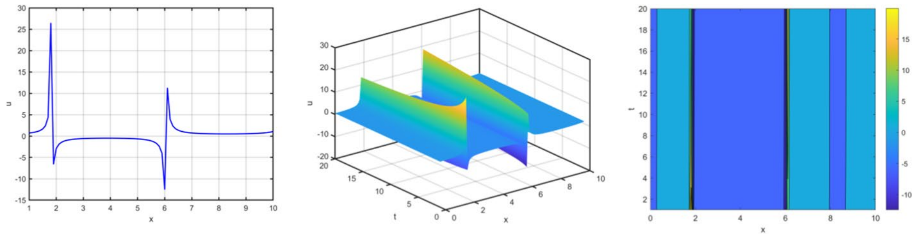

Fig. 4 Two-Dimensional, surface and contour plot for the solution $u_{1,11}$ for $t = 1$, $a = 0.1$, $b = 100$, $B = 0.5$, $c = -1$, $m = 0.5$, $\gamma = 0.9$, $\alpha = 0.1$.

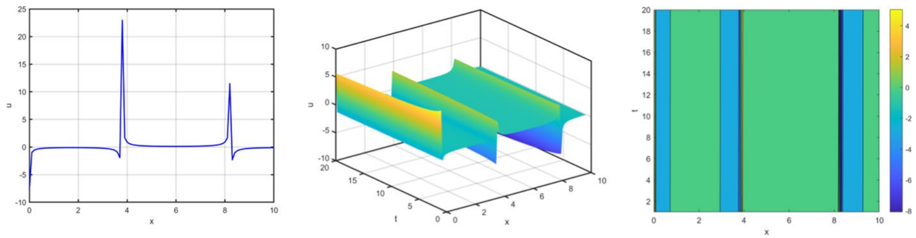

Fig. 5 Two-Dimensional, surface and contour plot for the solution $u_{1,20}$ for $t = 1$, $a = 1$, $b = 1$, $B = 0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1.B = 0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1$.

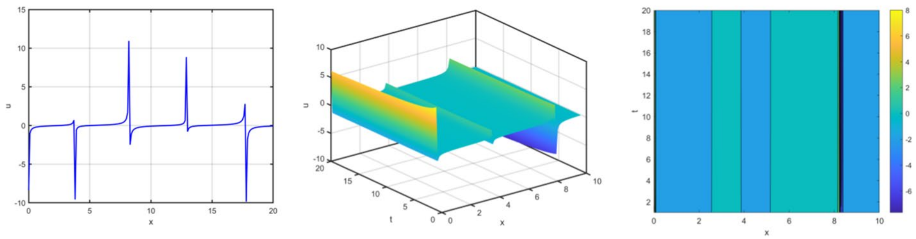

Fig. 6 Two-Dimensional, surface and contour plot for the solution $u_{1,22}$ for $t = 1$, $a = 1$, $b = 100$, $B = -0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.9$, $\alpha = 0.1$.

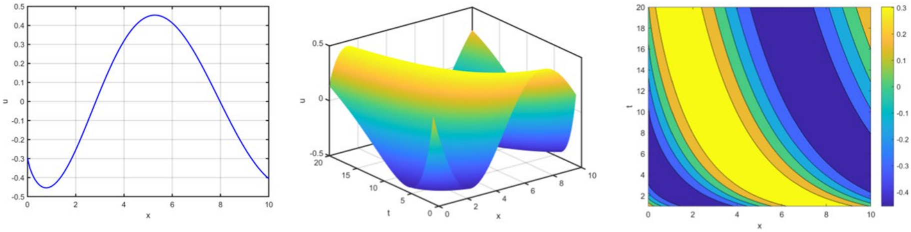

Fig. 7 Two-Dimensional, surface and contour plot for the solution $u_{1,27}$ for $t = 1$, $a = 1$, $b = 1$, $B = -0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1$.

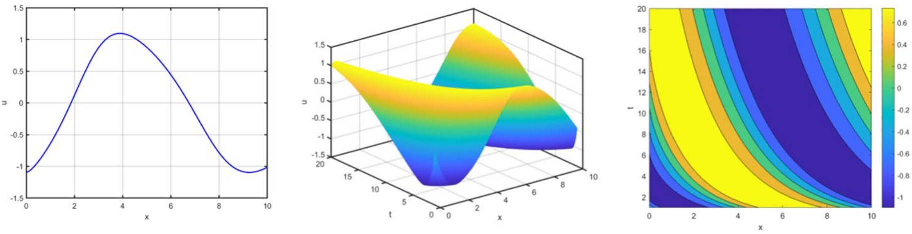

Fig. 8 Two-Dimensional, surface and contour plot for the solution $u_{1,29}$ for $t = 1$, $a = 1$, $b = 1$, $B = -0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1$.

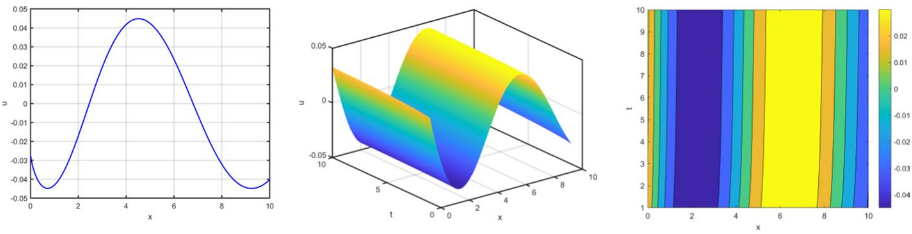

Fig. 9 Two-Dimensional, surface and contour plot for the solution $u_{1,33}$ for $t = 1$, $a = 1$, $b = 1$, $B = -0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1$.

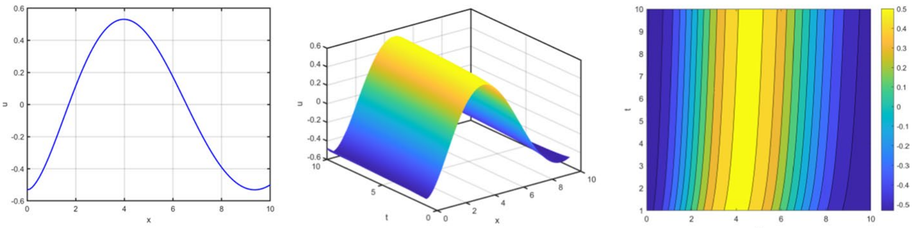

Fig. 10 Two-Dimensional, surface and contour plot for the solution $u_{1,37}$ for $t = 1$, $a = 0.1$, $b = 0.1$, $B = -0.2$, $c = -1$, $m = 0.5$, $\gamma = 0.8$, $\alpha = 0.1$.

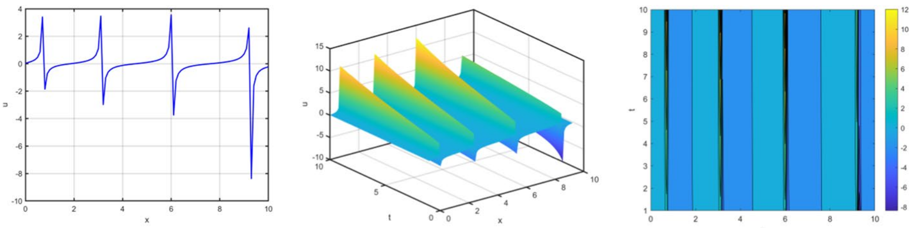

Fig. 11 Two-Dimensional, surface and contour plot for the solution $u_{1,40}$ for $t = 1$, $a = 1$, $b = 10$, $B = -0.1$, $c = -3$, $m = 0.1$, $\gamma = 0.8$, $\alpha = 0.1$.

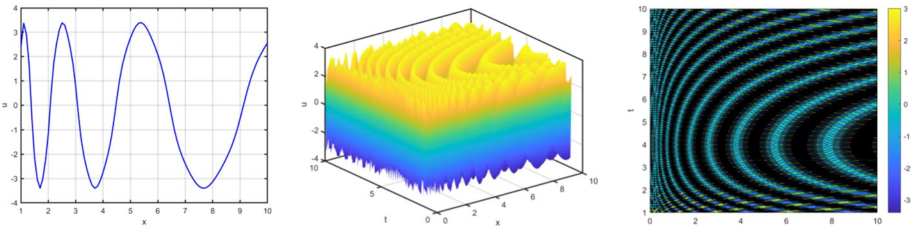

Fig. 12 Two-Dimensional, surface and contour plot for the solution $u_{1,40}$ for $t = 1$, $a = 1$, $b = 0.1$, $B = -0.5$, $c = -12$, $m = 0.9$, $\gamma = 0.8$, $\alpha = 0.1$.

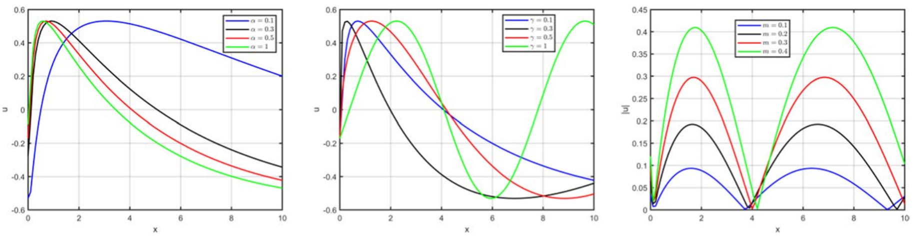

Fig. 13 $u_{1,40}$ for $t = 1$, $a = 1$, $b = 0.1$, $B = -0.2$, $c = -1$ for various values of α, γ, and m.

distinct perspective on the wave dynamics. Not only are the graphical representations useful as visual aids, but they also play an important role in validating the theoretical results that we had previously derived. We are able to verify the accuracy and dependability of the mathematical model as well as the solutions that were obtained by comparing these graphical outputs with the theoretical predictions. In Fig. 13, the left side figure represents variation of conformable parameter α, of time variable when the parameter α increases the amplitude of the solution decreases; the right side in Fig. 13, the middle figure represents variation of conformable parameter γ, of space variable when the parameter γ increases the amplitude of the solution increases, and m influences the periodicity and shape of the solutions. As m changes, it affects the waveform's peaks and troughs, indicating a transition from sinusoidal-like waves to sharper, possibly more solitary wave-like structures. As part of our effort, we made use of MATLAB, which is well-known for its reliability in mathematical computations and visualizations, in order to guarantee that the graphical representations we presented were accurate and clear. Due to the fact that this programmed is particularly skilled in charting complicated mathematical functions, it is an excellent option for illustrating elaborate wave patterns of solitons that are produced from the improved mKdV equation. Due to the fact that this tool is capable of displaying complex mathematical functions, the decision to employ it was one that was deliberated about. The purpose of our research is to bridge the gap between theoretical conceptions and a concrete, visual understanding of wave dynamics by utilizing this fusion of theoretical mathematics with computational visualization.

5 Conclusion

Within the scope of this exhaustive investigation, we had thoroughly investigated the use of the Jacobi Elliptic Function Expansion Method (JEFE) in order to generate accurate solutions for the improved modified Korteweg-de Vries equation with conformable derivative. As a result of our research, we had been able to successfully develop a wide range of solutions. These solutions included soliton, hyperbolic, and trigonometric forms, all of which were essential in comprehending the complexity of the equation. An essential component of our strategy was the implementation of the Jacobi elliptic function expansion (JEFE), in which we made use of a series expansion of degree n in Jacobi elliptic functions. It had been demonstrated that this novel approach was exceptionally adaptable, as it had been able to produce a wide variety of solutions, ranging from single-function to combined-function formulations. The effectiveness of the JEFE in producing a wide variety of solutions was especially noteworthy when compared to the effectiveness of other analytical methods that were utilized in the field. In order to gain a more in-depth understanding of the complexities of these solutions, we utilized MATLAB to plot a variety of surface and contour graphs in both two and three dimensions. With the help of this graphical depiction, we were able to gain a profound understanding of the dynamic behavior of the solutions that we developed. In addition, the results of our inquiry highlighted the extraordinary adaptability of the Jacobi elliptic function expansion strategy in the context of solving a wide variety of partial differential equations, including those of the conformable derivative type. It had not only been proved that this method was capable of addressing difficult mathematical problems, but it had also made a substantial contribution to the development of analytical procedures within the field. In conclusion, the findings of our research demonstrated that the Jacobi elliptic function expansion approach was a powerful instrument that could be utilized in the analytical evaluation of the improved mKdV equation on account of its versatility and potency. It was a shining example of how the landscape of mathematical problem-solving strategies was constantly shifting, notably in the field of partial differential equations.

Additionally, we have expressed a strong desire to investigate other structured approaches in the future so that we can advance our research. Bifurcation analysis, chaotic behaviors, sensitivity analysis, and soliton solutions of a generalized Schrodinger equation are some of the methodologies that fall under this category. In addition, we are interested in examining the solitary waves of coupled nonlinear Schrodinger equations by means of a generalized approach, as well as the periodic and solitary waves of the nonlinear Konno-Oono Model by means of generalized techniques and a generalized nonlinear Schrodinger equation involving mild no locality. In particular, we will be concentrating on analyzing the Jacobi elliptic function solutions and the modulation instability brought about by these models. The objective of this technique is to derive additional soliton waves of the governing model, so demonstrating our dedication to enhancing their comprehension and making contributions that are original to the field.

Author contributions AF: Writing: Original Draft, Writing: Review and Editing, Software, Conceptualisation. MIK: Writing: Review and Editing, Methodology, Formal Analysis.WXM: Revised and Editing.

Funding No funding.

Declarations

Conflict of interests The authors declare no competing interests.

References

Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
Abdulhameed, M., Muhammad, M.M., Gital, A.Y., Yakubu, D.G., Khan, I.: Effect of fractional derivatives on transient MHD flow and radiative heat transfer in a micro-parallel channel at high zeta potentials. Physica A: Stat. Mech. Appl. **519**, 42–71 (2019)
Alquran, M.: New interesting optical solutions to the quadratic–cubic Schrodinger equation by using the Kudryashov-expansion method and the updated rational sine–cosine functions. Opt. Quant. Electronics **54**(10), 666 (2022)
Alquran, M.: Classification of single-wave and bi-wavemotion through fourth-order equations generated from the Ito model. Phys. Scr. (2023)
Alquran, M., Alhami, R.: Analysis of lumps, single-stripe, breather-wave, and two-wave solutions to the generalized perturbed-KdV equation by means of Hirota's bilinear method. Nonlinear Dyn. **109**(3), 1985–1992 (2022)
Alquran, M., Ali, M., Jadallah, H.: New topological and non-topological unidirectional-wave solutions for the modified-mixed KdV equation and bidirectional-waves solutions for the Benjamin Ono equation using recent techniques. J. Ocean Eng. Sci. **7**(2), 163–169 (2022)
Alquran, M., Najadat, O., Ali, M., Qureshi, S.: New kink-periodic and convex–concave-periodic solutions to the modified regularized long wave equation by means of modified rational trigonometric–hyperbolic functions. Nonlinear Eng. **12**(1), 20220307 (2023a)
Alquran, M., Ali, M., Gharaibeh, F., Qureshi, S.: Novel investigations of dual-wave solutions to the Kadomtsev-Petviashvili model involving second-order temporal and spatial–temporal dispersion terms. Partial Differ. Equations Appl. Math. **8**, 100543 (2023b)
Bekir, A., Güner, Ö.: Exact solutions of nonlinear fractional differential equations by (G′/G)-expansion method. Chin. Phys. B **22**(11), 110202 (2013)
Chang, A., Sun, H., Zhang, Y., Zheng, C., Min, F.: Spatial fractional Darcy's law to quantify fluid flow in natural reservoirs. Physica A: Stat. Mech. Appl. **519**, 119–126 (2019)
Dubey, V.P., Kumar, R., Kumar, D.: Analytical study of fractional Bratu-type equation arising in electro-spun organic nanofibers elaboration. Physica A: Stat. Mech. Appl. **521**, 762–772 (2019)
Eslami, M., Rezazadeh, H.: The first integral method for Wu–Zhang system with conformable time-fractional derivative. Calcolo **53**, 475–485 (2016)
Gepreel, K.A., Omran, S.: Exact solutions for nonlinear partial fractional differential equations. Chin. Phys. B **21**(11), 110204 (2012)
Ghanbari, B.: Abundant soliton solutions for the Hirota-Maccari equation via the generalized exponential rational function method. Modern Phys. Lett. B **33**(09), 1950106 (2019)
Ghanbari, B., Baleanu, D.: New optical solutions of the fractional Gerdjikov-Ivanov equation with conformable derivative. Front. Phys. **8**, 167 (2020)
Ghanbari, B., Gómez-Aguilar, J.F.: Optical soliton solutions for the nonlinear Radhakrishnan–Kundu–Lakshmanan equation. Modern Phys. Lett. B **33**(32), 1950402 (2019a)
Ghanbari, B., Gómez-Aguilar, J.F.: New exact optical soliton solutions for nonlinear Schrödinger equation with second-order spatio-temporal dispersion involving M-derivative. Modern Phys. Lett. B **33**(20), 1950235 (2019b)
Ghanbari, B., Kuo, C.K.: New exact wave solutions of the variable-coefficient (1+ 1)-dimensional Benjamin-Bona-Mahony and (2+ 1)-dimensional asymmetric Nizhnik-Novikov-Veselov equations via the generalized exponential rational function method. Euro. Phys. J. Plus **134**(7), 334 (2019)
Goulart, A.G., Lazo, M.J., Suarez, J.M.S.: A new parameterization for the concentration flux using the fractional calculus to model the dispersion of contaminants in the Planetary Boundary Layer. Physica A: Stat. Mech. Appl. **518**, 38–49 (2019)

Guo, S., Mei, L., Li, Y., Sun, Y.: The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics. Phys. Lett. A **376**(4), 407–411 (2012)

Hirota, R., Satsuma, J.: Soliton solutions of a coupled Korteweg-de Vries equation. Phys. Lett. A **85**(8–9), 407–408 (1981)

Hwang, C. K., Lu, T. T.: The exact solutions of certain linear partial difference equations. arXiv preprint arXiv: 2301.09501 (2023).

Inc, M.: The approximate and exact solutions of the space-and time-fractional Burgers equations with initial conditions by variational iteration method. J. Math. Anal. Appl. **345**(1), 476–484 (2008)

Inc, M., Ergüt, M.: Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method. Appl. Math. E-Notes [electronic only] **5**, 89–96 (2005)

Inç, M., Evans, D.J.: A study for obtaining more solitary pattern solutions of fifth-order KdV-like equations. Int. J. Comp. Math. **81**(4), 473–482 (2004)

Jaradat, I., Alquran, M.: Construction of solitary two-wave solutions for a new two-mode version of the Zakharov-Kuznetsov equation. Mathematics **8**(7), 1127 (2020)

Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)

Khan, M.I., Asghar, S., Sabi'u, J.: Jacobi elliptic function expansion method for the improved modified kortwedge-de vries equation. Opt. Quant. Electr. **54**(11), 734 (2022)

Khater, M., Ghanbari, B.: On the solitary wave solutions and physical characterization of gas diffusion in a homogeneous medium via some efficient techniques. Euro. Phys. J. Plus **136**(4), 1–28 (2021)

Khresat, H., El-Ajou, A., Al-Omari, S., Alhazmi, S.E., Oqielat, M.A.N.: Exact and approximate solutions for linear and nonlinear partial differential equations via laplace residual power series method. Axioms **12**(7), 694 (2023)

Lu, B.: Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations. Phys. Lett. A **376**(28–29), 2045–2048 (2012)

Özkan, A., Özkan, E.M., Yildirim, O.: On exact solutions of some space–time fractional differential equations with m-truncated derivative. Fractal Fractional **7**(3), 255 (2023)

Qiao, J.M., Zhang, R.F., Yue, R.X., Rezazadeh, H., Seadawy, A.R.: Three types of periodic solutions of new (3+ 1)-dimensional Boiti–Leon–Manna–Pempinelli equation via bilinear neural network method. Math. Methods Appl. Sci. **9**, 5612–5621 (2022)

Řehák, P.: Superlinear solutions of sublinear fractional differential equations and regular variation. Fract. Calculus Appl. Anal. **26**(3), 989–1015 (2023)

Rezazadeh, H.: New solitons solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity. Optik **167**, 218–227 (2018)

Sait, S. A. N.: Extended Jacobi elliptic function solutions for general boussinesq systems. Rev. Mexicana de Física **69**(2 Mar–Apr), 021401-1 (2023).

Tang, B., He, Y., Wei, L., Zhang, X.: A generalized fractional sub-equation method for fractional differential equations with variable coefficients. Phys. Lett. A **376**(38–39), 2588–2590 (2012)

Wen, C., Zheng, B.: A new fractional sub-equation method for fractional partial differential equations. WSEAS Trans. Math. **12**(5), 564–571 (2013)

Younas, U., Ren, J., Akinyemi, L., Rezazadeh, H.: On the multiple explicit exact solutions to the double-chain DNA dynamical system. Math. Methods Appl. Sci. **46**(6), 6309–6323 (2023)

Zhang, S., Zhang, H.Q.: Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys. Lett. A **375**(7), 1069–1073 (2011)

Zheng, B.: (G′/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. Commun. Theor. Phys. **58**(5), 623 (2012)

Zheng, B.: A new fractional Jacobi elliptic equation method for solving fractional partial differential equations. Adv. Differ. Equations **2014**(1), 1–11 (2014)

Zheng, B., Wen, C.: Exact solutions for fractional partial differential equations by a new fractional sub-equation method. Adv. Differ. Equations **2013**, 1–12 (2013)

Authors and Affiliations

Aamir Farooq[1] · **Muhammad Ishfaq Khan**[2] · **Wen Xiu Ma**[1,3,4,5]

✉ Aamir Farooq
aamir88@yahho.com

Muhammad Ishfaq Khan
m.ishfaqkhan032@gmail.com

Wen Xiu Ma
mawx@cas.usf.edu

1 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, People's Republic of China

2 Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

3 Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

4 Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

5 Material Science Innovation and Modelling, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa