Results in Physics 54 (2023) 107126

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

The computation of Lie point symmetry generators, modulational instability, classification of conserved quantities, and explicit power series solutions of the coupled system

Waqas Ali Faridi [a,*], Abdullahi Yusuf [b,c,d], Ali Akgül [d,e,f], Ferdous M.O. Tawfiq [g], Fairouz Tchier [g], Rawya Al-deiakeh [h], Tukur A. Sulaiman [b,c,d], Ahmed M. Hassan [i], Wen-Xiu Ma [j,k,l]

[a] Department of Mathematics, University of Management and Technology, Lahore, Pakistan
[b] Near East University, Operational Research Center in Healthcare, Nicosia 99138, TRNC Mersin 10, Turkey
[c] Department of Mathematics, Federal University Dutse, Jigawa, Nigeria
[d] Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon
[e] Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey
[f] Department of Mathematics, Mathematics Research Center, Near East University, Near East Boulevard, Mersin 10, Nicosia 99138, Turkey
[g] Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh, Saudi Arabia
[h] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, United Arab Emirates
[i] Faculty of Engineering, Future University in Egypt, Egypt
[j] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang China
[k] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[l] School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

ARTICLE INFO

Keywords:
Lie symmetry analysis
Power series solution approach
Modulation instability
Conserved quantities
Coupled Chaffee–Infante reaction system

ABSTRACT

The well-known Chaffee–Infante reaction hierarchy is examined in this article along with its reaction–diffusion coupling. It has numerous variety of applications in modern sciences, such as electromagnetic wave fields, fluid dynamics, high-energy physics, ion-acoustic waves in plasma physics, coastal engineering, and optical fibres. The physical processes of mass transfer and particle diffusion might be expressed in this way. The Lie invariance criteria is taken into consideration while we determine the symmetry generators. The suggested approach produces the six dimensional Lie algebra, where translation symmetries in space and time are associated to mass conservation and conservation of energy respectively, the other symmetries are scaling or dilation. Additionally, similarity reductions are performed, and the optimal system of the sub-algebra should be quantified. There are an enormous number of exact solutions can construct for the traveling waves when the governing system is transformed into ordinary differential equations using the similarity transformation technique. The power series approach is also utilized for ordinary differential equations to obtain closed-form analytical solutions for the proposed diffusive coupled system. The stability of the model under the limitations is ensured by the modulation instability analysis. The reaction diffusion hierarchy's conserved vectors are calculated using multiplier methods using Lie Backlund symmetries. The acquired results are presented graphically in 2-D and 3-D to demonstrate the wave propagation behavior.

Introduction

The partial differential equations are organically reconnoitered within numerous areas of science like that engineering, chemistry, physics issues. The use of Mathematical simulation with non-linear PDEs have been imitated captivate when studying differential equations. One of the important tools for thoroughly interrogating the features of physical occurrences is the partial differential equation. The outstanding methods as further, exactly interpreting the intricate physical non-linear structure is the Schrödinger kind governing equation, that can be crucial in the domains of optics, fiber optics, telecommunication technology, and plasma technology [1–4]. Derivation of analytical solutions to Schrödinger equation is very significant examination domain whereas the analytical solutions poses a significant function in characterizing the tangible features for non-linear structure along-with applied mathematics [5,6]. The analysis of

* Corresponding author.
E-mail address: wa966142@gmail.com (W.A. Faridi).

https://doi.org/10.1016/j.rinp.2023.107126
Received 5 August 2023; Received in revised form 15 September 2023; Accepted 27 October 2023
Available online 30 October 2023

non-linear PDEs has grown in relevance within both excess in both applied as well as pure mathematics the past ten years. The manipulation of computer automation has given mathematicians access to new areas in the applied sciences. The non-linear systems which can be broadly utilize in engineering, physics and mathematical sciences have gaining popularity. This field is attracting the researchers and scientists because a lot work is dolloping in this field. Kaur and Wazwaz [7] developed the invariant solitons and performed the painleve test for the integrable system. Kaur and Gupta [8] investigated Kawahara equation and modified Kawahara equation by Lie symmetry approach and developed variety of soliton solutions. Kaur and Wazwaz [9] generated the Lie symmetry infinitesimals generators for Einstein's vacuum field equation. Wazwaz and Kaur [10] applied Hirota method on Boussinesq equations to construct the real and complex soliton solutions and also examined the integrability of the model.

Analytical solutions are the premier choice for establishing quantifiable forecasts [11]. Reliability and reliability of the analytical outcomes these are main source. That appears fore improve validity of the quantifiable evaluations that have been developed. In mathematical sciences, making quantitative predictions is a crucial activity that is typically carried out via resolving differential equations. The action of operating structure in the physics, biology and chemistry is explained using differential equations. These equations need to be solved to get numerical predictions that are quantitative. These are one or the other resolved numerically as well as analytically, [12]. Analytical approaches can be used to explain outcomes as functions of variables when it is possible to provide it. Then, quantitative forecasts are generated employing these analytical findings.

$$\begin{aligned} u_t &= 12u_{xx} + 24(u - u^2 v), \\ v_t &= -12v_{xx} - 24(v - v^2 u). \end{aligned} \tag{1}$$

We are not aware of a single, standard methodology which can be maneuvered to find the analytical results to all stiff non-linear partial differential equations in the current decade. Therefore, more efficient methods are always needed to identify the exact outcome of similar questions. Such methodologies have so deserving of desire.

There are so many techniques has been substantiated to perceive the analytical solutions for non-linear partial differential equations similar that, Kudryashov methodology [13,14], sine-Gordon expansion methodology [15,16], bilinear neural network methodology [17,18], extended simple equation methodology [19], F-expansion methodology [20,21], unified auxiliary equation methodology [22,23], $\frac{G'}{G}-$ expansion approach [24], Hirota bilinear methodology [25], the generalized exponential function methodology [26], and numerous furthers [27–31].

In this study, the lie symmetry analysis performed and infinitesimals generators developed. The power series solution constructed. But, there are numerous kinds of analytical exact solutions are still mystery for this system. There are many physical aspects and solutions are not developed, such as rogue waves, breathers, solitons interactions, bifurcation analysis, sensitivity and chaos analysis.

Lie infinitesimals algebra

Now, we examine Lie symmetry generators of CSGE for the reason of constructing the Cls.

Let us assume Lie point group of the infinitesimal transformations with one parameter acting on the x, t independent as well as u, v dependent factors related to Eq. (1) are as follows,

$$\begin{aligned} x^* &= O(\varpi^2) + \varpi \mathfrak{E}^x(t,x,u,v) + x, \\ t^* &= O(\varpi^2) + \varpi \mathfrak{E}^t(t,x,u,v) + t, \\ u^* &= O(\varpi^2) + \varpi \Upsilon^u(t,x,u,v) + u, \\ v^* &= O(\varpi^2) + \varpi \Upsilon^v(t,x,u,v) + v, \end{aligned} \tag{2}$$

where $\varpi \ll 1$ is very tiny Lie point group component and $\mathfrak{E}^x, \mathfrak{E}^t, \Upsilon^u$, and Υ^v are the Lie infinitesimals of the transformations that must be found for independent and dependent variables, respectively.

The symmetry generator associated to the Lie algebra of Eq. (1) has the formation,

$$\Gamma = \mathfrak{E}^x(t,x,u,v)\,\partial x + \mathfrak{E}^t(t,x,u,v)\,\partial t + \Upsilon^u(t,x,u,v)\,\partial u + \Upsilon^v(t,x,u,v)\,\partial v. \tag{3}$$

The symmetries within Eq. (1) are generated by the vector field (3). Eq. (1)'s invariance condition changes through Γ from being,

$$\mathbb{P}r^2\Gamma(\Delta) = 0, \tag{4}$$

when Δ is zero. Where $\mathbb{P}r^2$ is second prolongation of Γ it would be represented as,

$$\mathbb{P}r^2\Gamma = \Gamma + \Upsilon^{u,t}\frac{\partial}{\partial u_t} + \Upsilon^{u,xx}\frac{\partial}{\partial u_{xx}} + \Upsilon^{v,t}\frac{\partial}{\partial v_t} + \Upsilon^{v,xx}\frac{\partial}{\partial v_{xx}}, \tag{5}$$

with the coefficients

$$\Upsilon^i = D_i\left(\Upsilon - \sum_{\mathcal{J}=1}^{2}\mathfrak{E}^{\mathcal{J}}u^{\alpha}_{\mathcal{J}}\right) + \sum_{\mathcal{J}=1}^{2}\mathfrak{E}^{\mathcal{J}}u^{\alpha}_{i,\mathcal{J}},$$

$$\Upsilon^i = D_i\left(\Upsilon - \sum_{\mathcal{J}=1}^{2}\mathfrak{E}^{\mathcal{J}}v^{\alpha}_{\mathcal{J}}\right) + \sum_{\mathcal{J}=1}^{2}\mathfrak{E}^{\mathcal{J}}v^{\alpha}_{i,\mathcal{J}},$$

where $(j_1 \dots j_s)$, $1 \le j_s \le 2$, $1 \le s \le 2$. By concerning the 2nd prolongation (5) onto Eq. (1), then recapture the collection of determined equations. That obtained model of determined partial differential equation can be elucidated by implementing the computer algebra program `Maple` as well as secured necessary infinitesimal generators whenever,

$$\begin{aligned} \mathfrak{E}_x(t,x,u,v) &= xC_1 + tC_2 + C_3, \\ \mathfrak{E}_t(t,x,u,v) &= 2C_1 t + C_4, \\ \Upsilon_u(t,x,u,v) &= \frac{1}{24}u(1152tC_1 - C_2 x + 24C_5), \\ \Upsilon_v(t,x,u,v) &= -\frac{1}{24}v(1152tC_1 - C_2 x + 48C_1 + 24C_5), \end{aligned} \tag{6}$$

where the arbitrary constants are C_j, $(j = 1,2,3,4,5)$. The following vector fields produce the algebra of Lie point symmetries,

$$\begin{aligned} \Gamma_1 &= \partial_x, \\ \Gamma_2 &= \partial_t, \\ \Gamma_3 &= u\partial_u - v\partial_v, \\ \Gamma_4 &= -\frac{1}{24}ux\partial_u + \frac{1}{24}vx\partial_v + t\partial_x, \\ \Gamma_5 &= 48tu\partial_u + 2t\partial_t + x\partial_x - (48tv + 2v)\partial_v. \end{aligned} \tag{7}$$

While, for our suitability, we have to expand an commutator Table 1 via accepting inputs on now $[\Gamma_i, \Gamma_j] = \Gamma_i.\Gamma_j - \Gamma j.\Gamma i$ for Eqs. (1).

Table 1
Commutator table.

$[\Gamma_i,\Gamma_j]$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	0	0	$-\frac{1}{24}\Gamma_3$	Γ_1
Γ_2	0	0	0	Γ_1	$2\Gamma_2 + 48\Gamma_3$
Γ_3	0	0	0	0	0
Γ_4	$\frac{1}{24}\Gamma_3$	$-\Gamma_1$	0	0	$-\Gamma_4$
Γ_5	$-\Gamma_1$	$-2\Gamma_2 - 48\Gamma_3$	0	Γ_4	0

Lie symmetry group

Clearly, the infinite-dimensional Lie point algebra can be comprised for infinitesimal vector generators $\Gamma_i, 1 \le i \le 5$ (7) evolves some infinite continuous group with conversions of Eqs. (1). There is linear independence among the infinitesimal generators. However, one would be represent some infinitesimal symmetry generators of (7) as a linear combination of Γ_i properly, such that,

$$\mathcal{A} = \mathbb{B}_1 \Gamma_1 + \mathbb{B}_2 \Gamma_2 + \mathbb{B}_3 \Gamma_3 + \mathbb{B}_4 \Gamma_4 + \mathbb{B}_5 \Gamma_5 \tag{8}$$

The theory of Lie analysis can be stand on the examination of there invariance about one parameter sub-group of point transformations with infinitesimal generators interpreted as vector fields. In order to acquire analytical outcomes of (1), to find Lie symmetry groups, we should employ related symmetry algebras. In order to secure the Lie symmetry groups, the first several issues listed below must be resolved,

$$\begin{aligned}
\frac{d\tilde{x}}{d\varpi} &= \mathfrak{E}^x(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \\
\tilde{x}\,|_{\varpi=0} &= x. \\
\frac{d\tilde{t}}{d\varpi} &= \mathfrak{E}^t(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \\
\tilde{t}\,|_{\varpi=0} &= t. \\
\frac{d\tilde{u}}{d\varpi} &= \Upsilon^u(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \\
\tilde{u}\,|_{\varpi=0} &= u. \\
\frac{d\tilde{v}}{d\varpi} &= \Upsilon^v(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \\
\tilde{v}\,|_{\varpi=0} &= v.
\end{aligned}$$

Now, ϖ is a discretionary real parameter. So, Lie symmetry group transformations could be obtained as,

$$\mathcal{G}_i : (t, x, u, v) \longrightarrow (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}).$$

The group transformations acquire comparability with upper symmetry algebra,

Category 1: Here, if we choose symmetry point algebra $\Gamma_1 = \partial_x$, as well it is a symmetry group transform,

$$\mathcal{G}_1 : (t, x, u, v) \longrightarrow (t, x + \varpi, u, v).$$

Category 2: Here, if we choose the symmetry algebra $\Gamma_2 = \partial_t$, as well it is a symmetry group transform,

$$\mathcal{G}_2 : (t, x, u, v) \longrightarrow (t + \varpi, x, u, v).$$

Category 3: Here, if we choose the symmetry algebra $\Gamma_3 = u\partial_u - v\partial_v$, as well it is a symmetry group transform,

$$\mathcal{G}_3 : (t, x, u, v) \longrightarrow (t, x, ue^{\varpi}, v\,e^{-\varpi}).$$

Category 4: Here, if we choose the symmetry algebra $\Gamma_4 = -\frac{1}{24}ux\partial_u + \frac{1}{24}vx\partial_v + t\partial_x$, then it is a symmetry group transform,

$$\mathcal{G}_4 : (t, x, u, v) \longrightarrow (x + t\varpi, t, ue^{-\frac{1}{24}x\varpi}, ve^{\frac{1}{24}x\varpi}).$$

Category 5: Here, if we choose the symmetry algebra $\Gamma_5 = 48tu\partial_u + 2t\partial_t + x\partial_x - (48tv + 2v)\partial_v$, as well it is a symmetry group transform,

$$\mathcal{G}_5 : (t, x, u, v) \longrightarrow (xe^{\varpi}, te^{2\varpi}, ue^{48t\varpi}, ve^{-(48t+2)\varpi}).$$

A linear combination of generators can create there are many subalgebras in this Lie algebra. $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, and Γ_5. Moreover similar transformation can be associated with two subalgebras' invariant solutions if there are analogous, that is, there have been similar conjugate within the symmetry group. As a result, placing all similar subalgebras in one class and electing a illustrative for several class is adequate. The combination of all their illustrative constitutes an optimal structure.

Optimal system and similarity reduction to system (1)

Optimal system

An optimal system of Lie point symmetries, also known as a "optimal symmetry group", is a collection of symmetries that provides the most concise and complete description of the invariance properties of a particular system of differential equations. It is a concept that is frequently used in the field of differential equations and symmetry analysis, particularly when looking for exact solutions to differential equations. It could be observed through Table 1 while, the generated vector fields Γ_1, Γ_2, along with Γ_3, applies for an abelian algebra's configuration. So, it could be represented these vector fields like that,

$$\mathcal{L}_1 = \langle \Gamma_1 \rangle,$$

$$\mathcal{L}_2 = \langle \Gamma_2 \rangle,$$

$$\mathcal{L}_3 = \langle \Gamma_1 + b_2\Gamma_2 \rangle,$$

$$\mathcal{L}_4 = \langle \Gamma_1 + b_2\Gamma_2 + b_3\Gamma_3 \rangle.$$

$\mathcal{L}_1 = \langle \Gamma_1 \rangle = \frac{\partial}{\partial x}$

Here, the corresponding Lagrange equation is,

$$\begin{aligned}
&\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}, \\
&\varphi = t, \qquad u(x,t) = \mathcal{F}(\varphi), \\
&\varphi = t, \qquad v(x,t) = \mathcal{H}(\varphi).
\end{aligned} \tag{9}$$

Using an Eq. (9) with the partial differential equation (1). So, we obtained a ordinary differential model while,

$$\begin{aligned}
&\mathcal{F}'(\varphi) = 24(\mathcal{F} - \mathcal{F}^2\mathcal{H}), \\
&\mathcal{H}'(\varphi) = -24(\mathcal{H} - \mathcal{H}^2\mathcal{F}).
\end{aligned} \tag{10}$$

However, in this scenario, the only similarity variable is the temporal component t. As a result, we can determine the solutions to Eq. (10) in terms of variable t, which may not actually correspond to occurrences that are interesting from a physical perspective. So, we will not go into depth here.

$\mathcal{L}_2 = \langle \Gamma_1 \rangle = \frac{\partial}{\partial t}$

Here, the corresponding Lagrange equation is,

$$\begin{aligned}
&\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} = \frac{dv}{0}, \\
&\varphi = x, \qquad u(x,t) = \mathcal{F}(\varphi), \\
&\varphi = x, \qquad v(x,t) = \mathcal{H}(\varphi).
\end{aligned} \tag{11}$$

Using an Eq. (11) with the partial differential equation (1). So, we obtained a ordinary differential model while,

$$\begin{aligned}
&\mathcal{F}''(\varphi) + 24(\mathcal{F} - \mathcal{F}^2\mathcal{H}) = 0, \\
&\mathcal{H}''(\varphi) + 24(\mathcal{H} - \mathcal{H}^2\mathcal{F}) = 0.
\end{aligned} \tag{12}$$

However, in this scenario, the only similarity variable is the spatio component x. As a result, we can determine the solutions to Eq. (12) in terms of variable x, which may not actually correspond to occurrences that are interesting from a physical perspective. So, we will not go into depth here.

$\mathcal{L}_3 = \langle \Gamma_1 + a_2\Gamma_2 \rangle = \frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}$

Here, the corresponding Lagrange equation is,

$$\begin{aligned}
&\frac{dx}{1} = \frac{dt}{a_2} = \frac{du}{0} = \frac{dv}{0}, \\
&\varphi = a_2 x - t, \qquad u(x,t) = \mathcal{F}(\varphi), \\
&\varphi = a_2 x - t, \qquad v(x,t) = \mathcal{H}(\varphi).
\end{aligned} \tag{13}$$

Using an Eq. (13) with the partial differential equation (1). So, we acquire a ordinary differential model while,

$$\begin{aligned}
&\mathcal{F}'(\varphi) = -12a_2\mathcal{F}''(\varphi) - 24(\mathcal{F}(\varphi) - \mathcal{F}^2(\varphi)\mathcal{H}(\varphi)), \\
&\mathcal{H}'(\varphi) = 12a_2\mathcal{H}''(\varphi) + 24(\mathcal{H}(\varphi) - \mathcal{H}^2(\varphi)\mathcal{F}(\varphi)).
\end{aligned} \tag{14}$$

$\mathcal{L}_4 = \langle \Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \rangle = \frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t} + a_3 u\frac{\partial}{\partial u} - a_3 v\frac{\partial}{\partial v}$

Here, the corresponding Lagrange equation is,

$$\frac{dx}{1} = \frac{dt}{a_2} = \frac{du}{a_3 u} = \frac{dv}{-a_3 v},$$
$$\varphi = a_2 x - t, \qquad u(x,t) = \mathcal{F}(\varphi) + e^{a_3 x},$$
$$\varphi = a_2 x - t, \qquad v(x,t) = \mathcal{H}(\varphi) + e^{-a_3 x}. \tag{15}$$

Using an Eq. (15) to the partial differential equation (1). So, we acquire a ordinary differential model while,

$$\mathcal{F}'(\varphi) = -12a_2^2\mathcal{F}''(\varphi) - 12a_3^2 e^{a_3 x} - 24(\mathcal{F}(\varphi) + e^{a_3 x} - (\mathcal{F}(\varphi) + e^{a_3 x})^2\mathcal{H}(\varphi)),$$
$$\mathcal{H}'(\varphi) = 12a_2^2\mathcal{H}''(\varphi) + 12a_3^2 e^{-a_3 x} + 24(\mathcal{H}(\varphi) + e^{-a_3 x} - (\mathcal{H}(\varphi) + e^{-a_3 x})^2\mathcal{F}(\varphi)). \tag{16}$$

Analytical solution of the system

The developed infinitesimals Lie symmetry points are executed to construct the next traveling wave variables and applied on the considered system of equations (1) to obtain ordinary differential equations. To find nonlinear propagating wave profiles of a system (1), we are considering the ordinary system (14) and applying the power series solution approach.

Series solutions of the system (14)

We get the following equation from the first equation of system (14),

$$\mathcal{H}(\varphi) = \frac{\mathcal{F}'(\varphi) + 12\,a_2\mathcal{F}''(\varphi) + 24\,\mathcal{F}(\varphi)}{24\,(\mathcal{F}(\varphi))^2} \tag{17}$$

Eq. (17) can be inserted into the second equation of model (14) and acquire the following unique ordinary differential equation,

$$144a_2^2\mathcal{F}^2\mathcal{F}'''' - 576a_2^2\mathcal{F}\mathcal{F}'\mathcal{F}''' - 432a_2^2\mathcal{F}(\mathcal{F}'')^2 + 864a_2^2(\mathcal{F}')^2\mathcal{F}'' - 576a_2\mathcal{F}^2\mathcal{F}'' + 576a_2\mathcal{F}(\mathcal{F}')^2 - 72a_2\mathcal{F}\mathcal{F}'\mathcal{F}'' + 72a_2(\mathcal{F}')^3 - \mathcal{F}^2\mathcal{F}'' + \mathcal{F}(\mathcal{F}')^2 = 0 \tag{18}$$

Suppose that series outcome of Eq. (18) is following in the sense of power series,

$$\mathcal{F}(\varphi) = \sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}, \tag{19}$$

thus we have,

$$\mathcal{F}'(\varphi) = \sum_{\mathfrak{K}=0}^{\infty} \mathfrak{K} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}},$$
$$\mathcal{F}''(\varphi) = \sum_{\mathfrak{K}=0}^{\infty} (\mathfrak{K}+1)(\mathfrak{K}+2)\,b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}},$$
$$\mathcal{F}'''(\varphi) = \sum_{\mathfrak{K}=0}^{\infty} (\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+3)\,b_{\mathfrak{K}+3}\varphi^{\mathfrak{K}},$$
$$\mathcal{F}''''(\varphi) = \sum_{\mathfrak{K}=0}^{\infty} (\mathfrak{K}+3)(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+4)\,b_{\mathfrak{K}+4}\varphi^{\mathfrak{K}}, \tag{20}$$

and

$$\mathcal{F}^2(\varphi) = \sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right) \tag{21}$$

The values (19), (20) and (21) are plugging into Eq. (18), and get,

$$\begin{aligned}
&-576\,a_2{}^2\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\\
&\times\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+3}\varphi^{\mathfrak{K}}(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+3) - 432\,a_2{}^2\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\\
&\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\right)^2\\
&+864\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2{}^2\\
&+576\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 a_2\\
&-72\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}n\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2\\
&+144\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)\\
&\times\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+4}\varphi^{\mathfrak{K}}(\mathfrak{K}+3)(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+4)\,a_2{}^2 + 72\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^3 a_2\\
&+\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2\\
&-576\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2\\
&-\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2) = 0.
\end{aligned} \tag{22}$$

We acquire that value for b_4 through (22) while differentiate the coefficients of $n = 0$.

$$b_4 = \frac{b_2\left(864\,a_2{}^2 b_2 + 576\,a_2 b_0 + b_0\right)}{1728\,a_2{}^2 b_0}, \tag{23}$$

similarly, we have for $\mathfrak{K} \geq 1$,

$$\begin{aligned}
b_{\mathfrak{K}+4} = &-\frac{1}{144\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)(\mathfrak{K}+3)(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+4)\,a_2^2}\times\\
&\left(-576\,a_2{}^2\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\,b_{\mathfrak{K}+3}(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+3) - 432\,a_2{}^2 b_{\mathfrak{K}}\right.\\
&\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\right)^2\\
&+864\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2{}^2\\
&+576\,b_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 a_2\\
&-72\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2 +\\
&72\,b_{\mathfrak{K}+1}\mathfrak{K}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 a_2 + b_{\mathfrak{K}}\varphi\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2\\
&-576\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_k b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)\\
&\left.b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2 - \sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\right).
\end{aligned} \tag{24}$$

Now, the constants b_0, b_1, b_2, and b_3 could be random. The one of kind strategy is that the unlike kinds for sequence $\{b_n\}_{\mathfrak{K}=0}^{\infty}$ could be solved consecutively from Eq. (24). So, power series solution of the Eq. (18) would demonstrated while,

$$\mathcal{F}(\varphi) = b_0 + b_1\varphi + b_2\varphi^2 + b_3\varphi^3 + b_4\varphi^4 + \sum_{n=1}^{\infty} b_{\mathfrak{K}+4}\varphi^{\mathfrak{K}+4}. \tag{25}$$

$$
\begin{aligned}
v(x,t) = {} & \frac{1}{24\left(b_0+b_1(a_2x-t)+b_2(a_2x-t)^2+b_3(a_2x-t)^3+b_4(a_2x-t)^4+\sum_{\mathfrak{K}=1}^{\infty} b_{\mathfrak{K}+4}(a_2x-t)^{\mathfrak{K}+4}\right)^2}\times \\
& \Bigg(24\left(b_0+b_1(a_2x-t)+b_2(a_2x-t)^2+b_3(a_2x-t)^3+b_4(a_2x-t)^4+\sum_{\mathfrak{K}=1}^{\infty} b_{\mathfrak{K}+4}(a_2x-t)^{\mathfrak{K}+4}\right)+ \\
& b_1+2\,b_2(a_2x-t)+3\,b_3(a_2x-t)^2+4\,b_4(a_2x-t)^3+\sum_{\mathfrak{K}=1}^{\infty} b_{\mathfrak{K}+4}\,(\mathfrak{K}+4)\,(a_2x-t)^{\mathfrak{K}+3}+ \\
& 12\,a_2\left(2\,b_2+6\,b_3(a_2x-t)+12\,b_4(a_2x-t)^2+\sum_{\mathfrak{K}=1}^{\infty} b_{\mathfrak{K}+4}\left((\mathfrak{K}+4)^2-(\mathfrak{K}+4)\right)(a_2x-t)^{\mathfrak{K}+2}\right)\Bigg).
\end{aligned}
\tag{27}
$$

Box I.

The values of a_4 and $a_{\mathfrak{K}+4}$ are inserting into Eq. (25),

$$
\begin{aligned}
u(x,t) = {} & b_0+b_1\varphi+b_2\varphi^2+b_3\varphi^3+\left(\frac{b_2\left(864\,a_2{}^2b_2+576\,a_2b_0+b_0\right)}{1728\,a_2{}^2b_0}\right)\varphi^4+ \\
& \sum_{\mathfrak{K}=1}^{\infty}\Bigg(-\frac{1}{144\,\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)(\mathfrak{K}+3)(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+4)\,a_2{}^2}\times \\
& \Bigg(-576\,a_2{}^2\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}b_{\mathfrak{K}+3}(\mathfrak{K}+2)(\mathfrak{K}+1)(\mathfrak{K}+3) \\
& -\,432\,a_2{}^2b_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+2}\varphi^{\mathfrak{K}}(\mathfrak{K}+1)(\mathfrak{K}+2)\right)^2 \\
& +\,864\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2{}^2 \\
& +\,576\,b_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 a_2- \\
& 72\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}}\varphi^{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{n}\mathfrak{K}b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2 \\
& +\,72\,b_{\mathfrak{K}+1}\mathfrak{K}\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}\mathfrak{K}\right)^2 a_2+ \\
& b_{\mathfrak{K}}\varphi\left(\sum_{\mathfrak{K}=0}^{\infty} b_{\mathfrak{K}+1}\varphi^{\mathfrak{K}}n\right)^2-576\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_kb_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\,a_2- \\
& \sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} b_{\mathfrak{K}}b_{\mathfrak{K}-k}\varphi^{\mathfrak{K}}\right)b_{\mathfrak{K}+2}(\mathfrak{K}+1)(\mathfrak{K}+2)\Bigg)\Bigg)\varphi^{\mathfrak{K}+4},
\end{aligned}
\tag{26}
$$

where $\varphi = a_2x-t$ (see Figs. 1 and 2).

Since, by reference of Eq. (17), one could be shown the solution for v while, (see Box I)

Convergence analysis of the power series solution

In this section, we will prove the convergence of the power series solution (19) for Eq. (18).

$$
\begin{aligned}
b_{\mathfrak{K}+4} = {} & -\frac{M}{\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}}|b_{\mathfrak{K}}||b_{\mathfrak{K}-k}|\right)} \\
& \times\Bigg(-\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}}|\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}||b_{\mathfrak{K}+3}|-|b_{\mathfrak{K}}|\left(\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+2}|\right)^2 \\
& +\left(\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}|\right)^2|b_{\mathfrak{K}+2}| \\
& +\,|b_{\mathfrak{K}}|\left(\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}|\right)^2-\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}}|\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}||b_{\mathfrak{K}+2}|+|b_{\mathfrak{K}|+1}|\left(\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}|\right)^2 \\
& +\,|b_{\mathfrak{K}}|\left(\sum_{\mathfrak{K}=0}^{\infty}|b_{\mathfrak{K}+1}|\right)^2-\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}}|b_k||b_{\mathfrak{K}-k}|\right) \\
& |b_{\mathfrak{K}+2}|-\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}}|b_{\mathfrak{K}}||b_{\mathfrak{K}-k}|\right)|b_{\mathfrak{K}+2}|\Bigg).
\end{aligned}
\tag{28}
$$

where

$$M = max\left[\frac{144}{11},\frac{108}{11},\frac{116}{11},\frac{144}{11a_2},\frac{18}{11a_2}\right]$$

for $\mathfrak{K}\geq 4$, $b_{\mathfrak{K}} = \sum_{i=0}^{\mathfrak{K}-4} b_{i+1}b_{\mathfrak{K}-i-2}$.

Now define a new power series $\mathfrak{A}(\beta) = \sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}}\beta^{\mathfrak{K}}$ and suppose $a_j = |b_j|$ for $j = 0, 1, 2, \ldots$, then we have,

$$
\begin{aligned}
a_{\mathfrak{K}+4} = {} & -\frac{M}{\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} a_{\mathfrak{K}}a_{\mathfrak{K}-k}|\right)} \\
& \times\Bigg(-\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}a_{\mathfrak{K}+3}|-a_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+2}\right)^2+\left(\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}\right)^2 a_{\mathfrak{K}+2} \\
& +\,a_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}\right)^2-\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}}\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}a_{\mathfrak{K}+2} \\
& +\,a_{\mathfrak{K}|+1}\left(\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}\right)^2+a_{\mathfrak{K}}\left(\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}+1}\right)^2-\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} a_ka_{\mathfrak{K}-k}\right) \\
& a_{\mathfrak{K}+2}-\sum_{\mathfrak{K}=0}^{\infty}\left(\sum_{k=0}^{\mathfrak{K}} a_{\mathfrak{K}}a_{\mathfrak{K}-k}\right)a_{\mathfrak{K}+2}\Bigg).
\end{aligned}
\tag{29}
$$

It can be noticed that $|b_j| \leq a_j$ thus, $\varphi = \mathfrak{A}(\beta) = \sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}}\beta^{\mathfrak{K}}$ is a majorant series. Thus, we need to prove that it has positive radius of convergence and rewrite the series as,

$$\sum_{\mathfrak{K}=0}^{\infty} a_{\mathfrak{K}}\beta^{\mathfrak{K}} = a_0+a_1\beta+a_2\beta^2+a_3\beta^3+(\varphi-a_0)(\varphi-a_0-a_1\beta)\beta. \tag{30}$$

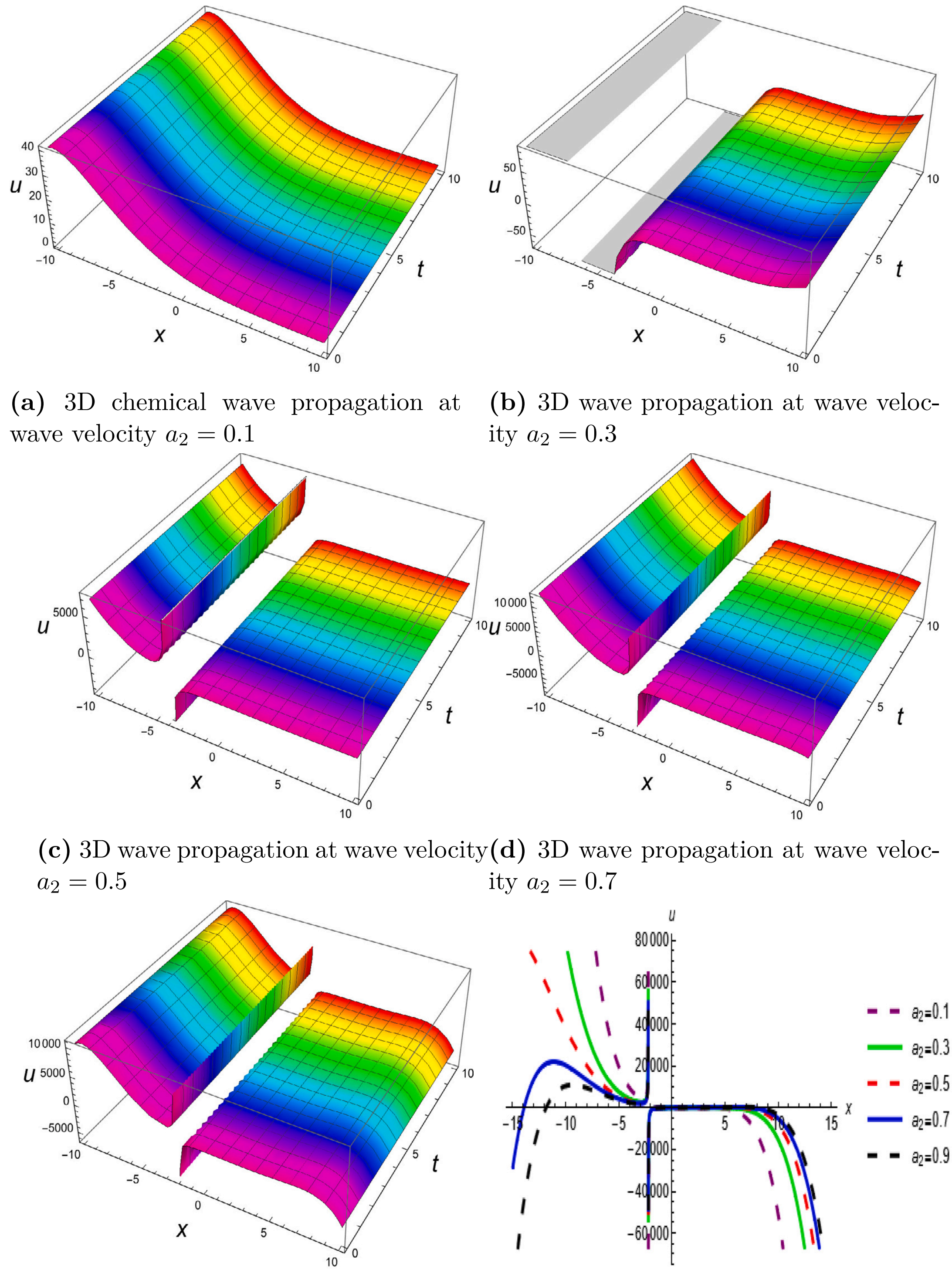
Then we consider the implicit function with respect to the independent variable β,

$$\lambda(\beta,\varphi) = \varphi a_0-a_1\beta-a_2\beta^2-a_3\beta^3-(\varphi-a_0)(\varphi-a_0-a_1\beta)\beta. \tag{31}$$

It is easy to verify that $\lambda(\beta,\varphi)$ is analytic in the neighborhood of $(0, a_0)$, where $\lambda(0, a_0) = 0$ and $\lambda'_{\varphi}(0, a_0) = 1 \neq 0$. Based on implicit function theorem, we find that φ is analytic and convergent in the neighborhood of the point $(0, a_0)$. Therefore, the power series solution is convergent by the method of majorants.

Modulation instability assessment

In this portion, the aim is to generate the modulation instability (MI) gain of steady-state solution of the governing structure (1) through the virtue of the linear stability analysis. The MI can consist of the



(a) 3D chemical wave propagation at wave velocity $a_2 = 0.1$

(b) 3D wave propagation at wave velocity $a_2 = 0.3$

(c) 3D wave propagation at wave velocity $a_2 = 0.5$

(d) 3D wave propagation at wave velocity $a_2 = 0.7$

(e) 3D wave propagation at wave velocity $a_2 = 0.9$

(f) 2D influence of wave velocity

Fig. 1. This figure is presenting the behavior of solution (26).

exponential growth of the small-scale perturbations in the stage of optical waves or the amplitude. That is essential to examine into non-linear wave physics.

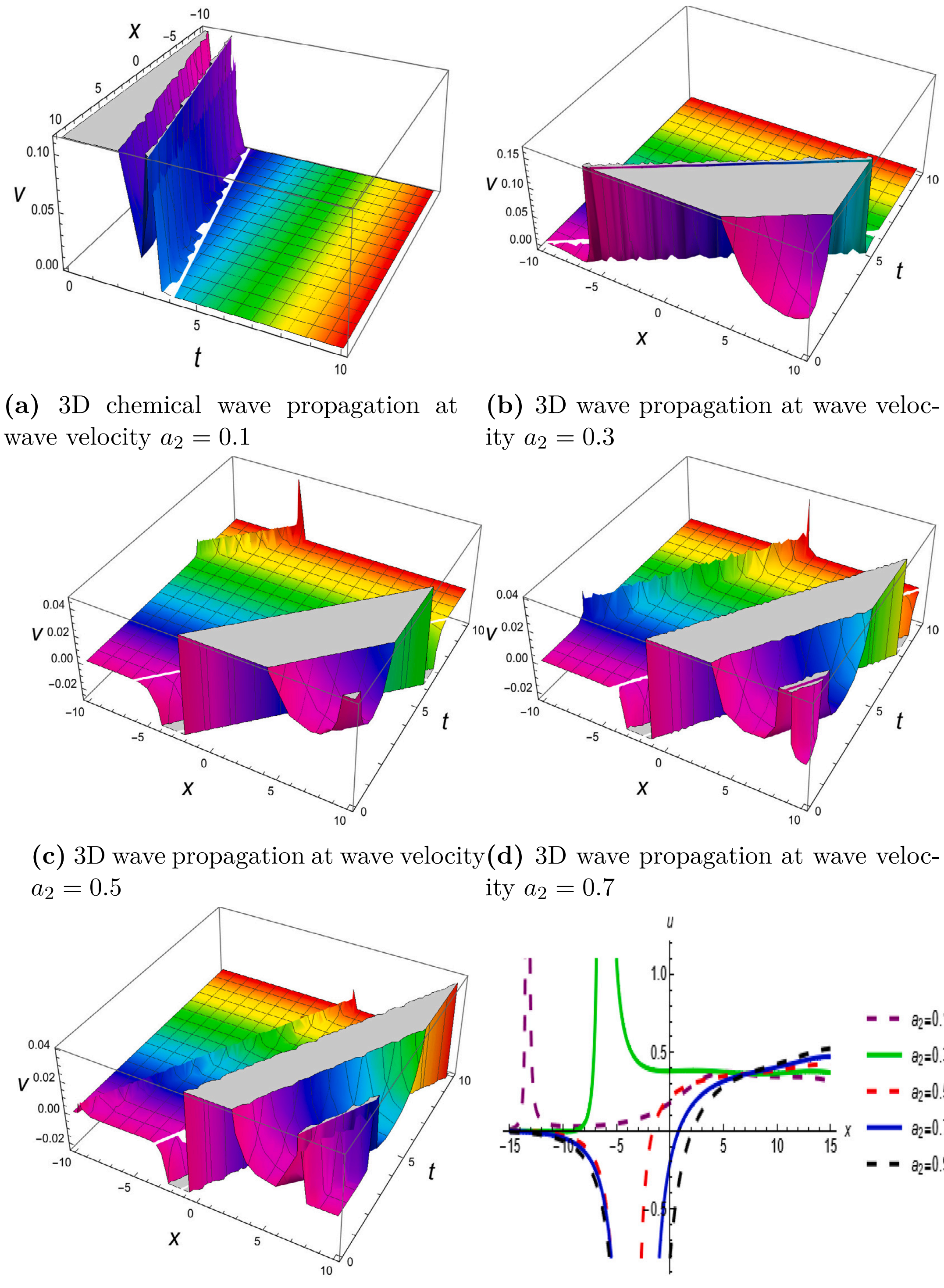
Let us suppose that steady-state solution in order to achieve stability analysis,

$$u = \mathcal{A}_0,\ \text{and}\ v = \mathcal{B}_0, \tag{32}$$

where, $\mathcal{A}_0$ and $\mathcal{B}_0$ are the initial incidence power (real constant-amplitudes). Moreover, that outcomes (32) are changing into stationary perturbed outcomes while,

$$u = \mathcal{A}_0 + \sigma\,\varphi(x,t),\ \text{and}\ v = \mathcal{B}_0 + \sigma\,\mathfrak{E}(x,t), \tag{33}$$

here φ along-with $\mathfrak{E}$ are real functions of x, t as well as the perturbation coefficient parameter is $\sigma \ll 1$. These disturbance equations have been



(a) 3D chemical wave propagation at wave velocity $a_2 = 0.1$

(b) 3D wave propagation at wave velocity $a_2 = 0.3$

(c) 3D wave propagation at wave velocity $a_2 = 0.5$

(d) 3D wave propagation at wave velocity $a_2 = 0.7$

(e) 3D wave propagation at wave velocity $a_2 = 0.9$

(f) 2D influence of wave velocity

Fig. 2. This figure is presenting the behavior of solution (27). .

generated, the perturbed stationary solutions into the PDE system (1),

$$\begin{aligned}
&24\left(A_0+\sigma\varphi(x,t)\right)^2\left(B_0+\sigma\mathfrak{E}(x,t)\right)-24A_0\\
&-24\sigma\varphi(x,t)+\sigma\frac{\partial}{\partial t}\varphi(x,t)-12\sigma\frac{\partial^2}{\partial x^2}\varphi(x,t)=0.\\
&-24\left(A_0+\sigma\varphi(x,t)\right)\left(B_0+\sigma\mathfrak{E}(x,t)\right)^2+24\,B_0\\
&+24\,\sigma\mathfrak{E}(x,t)+\sigma\frac{\partial}{\partial t}\mathfrak{E}(x,t)+12\,\sigma\frac{\partial^2}{\partial x^2}\mathfrak{E}(x,t)=0.
\end{aligned} \tag{34}$$

These disturbance equations (34) could have been expressed likewise after linearization,

$$\begin{aligned}
&24A_0^2B_0+24A_0^2\sigma\mathfrak{E}(x,t)+48A_0B_0\sigma\varphi(x,t)-24A_0\\
&-24\sigma\varphi(x,t)+\sigma\frac{\partial}{\partial t}\varphi(x,t)-12\sigma\frac{\partial^2}{\partial x^2}\varphi(x,t)=0.\\
&-24A_0B_0^2-24B_0^2\sigma\varphi(x,t)-48A_0B_0\sigma\mathfrak{E}(x,t)\\
&+24\,B_0+24\,\sigma\mathfrak{E}(x,t)+\sigma\frac{\partial}{\partial t}\mathfrak{E}(x,t)+12\,\sigma\frac{\partial^2}{\partial x^2}\mathfrak{E}(x,t)=0.
\end{aligned} \tag{35}$$

$$\begin{pmatrix} 48A_0B_0\sigma - 24\sigma - i\sigma\mathcal{W} + 12\kappa^2\sigma & 0 & 24A_0^2\sigma & 0 \\ -24A_0^2\sigma & 0 & -48A_0B_0\sigma - 24\sigma + i\sigma\mathcal{W} + 12\kappa^2\sigma & 0 \\ 0 & 48A_0B_0\sigma + 24\sigma - i\sigma\mathcal{W} - 12\kappa^2\sigma & 0 & 24A_0^2\sigma \\ 0 & 24B_0^2\sigma & 0 & 48A_0B_0\sigma + 24\sigma + i\sigma\mathcal{W} - 12\kappa^2\sigma \end{pmatrix} \quad (38)$$

Box II.

At this time, we would be present a $\varphi(x,t)$ along-with $\mathfrak{E}(x,t)$, which includes,

$$\begin{aligned} \varphi(x,t) &= \mathcal{A}_1 \exp^{i(\kappa x-\mathcal{W}t)} + \mathcal{B}_1 \exp^{-i(\mathcal{K}x-\mathcal{W}t)}, \\ \mathfrak{E}(x,t) &= \mathcal{A}_2 \exp^{i(\kappa x-\mathcal{W}t)} + \mathcal{B}_2 \exp^{-i(\kappa x-\mathcal{W}t)}. \end{aligned} \quad (36)$$

In order to create a structure of homogeneous equations, the functions (36) must be substituted within (35),

$$\begin{aligned} &24A_0^2\sigma\mathcal{A}_2 + 48A_0B_0\sigma\mathcal{A}_1 - 24\,\sigma\,\mathcal{A}_1 - i\sigma\,\mathcal{A}_1\mathcal{W} + 12\,\kappa^2\sigma\,\mathcal{A}_1 = 0, \\ &-24B_0^2\sigma\mathcal{A}_1 - 48A_0B_0\sigma\mathcal{A}_2 - 24\,\sigma\,\mathcal{A}_2 + i\sigma\,\mathcal{A}_2\mathcal{W} + 12\,\kappa^2\sigma\,\mathcal{A}_2 = 0, \\ &24A_0^2\sigma\mathcal{B}_2 + 48A_0B_0\sigma\mathcal{B}_1 + 24\,\sigma\,\mathcal{B}_1 - i\sigma\,\mathcal{B}_1\mathcal{W} - 12\,\kappa^2\sigma\,\mathcal{B}_1 = 0, \\ &-24B_0^2\sigma\mathcal{B}_1 - 48A_0B_0\sigma\mathcal{B}_2 + 24\,\sigma\,\mathcal{B}_2 + i\sigma\,\mathcal{B}_2\mathcal{W} - 12\,\kappa^2\sigma\,\mathcal{B}_2 = 0. \end{aligned} \quad (37)$$

The following, $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{A}_2$, and $\mathcal{B}_2$, is a possible way to write the coefficient matrix of the system (37), (see Box II) This coefficient matrix (38) has the non-trivial answers, whenever determinant disappears. The dispersion relation developed through enlarging the coefficient matrix's determinant,

$$\begin{aligned} &-\mathcal{W}^4\imath^4\sigma^4 + 96\,\mathcal{W}^3\imath^3\sigma^4A_0B_0 + 288\,\mathcal{W}^2\imath^2\kappa^4\sigma^4 \\ &-1152\,\mathcal{W}^2\imath^2\kappa^2\sigma^4A_0B_0 + 576\,\mathcal{W}^2\imath^2\sigma^4{A_0}^4 - \\ &576\,\mathcal{W}^2\imath^2\sigma^4{A_0}^2{B_0}^2 - 13824\,W\imath\,\kappa^4\sigma^4A_0B_0 \\ &+110592\,W\imath\,\kappa^2\sigma^4{A_0}^2{B_0}^2 - 165888\,W\imath\,\sigma^4{A_0}^3{B_0}^3 - \\ &20736\,\kappa^8\sigma^4 + 165888\,\kappa^6\sigma^4A_0B_0 - 82944\,\kappa^4\sigma^4{A_0}^4 \\ &+82944\,\kappa^4\sigma^4{A_0}^2{B_0}^2 + 663552\,\kappa^2\sigma^4{A_0}^5B_0 - \\ &2654208\,\kappa^2\sigma^4{A_0}^3{B_0}^3 - 995328\,\sigma^4{A_0}^6{B_0}^2 + 3981312\,\sigma^4{A_0}^4{B_0}^4 \\ &-1152\,\mathcal{W}^2\imath^2\kappa^2\sigma^4 + 2304\,\mathcal{W}^2\imath^2\sigma^4A_0B_0 \\ &+55296\,W\imath\,\kappa^2\sigma^4A_0B_0 - 221184\,W\imath\,\sigma^4{A_0}^2{B_0}^2 + 165888\,\kappa^6\sigma^4 \\ &-995328\,\kappa^4\sigma^4A_0B_0 + 331776\,\kappa^2\sigma^4{A_0}^4 \\ &-331776\,\kappa^2\sigma^4{A_0}^2{B_0}^2 - 1327104\,\sigma^4{A_0}^5B_0 + 5308416\,\sigma^4{A_0}^3{B_0}^3 \\ &+1152\,\mathcal{W}^2\imath^2\sigma^4 - 55296\,W\imath\,\sigma^4A_0B_0 \\ &-497664\,\kappa^4\sigma^4 + 1990656\,\kappa^2\sigma^4A_0B_0 - 331776\,\sigma^4{A_0}^4 + \\ &331776\,\sigma^4{A_0}^2{B_0}^2 + 663552\,\kappa^2\sigma^4 - 1327104\,\sigma^4A_0B_0 - 331776\,\sigma^4 = 0 \end{aligned} \quad (39)$$

It can be seen that the linked nonlinear system is modulational suitable in each wavenumber κ if and only if the four roots $\mathcal{W}$ of (39) are all positive real values. However, obtaining the roots of (39) is It is not that easy, we must use the existing efficient analytical formulations and related requirements for fourth-order polynomial roots. The solution of the dispersion relation (39) is acquired,

$$\mathcal{W} = \pm i\left(48\,A_0B_0 + 12\sqrt{\kappa^4 + 4\,{A_0}^4 - 4\,\kappa^2 + 4}\right). \quad (40)$$

So, it could be think about this, linked network modulation instability (1) materializes when either,

$$\sqrt{\kappa^4 + 4\,{A_0}^4 - 4\,\kappa^2 + 4} < 0.$$

Modulation instability is a phenomenon that occurs in certain wave systems, such as optical or water waves, where a continuous wave (CW) or monochromatic wave evolves into a more complex waveform due to small perturbations. In this study, modulational instability analysis is particularly important because it allows for the generation of new frequency components by amplifying perturbations in laser beams. This property has a wide range of applications, including telecommunications, spectroscopy, and the creation of supercontinuum sources. The considered model also have application in fluid dynamics. Thus, Modulational instability analysis is critical in fluid dynamics for understanding the formation of rogue waves in the ocean. These are large, dangerous waves that can appear seemingly out of nowhere. Predicting when and where rogue waves will occur can have serious consequences for maritime safety.

Lie backlund

It is possible to consider the Lie-Backlund transformation group to be a tangent transformation group. It is intended to be an analog of the one-parameter group of continuous symmetry transformations. The vector field form may be utilized to develop the governing equation's Lie-Backlund symmetry generator,

$$Q = \Upsilon^{\chi}(x,t,u,v,u_x,v_x,u_{xx},v_{xx}), \quad (41)$$

where Q holds for $Q^{(2)}\Delta_i/_{\Delta_i} = 0$. The Over-determined systems are uncovered by manipulating the fourth extension to the governing equation. The typical outcomes of the over-determined structure are:

$$\begin{aligned} \Upsilon_u &= 48C_6tu^2v + 2C_3u^2v + 6C_5uu_xv + 24C_4tu_x \\ &+ C4ux - 24C_6tu_{xx} - C_6u_{xx} \\ &+ C_1u + C_2u_x - C_3u_{xx} - C_5u_{xxx}, \\ \Upsilon_v &= -48C_6tv^2u - 2C_3v^2v + 6C_5vv_xu + 24C_4tv_x \\ &- C4vx + 24C_6tv_{xx} - C_6v_{xx} \\ &- C_1v + C_2v_x + C_3v_{xx} - C_5v_{xxx} - 2C_6v. \end{aligned} \quad (42)$$

where arbitrary constants are C_1, C_2, C_3, C_4, and C_5. After that, vector fields are used to generate the algebra of Lie point symmetries.

$$\begin{aligned} \Gamma_1 &= u\partial_u - v\partial_v, \\ \Gamma_2 &= u_x\partial_u + v_x\partial_v, \\ \Gamma_3 &= (2u^2v - u_{xx})\partial_u + (-2v^2u + v_{xx})\partial_v, \\ \Gamma_4 &= (24tu_x + u_x)\partial_u + (24tv_x - v_x)\partial_v, \\ \Gamma_5 &= (6uu_xv - u_{xxx})\partial_u + (6vv_xu - v_{xxx})\partial_v, \\ \Gamma_6 &= (48tu^2v - 24tu_{xx} - u_{xx})\partial_u + (-48tv^2u + 24tv_{xx} - v_{xx} - 2v)\partial_v. \end{aligned} \quad (43)$$

Conservation laws

The relationship between symmetry and conserved quantities is a fundamental concept in modern physics. Emmy Noether established Noether's theorem, which states that for every continuous symmetry in a physical system, there is a conserved quantity. Conservation laws, which are fundamental physics precepts, describe the behavior of physical systems. These laws state that specific quantities, such as mass, energy, momentum, and charge, are conserved over time, which means they cannot be created or destroyed but can only be transferred from one system to another. The principle of mass conservation, for example, states that even if mass is redistributed within a closed system, the total mass of the system will always remain constant over time. According to the conservation of energy principle, the total energy of a closed system is constant, implying that energy cannot be created or destroyed, but only converted from one form to another. The conservation of momentum principle states that the total momentum of a closed system remains constant over time, implying that momentum cannot be created or destroyed but can only be transferred between objects within the system.

Theorem 1. *Let Eq. (73) be a symmetry (point, contact or Bäcklund) of Eq. (74) with Lagrangian equation (76). The conservation equation then becomes satisfied by Eq. (74)*

$$D_i(T^i) = 0, \tag{44}$$

where

$$\begin{aligned} T^i &= \mathfrak{E}^i \mathcal{L} + W^{\overline{\alpha}}\left[\frac{\partial \mathcal{L}}{\partial \psi_i^{\overline{\alpha}}} - D_j\left(\frac{\partial \mathcal{L}}{\partial \psi_{ij}^{\overline{\alpha}}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}^{\overline{\alpha}}}\right) - \cdots\right] \\ &+ D_j(W^{\overline{\alpha}})\left[\frac{\partial \mathcal{L}}{\partial \psi_{ij}^{\overline{\alpha}}} - D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}^{\overline{\alpha}}}\right) + \cdots\right] + D_j D_k(W^{\overline{\alpha}})\left[\frac{\partial \mathcal{L}}{\partial \psi_{ijk}^{\overline{\alpha}}} - \cdots\right], \end{aligned} \tag{45}$$

and

$$W^{\overline{\alpha}} = Y^{\overline{\alpha}} - \mathfrak{E}^j \psi_j^{\overline{\alpha}}. \tag{46}$$

These conserved vectors become T^i. Now these conserved vectors may be gained through this fact and Theorem 6.1. $(T_i^x; T_i^t)$ $(i = 1, 2, \ldots, 6)$. The conservation equation is going to be satisfied through the vectors.

Below is utilizing theorem 6.1, Eq. (92) we derive the set of non-trivial conserved vectors associated with each of the vector fields.

Conserved vectors using point symmetries

- Lie point symmetry generator $\Gamma_1 = \partial_x$ bring in the conserved vector along-with components

$$\begin{aligned} \mathbf{T_1^x} &= -12u_x z_x + u_t z + 24v\left(u^2 z + w\right) - 24uv^2 w - 24uz + 12v_x w_x + v_t w, \\ \mathbf{T_1^t} &= zu_x - wv_x. \end{aligned} \tag{47}$$

- Lie point symmetry generator $\Gamma_2 = \partial_t$ bring in the conserved vector along-with components

$$\begin{aligned} \mathbf{T_2^x} &= 12\left(-u_t(x,t)z_x + u_{xt}z + v_t w_x - v^{xt} w\right), \\ \mathbf{T_2^t} &= 12\left(-u_{xx}z + 2v\left(u^2 z + w\right) - 2uv^2 w - 2uz + v_{xx} w\right). \end{aligned} \tag{48}$$

- Lie point symmetry generator $\Gamma_3 = u\partial_u + v\partial_v$ bring in the conserved vector along-with components

$$\begin{aligned} \mathbf{T_3^x} &= 12\left(u_x z - u z_x + v_x w - v w_x\right), \\ \mathbf{T_3^t} &= vw - uz. \end{aligned} \tag{49}$$

- Lie point symmetry generator $\Gamma_4 = -ux\partial_u + vx\partial_v + 24t\partial_x$ bring in the conserved vector along-with components

$$\begin{aligned} \mathbf{T_4^x} &= 12\Big(-24tu_x z_x + 2tu_t z + xu_x z + v\left(48tu^2 z - xw_x + (48t+1)w\right) \\ &- 48tuv^2 w + u\left(-xz_x - 48tz + z\right) + 24tv_x w_x + 2tv_t w + xv_x w\Big), \\ \mathbf{T_4^t} &= \left(xv - 24tv_x\right) w - \left(24tu_x + xu\right) z. \end{aligned} \tag{50}$$

- Lie point symmetry generator $\Gamma_5 = -48tv\partial_v + 2t\partial_t + x\partial_x + (48tu - 2u)\partial_u$ bring in the conserved vector along-with components

$$\begin{aligned} \mathbf{T_5^x} &= xz\left(u_t - 12u_{xx} + 24u^2 v - 24u\right) + 12\left(-2tu_t - xu_x + (48t-2)u\right) z_x \\ &- 12\left((48t-3)u_x - 2tu_{(x,t)} - xu_{xx}\right) z + xw\left(-24uv^2 + v_t + 12v_{xx} + 24v\right) \\ &+ 12\left(2tv_t + xv_x + 48tv\right) w_x - 12\left((48t+1)v_x + 2tv_{xt} + xv_{xx}\right) w, \\ \mathbf{T_5^t} &= -\left(xu_x + 24tu_{xx}\right) z - 2u\left(24tv^2 w + z\right) + 48tu^2 vz + \left(24tv_{xx} - xv_x\right) w. \end{aligned} \tag{51}$$

Conserved vectors using Lie backlund symmetries

- Lie Backlund symmetry generator $\Gamma_1 = u\partial_u - v\partial_v$ yields

$$\begin{aligned} \mathbf{T_1^x} &= 12\left(-z\left(u_x + u_x\right) + z_x u - w\left(v_x + v_x\right) + w_x v\right), \\ \mathbf{T_1^t} &= zu - wv. \end{aligned} \tag{52}$$

- Lie Backlund symmetry generator $\Gamma_2 = u_x\partial_u + v_x\partial_v$ yields

$$\begin{aligned} \mathbf{T_2^x} &= 12\left(u_x z_x - u_{xx} z - v_x w_x + v_{xx} w\right), \\ \mathbf{T_2^t} &= u_x z + v_x w. \end{aligned} \tag{53}$$

- Lie Backlund symmetry generator $\Gamma_3 = (2u^2 v - u_{xx})\partial_u + (-2v^2 u + v_{xx})\partial_v$ yields

$$\begin{aligned} \mathbf{T_3^x} &= 12\Big(2uv\left(w_x v - 2\left(z\left(u_x + u_x\right) + w\left(v_x + v_x\right)\right)\right) \\ &+ w\left(-2\left(u_x + u_x\right) v^2 + v_{xxx} + v_{xxx}\right) \\ &+ zu_{xxx} + 2u^2\left(z_x v - z\left(v_x + v_x\right)\right) - z_x u_{xx} + zu_{xxx} - w_x v_{xx}\Big), \\ \mathbf{T_3^t} &= w\left(v_{xx} - 2uv^2\right) + z\left(2u^2 v - u_{xx}\right) \end{aligned} \tag{54}$$

- Lie Backlund symmetry generator $\Gamma_4 = (24tu_x + u_x)\partial_u + (24tv_x - v_x)\partial_v$ yields

$$\begin{aligned} \mathbf{T_4^x} &= 12\left((24t+1)u_1 z_x - (24t+1)u_{xx} z - (24t-1)v_x w_x + (24t-1)v_{xx} w\right), \\ \mathbf{T_4^t} &= (24t+1)u_x z + (24t-1)v_x w. \end{aligned} \tag{55}$$

- Lie Backlund symmetry generator $\Gamma_5 = (6uu_x v - u_{xxx})\partial_u + (6vv_x - v_{xxx})\partial_v$ yields

$$\begin{aligned} \mathbf{T_5^x} &= 12\Big(-6u\left(v\left(-u_x z_x + u_{xx} z + v_x w_x - v_{xx} w\right) + \left(v_x + v_x\right)\left(u_x z - v_x w\right)\right) \\ &+ 6v_x w u_x v \\ &+ z\left(-6u_x\left(u_x + u_x\right) v + u_{xxxx} + u_{xxxx}\right) \\ &+ 6v_x w u_x v - z_x u_{xxx} - wv_{xxxx}(x,t) + w_x v_{xxx} - wv_{xxxx}\Big), \\ \mathbf{T_5^t} &= z\left(6u_x uv - u_{xxx}\right) + w\left(6v_x uv - v_{xxx}\right). \end{aligned} \tag{56}$$

- Lie Backlund symmetry generator $\Gamma_6 = (48tu^2 v - 24tu_{xx} - u_{xx})\partial_u + (-48tv^2 u + 24tv_{xx} - v_{xx} - 2v)\partial_v$ yields

$$\begin{aligned} \mathbf{T_6^x} &= 12\Big(w\left(-48t\left(u_x + u_x\right) v^2 - 96tu\left(v_x + v_x\right) v - v_{xxx}\right. \\ &- \left.24t\left(v_{xxx} + v_{xxx}\right) - 2\left(v_x + v_x\right) - v_{xxx}\right) \\ &- z\left(96t\left(u_x + u_x\right) uv - (24t+1)\left(u_{xxx} + u_{xxx}\right) + 48tu^2\left(v_x + v_x\right)\right) \\ &+ w_x\left(48tuv^2 + 2v + (24t+1)v_{xx}\right) + z_x\left(48tu^2 v - (24t+1)u_{xx}\right)\Big), \\ \mathbf{T_6^t} &= z\left(48tu^2 v - (24t+1)u_{xx}\right) - w\left(48tuv^2 + 2v + (24t+1)v_{xx}\right). \end{aligned} \tag{57}$$

Conserved vectors using multipliers

Using the multiplier approach as described in [32], To assemble the conservation laws for the governing system. From the determining equation, the following 1st-order multipliers are derived:

$\Lambda^1(x, t, u, v, u_x, v_x, u_t, v_t)$, and $\Lambda^2(x, t, u, v, u_x, v_x, u_t, v_t)$ as long as the model which is given by

$$\begin{aligned} \Lambda^1 &= \frac{1}{24}(C_1 x + 24C_4)v + \frac{1}{24}(-24C_1 t - 24C_3)vx - C_2 v_t, \\ \Lambda^2 &= \frac{1}{24}(C_1 x + 24C_4)u + \frac{1}{24}(24C_1 t + 24C_3)ux + C_2 u_t, \end{aligned} \tag{58}$$

C_1, C_2, C_3 and C_4 are unknown constants. The multipliers and Cls based on the free constants are given by

(1)

$$\Lambda^1 = \frac{1}{24}v_x - tv_x, \ \Lambda^2 = \frac{1}{24}u_x + tu_x, \tag{59}$$

$$\begin{aligned} \mathbf{T^x} &= -12tv^2u^2 + 24tuv + tuv_t + \frac{1}{2}v_xu_x + \frac{1}{2}uv + 12u_xtv_x - \frac{1}{2}u_xv_x, \\ \mathbf{T^t} &= -tuv_x, \end{aligned} \tag{60}$$

(2)

$$\Lambda^1 = -v_t, \ \Lambda^2 = u_t. \tag{61}$$

$$\begin{aligned} \mathbf{T^x} &= 12u_tv_x + 12u_xv_t, \\ \mathbf{T^t} &= -12u_xv_x, \end{aligned} \tag{62}$$

(3)

$$\Lambda^1 = -v_x, \ \Lambda^2 = u_x. \tag{63}$$

$$\begin{aligned} \mathbf{T^x} &= -12u^2v^2 + 24uv + uv_t + 12u_xv_x, \\ \mathbf{T^t} &= -uv_x. \end{aligned} \tag{64}$$

(4)

$$\Lambda^1 = -v, \ \Lambda^2 = u. \tag{65}$$

$$\begin{aligned} \mathbf{T^x} &= 12uv_x - 12u_xv, \\ \mathbf{T^t} &= 0. \end{aligned} \tag{66}$$

The conservation laws providing the guarantee of existence of integral invariants for the coupled Chaffee–Infante diffusion-reaction system and also ensured the conservation of mass, momentum, and energy their stability and persistence of these quantities in a closed system.

Conclusion

This study has taken into account the nonlinear mathematically coupled Chaffee–Infante diffusion-reaction system. We have explored the reaction–diffusion hierarchy's infinitesimal generators and used them to create the most effective system of subalgebras. Also calculated are the symmetry reductions on those vector fields which are a constituent of an optimal system. Preceding symmetry reduction, non-linear PDEs are changed within non-linear ODEs, and the power series procedure is applied to draw out the particular power series solutions of these ordinary differential equations. The convergence of the power series outcomes is shown. The right relevant values are chosen for the involved free parameters in order to explain the pictorial characteristic of wave propagation utilizing the computed analytical outcomes. Moreover, we have established conserved vectors for the governing equation using different approaches such as Lie symmetry, Backlund symmetry, and multiplier. The developed solutions can be reliable when interpreting the visual information of the nonlinear model.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

Acknowledgments

Researchers Supporting Project number (RSP2023R440), King Saud University, Riyadh, Saudi Arabia.

Funding

The authors have not disclosed any funding.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent

Informed consent was obtained from all individual participants included in the study.

References

[1] Gao W, Ismael HF, Bulut H, Baskonus HM. Instability modulation for the (2+1)-dimension paraxial wave equation and its new optical soliton solutions in Kerr media. Phys Scr 2020;95(3):035207.
[2] Faridi WA, Bakar MA, Myrzakulova Z, Myrzakulov R, Akgül A, El Din SM. The formation of solitary wave solutions and their propagation for Kuralay equation. Results Phys 2023;52:106774.
[3] Faridi WA, Bakar MA, Akgül A, Abd El-Rahman M, El Din SM. Exact fractional soliton solutions of thin-film ferroelectric material equation by analytical approaches. Alex Eng J 2023;78:483–97.
[4] Faridi WA, Asghar U, Asjad MI, Zidan AM, Eldin SM. Explicit propagating electrostatic potential waves formation and dynamical assessment of generalized Kadomtsev–Petviashvili modified equal width-Burgers model with sensitivity and modulation instability gain spectrum visualization. Results Phys 2023;44:106167.
[5] Gao W, Ismael HF, Husien AM, Bulut H, Baskonus HM. Optical soliton solutions of the cubic-quartic nonlinear Schrödinger and resonant nonlinear Schrödinger equation with the parabolic law. Appl Sci 2019;10(1):219.
[6] Ali KK, Yilmazer R, Yokus A, Bulut H. Analytical solutions for the (3+1)-dimensional nonlinear extended quantum Zakharov–Kuznetsov equation in plasma physics. Phys A: Stat Mech Appl 2020;548:124327.
[7] Kaur L, Wazwaz AM. Painlevé analysis and invariant solutions of generalized fifth-order nonlinear integrable equation. Nonlinear Dyn 2018;94:2469–77.
[8] Kaur L, Gupta RK. Kawahara equation and modified Kawahara equation with time dependent coefficients: Symmetry analysis and generalized-expansion method. Math Methods Appl Sci 2013;36(5):584–600.
[9] Kaur L, Wazwaz AM. Einstein's vacuum field equation: Painlevé analysis and Lie symmetries. Waves Random Complex Media 2021;31(2):199–206.
[10] Wazwaz AM, Kaur L. New integrable Boussinesq equations of distinct dimensions with diverse variety of soliton solutions. Nonlinear Dyn 2019;97:83–94.
[11] Babelon O, Bernard D, Talon M. Introduction to classical integrable systems. Cambridge University Press; 2003.
[12] Ardourel V, Jebeile J. On the presumed superiority of analytical solutions over numerical methods. Eur J Philos Sci 2017;7(2):201–20.
[13] Zafar A, Shakeel M, Ali A, Akinyemi L, Rezazadeh H. Optical solitons of nonlinear complex Ginzburg–Landau equation via two modified expansion schemes. Opt Quantum Electron 2022;54(1):1–5.
[14] Srivastava HM, Baleanu D, Machado JA, Osman MS, Rezazadeh H, Arshed S, et al. Traveling wave solutions to nonlinear directional couplers by modified Kudryashov method. Phys Scr 2020;95(7):075217.
[15] Fahim MR, Kundu PR, Islam ME, Akbar MA, Osman MS. Wave profile analysis of a couple of (3+ 1)-dimensional nonlinear evolution equations by sine-Gordon expansion approach. J Ocean Eng Sci 2022;7(3):272–9.
[16] Abdul Kayum M, Ali Akbar M, Osman MS. Stable soliton solutions to the shallow water waves and ion-acoustic waves in a plasma. Waves Random Complex Media 2022;32(4):1672–93.
[17] Zhang R, Bilige S, Chaolu T. Fractal solitons, arbitrary function solutions, exact periodic wave and breathers for a nonlinear partial differential equation by using bilinear neural network method. J Syst Sci Complexity 2021;34(1):122–39.
[18] Zhang RF, Bilige S. Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation. Nonlinear Dyn 2019;95(4):3041–8.
[19] Khater M, Jhangeer A, Rezazadeh H, Akinyemi L, Akbar MA, Inc M, et al. New kinds of analytical solitary wave solutions for ionic currents on microtubules equation via two different techniques. Opt Quantum Electron 2021;53(11):1–27.
[20] Karaman B. The use of improved-F expansion method for the time-fractional Benjamin–Ono equation. Rev Real Acad de Ciencias Exactas, Físicas y Nat. Ser A. Mat 2021;115(3):1–7.
[21] Yıldırım Y. Optical solitons with Biswas–Arshed equation by F-expansion method. Optik 2021;227:165788.
[22] Zayed EM, Shohib RM. Solitons and other solutions for two higher-order nonlinear wave equations of KdV type using the unified auxiliary equation method. Acta Phys Polonica, A. 2019;136(1).

[23] Zayed EM, Gepreel KA, Shohib RM, Alngar ME, Yıldırım Y. Optical solitons for the perturbed Biswas-Milovic equation with Kudryashov's law of refractive index by the unified auxiliary equation method. Optik 2021;230:166286.

[24] Ismael HF, Bulut H, Baskonus HM. Optical soliton solutions to the Fokas–Lenells equation via sine-Gordon expansion method and $(m+(G'/G))-$ expansion method. Pramana 2020;94(1):1–9.

[25] Abdulkadir Sulaiman T, Yusuf A. Dynamics of lump-periodic and breather waves solutions with variable coefficients in liquid with gas bubbles. Waves Random Complex Media 2021;6:1–4.

[26] Khodadad FS, Mirhosseini-Alizamini SM, Günay B, Akinyemi L, Rezazadeh H, Inc M. Abundant optical solitons to the Sasa-Satsuma higher-order nonlinear Schrödinger equation. Opt Quantum Electron 2021;53(12):1–7.

[27] Ghanbari B, Kumar S, Niwas M, Baleanu D. The Lie symmetry analysis and exact Jacobi elliptic solutions for the Kawahara–KdV type equations. Results Phys 2021;23:104006.

[28] Lü D. Jacobi elliptic function solutions for two variant Boussinesq equations. Chaos, Solitons Fractals 2005;24(5):1373–85.

[29] Baskonus HM, Sulaiman TA, Bulut H. On the exact solitary wave solutions to the long-short wave interaction system. In: ITM web of conferences 2018, Vol. 22. EDP Sciences; p. 01063.

[30] Rehman SU, Yusuf A, Bilal M, Younas U, Younis M, Sulaiman TA. Application of (G^1/G^2)-expansion method to microstructured solids, magneto-electro-elastic circular rod and (2+ 1)-dimensional nonlinear electrical lines. Math Eng, Sci Aerosp (MESA) 2020;11(4).

[31] Zhang RF, Li MC, Yin HM. Rogue wave solutions and the bright and dark solitons of the (3+ 1)-dimensional Jimbo–Miwa equation. Nonlinear Dyn 2021;103(1):1071–9.

[32] Aliyu AI, Tchier F, Inç M, Yusuf A, Baleanu D. Dynamics of optical solitons, multipliers and conservation laws to the nonlinear Schrodinger equation in (2+ 1)-dimensions with non-Kerr law nonlinearity. J Modern Opt 66(2):136–42.