The computation of Lie point symmetry generators, modulational instability, classification of conserved quantities, and explicit power series solutions of the coupled system

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The well-known Chaffee-Infante reaction hierarchy is examined in this article along with its reaction-diffusion coupling. It has numerous applications in modern sciences, such as electromagnetic wave fields, fluid dynamics, high-energy physics, ion-acoustic waves in plasma physics, coastal engineering, and optical fibres. The physical processes of mass transfer and particle diffusion might be expressed in this way. The Lie invariance criteria is taken into consideration while we determine the symmetry generators. The suggested approach produces the six dimensional Lie algebra, where translation symmetries in space and time are associated to mass conservation and conservation of energy respectively, the other symmetries are scaling or dilation. Additionally, similarity reductions are performed, and the optimal system of the sub-algebra should be quantified. There are an enormous number of exact solutions can construct for the traveling waves when the governing system is transformed into ordinary differential equations using the similarity transformation technique. The power series approach is also utilized for ordinary differential equations to obtain closed-form analytical solutions for the proposed diffusive coupled system. The stability of the model under the limitations is ensured by the modulation instability analysis. The reaction diffusion hierarchy’s conserved vectors are calculated using multiplier methods using Lie Backlund symmetries. The acquired results are presented graphically in 2-D and 3-D to demonstrate the wave propagation behavior.

Introduction

The partial differential equations are organically reconnoitered within numerous areas of science like that engineering, chemistry, physics issues. The use of Mathematical simulation with non-linear PDEs have been imitated captivate when studying differential equations. One of the important tools for thoroughly interrogating the features of physical occurrences is the partial differential equation. The outstanding methods as further, exactly interpreting the intricate physical non-linear structure is the Schrödinger kind governing equation, that can be crucial in the domains of optics, fiber optics, telecommunication technology, and plasma technology [1–4]. Derivation of analytical solutions to Schrödinger equation is very significant examination domain whereas the analytical solutions poses a significant function in characterizing the tangible features for non-linear structure along-with applied mathematics [5,6].
non-linear PDEs has grown in relevance within both excess in both applied as well as pure mathematics the past ten years. The manipulation of computer automation has given mathematicians access to new areas in the applied sciences. The non-linear systems which can be broadly utilize in engineering, physics and mathematical sciences have gaining popularity. This field is attracting the researchers and scientists because a lot work is dolloping in this field. Kaur and Wazwaz [7] developed the invariant solitons and performed the painleve test for the integrable system. Kaur and Gupta [8] investigated Kawahara equation and modified Kawahara equation by Lie symmetry approach and developed variety of soliton solutions. Kaur and Wazwaz [9] generated the Lie symmetry infinitesimals generators for Einstein’s vacuum field equation. Wazwaz and Kaur [10] applied Hirota method on Boussinesq equations to construct the real and complex soliton solutions and also examined the integrability of the model.

Analytical solutions are the premier choice for establishing quantifiable forecasts [11]. Reliability and reliability of the analytical outcomes these are main source. That appears fore improve validity of the quantifiable evaluations that have been developed. In mathematical sciences, making quantitative predictions is a crucial activity that is typically carried out via resolving differential equations. The action of operating structure in the physics, biology and chemistry is explained using differential equations. These equations need to be solved to get numerical predictions that are quantitative. These are one or the other resolved numerically as well as analytically, [12]. Analytical approaches can be used to explain outcomes as functions of variables when it is possible to provide it. Then, quantitative forecasts are generated employing these analytical findings.

\[
\begin{align*}
    u_t &= 12uv_x + 24(u - u^2)v, \\
    v_t &= -12v_{xx} - 24(v - v^3)u.
\end{align*}
\]

We are not aware of a single, standard methodology which can be maneuvered to find the analytical results to all stiff non-linear partial differential equations in the current decade. Therefore, more efficient methods are always needed to identify the exact outcome of similar questions. Such methodologies have so deserving of desire. There are so many techniques has been substantiated to perceive the analytical solutions for non-linear partial differential equations similar that, Kudryashov methodology [13,14], sine-Gordon expansion methodology [15,16], bilinear linear network methodology [17,18], extended simple equation methodology [19], F-expansion methodology [20,21], unified auxiliary equation methodology [22,23], \( \frac{D}{\partial t} \) - expansion approach [24], Hirota bilinear methodology [25], the generalized exponential function methodology [26], and numerous further [27–31].

In this study, the lie symmetry analysis performed and infinitesimals generators developed. The power series solution constructed. But, there are numerous kinds of analytical exact solutions are still mystery for this system. There are many physical aspects and solutions are not developed, such as rogue waves, breathers, solitons interactions, bifurcation analysis, sensitivity and chaos analysis.

### Lie infinitesimals algebra

Now, we examine Lie symmetry generators of CSGE for the reason of constructing the Cls.

Let us assume Lie point group of the infinitesimal transformations with one parameter acting on the \( x,t \) independent as well as \( u,v \) dependent factors related to Eq. (1) are as follows,

\[
\begin{align*}
    x^\nu &= O(\nu^2) + \nu E^x(t, x, u, v) + x, \\
    t^\nu &= O(\nu^2) + \nu E^t(t, x, u, v) + t, \\
    u^\nu &= O(\nu^2) + \nu Y^u(t, x, u, v) + u, \\
    v^\nu &= O(\nu^2) + \nu Y^v(t, x, u, v) + v,
\end{align*}
\]

where \( \nu \ll 1 \) is very tiny Lie point group component and \( E^x, E^t, Y^u, \) and \( Y^v \) are the Lie infinitesimals of the transformations that must be found for independent and dependent variables, respectively.

The symmetry generator associated to the Lie algebra of Eq. (1) has the formation,

\[
\Gamma = \mathfrak{e}^x(t, x, u, v) \partial x + \mathfrak{e}^t(t, x, u, v) \partial t + Y^u(t, x, u, v) \partial u + Y^v(t, x, u, v) \partial v.
\]

(3)

The symmetries within Eq. (1) are generated by the vector field (3). Eq. (1)’s invariance condition changes through \( \Gamma \) from being,

\[
\mathcal{S}^2 \Gamma(\Delta) = 0,
\]

(4)

when \( \Delta \) is zero. Where \( \mathcal{S}^2 \) is second prolongation of \( \Gamma \) it would be represented as,

\[
\mathcal{S}^2 \Gamma = \Gamma + Y^{u_{\alpha}} \frac{\partial}{\partial u_{\alpha}} + Y^{v_{\alpha \beta}} \frac{\partial}{\partial v_{\alpha \beta}} + Y^{v_{\alpha}} \frac{\partial}{\partial v_{\alpha}} + Y^{v_{\alpha \beta}} \frac{\partial}{\partial v_{\alpha \beta}},
\]

(5)

with the coefficients

\[
\begin{align*}
    Y^{u_i} &= D_i \left( Y - \sum_{j=1}^2 \mathfrak{e}^j u^j, \right) + \sum_{i=1}^2 \mathfrak{e}^i u^i, \\
    Y^{v_i} &= D_i \left( Y - \sum_{j=1}^2 \mathfrak{e}^j v^j, \right) + \sum_{i=1}^2 \mathfrak{e}^i v^i,
\end{align*}
\]

where \((j_1 \ldots j_s, 1 \leq j_s \leq 2, 1 \leq s \leq 2\). By concerning the 2nd prolongation (5) onto Eq. (1), then recapture the collection of determined equations. That obtained model of determined partial differential equation can be elucidated by implementing the computer algebra program Maple as well as secured necessary infinitesimal generators whenever, \( E_{\alpha_j}(t, x, u, v) = x C_{j_1} + t C_{j_2} + C_{j_3}, \)
\( E_{\alpha_j}(t, x, u, v) = 2C_{j_1} + C_4, \)

\[
\begin{align*}
    Y_{\alpha_j}(t, x, u, v) &= \frac{1}{24} u(1152C_{2j} - C_2x + 24C_{2j}), \\
    Y_{\alpha_j}(t, x, u, v) &= -\frac{1}{24} u(1152C_{1j} - C_2x + 48C_{1j} + 24C_{3j}),
\end{align*}
\]

where the arbitrary constants are \( C_{j_i} \), \( (j = 1, 2, 3, 4, 5) \). The following vector fields produce the algebra of Lie point symmetries,

\[
\begin{align*}
    F_1 &= \partial_x, \\
    F_2 &= \partial_t, \\
    F_3 &= u \partial_u - v \partial_v, \\
    F_4 &= -\frac{1}{24} u x \partial_u + \frac{1}{24} u x \partial_x + t \partial_t, \\
    F_5 &= 48u \partial_u + 2t \partial_t + x \partial_x - (48u + 2v) \partial_v.
\end{align*}
\]

While, for our suitability, we have to expand a commutator Table 1 via accepting inputs on now \([\Gamma_i, \Gamma_j] = \Gamma_i \Gamma_j - \Gamma_j \Gamma_i\) for Eqs. (1).

### Lie symmetry group

Clearly, the infinite-dimensional Lie point algebra can be comprised for infinitesimal vector generators \( \Gamma_i, 1 \leq i \leq 5 \) (7) evolves some infinite continuous group with conversions of Eqs. (1). There is linear independence among the infinitesimal generators. However, one would
be represent some infinitesimal symmetry generators of (7) as a linear combination of $\mathcal{G}_1$, properly, such that,

$$A = \mathbb{B}_1 \mathcal{G}_1 + \mathbb{B}_2 \mathcal{G}_2 + \mathbb{B}_3 \mathcal{G}_3 + \mathbb{B}_4 \mathcal{G}_4 + \mathbb{B}_5 \mathcal{G}_5$$  \hspace{1cm} (8)

The theory of Lie analysis can be stand on the examination of there invariance about one parameter sub-group of point transformations with infinitesimal generators interpreted as vector fields. In order to acquire analytical outcomes of (1), to find Lie symmetry groups, we should employ related symmetry algebras. In order to secure the Lie analysis, particularly when looking for exact solutions to differential equations, it is a concept that is frequently used in the field of differential equations and symmetry analysis, particularly when looking for exact solutions to differential equations. It could be observed through Table 1 while, the generated vector fields $\mathcal{G}_1, \mathcal{G}_2$, along with $\mathcal{G}_3$, applies for an abelian algebra’s configuration. So, it could be represented these vector fields like that,

$$\mathcal{L}_1 = \langle \mathcal{G}_1 \rangle,$$

$$\mathcal{L}_2 = \langle \mathcal{G}_2 \rangle,$$

$$\mathcal{L}_3 = \langle \mathcal{G}_1 + b_2 \mathcal{G}_2 \rangle,$$

$$\mathcal{L}_4 = \langle \mathcal{G}_1 + b_2 \mathcal{G}_2 + b_1 \mathcal{G}_3 \rangle.$$

$$\mathcal{L}_5 = \langle \mathcal{G}_1 + \frac{a_2}{a_1} \mathcal{G}_2 \rangle,$$

Here, the corresponding Lagrange equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}.$$  \hspace{1cm} (9)

Using an Eq. (9) with the partial differential equation (1). So, we obtained a ordinary differential model while,

$$F''(\phi) = 24(F - F^2)H,$$

$$H'(\phi) = 24(H - H^2)F.$$  \hspace{1cm} (10)

However, in this scenario, the only similarity variable is the temporal component $t$. As a result, we can determine the solutions to Eq. (10) in terms of variable $t$, which may not actually correspond to occurrences that are interesting from a physical perspective. So, we will not go into depth here.

$$\mathcal{L}_2 = \langle \mathcal{G}_1 \rangle = \frac{1}{2},$$

Here, the corresponding Lagrange equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}.$$  \hspace{1cm} (11)

Using an Eq. (11) with the partial differential equation (1). So, we obtained a ordinary differential model while,

$$F''(\phi) + 24(F - F^2)H = 0,$$

$$H''(\phi) + 24(H - H^2)F = 0.$$  \hspace{1cm} (12)

In this scenario, the only similarity variable is the spatio component $x$. As a result, we can determine the solutions to Eq. (12) in terms of variable $x$, which may not actually correspond to occurrences that are interesting from a physical perspective. So, we will not go into depth here.

$$\mathcal{L}_3 = \langle \mathcal{G}_1 + a_2 \mathcal{G}_2 \rangle = \frac{a_2}{a_1},$$

Here, the corresponding Lagrange equation is,

$$\frac{dx}{a_2} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}.$$  \hspace{1cm} (13)

Using an Eq. (13) with the partial differential equation (1). So, we acquire an ordinary differential model while,

$$F''(\phi) = -12a_2F''(\phi) - 24(F' - F^2)H(\phi),$$

$$H'(\phi) = 12a_2H''(\phi) + 24(H(\phi) - H^2F')F(\phi).$$  \hspace{1cm} (14)
\[ \mathcal{L}_t = (\Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3) = \frac{\partial}{\partial t} + a_2 \frac{\partial}{\partial x} + a_3 u \frac{\partial}{\partial t} - a_3 \frac{\partial}{\partial x} \]

Here, the corresponding Lagrange equation is,

\[
\frac{dx}{\lambda} = \frac{dt}{\lambda} = \frac{du}{\lambda u} = \frac{dv}{\lambda v},
\]

\[
\varphi = a_2 x - t, \quad u(x, t) = F(\varphi) + e^{a_3 x}
\]

\[
\varphi = a_3 x - t, \quad v(x, t) = H(\varphi) + e^{-a_3 x}
\]

Using an Eq. (15) to the partial differential equation (1), So, we acquire a ordinary differential model while,

\[
F'(\varphi) = -12a_2^2 F''(\varphi) - 12a_2^2 e^{a_3 x} - 24(F(\varphi) + e^{a_3 x})
\]

\[
H'(\varphi) = 12a_2^2 H'(\varphi) + 12a_2 e^{-a_3 x} + 24(H(\varphi) + e^{-a_3 x})
\]

Analytical solution of the system

The developed infinitesimals Lie symmetry points are executed to construct the next traveling wave profiles and applied on the considered system of equations (1) to obtain ordinary differential equations. To find nonlinear propagating wave profiles of a system (1), we are considering the ordinary system (14) and applying the power series solution approach.

Series solutions of the system (14)

We get the following equation from the first equation of system (14),

\[
H(\varphi) = \frac{F'(\varphi) + 12a_2 F''(\varphi) + 24 F(\varphi)}{24(F(\varphi))^2}
\]

Eq. (17) can be inserted into the second equation of model (14) and acquire the following unique ordinary differential equation,

\[
144a_2^2 F'^{2m} - 576a_2^2 FF''^{m} - 432a_2^2 F''^{2m} + 864a_2^2 (F')^2 - 576a_2 F(F')^2 - 72a_2 F'F'' + 72a_2 (F')^3 - F^{2m} + F(F')^2 = 0
\]

Suppose that series outcome of Eq. (18) is following in the sense of power series,

\[
F(\varphi) = \sum_{k=0}^{\infty} b_k \varphi^k,
\]

thus we have,

\[
F'(\varphi) = \sum_{k=0}^{\infty} b_k k \varphi^{k-1},
\]

\[
F''(\varphi) = \sum_{k=0}^{\infty} b_k (k+1) \varphi^{k-1},
\]

\[
F'''(\varphi) = \sum_{k=0}^{\infty} b_k (k+2) \varphi^{k-1},
\]

\[
F^{(4)}(\varphi) = \sum_{k=0}^{\infty} b_k (k+3) \varphi^{k-1},
\]

and

\[
F^2(\varphi) = 1 + \sum_{k=0}^{\infty} \left( b_k b_{k+1} \varphi^{k} \right) (k+1)
\]

The values (19), (20) and (21) are plugging into Eq. (18), and get,

\[
-576 a_2^2 \sum_{k=0}^{\infty} b_k \varphi^k \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi
\]

\[
\times \left( \sum_{k=0}^{\infty} b_{k+3} \varphi^k (k+2) (k+1) + 342 a_2^2 \sum_{k=0}^{\infty} b_k \varphi^k \right)
\]

\[
+ 864 \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \right)^2
\]

\[
+ 576 \sum_{k=0}^{\infty} b_k \varphi^k \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right)^2 a_2
\]

\[
- 72 \sum_{k=0}^{\infty} b_k \varphi^k \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \right) (k+1) \varphi^2
\]

\[
+ 144 \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} b_k b_{k+1} \varphi^k \right)^2
\]

\[
\times \left( \sum_{k=0}^{\infty} b_{k+3} \varphi^k (k+3) (k+2) (k+1) (k+4) \varphi^2 + 72 \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right)^3 a_2
\]

\[
+ \sum_{k=0}^{\infty} b_k \varphi^k \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right)^2 \right)
\]

\[
- 576 \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} b_k b_{k+2} \varphi^k \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \right) (k+1) \varphi^2
\]

\[
- \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} b_k b_{k+1} \varphi^k \varphi \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \right) (k+1) \varphi^2 = 0
\]

We acquire that value for \( b_4 \) through (22) while differentiate the coefficients of \( n = 0 \).

\[
b_4 = \frac{b_3 \left( 864 a_2^2 b_2 + 576 a_2 b_0 + b_1 \right)}{1728 a_2^2 b_3},
\]

Similarly, we have for \( \lambda \geq 1 \),

\[
b_{\lambda+1} = -\frac{1}{144 \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} b_k b_{\lambda} \varphi^k \right) \left( \lambda+3 \right) \left( \lambda+2 \right) \left( \lambda+1 \right) \varphi \left( \lambda+4 \right) \varphi^2}
\]

\[
\times \left( -576 a_2^2 \sum_{k=0}^{\infty} b_k \varphi^k \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right) \left( \lambda+3 \right) \left( \lambda+2 \right) \left( \lambda+1 \right) \varphi \left( \lambda+4 \right) \varphi^2
\]

\[
+ 864 \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right) \left( \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \right)^2 a_2
\]

\[
+ 576 b_{\lambda} \left( \sum_{k=0}^{\infty} b_k \varphi^k \varphi \right)^2 \varphi
\]

\[
- 72 \sum_{k=0}^{\infty} b_k \varphi^k \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \left( \lambda+2 \right) \varphi
\]

\[
+ 72 b_{\lambda+1} \varphi \left( \sum_{k=0}^{\infty} b_{k+1} \varphi^k \varphi \right) \varphi
\]

\[
- 576 \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} b_k b_{\lambda-k} \varphi^k \right) \sum_{k=0}^{\infty} b_{k+2} \varphi^k \varphi \left( \lambda+2 \right) \varphi
\]

Now, the constants \( b_0, b_1, b_2 \) and \( b_3 \) could be random. The one of kind strategy is that the unlike kinds for sequence \( b_{\lambda} \) could be solved consecutively from Eq. (24). So, power series solution of the Eq. (18) would demonstrated while,

\[
F(\varphi) = b_0 + b_1 \varphi + b_2 \varphi^2 + b_3 \varphi^3 + b_4 \varphi^4 + \sum_{n=1}^{\infty} b_{\lambda+n} \varphi^{\lambda+n}.\]
\[
\begin{align*}
\psi(x,t) &= \frac{1}{24} \left( b_0 + b_1 (a_2 x - t) + b_2 (a_2 x - t)^2 + b_3 (a_2 x - t)^3 + b_4 (a_2 x - t)^4 + \sum_{k=5}^{\infty} b_{4k+4} (a_2 x - t)^{4k+4} \right) \\
&\quad \left( 24 \left( b_0 + b_1 (a_2 x - t) + b_2 (a_2 x - t)^2 + b_3 (a_2 x - t)^3 + b_4 (a_2 x - t)^4 + \sum_{k=5}^{\infty} b_{4k+4} (a_2 x - t)^{4k+4} \right) \right) + \\
&\quad b_1 + 2b_2 (a_2 x - t) + 3b_3 (a_2 x - t)^2 + 4b_4 (a_2 x - t)^3 + \sum_{k=5}^{\infty} b_{4k+4} (a_2 x - t)^{4k+4} + \\
&\quad 12a_2 \left( 2b_2 + 6b_3 (a_2 x - t) + 12b_4 (a_2 x - t)^2 + \sum_{k=5}^{\infty} b_{4k+4} ((a_2 x - t)^2 - (a_2 x - t) (a_2 x - t)^2) \right).
\end{align*}
\]

It can be noticed that \(\sum_{k=0}^{\infty} a_k \) is analytic and convergent in the neighborhood \( |x| \leq a \), thus, \(\varphi = A(\beta) = \sum_{k=0}^{\infty} a_k \beta^k \) is a majorant series. Hence, we need to prove that it has positive radius of convergence and rewrite the series as,

\[
\sum_{k=0}^{\infty} a_k \beta^k = a_0 + a_1 \beta + a_2 \beta^2 + a_3 \beta^3 + (\varphi - a_0) (\varphi - a_0 - a_1 \beta). \quad \text{(30)}
\]

It is easy to verify that \(\dot{\lambda}(\beta, \varphi)\) is analytic in the neighborhood of \((0, a_0)\), where \(\dot{\lambda}(0, a_0) = 0\) and \(\dot{\lambda}'(0, a_0) = 1 \neq 0\). Based on implicit function theorem, we find that \(\varphi\) is analytic and convergent in the neighborhood of the point \((0, a_0)\). Therefore, the power series solution is convergent by the method of majorants.

**Modulation instability assessment**

In this portion, the aim is to generate the modulation instability (MI) gain of steady-state solution of the governing structure (1) through the virtue of the linear stability analysis. The MI can consist of the

\[
\dot{\lambda}(\beta, \varphi) = \varphi a_0 - a_1 \beta - a_2 \beta^2 - a_3 \beta^3 - (\varphi - a_0) (\varphi - a_0 - a_1 \beta). \quad \text{(31)}
\]
exponential growth of the small-scale perturbations in the stage of optical waves or the amplitude. That is essential to examine into non-linear wave physics.

Let us suppose that steady-state solution in order to achieve stability analysis,

$$u = A_0, \quad v = B_0,$$

where, $A_0$ and $B_0$ are the initial incidence power (real constant-amplitudes). Moreover, that outcomes (32) are changing into stationary perturbed outcomes while,

$$u = A_0 + \sigma \varphi(x,t), \quad v = B_0 + \sigma \xi(x,t),$$

where $\varphi$ along-with $\xi$ are real functions of $x, t$ as well as the perturbation coefficient parameter is $\sigma \ll 1$. These disturbance equations have been

Fig. 1. This figure is presenting the behavior of solution (26).
Fig. 2. This figure is presenting the behavior of solution (27).

(a) 3D chemical wave propagation at wave velocity $a_2 = 0.1$

(b) 3D wave propagation at wave velocity $a_2 = 0.3$

(c) 3D wave propagation at wave velocity $a_2 = 0.5$

(d) 3D wave propagation at wave velocity $a_2 = 0.7$

(e) 3D wave propagation at wave velocity $a_2 = 0.9$

(f) 2D influence of wave velocity

generated, the perturbed stationary solutions into the PDE system (1),

\[ 24 \left( A_0 + \sigma \varphi(t) \right)^2 \left( B_0 + \sigma E(t) \right) - 24 A_0 \]
\[ - 24 \sigma \varphi(t) + \sigma \frac{\partial}{\partial t} \varphi(t) - 12 \sigma \frac{\partial^2}{\partial x^2} \varphi(t) = 0. \]
\[ - 24 \left( A_0 + \sigma \varphi(t) \right) \left( B_0 + \sigma E(t) \right)^2 + 24 B_0 \]
\[ + 24 \sigma E(t) + \sigma \frac{\partial}{\partial t} E(t) + 12 \sigma \frac{\partial^2}{\partial x^2} E(t) = 0. \]  

These disturbance equations (34) could have been expressed likewise after linearization,

\[ 24 A_0^2 B_0 + 24 A_0 \sigma E(t) + 48 A_0 B_0 \sigma \varphi(t) - 24 A_0 \]
\[ - 24 \sigma \varphi(t) + \sigma \frac{\partial}{\partial t} \varphi(t) - 12 \sigma \frac{\partial^2}{\partial x^2} \varphi(t) = 0. \]
\[ - 24 A_0 B_0^2 - 24 B_0^2 \sigma \varphi(t) - 48 A_0 B_0 \sigma E(t) \]

\[ + 24 B_0 + 24 \sigma E(t) + \sigma \frac{\partial}{\partial t} E(t) + 12 \sigma \frac{\partial^2}{\partial x^2} E(t) = 0. \]
The considered model also have application in fluid dynamics. Thus, Modulational instability analysis is critical in fluid dynamics for understanding the formation of rogue waves in the ocean. These are large, dangerous waves that can appear seemingly out of nowhere. Predicting when and where rogue waves will occur can have serious consequences for maritime safety.

**Lie backlund**

It is possible to consider the Lie-Backlund transformation group to be a tangent transformation group. It is intended to be an analog of the one-parameter group of continuous symmetry transformations. The vector field form may be utilized to develop the governing equation’s Lie-Backlund symmetry generator,

\[ Q = Y' \times \sigma(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}) \]  

where \( Q \) holds for \( Q^2 \Delta / \Delta = 0 \). The Over-determined systems are uncovered by manipulating the fourth extension to the governing equation. The typical outcomes of the over-determined structure are:

\[ Y_u = 48C_9 u^2 v + 2C_2 u^2 v^2 + 6C_5 u w^2 v + 24C_4 t u \]
\[ + C_4 u x + 24C_3 u x u_x - C_1 u x \]
\[ + C_1 u + C_1 u_x + C_1 v_x - C_1 u_{xx} \]
\[ - 2C_1 v + C_1 v_x \]
\[ - 2C_1 v + C_1 v_x + C_1 v_{xx} - 2C_1 v \]

where arbitrary constants are \( C_1, C_2, C_3, C_4, \) and \( C_5 \). After that, vector fields are used to generate the algebra of Lie point symmetries.

\[ F_1 = u \partial_u - \sigma v \partial_v, \]
\[ F_2 = u \partial_u + v \partial_v, \]
\[ F_3 = (2u^2 v - u_{xx}) \partial_u + (-2u^2 - u_{xx}) \partial_v, \]
\[ F_4 = (24u_{xx} + u_x) \partial_u + (24u_{xx} - v_x) \partial_v, \]
\[ F_5 = (6u_{xx} - u_{xxx}) \partial_u + (6u_{xx} - v_{xxx}) \partial_v. \]

**Conservation laws**

The relationship between symmetry and conserved quantities is a fundamental concept in modern physics. Emmy Noether established Noether’s theorem, which states that for every continuous symmetry in a physical system, there is a conserved quantity. Conservation laws, which are fundamental physics precepts, describe the behavior of physical systems. These laws state that specific quantities, such as mass, energy, momentum, and charge, are conserved over time, which means they cannot be created or destroyed but can only be transferred from one system to another. The principle of mass conservation, for example, states that even if mass is redistributed within a closed system, the total mass of the system will always remain constant over time.

According to the conservation of energy principle, the total energy of a closed system is constant, implying that energy cannot be created or destroyed, but only converted from one form to another. The conservation of momentum principle states that the total momentum of a closed system remains constant over time, implying that momentum cannot be
Conserved vectors using point symmetries

- Lie point symmetry generator $Γ_1 = \partial_u$ bring in the conserved vector along-with components
  
  $T_1 = -12u_1 z_x + u_x z + 24v (u^2 z + w) - 24uw^2 w - 24uxz + 12v_x w_x + v_x w,$
  
  $T'_1 = 2u_1 - 4w.$

- Lie point symmetry generator $Γ_2 = \partial_v$ bring in the conserved vector along-with components
  
  $T_2 = 12 (−u_2 (x, t) + \frac{3}{2} u z + v_x w_y - v w^3 x,$
  
  $T'_2 = 12 (−u_2 x z + 2v (u^2 z + w) - 2u^2 v^2 w - 2uxz + 4v_x w).$

- Lie point symmetry generator $Γ_3 = u_i \partial_x + v_i \partial_v$ bring in the conserved vector along-with components
  
  $T_3 = 12 (u z - u x z + v_x w - uw_x),$
  
  $T'_3 = w v - u z.$

- Lie point symmetry generator $Γ_4 = -ux \partial_x + vx \partial_v + 24 \partial_u$ bring in the conserved vector along-with components
  
  $T_4 = 12 \left(−24u z x_x + 2u_x z + x u z + v \left(48u^2 z - x w_x + (48t + 11) u\right)\right.\left.− 48tu^2 w + u \left(−x z_x - 48r + z\right) + 24v x w + 2t v w + x_v w \right),$
  
  $T'_4 = (x v - 24t u) w - (24u_x + x u) z.$

- Lie point symmetry generator $Γ_5 = 48v \partial_x + 2v \partial_v + x \partial_u + (48u - 2v) \partial_u$ bring in the conserved vector along-with components
  
  $T_5 = xz (u_1 - 12u_x z + 24u u^2 v - 24u) + 12 (−2u - x u_z + (48t - 2u) z_x - 12 (48r - 3) u_x - 2u z_x) z + x w \left(−48u^2 v - v_i + 12 v_x z + 24v\right)\right.$
  
  $\left.+ 12 (2v_i + x v_x + 48v t) w_x v - 12 (48t + 1)v_i + 2v_x v + x v x) w,\right.$
  
  $T'_5 = (−(x u + 24u z)_x) z - 2u (24r^2 + z) + 48tu^2 + x (24v x z - x v v) w.$

Conserved vectors using Lie backlund symmetries

- Lie Backlund symmetry generator $Γ_1 = u_i \partial_x - v_i \partial_v$ yields
  
  $T_1 = 12 (-z (u_i + u_v) + z u - w (v_i + v) + w v v),$
  
  $T'_1 = 2u_i - uv.$

- Lie Backlund symmetry generator $Γ_2 = u_x \partial_u + v_x \partial_v$ yields
  
  $T_2 = 12 (u z_x - u x z - v_i w_x + v_x w),$
  
  $T'_2 = u z + v x.$

- Lie Backlund symmetry generator $Γ_3 = (2u^2 v - u_x z) \partial_u + (-2u^2 u + v_x z) \partial_v$ yields
  
  $T_3 = 12 \left(2w (w v - 2 (z (u_i + u_v) + w (v_i + v_x))))\right.$
  
  $\left.+ w (-2 (u_i + u_v) c^2 + v_x x + v v x)\right.$
  
  $\left.+ z u x x + 2u^2 (z v_i - z (v_i + v_x)) - z u x x + zw_x x - w_i v_x\right),$
  
  $T'_3 = w (v x - 2u^2) + z (2u^2 - u x x).$

- Lie Backlund symmetry generator $Γ_4 = (24u x + u_v) \partial_u + (24v x - v_v) \partial_v$ yields
  
  $T_4 = 12 ((24t + 1) u_i z_x - (24t + 1) u_x z - (24t - 1) u_x w_x + (24t - 1) v_x w_i),$
  
  $T'_4 = (24t + 1) u_i z + (24t - 1) i u.$

- Lie Backlund symmetry generator $Γ_5 = (6u u_v - v_x x) \partial_u + (6v i_v - v_x x) \partial_v$ yields
  
  $T_5 = 12 \left(−6 u \left(−u z_x + u x_z + v_i w_x - v_x w\right) + (v_i + v) \left(u z - v v_x v\right)\right.$
  
  $\left.+ 6w w u v, v\right.$
  
  $\left.+ z (-6 u_i (u_i + u_v) v + u x x x + u x x x),\right.$
  
  $\left.+ 6w w u v - z u x x x - w w u x x x (x) + w v x x - w x x x\right),$
  
  $T'_5 = z (6u v u - u v v) + w (6w u v - v x x v).$

- Lie Backlund symmetry generator $Γ_6 = (48u^2 v - 24u x x z - u_x x) \partial_u + (−48u^2 u + 24v x x x - 24 u x x - 2u) \partial_v$ yields
  
  $T_6 = 12 \left(w (−48 u_i + u_v) c^2 - 96 u (v_i + v_x) v - v x x x\right.$
  
  $\left.- 24 t \left(v x x x + v x x x\right) - 2 \left(v_i + v\right) v_x x\right.$
  
  $\left.- z \left(96 u_i (u_i + u_v) v - (24t + 1) (u x x x + u x x) + 48u^2 v (v_i + v)\right)\right.$
  
  $\left.+ w \left(48u v^2 + z (24t + 1) u x x x) + v_i (48u v^2 - (24t + 1) u x x)\right)\right.$
  
  $\left.+ w (48u v^2 + 2w + (24t + 1) u x x x)\right),$
  
  $T'_6 = z (48u v^2 - (24t + 1) u x x) - w (48u v^2 + 2w + (24t + 1) u x x).$

Conserved vectors using multipliers

Using the multiplier approach as described in [32], To assemble the conservation laws for the governing system. From the determining equation, the following 1st-order multipliers are derived:

$A^1 (x, t, u, v, u_v, v_x, u_i, v_i)$ and $A^2 (x, t, u, v, u_v, v_x, u_i, v_i)$ as long as the model which is given by

$A^1 = \frac{1}{24} (C_1 x + 24C_4 x^3) v + \frac{1}{24} (−24C_1 t - 24C_4) i x - C_2 v v,$

$A^2 = \frac{1}{24} (C_1 x + 24C_4 x^3) u + \frac{1}{24} (24C_1 t + 24C_4) ux + C_2 u.$
The conservation laws providing the guarantee of existence of integral invariants for the coupled Chaffee–Infante diffusion-reaction system and also ensured the conservation of mass, momentum, and energy their stability and persistence of these quantities in a closed system.

Conclusion

This study has taken into account the nonlinear mathematically coupled Chaffee–Infante diffusion-reaction system. We have explored the reaction–diffusion hierarchy's infinitesimal generators and used them to create the most effective system of subalgebras. Also calculated are the symmetry reductions on those vector fields which are a constituent of an optimal system. Preceding symmetry reduction, calculated are the symmetry reductions on those vector fields which are a constituent of an optimal system. Preceding symmetry reduction, calculated are the symmetry reductions on those vector fields which are a constituent of an optimal system. Preceding symmetry reduction, calculated are the symmetry reductions on those vector fields which are a constituent of an optimal system.

Declarations of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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