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Modulational instability, optical solitons and travelling wave solutions to two nonlinear models in birefringent fibres with and without four-wave mixing terms

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Abstract. The paper aims to construct optical solitons and travelling wave solutions to two birefringent nonlinear models which consist of two-component form of vector solitons in optical fibre: the Biswas–Arshed model with Kerr-type nonlinearity and without four-wave mixing terms and the nonlinear Schrödinger equation with quadratic-cubic law of refractive index along with four-wave mixing terms. These nonlinear Schrödinger equations are applied in many physical and engineering fields. Optical solitons are considered in the context of photonic crystal fibres, couplers, polarisation-preserving fibres, metamaterials, birefringent fibres, and so on. Two reliable integration architectures, namely, the extended simplest equation method and the generalised sub-ODE approach, are adopted. As a result, bright soliton, kink and dark soliton, singular soliton, hyperbolic wave, a periodic wave, elliptic function solutions of Weierstrass and Jacobian types, and other travelling wave solutions, such as breather solutions and optical rogons, are derived, together with the existence conditions. In addition, the amplitude and intensity diagrams are portrayed by taking appropriate values for a few selected solutions. Furthermore, based on linear stability analysis, the modulation instability was explored for the obtained steady-state solutions. The reported results of this paper can enrich the dynamical behaviours of the two considered nonlinear models and can be useful in many scientific fields, such as mathematical physics, mathematical biology, telecommunications, engineering and optical fibres. This study confirms that the proposed approaches are sufficiently effective in extracting a variety of analytical solutions to other nonlinear models in both engineering and science.

Keywords. Traveling wave solution; birefringent fibre models; the simplest equation method; the sub-ODE method; optical soliton.

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1. Introduction

In natural and applied sciences, several nonlinear phenomena, such as optical fibres, acoustics, fluid dynamics, plasma physics, mechanics, biology, biophysics, solid-state physics, Higgs mechanism, propagation of shallow-water waves, thermodynamics and mathematical finance, are described by partial differential equations (PDEs). Many researchers have been interested in investigating the nonlinear Shrödinger (NLS) equation for the last few decades since the NLS equation has numerous applications in optical fibres, plasma and other fields of science and engineering [1–3].

There exist many approaches to investigate nonlinear equations in engineering and science, such as the Hirota's bilinear approach [4,5], the trigonometric function series method [6], the modified mapping method [7], the modified trigonometric function series method [8,9], the bifurcation method [10,11], the tanh–coth method [12], the Jacobi elliptic function method

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[13,14], the exp-function method [15], the F-expansion method [16], the mapping method [17], new ϕ^6-model expansion method [18], the unified Riccati equation expansion method [19], the modified simple equation method [20], the extended simplest equation method [20,21], the generalised sub-ODE method [22–24], the new extended auxiliary equation method [25,26], the transformed rational function method [27] and the multiple exp-function method [28]. Very recently, N-soliton solutions have been explored for local integrable equations (see, e.g., [29]) and nonlocal integrable equations (see, e.g., [30–32]). Some other related works on recent developments in the solitary wave solutions of nonlinear equations based on various methods and interesting physical applications are listed in [33–41].

The idea of solving nonlinear PDE via most of these techniques is to reduce it to a nonlinear ordinary differential equation (ODE) and hence solve it by the procedures of these approaches, leading to the exact solutions to the original PDE under consideration.

Therefore, the objective of this paper is to apply the extended simplest equation and the generalised sub-ODE methods to extract optical soliton solutions and other solutions to two models, namely, the Biswas–Arshed model with Kerr-type nonlinearity in birefringent fibres without four-wave mixing (FWM) terms and the NLS equation in birefringent fibres with quadratic–cubic law of refractive index along with FWM, respectively.

To this end, the current article is structured as follows: In §2, the descriptions of the used approaches are presented. In §3 and 4, optical soliton solutions and other solutions of the two considered models are derived via a variety of applications of these novel approaches. In §5, graphical discussion of the obtained solutions is presented. In §6, the modulation instability of the obtained steady-state solutions of this paper is analysed. In §7, conclusions are drawn.

2. Preliminaries

We take a nonlinear PDE with two independent variables x, t and one dependent variable u as

$$M(u,\ u_x,\ u_t, u_{xx}, u_{xt},\ u_{tt},\ \ldots) = 0, \tag{1}$$

where M denotes a polynomial in $u(x,t)$ and its partial derivatives, including highest-order derivatives and nonlinear terms.

Let the transformation

$$u(x,t) = U(\hbar), \hbar = x - \omega_1 t, \tag{2}$$

where ω_1 is a constant representing the speed wave, to simplify eq. (1) to the ODE

$$N(U(\hbar), U^{'}(\hbar),\ U^{''}(\hbar),\ \ldots) = 0, \tag{3}$$

where N is a polynomial in $U(\hbar)$ as well as its total derivatives with respect to $\hbar$.

2.1 *The extended simplest equation method*

For the method of extended simplest equation, consider the following projective Riccati equations:

$$r'(\hbar) = -r(\hbar)s(\hbar), s'(\hbar) = -\ s^2(\hbar) + \Lambda\ r(\hbar) -\ \lambda, \tag{4}$$

where Λ and λ are constants. Let

$$r(\hbar) = \frac{1}{\tau(\hbar)},\quad s(\hbar) = \frac{\tau'(\hbar)}{\tau(\hbar)}, \tag{5}$$

where $\tau(\hbar)$ is the solution of the following ODE:

$$\tau''(\hbar) + \lambda\tau(\hbar) = \Lambda. \tag{6}$$

It is a common knowledge that eq. (6) possesses distinct types of the following solutions:

Type 1. If $\lambda \prec 0$, then we have the hyperbolic function solutions

$$\tau(\hbar) = S_1\ \cosh(\sqrt{-\lambda}\,\hbar) + S_2\ \sinh(\sqrt{-\lambda}\,\hbar) + \Lambda/\lambda \tag{7}$$

and hence

$$\left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right)^2 = (\lambda\, S_1^2 - \lambda\, S_2^2 - \Lambda^2/\lambda)\left(\frac{1}{\tau(\hbar)}\right)^2 - \varepsilon + 2\Lambda/\tau(\hbar). \tag{8}$$

Type 2. If $\lambda \succ 0$, then the trigonometric function solutions follow as

$$\tau(\hbar) = S_1\ \cos(\sqrt{\lambda}\,\hbar) + S_2\ \sin(\sqrt{\lambda}\,\hbar) + \Lambda/\lambda \tag{9}$$

and hence

$$\left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right)^2 = (\lambda\, S_1^2 + \lambda\, S_2^2 - \Lambda^2/\lambda)\left(\frac{1}{\tau(\hbar)}\right)^2 - \varepsilon + 2\Lambda/\tau(\hbar). \tag{10}$$

Type 3. If $\lambda = 0$, then we have the following solutions

$$\tau(\hbar) = \frac{\Lambda}{2}\ \hbar^2 + S_1\,\hbar + S_2, \tag{11}$$

and hence

$$\left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right)^2 = (\ S_1^2 - 2\ \Lambda S_2)\left(\frac{1}{\tau(\hbar)}\right) + 2\Lambda/\tau(\hbar). \tag{12}$$

where S_1 and S_2 are arbitrary constants.

Step 1. Assume that eq. (3) admits the proposed extended solution

$$U(\hbar) = \sum_{m=0}^{M} A_m \left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right)^m + \sum_{n=0}^{M-1} B_n \left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right)^n \left(\frac{1}{\tau(\hbar)}\right), \quad (13)$$

where A_m $(m = 0, 1, 2, \ldots, M)$ and B_n $(n = 0, 1, 2, \ldots, M-1)$ are constants to be determined provided $A_M B_{M-1} \neq 0$.

Step 2. By substituting (13) into (3) and using eqs (8), (10) and (12), we get a set of algebraic equations to be solved for the unknowns A_m, B_n, ω_1, λ and Λ.

Step 3. Using Mathematica or Maple software, we can solve this system of algebraic equations and then we attain the exact solutions of eq. (1) utilising the solutions provided in eqs (7), (9) and (11).

2.2 *The generalised sub-ODE method*

In this context, Zi-Liang Li [22] has proposed, for the first time, a generalised sub-ODE with higher-order nonlinear terms of the form

$$\Psi'^2(\hbar) = A\Psi^{2-2q}(\hbar) + B\Psi^{2-q}(\hbar) + C\Psi^2(\hbar) + D\Psi^{2+q}(\hbar) + E\Psi^{2+2q}(\hbar), q \succ 0, \quad (14)$$

where $\Psi'(\hbar) = \mathrm{d}\Psi/\mathrm{d}\hbar$, A, B, C, D and E are constants.

The solutions of the generalised sub-ODE eq. (14) are provided in [22–24]. Therefore, we can present the framework of the generalised sub-ODE method in the following steps:

Step 1. Assume that eq. (3) has the formal solution

$$U(\hbar) = \Lambda\Psi^N(\hbar), \quad \Lambda \succ 0, \quad (15)$$

where N is a parameter and $\psi(\hbar)$ satisfies eq. (14). In (15), we calculate N by using the homogeneous balance method which is detailed as follows:

$$Deg(U) = N,$$
$$Deg(U^2) = 2N, \ldots, Deg(U') = N + q,$$
$$Deg(U'') = N + 2q, \ldots. \quad (16)$$

Step 2. Replacing eq. (15) by eq. (3) along with eq. (14) and gathering all the coefficients of $\Psi^{Nj}(\hbar)\left[\Psi'(\hbar)\right]^{\ell}$ $(\ell = 0, 1; j = 0, 1, 2, 3, \ldots)$ and equating them to zero, we receive a system of algebraic equations with respect to ω_1, A, B, C, D, E and Λ.

Step 3. With the assistance of the solutions of eq. (14) presented in [22–24], we can extract optical soliton solutions and other solutions of the nonlinear PDE (1) under investigation.

3. Biswas–Arshed model with Kerr-type nonlinearity in birefringent fibres without FWM

3.1 *The governing model*

Biswas and Arshad [42] recently devised the Biswas–Arshed equation (BAE), which is a more expanded version of the Schrödinger equation. The BAE is one of the most well-known models in the telecommunications sector. The Biswas–Arshed equation for Kerr law [43–52] in polarisation-preserving fibres is

$$iU_t + \alpha_1 U_{xx} + \alpha_2 U_{xt} + i(\beta_1 U_{xxx} + \beta_2 U_{xxt}) = i\left[\delta\left(|U|^2 U\right)_x + \epsilon\left(|U|^2\right)_x U + \eta|U|^2 U_x\right], \quad (17)$$

where $U(x, t)$ is the complex valued function representing the wave pattern. The coefficients of α_1 and α_2 provide sequentially that group velocity dispersion and spatio-temporal dispersion exist in order. Also, the coefficients of β_1 and β_2 provide sequentially that third-order dispersion and third-order spatio-temporal dispersion exist in order. The coefficients of δ, ϵ and η provide self-steepening and nonlinear dispersions in order.

It is well-known [43–52] that Biswas–Arshed model in birefringent fibres without FWM is written in the form

$$\begin{aligned} iU_t + \alpha_1 U_{xx} + \beta_1 U_{xt} + i(\gamma_1 U_{xxx} + \Lambda_1 U_{xxt}) &= i\left[\delta_1\left(|U|^2 U\right)_x + \sigma_1\left(|V|^2 V\right)_x\right] \\ &+ i\left[\varepsilon_1\left(|U|^2\right)_x + \tau_1\left(|V|^2\right)_x\right] U \\ &+ i\left[\eta_1|U|^2 + r_1|V|^2\right] U_x, \end{aligned} \quad (18)$$

$$\begin{aligned} iV_t + \alpha_2 V_{xx} + \beta_2 V_{xt} + i(\gamma_2 V_{xxx} + \Lambda_2 V_{xxt}) &= i\left[\delta_2\left(|V|^2 V\right)_x + \sigma_2\left(|U|^2 U\right)_x\right] \\ &+ i\left[\varepsilon_2\left(|V|^2\right)_x + \tau_2\left(|U|^2\right)_x\right] V \\ &+ i\left[\eta_2|V|^2 + r_2|U|^2\right] V_x, \end{aligned} \quad (19)$$

where the complex valued functions $U(x, t)$ and $V(x, t)$ denote the wave pattern, the coefficients α_i and β_i $(i = 1, 2)$ represent the group velocity dispersion and spatio-temporal dispersion respectively, while the coefficients γ_i and Λ_i guarantee the third-order dispersion and third-order spatio-temporal dispersion, respectively. Also, the coefficients $\delta_i, \sigma_i, \varepsilon_i, \tau_i, \eta_i$ and r_i secure self-steepening and nonlinear dispersions. One of the most prominent models in the telecommunications sector during a possible slow down is the Biswas–Arshed model.

The most notable aspect of this model is that it ignores self-phase modulation and has negligible group velocity dispersion. In addition, both second-order and third-order spatio-temporal descriptions are included in the model to compensate for low group velocity dispersion. Therefore, extracting optical solitons and travelling wave solutions of this model is of prime importance. We have also addressed other types of nonlinearity of this model in refs [53–56].

3.2 *Mathematical analysis*

To consider the Biswas–Arshed coupled systems (18) and (19), in birefringent fibres without FWM, we assume that

$$U(x,t) = G_1(\hbar)\exp\left[i\Omega(x,t)\right], \tag{20}$$

$$V(x,t) = G_2(\hbar)\exp\left[i\Omega(x,t)\right] \tag{21}$$

and

$$\hbar = x - \omega t + \hbar_0, \quad \Omega(x,t) = -\kappa x + C_0 t + \Omega_0, \tag{22}$$

where $\omega,\ \kappa,\ C_0$ and Ω_0 are constants that are not equal to zero representing some physical concepts of the model, such as soliton velocity, soliton frequency, wave number and phase constant, respectively and $\hbar_0$ is an arbitrary constant.

The function $\Omega(x,t)$, which is a real function, denotes the phase component of the soliton, while the real functions $G_i(\hbar)$ $(i = 1, 2)$ represent the pulse shape of the solitons. Using (20) and (21) with (22) in eqs (18) and (19) and extracting the real and imaginary terms, the following equations are available:

$$\begin{aligned}&[\alpha_s - \omega\beta_s + 3\kappa\gamma_s - (2\kappa\omega + C_0)\Lambda_s]G_s''(\hbar)\\&\quad + [C_0\kappa(\beta_s + \kappa\Lambda_s) - \gamma_s\kappa^3 - C_0 - \alpha_s\kappa^2]G_s(\hbar)\\&\quad - \kappa(\delta_s + \eta_s)G_s^3(\hbar)\\&\quad - \kappa\sigma_s G_{s^*}^3(\hbar) - \kappa r_s G_s(\hbar)G_{s^*}^2(\hbar) = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}&[\gamma_s - \omega\Lambda_s]G_s'''(\hbar) + [\omega\kappa\beta_s + C_0\beta_s - \omega - 2\kappa\alpha_s\\&\quad - 3\gamma_s\kappa^2 + \omega\kappa^2\Lambda_s + 2C_0\kappa\Lambda_s]G_s'(\hbar)\\&\quad - (2\varepsilon_s + \eta_s + 3\delta_s){G_s}^2(\hbar)G_s'(\hbar)\\&\quad - 2\tau_s G_s(\hbar)G_{s^*}(\hbar){G_{s^*}}'(\hbar)\\&\quad - r_s G_s'(\hbar)G_{s^*}^2(\hbar) - 3\sigma_s {G_{s^*}}'(\hbar)G_{s^*}^2(\hbar) = 0,\end{aligned} \tag{24}$$

where $s = 1, 2$ and $s^* = 3 - s$. By the balancing procedure, we have $G_s(\hbar) = G_{s^*}(\hbar)$. Therefore, from eq. (24), we obtain

$$\begin{aligned}&[\gamma_s - \omega\Lambda_s]G_s'''(\hbar) + [\omega\kappa\beta_s + C_0\beta_s - \omega - 2\kappa\alpha_s\\&\quad - 3\gamma_s\kappa^2 + \omega\kappa^2\Lambda_s + 2C_0\kappa\Lambda_s]G_s'(\hbar)\\&\quad - (2\varepsilon_s + \eta_s + 3\delta_s + 2\tau_s + r_s\\&\quad + 3\sigma_s)G_s^2(\hbar)G_s'(\hbar) = 0.\end{aligned} \tag{25}$$

Putting the linearly independent function coefficients in eq. (25) to zero, we thus have

$$\omega = \frac{3\gamma_s\kappa^2 - 2C_0\kappa\Lambda_s + 2\kappa\alpha_s - C_0\beta_s}{\kappa\beta_s + \Lambda_s\kappa^2 - 1} \tag{26}$$

and

$$\omega = \frac{\gamma_s}{\Lambda_s}, \tag{27}$$

together with the constraint conditions

$$2\varepsilon_s + \eta_s + 3\delta_s + 2\tau_s + r_s + 3\sigma_s = 0, \ \ s = 1,\ 2. \tag{28}$$

Equations (26) and (27) give also the constraints

$$\begin{aligned}&3\Lambda_s\gamma_s\kappa^2 - 2C_0\kappa\Lambda_s^2 + 2\alpha_s\Lambda_s\kappa - C_0\beta_s\Lambda_s\\&\quad - \kappa\beta_s\gamma_s - \kappa^2\Lambda_s\gamma_s + \gamma_s = 0,\ s = 1,\ 2.\end{aligned} \tag{29}$$

Therefore, from eq. (23), we get the following ODE:

$$\begin{aligned}&[\alpha_s - \omega\beta_s + 3\kappa\gamma_s - (2\kappa\omega + C_0)\Lambda_s]G_s''(\hbar)\\&\quad - [-C_0\kappa(\beta_s + \kappa\Lambda_s) + \gamma_s\kappa^3 + C_0 + \alpha_s\kappa^2]G_s(\hbar)\\&\quad - \kappa(\delta_s + \eta_s + r_s + \sigma_s)G_s^3(\hbar) = 0.\end{aligned} \tag{30}$$

Hence, we aim to solve eq. (30) to find the optical solitons and travelling wave solutions of the Biswas–Arshed model in birefringent fibres without FWM, namely the system of eqs (18) and (19). To this end, let us rewrite eq. (30) in the form

$$\Delta_{0s}G_s''(\hbar) + \Delta_{1s}G_s(\hbar) + \Delta_{3s}G_s^3(\hbar) = 0, \tag{31}$$

where

$$\begin{cases}\Delta_{0s} = \alpha_s - \omega\beta_s + 3\kappa\gamma_s - (2\kappa\omega + C_0)\Lambda_s,\\ \Delta_{1s} = -\left[-C_0\kappa(\beta_s + \kappa\Lambda_s) + \gamma_s\kappa^3 + C_0 + \alpha_s\kappa^2\right],\\ \Delta_{3s} = -\kappa(\delta_s + \eta_s + r_s + \sigma_s).\end{cases} \tag{32}$$

3.3 *Solutions of eq.* (31) *via the extended simplest equation method*

According to the regime of the extended simplest equation method, we have the balance number $M = 1$, by adopting the homogeneous balance technique to eq. (31). Therefore, eq. (31) has the general solution

$$G_s(\hbar) = A_0 + A_1\left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right) + B_1\left(\frac{1}{\tau(\hbar)}\right), \tag{33}$$

where A_0, A_1 and B_1 are constants, provided $A_1B_1 \neq 0$ and the function $\tau(\hbar)$ satisfies the linear ODE (6). Accordingly, we have the following distinct types of solutions:

Type 1. If $\lambda \prec 0$. Here, we put (33) into eq. (31) and using eqs (6) and (8), setting all terms with the same order of $(1/\tau(\hbar))^j \left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ $(i = 0, 1, 2, 3, 4;\ j = 0, 1, 2, 3)$ to zero, we obtain nonlinear algebraic equations. With the assistance of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$A_0 = 0,\ A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}},$$

$$B_1 = \theta\sqrt{\frac{4\Delta_{1s}^2(S_1^2 - S_2^2) - \Lambda^2\Delta_{0s}^2}{4\Delta_{1s}\Delta_{3s}}},$$

$$\lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}},\ \Lambda = \Lambda, \tag{34}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{1s}\Delta_{3s}(4\Delta_{1s}^2(S_1^2 - S_2^2) - \Lambda^2\Delta_{0s}^2) \succ 0$ and $\theta = \pm 1$.

From (7), (20), (21), (33) and (34), we have the hyperbolic solutions of eqs (18) and (19) in the form

$$\begin{cases} U(x,t) = \theta\left(\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\left[\frac{S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \frac{\Lambda\Delta_{01}}{2\Delta_{11}}}\right]\right. \\ \qquad \left. + \frac{\sqrt{\frac{4\Delta_{11}^2(S_1^2-S_2^2)-\Lambda^2\Delta_{01}^2}{4\Delta_{11}\Delta_{31}}}}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \frac{\Lambda\Delta_{01}}{2\Delta_{11}}}\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \theta\left(\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\left[\frac{S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \frac{\Lambda\Delta_{02}}{2\Delta_{12}}}\right]\right. \\ \qquad \left. + \frac{\sqrt{\frac{4\Delta_{12}^2(S_1^2-S_2^2)-\Lambda^2\Delta_{02}^2}{4\Delta_{12}\Delta_{32}}}}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \frac{\Lambda\Delta_{02}}{2\Delta_{12}}}\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{35}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{1s}\Delta_{0s} \succ 0$ and $\theta = \pm 1$.

Particularly, if $S_1 = 0$, $S_2 \neq 0$ and $\Lambda = 0$ in (35), we have the combo singular soliton solutions in the form

$$\begin{cases} U(x,t) = \left(\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\left[\operatorname{csch}\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)\right.\right. \\ \qquad \left.\left. + \coth\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)\right]\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\left[\operatorname{csch}\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)\right.\right. \\ \qquad \left.\left. + \coth\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)\right]\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}. \end{cases} \tag{36}$$

Output 2.

$$A_0 = 0,\quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}},$$

$$B_1 = \theta\sqrt{-\frac{\Delta_{1s}}{\Delta_{3s}}(S_1^2 - S_2^2)},\ \lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}},\quad \Lambda = 0, \tag{37}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{1s}\Delta_{3s}(S_1^2 - S_2^2) \prec 0$ and $\theta = \pm 1$.

From (7), (20), (21), (33) and (37), we attain the following solutions of hyperbolic type of eqs (18) and (19) in the form

$$\begin{cases} U(x,t)=\left(\dfrac{\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\left[S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)+\sqrt{-(S_1^2-S_2^2)}\right]}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)}\right)\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t)=\left(\dfrac{\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\left[S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)+\sqrt{-(S_1^2-S_2^2)}\right]}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)}\right)\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{38}$$

provided $\Delta_{1s}\Delta_{3s}\prec 0$, $\Delta_{1s}\Delta_{0s}\succ 0$ and $\theta=\pm 1$.

Output 3.

$$A_0=0,\quad A_1=0,\quad B_1=\theta\sqrt{\frac{2\Delta_{1s}}{\Delta_{3s}}(S_1^2-S_2^2)},$$
$$\lambda=\frac{\Delta_{1s}}{\Delta_{0s}},\quad \Lambda=0, \tag{39}$$

provided $\Delta_{1s}\Delta_{3s}(S_1^2-S_2^2)\succ 0$ and $\theta=\pm 1$. From (7), (20), (21), (33) and (39), we attain the following solutions of hyperbolic type:

$$\begin{cases} U(x,t)=\left(\dfrac{\theta\sqrt{-\frac{2\Delta_{11}}{\Delta_{31}}(S_1^2-S_2^2)}}{S_1\cosh\left(\hbar\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}\right)+S_2\sinh\left(\hbar\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}\right)}\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t)=\left(\dfrac{\theta\sqrt{-\frac{2\Delta_{12}}{\Delta_{32}}(S_1^2-S_2^2)}}{S_1\cosh\left(\hbar\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}\right)+S_2\sinh\left(\hbar\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}\right)}\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{40}$$

provided $\Delta_{0s}\Delta_{1s}\prec 0$, $\Delta_{1s}\Delta_{3s}(S_1^2-S_2^2)\prec 0$ and $\theta=\pm 1$.

If $S_1=0$ and $S_2\neq 0$, in (40), we attain the following solutions of singular type:

$$\begin{cases} U(x,t)=\left(\theta\sqrt{\frac{2\Delta_{11}}{\Delta_{31}}}\left[\operatorname{csch}\left(\hbar\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}\right)\right]\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t)=\left(\theta\sqrt{\frac{2\Delta_{12}}{\Delta_{32}}}\left[\operatorname{csch}\left(\hbar\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}\right)\right]\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{41}$$

and if $S_2=0$ and $S_1\neq 0$, in (40), we have the bright soliton solution in the form

$$\begin{cases} U(x,t)=\left(\theta\sqrt{-\frac{2\Delta_{11}}{\Delta_{31}}}\left[\operatorname{sech}\left(\hbar\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}\right)\right]\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t)=\left(\theta\sqrt{-\frac{2\Delta_{12}}{\Delta_{32}}}\left[\operatorname{sech}\left(\hbar\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}\right)\right]\right)\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}. \end{cases} \tag{42}$$

Output 4.

$$A_0=0,$$
$$A_1=\theta\sqrt{-\frac{2\Delta_{0s}}{\Delta_{3s}}},\quad B_1=0,$$
$$\lambda=-\frac{\Delta_{1s}}{2\Delta_{0s}},\quad \Lambda=0, \tag{43}$$

provided $\Delta_{0s}\Delta_{3s}\prec 0$ and $\theta=\pm 1$.

From (7), (20), (21), (33) and (43), we attain the following solutions of hyperbolic type of eqs (18) and (19):

$$\begin{cases} U(x,t)=\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\times\left[\dfrac{S_1\sinh\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)+S_2\cosh\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)+S_2\sinh\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)}\right]\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t)=\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\times\left[\dfrac{S_1\sinh\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)+S_2\cosh\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)+S_2\sinh\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)}\right]\times\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{44}$$

provided $\Delta_{1s}\Delta_{3s}\prec 0$, $\Delta_{0s}\Delta_{1s}\succ 0$ and $\theta=\pm 1$.

If $S_1 = 0$ and $S_2 \neq 0$, in (44), we attain the following dark soliton solutions of singular type:

$$\begin{cases} U(x,t) &= \left\{\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\coth\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)\right\} \\ &\quad\times e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) &= \left\{\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\coth\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)\right\} \\ &\quad\times e^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{45}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \succ 0$ and $\theta = \pm 1$.

If $S_2 = 0$ and $S_1 \neq 0$, in (44), we have the dark soliton solution

$$\begin{cases} U(x,t) &= \left\{\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\tanh\left(\hbar\sqrt{\frac{\Delta_{11}}{2\Delta_{01}}}\right)\right\} \\ &\quad\times e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) &= \left\{\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\tanh\left(\hbar\sqrt{\frac{\Delta_{12}}{2\Delta_{02}}}\right)\right\} \\ &\quad\times e^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{46}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \succ 0$ and $\theta = \pm 1$.

Output 5.

$$\begin{aligned} &A_0 = 0, \\ &A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}}, \\ &B_1 = 0, \\ &\Lambda = \frac{2\theta\Delta_{1s}}{\Delta_{0s}}\sqrt{-(S_1^2 - S_2^2)}, \\ &\lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}}, \end{aligned} \tag{47}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$ and $\theta = \pm 1$.

From (7), (20), (21), (33) and (47), we attain the following solutions of hyperbolic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}}\left[\frac{S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \theta\sqrt{S_1^2 - S_2^2}}\right]e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) = \theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}}\left[\frac{S_1\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)}{S_1\cosh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\sinh\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \theta\sqrt{S_1^2 - S_2^2}}\right]e^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{48}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \succ 0$ and $\theta = \pm 1$.

If $S_2 = 0$ and $S_1 \neq 0$, in (48), we have the combo singular soliton solution in the form

$$\begin{cases} U(x,t) = \theta\sqrt{-\frac{\Delta_{11}}{\Delta_{31}}} \\ \qquad\times\left[\frac{1}{\coth\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \theta\,\mathrm{csch}\left(\hbar\sqrt{\frac{2\Delta_{11}}{\Delta_{01}}}\right)}\right] \\ \qquad\times e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) = \theta\sqrt{-\frac{\Delta_{12}}{\Delta_{32}}} \\ \qquad\times\left[\frac{1}{\coth\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \theta\,\mathrm{csch}\left(\hbar\sqrt{\frac{2\Delta_{12}}{\Delta_{02}}}\right)}\right] \\ \qquad\times e^{i(-\kappa x+C_0t+\Omega_0)}. \end{cases} \tag{49}$$

Type 2. If $\lambda \succ 0$. Here, we put (33) into eq. (31) and using eqs (6) and (10), setting all terms with the same order of $(1/\tau(\hbar))^j\ \left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ ($i = 0, 1, 2, 3, 4$; $j = 0, 1, 2, 3$) to zero, we attain nonlinear algebraic equations. With the assistance of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$\begin{aligned} &A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}}, \\ &B_1 = \theta\sqrt{\frac{4\Delta_{1s}^2(S_1^2 + S_2^2) - \Lambda^2\Delta_{0s}^2}{4\Delta_{1s}\Delta_{3s}}}, \\ &\lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}}, \quad \Lambda = \Lambda, \end{aligned} \tag{50}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{1s}\Delta_{3s}(4\Delta_{1s}^2(S_1^2+S_2^2) - \Lambda^2\Delta_{0s}^2) \succ 0$ and $\theta = \pm 1$. From (9), (20), (21), (33) and (50), we obtain the following solution of periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\left(\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)-\frac{\Lambda\Delta_{01}}{2\Delta_{11}}}\right]\right. \\ \left. +\dfrac{\sqrt{\frac{4\Delta_{11}^2(S_1^2+S_2^2)-\Lambda^2\Delta_{01}^2}{4\Delta_1\Delta_3}}}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)-\frac{\Lambda\Delta_{01}}{2\Delta_{11}}}\right)\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \theta\left(\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)-\frac{\Lambda\Delta_{02}}{2\Delta_{12}}}\right]\right. \\ \left. +\dfrac{\sqrt{\frac{4{\Delta_{11}}^2(S_1^2+S_2^2)-\Lambda^2\Delta_{01}^2}{4\Delta_1\Delta_3}}}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)-\frac{\Lambda\Delta_{02}}{2\Delta_{12}}}\right)\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{51}$$

provided $\Delta_{1s}\Delta_{3s} \succ 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

If $S_2 = 0$, $S_1 \neq 0$ and $\Lambda = 0$, in (51), we obtain the following solutions of the periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\sec\left(\hbar\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{01}}}\right)\right. \\ \qquad\left. -\tan\left(\hbar\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{01}}}\right)\right]\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\left[\sec\left(\hbar\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{02}}}\right)\right. \\ \qquad\left. -\tan\left(\hbar\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{02}}}\right)\right]\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}. \end{cases} \tag{52}$$

Output 2.

$$A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}},$$

$$B_1 = \theta\sqrt{\frac{\Delta_{1s}}{\Delta_{3s}}(S_1^2+S_2^2)},$$

$$\lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}}, \quad \Lambda = 0, \tag{53}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{1s}\Delta_{3s}(S_1^2+S_2^2) \succ 0$ and $\theta = \pm 1$.

From (9), (20), (21), (33) and (53), we attain the following solutions of the periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+\sqrt{S_1^2+S_2^2}}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)}\right]\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+\sqrt{S_1^2+S_2^2}}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)+S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)}\right]\mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{54}$$

provided $\Delta_{1s}\Delta_{3s} \succ 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

If $S_1 = 0$ and $S_2 \neq 0$ in (54), we obtain the periodic wave solutions in the form

$$\begin{cases} U(x,t) = \theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\cot\left(\hbar\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{01}}}\right) + \csc\left(\hbar\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{01}}}\right)\right]\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\left[\cot\left(\hbar\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{02}}}\right) + \csc\left(\hbar\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{02}}}\right)\right]\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{55}$$

but if $S_2 = 0$ and $S_1 \neq 0$ in (54), we obtain the periodic wave solution.

Output 3.

$$A_0 = 0, \quad A_1 = 0, \quad B_1 = \theta\sqrt{-\frac{2\Delta_{1s}}{\Delta_{3s}}(S_1^2 + S_2^2)},$$
$$\lambda = \frac{\Delta_{1s}}{\Delta_{0s}}, \quad \Lambda = 0, \tag{56}$$

provided $\Delta_{1s}\Delta_{3s}(S_1^2 + S_2^2) \prec 0$ and $\theta = \pm 1$. From (9), (20), (21), (33) and (56), we obtain the following solutions of the periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \left(\dfrac{\theta\sqrt{-\frac{2\Delta_{11}}{\Delta_{31}}(S_1^2 + S_2^2)}}{S_1\cos\left(\hbar\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}\right) + S_2\sin\left(\hbar\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}\right)}\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\dfrac{\theta\sqrt{-\frac{2\Delta_{12}}{\Delta_{32}}(S_1^2 + S_2^2)}}{S_1\cos\left(\hbar\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}\right) + S_2\sin\left(\hbar\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}\right)}\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{57}$$

provided $\Delta_{0s}\Delta_{1s} \succ 0$, $\Delta_{1s}\Delta_{3s}(S_1^2 + S_2^2) \prec 0$ and $\theta = \pm 1$.

If $S_1 = 0$ and $S_2 \neq 0$, in (57), we have the singular periodic solution in the form

$$\begin{cases} U(x,t) = \left(\theta\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{31}}}\left[\csc\left(\hbar\sqrt{\dfrac{\Delta_{11}}{\Delta_{01}}}\right)\right]\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\theta\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{32}}}\left[\csc\left(\hbar\sqrt{\dfrac{\Delta_{12}}{\Delta_{02}}}\right)\right]\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{58}$$

but, if $S_2 = 0$ and $S_1 \neq 0$, in (57), we have the periodic solution in the form

$$\begin{cases} U(x,t) = \left(\theta\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{31}}}\left[\sec\left(\hbar\sqrt{\dfrac{\Delta_{11}}{\Delta_{01}}}\right)\right]\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\theta\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{32}}}\left[\sec\left(\hbar\sqrt{\dfrac{\Delta_{12}}{\Delta_{02}}}\right)\right]\right) \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}. \end{cases} \tag{59}$$

Output 4.

$$A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{2\Delta_{0s}}{\Delta_{3s}}},$$
$$B_1 = 0, \lambda = -\frac{\Delta_{1s}}{2\Delta_{0s}}, \quad \Lambda = 0, \tag{60}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$ and $\theta = \pm 1$.

From (9), (20), (21), (33) and (60), we attain the following solutions of the periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}} \times \left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{\Delta_{11}}{2\Delta_{01}}}\right) + S_2\cos\left(\hbar\sqrt{-\frac{\Delta_{11}}{2\Delta_{01}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{\Delta_{11}}{2\Delta_{01}}}\right) + S_2\sin\left(\hbar\sqrt{-\frac{\Delta_{11}}{2\Delta_{01}}}\right)}\right] \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}} \times \left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{\Delta_{12}}{2\Delta_{02}}}\right) + S_2\cos\left(\hbar\sqrt{-\frac{\Delta_{12}}{2\Delta_{02}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{\Delta_{12}}{2\Delta_{02}}}\right) + S_2\sin\left(\hbar\sqrt{-\frac{\Delta_{12}}{2\Delta_{02}}}\right)}\right] \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{61}$$

provided $\Delta_{1s}\Delta_{3s} \succ 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

If $S_1 = 0$ and $S_2 \neq 0$, in (61), we have the periodic solution, but if we set $S_2 = 0$ and $S_1 \neq 0$, in (61), again we have attain the solution of the periodic type.

Output 5.

$$A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}},$$

$$B_1 = 0,$$

$$\lambda = -\frac{2\Delta_{1s}}{\Delta_{0s}}, \quad \Lambda = \frac{2\theta\Delta_{1s}}{\Delta_{0s}}\sqrt{S_1^2 + S_2^2}, \tag{62}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$ and $\theta = \pm 1$.

From (9), (20), (21), (33) and (62), we have the following periodic solutions of eqs (18) and (19):

$$\begin{cases} U(x,t) = \theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right) + S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \theta\sqrt{S_1^2+S_2^2}}\right] \\ \quad \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\left[\dfrac{-S_1\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)}{S_1\cos\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right) + S_2\sin\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \theta\sqrt{S_1^2+S_2^2}}\right] \\ \quad \times \mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{63}$$

provided $\Delta_{1s}\Delta_{3s} \succ 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

If $S_1 = 0$ and $S_2 \neq 0$, in (63), then we thus obtain the following solutions of the periodic type:

$$\begin{cases} U(x,t) = \left(\theta\sqrt{\dfrac{\Delta_{11}}{\Delta_{31}}}\left[\left[\dfrac{1}{\tan\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \theta\sec\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)}\right]\right]\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\theta\sqrt{\dfrac{\Delta_{12}}{\Delta_{32}}}\right. \\ \quad \times \left.\left[\left[\dfrac{1}{\tan\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \theta\sec\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)}\right]\right]\right)\mathrm{e}^{i(-\kappa x + C_0 t + \Omega_0)}, \end{cases} \tag{64}$$

but, if $S_2 = 0$ and $S_1 \neq 0$, in (63), we have the singular periodic solutions in the form

$$\begin{cases} U(x,t) = \left(\theta\sqrt{\frac{\Delta_{11}}{\Delta_{31}}} \times \left[\frac{1}{\cot\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right) - \theta csc\left(\hbar\sqrt{-\frac{2\Delta_{11}}{\Delta_{01}}}\right)}\right]\right) \\ \quad \times e^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\theta\sqrt{\frac{\Delta_{12}}{\Delta_{32}}} \times \left[\frac{1}{\cot\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right) - \theta csc\left(\hbar\sqrt{-\frac{2\Delta_{12}}{\Delta_{02}}}\right)}\right]\right) \\ \quad \times e^{i(-\kappa x + C_0 t + \Omega_0)}. \end{cases} \tag{65}$$

Type 3. If $\lambda = 0$.
Here, we put (33) into eq. (31) and using eqs (6) and (12), setting all terms with the same order of $(1/\tau(\hbar))^j \left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ ($i = 0, 1, 2, 3, 4$; $j = 0, 1, 2, 3$) to zero, we receive a system of algebraic equations. With the assistance of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$\begin{aligned} &A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}}, \\ &B_1 = \theta\sqrt{-\frac{\Delta_{0s}(S_1^2 - 2\Lambda S_2)}{2\Delta_{3s}}}, \quad \Delta_{1s} = 0, \\ &\Lambda = \Lambda, \end{aligned} \tag{66}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{3s}(S_1^2 - 2\Lambda S_2) \prec 0$ and $\theta = \pm 1$.

From (11), (20), (21), (33) and (66), we have the following solutions of eqs (18) and (19):

$$\begin{cases} U(x,t) = \left(\frac{\theta\sqrt{-\frac{\Delta_{01}}{2\Delta_{31}}}}{\frac{\Lambda}{2}\hbar^2 + S_1\hbar + S_2}\right) \times \left[\Lambda\hbar + S_1 + \sqrt{S_1^2 - 2\Lambda S_2}\right] e^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = \left(\frac{\theta\sqrt{-\frac{\Delta_{02}}{2\Delta_{32}}}}{\frac{\Lambda}{2}\hbar^2 + S_1\hbar + S_2}\right) \times \left[\Lambda\hbar + S_1 + \sqrt{S_1^2 - 2\Lambda S_2}\right] e^{i(-\kappa x + C_0 t + \Omega_0)}. \end{cases} \tag{67}$$

Output 2.

$$\begin{aligned} &A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{\Delta_{0s}}{2\Delta_{3s}}}, \quad B_1 = 0, \\ &\Delta_{1s} = 0, \Lambda = \frac{S_1^2}{2S_2}, \end{aligned} \tag{68}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$, $\Delta_{1s} = 0$ and $\theta = \pm 1$.

From (11), (20), (21), (33) and (68), we have the following solutions of eqs (18) and (19):

$$\begin{cases} U(x,t) = 2\theta\sqrt{-\frac{\Delta_{01}}{2\Delta_{31}}}\left(\frac{S_1^2\hbar + 2S_1S_2}{S_1^2\hbar^2 + 4S_1S_2\hbar + 4S_2^2}\right) \\ \quad \times e^{i(-\kappa x + C_0 t + \Omega_0)}, \\ V(x,t) = 2\theta\sqrt{-\frac{\Delta_{02}}{2\Delta_{32}}}\left(\frac{S_1^2\hbar + 2S_1S_2}{S_1^2\hbar^2 + 4S_1S_2\hbar + 4S_2^2}\right) \\ \quad \times e^{i(-\kappa x + C_0 t + \Omega_0)}. \end{cases} \tag{69}$$

3.4 *Solutions of eq. (31) via the generalised sub-ODE equation method*

Balancing $G_s''(\hbar)$ and $G_s^3(\hbar)$ in eq. (31), we have $N + 2q = 3N$ which yields $N = q$. Hence, eq. (31) takes the essential solution

$$G_s(\hbar) = \Lambda\Psi(\hbar)^q, \Lambda \succ 0, \tag{70}$$

where $\Psi(\hbar)$ satisfies eq. (14).

The following set of algebraic equations is obtained by substituting (70) along with (14) into eq. (31) and gathering all the coefficients of $\Psi^{jq}(\hbar)\left[\Psi'(\hbar)\right]^r$ ($j = 0, 1, 2, 3$; $r = 0, 1$) as

$$\Psi(\hbar)^0 = \Delta_{0s}Bq^2 = 0, \tag{71a}$$

$$\Psi(\hbar)^q = \Delta_{0s}Cq^2 + \Delta_{1s} = 0, \tag{71b}$$

$$\Psi(\hbar)^{2q} = \Delta_{0s}Dq^2 = 0, \tag{71c}$$

$$\Psi(\hbar)^{3q} = 2\Delta_{0s}Eq^2 + \Delta_{3s}\Lambda^2 = 0. \tag{71d}$$

On solving eqs (71a)–(71d), we get

$$A=0,\quad B=0,\quad D=0,$$
$$\Lambda=\theta q\sqrt{-\frac{2\Delta_{0s}E}{\Delta_{3s}}},\quad C=\frac{-\Delta_{1s}}{\Delta_{0s}q^2},\tag{72}$$

provided $\Delta_{0s}\Delta_{3s}E\prec 0$ and $\theta=\pm1$.

With regard to the solutions of eq. (14) listed in [22–24], we thus obtain the following variety of analytical solutions drawn below:

Type 1. Since, $B=D=0$, then the following cases arise:

Case I. Equation (14) has the solution

$$\Psi(\hbar)=\left[\frac{1}{E}\wp(q\hbar,g_2,g_3)-\frac{C}{3E}\right]^{\frac{1}{2q}},\tag{73}$$

where

$$g_2=\frac{4C^2-12AE}{3}$$

and

$$g_3=\frac{4C(-2C^2+9AE)}{27}.$$

Here, $\wp(q\hbar,g_2,g_3)$ is known as Weierstrass elliptic function which satisfies the ODE: $\wp'^2=4\wp^3-g_2\wp-g_3$, where g_2 and g_3 are called invariants of the Weierstrass elliptic function, in which $' := \mathrm{d}/\mathrm{d}\hbar$.

Since $A=0$, we get Weierstrass elliptic function solutions to the coupled systems (18) and (19) specified in the form

$$\begin{cases}U(x,t)\\ =\left[\theta\sqrt{-\dfrac{2(3\Delta_{01}\wp(x-\omega t+\hbar_0,g_2,g_3)+\Delta_{11})}{3\Delta_{31}}}\right]\\ \times\,\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)},\\ V(x,t)\\ =\left[\theta\sqrt{-\dfrac{2(3\Delta_{02}\wp(x-\omega t+\hbar_0,g_2,g_3)+\Delta_{12})}{3\Delta_{32}}}\right]\\ \times\,\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)},\end{cases}\tag{74}$$

provided $\Delta_{3s}\prec 0$ and $\theta=\pm1$, where $g_2=4\Delta_{1s}^2/3\Delta_{0s}^2$ and $g_3=8\Delta_{1s}^3/27\Delta_{0s}^3$.

The Weierstrass elliptic function, in particular, can be expressed in terms of the Jacobi elliptic functions *cn* and *ns* as [48]

$$\wp(q\hbar,g_2,g_3)=e_2-(e_2-e_3)cn^2(R\hbar;m),\tag{75}$$
$$\wp(q\hbar,g_2,g_3)=[e_3+(e_1-e_3)ns^2(R\hbar;m)],\tag{76}$$

where $R=\sqrt{e_1-e_3}$, $m^2=(e_2-e_3)/(e_1-e_3)$ is the modulus of the Jacobi elliptic functions *cn* and $ns=1/sn$ such that $0\prec m\prec 1$, e_i $(i=1,2,3;\ e_1\geq e_2\geq e_3)$ represent the three roots of the cubic equation

$$4z^3-g_2z+g_3=0.\tag{77}$$

Therefore, solution (74) can be rewritten as

$$\begin{cases}U(x,t)=\left[\theta\sqrt{-\dfrac{2(3\Delta_{01}[e_2-(e_2-e_3)cn^2(R\hbar;m)]+\Delta_{11})}{3\Delta_{31}}}\right]\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)},\\ V(x,t)=\left[\theta\sqrt{-\dfrac{2(3\Delta_{02}[e_2-(e_2-e_3)cn^2(R\hbar;m)]+\Delta_{12})}{3\Delta_{32}}}\right]\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)},\end{cases}\tag{78}$$

$$\begin{cases}U(x,t)=\left[\theta\sqrt{-\dfrac{2(3\Delta_{01}[e_3+(e_1-e_3)ns^2(R\hbar;m)]+\Delta_{11})}{3\Delta_{31}}}\right]\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)},\\ V(x,t)=\left[\theta\sqrt{-\dfrac{2(3\Delta_{02}[e_3+(e_1-e_3)ns^2(R\hbar;m)]+\Delta_{12})}{3\Delta_{32}}}\right]\mathrm{e}^{i(-\kappa x+C_0t+\Omega_0)}.\end{cases}\tag{79}$$

If $m \to 1$, (i.e. $e_2 \to e_1$), then $cn(R\hbar; m) \to \operatorname{sech}(R\hbar)$ and $ns(R\hbar; m) \to \coth(R\hbar)$, and therefore we obtain soliton solutions of bright and singular types of eqs (18) and (19) as

$$\begin{cases} U(x,t) = \left[\theta\sqrt{-\dfrac{2(3\Delta_{01}[e_1-(e_1-e_3)\operatorname{sech}^2(R\hbar)]+\Delta_{11})}{3\Delta_{31}}}\right] e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{-\dfrac{2(3\Delta_{02}[e_1-(e_1-e_3)\operatorname{sech}^2(R\hbar)]+\Delta_{12})}{3\Delta_{32}}}\right] e^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{80}$$

$$\begin{cases} U(x,t) = \left[\theta\sqrt{-\dfrac{2(3\Delta_{01}[e_3+(e_1-e_3)\coth^2(R\hbar)]+\Delta_{11})}{3\Delta_{31}}}\right] e^{i(-\kappa x+C_0t+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{-\dfrac{2(3\Delta_{02}[e_3+(e_1-e_3)\coth^2(R\hbar)]+\Delta_{12})}{3\Delta_{32}}}\right] e^{i(-\kappa x+C_0t+\Omega_0)}, \end{cases} \tag{81}$$

respectively.

Case II. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{3\sqrt{E^{-1}}\wp'(q\hbar, g_2, g_3)}{6\wp(q\hbar, g_2, g_3)+C}\right]^{\frac{1}{p}}, \quad g_2 = \frac{C^2}{12}+AE,$$
$$g_3 = \frac{C(36AE-C^2)}{216}. \tag{82}$$

Since $A = 0$, we attain solutions of Weierstrass elliptic function type of eqs (18) and (19) as

$$\begin{cases} U(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{01}^3}{\Delta_{31}}}\left(\dfrac{3\wp'(x-\omega t+\hbar_0, g_2, g_3)}{6\Delta_{01}\wp(x-\omega t+\hbar_0, g_2, g_3)-\Delta_{11}}\right)\right] e^{i(-kx+ct+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{02}^3}{\Delta_{32}}}\left(\dfrac{3\wp'(x-\omega t+\hbar_0, g_2, g_3)}{6\Delta_{02}\wp(x-\omega t+\hbar_0, g_2, g_3)-\Delta_{12}}\right)\right] e^{i(-kx+ct+\Omega_0)}, \end{cases} \tag{83}$$

provided $\Delta_{0s}\Delta_{3s} \prec 0$ and $\theta = \pm 1$, where

$$g_2 = \frac{\Delta_{1s}^2}{12\Delta_{0s}^2} \quad \text{and} \quad g_3 = \frac{\Delta_{1s}^3}{216\Delta_{0s}^3}.$$

Type 2. Since $A = B = 0$, then we have the following cases:

Case I. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{2C\operatorname{sech}^2\left(\frac{q\sqrt{C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}+D)\operatorname{sech}^2\left(\frac{p\sqrt{C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$
$$C \succ 0, \quad \Delta = D^2-4CE \succ 0. \tag{84}$$

Since $D = 0$, we attain the solutions of hyperbolic function type of eqs (18) and (19) as

$$\begin{cases} U(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{31}}} \right. \\ \quad \times \left.\left(\dfrac{1-\tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}{1+\tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{32}}} \right. \\ \quad \times \left.\left(\dfrac{1-\tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}{1+\tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{85}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

Case II. Equation (14) provides the solution

$$\Psi(\hbar) = \left[\frac{2C\,\mathrm{csch}^2\left(\frac{\theta q\sqrt{C}}{2}\hbar\right)}{2\sqrt{\Delta}+(\sqrt{\Delta}-D)\,\mathrm{csch}^2\left(\frac{\theta q\sqrt{C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$
$$C \succ 0, \quad \Delta = D^2-4CE \succ 0, \quad \theta = \pm 1. \tag{86}$$

Since $D = 0$, we attain the solutions of the hyperbolic function type of eqs (18) and (19) as

$$\begin{cases} U(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{11}}{\Delta_{31}}} \right. \\ \quad \times \left.\left(\dfrac{-1+\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}{1+\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{-\dfrac{2\Delta_{12}}{\Delta_{32}}} \right. \\ \quad \times \left.\left(\dfrac{-1+\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}{1+\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{87}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

Case III. Equation (14) provides the solution

$$\Psi(\hbar) = \left[-\frac{2C\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}-D)\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$
$$C \prec 0, \quad \Delta = D^2-4CE \succ 0. \tag{88}$$

Since $D = 0$, we attain the following wave solutions of the periodic type of eqs (18) and (19) as

$$\begin{cases} U(x,t) = \left[\theta\sqrt{\dfrac{-2\Delta_{11}}{\Delta_{31}}} \right. \\ \quad \times \left.\left(\dfrac{1+\tan^2\left(\frac{1}{2}\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}{1-\tan^2\left(\frac{1}{2}\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left[\theta\sqrt{\dfrac{-2\Delta_{12}}{\Delta_{32}}} \right. \\ \quad \times \left.\left(\dfrac{1+\tan^2\left(\frac{1}{2}\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}{1-\tan^2\left(\frac{1}{2}\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}\right)\right] \\ \quad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{89}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \succ 0$ and $\theta = \pm 1$.

Case IV. Equation (14) provides the solution

$$\Psi(\hbar) = \left[\frac{2C\csc^2\left(\frac{\theta q\sqrt{-C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}+D)\csc^2\left(\frac{\theta q\sqrt{-C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$
$$C \prec 0, \quad \Delta = D^2-4CE \succ 0, \quad \theta = \pm 1. \tag{90}$$

Since $D = 0$, we attain the following wave solutions of the singular periodic type of eqs (18) and (19):

$$\begin{cases} U(x,t) = \left[\theta_1 \sqrt{\dfrac{-2\Delta_{11}}{\Delta_{31}}} \right. \\ \qquad \times \left. \left(\dfrac{1+\cot^2\left(\frac{\theta}{2}\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)}{1-\cot^2\left(\frac{\theta}{2}\sqrt{\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)\right)} \right) \right] \\ \qquad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left[\theta_1 \sqrt{\dfrac{-2\Delta_{12}}{\Delta_{32}}} \right. \\ \qquad \times \left. \left(\dfrac{1+\cot^2\left(\frac{\theta}{2}\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)}{1-\cot^2\left(\frac{\theta}{2}\sqrt{\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)\right)} \right) \right] \\ \qquad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{91}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$, $\Delta_{0s}\Delta_{1s} \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$.

Case V. Equation (14) has the solution

$$\Psi(\hbar) = \left[-\frac{C\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)}{D + 2\theta\sqrt{-CE}\tan\left(\frac{q\sqrt{-C}}{2}\hbar\right)} \right]^{\frac{1}{q}},$$
$$C \prec 0, \quad E \succ 0, \quad \theta = \pm 1. \tag{92}$$

$$\begin{cases} U(x,t) = \left[\sqrt{\dfrac{-\Delta_{11}}{2\Delta_{31}}} \sec\left(\dfrac{1}{2}\sqrt{\dfrac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0) \right) \right. \\ \qquad \times \left. \csc\left(\dfrac{1}{2}\sqrt{\dfrac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0) \right) \right] \\ \qquad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left[\sqrt{\dfrac{-\Delta_{12}}{2\Delta_{32}}} \sec\left(\dfrac{1}{2}\sqrt{\dfrac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0) \right) \right. \\ \qquad \times \left. \csc\left(\dfrac{1}{2}\sqrt{\dfrac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0) \right) \right] \\ \qquad \times \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{93}$$

provided $\Delta_{1s}\Delta_{3s} \prec 0$ and $\Delta_{0s}\Delta_{1s} \succ 0$.

Case VI. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{4Cq^2\mathrm{e}^{\theta q\sqrt{C}\hbar}}{\left(\mathrm{e}^{\theta q\sqrt{C}\hbar} - Dq^2\right)^2 - 4CEq^4} \right]^{\frac{1}{q}},$$
$$C \succ 0, \quad \theta = \pm 1. \tag{94}$$

Since $D = 0$, we obtain the solutions of the following form:

$$\begin{cases} U(x,t) = \left(\dfrac{4\Delta_{11}\theta q\sqrt{\frac{-2\Delta_{01}E}{\Delta_{31}}}}{\Delta_{01}\mathrm{e}^{\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)} + 4\Delta_{11}Eq^2\mathrm{e}^{-\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)}} \right) \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left(\dfrac{4\Delta_{12}\theta q\sqrt{\frac{-2\Delta_{02}E}{\Delta_{32}}}}{\Delta_{02}\mathrm{e}^{\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)} + 4\Delta_{12}Eq^2\mathrm{e}^{-\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)}} \right) \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{95}$$

provided $\Delta_{0s}\Delta_{3s}E \prec 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

Since $D = 0$, we attain the following wave solutions of the periodic type of eqs (18) and (19) as

Case VII. Equation (14) provides the solution

$$\Psi(\hbar) = \left[\frac{4Cq^2\mathrm{e}^{\theta q\sqrt{C}\hbar}}{-1 + 4CEq^4\mathrm{e}^{2\theta q\sqrt{C}\hbar}} \right]^{\frac{1}{q}},$$
$$C \succ 0, \quad D = 0, \quad \theta = \pm 1. \tag{96}$$

Then, we have

$$\begin{cases} U(x,t) = \left(\dfrac{4\Delta_{11}\theta q\sqrt{\frac{-2\Delta_{01}E}{\Delta_{31}}}}{\Delta_{01}\mathrm{e}^{-\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)} + 4\Delta_{11}Eq^2\mathrm{e}^{\theta\sqrt{-\frac{\Delta_{11}}{\Delta_{01}}}(x-\omega t+\hbar_0)}} \right) \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \\ V(x,t) = \left(\dfrac{4\Delta_{12}\theta q\sqrt{\frac{-2\Delta_{02}E}{\Delta_{32}}}}{\Delta_{02}\mathrm{e}^{-\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)} + 4\Delta_{12}Eq^2\mathrm{e}^{\theta\sqrt{-\frac{\Delta_{12}}{\Delta_{02}}}(x-\omega t+\hbar_0)}} \right) \mathrm{e}^{i(-\kappa x+C_0 t+\Omega_0)}, \end{cases} \tag{97}$$

provided $\Delta_{0s}\Delta_{3s}E \prec 0$, $\Delta_{0s}\Delta_{1s} \prec 0$ and $\theta = \pm 1$.

4. NLSE in birefringent fibres with quadratic–cubic (QC) law of refractive index along with FWM

4.1 *The governing model*

The governing NLSE with quadratic–cubic nonlinearity for polarisation-preserving fibres has the form [57–60]

$$i\psi_t + a\psi_{xx} + (k_1|\psi| + k_2|\psi|^2)\psi = 0, \tag{98}$$

where $\psi(x,t)$ is the complex-valued wave function which stands for optical solitons, a signifies the coefficient of group velocity dispersion and k_1 and k_2 represent quadratic and cubic nonlinearities, while $i = \sqrt{-1}$.

In birefringent fibres with pulses divided into two, then the corresponding coupled vector NLSE with quadratic cubic law of refractive index along with FWM can be specified in the form

$$i\psi_t + a_1\psi_{xx} + k_1\psi\sqrt{|\psi|^2 + |\phi|^2 + \psi\phi^* + \psi^*\phi} + (\ell_1|\psi|^2 + m_1|\phi|^2)\psi + n_1\phi^2\psi^* = 0, \tag{99}$$

$$i\phi_t + a_2\phi_{xx} + k_2\phi\sqrt{|\phi|^2 + |\psi|^2 + \psi^*\phi + \psi\phi^*} + (\ell_2|\phi|^2 + m_2|\psi|^2)\phi + n_2\psi^2\phi^* = 0, \tag{100}$$

where ℓ_i and m_i $(i = 1, 2)$ are from self-phase modulation and cross-phase modulation, respectively, while n_i give the effect of FWM. Here * denotes the complex conjugate. The QC nonlinearity first appeared in 1994 [56] and the references therein. Other types of nonlinearity can be found in refs [61–63].

4.2 *Mathematical analysis*

Gomez-Aguilar *et al* [57] studied the system of eqs (99) and (100) by a variety of analytical methods. Therefore, we also conduct our analysis to further investigate the coupled systems (99) and (100) by using the proposed approaches of this paper to extract its other travelling wave solutions and other optical solitons in a concise manner. To this aim, we introduce the following wave transformation:

$$\psi(x,t) = H_1(\hbar)\mathrm{e}^{iG(x,t)}, \tag{101}$$

$$\phi(x,t) = H_2(\hbar)\mathrm{e}^{iG(x,t)} \tag{102}$$

and

$$\hbar = x - \beta t, \tag{103}$$

$$G(x,t) = -\kappa x + \omega t + \tau, \tag{104}$$

where β, κ, ω and τ are non-zero constants to be evaluated which stand for the velocity of soliton, frequency of solitons for the two components, soliton wave number and phase constants, respectively. Also, $H_1(\hbar)$, $H_2(\hbar)$ are real functions denoting the amplitudes of solitons for the two components while $G(x,t)$ denotes phase component of the soliton. Replacing eqs (101) and (102) into eqs (99) and (100) and separating into real and imaginary parts, we have

$$a_s H_s'' - (\omega + a_s\kappa^2)H_s + k_s H_s^2 + k_s H_s H_{s^*} + \ell_s H_s^3 + (m_s + n_s)H_s H_{s^*}^2 = 0, \tag{105}$$

$$(2a_s\kappa + \beta)\, H_s' = 0, \tag{106}$$

for $s = 1, 2$ and $s^* = 3 - s$. From eq. (106), the velocity of the soliton is obtained as

$$\beta = -2a_s\kappa. \tag{107}$$

On comparing the formulas of soliton velocity, we then have

$$a_1 = a_2 = a. \tag{108}$$

Therefore, soliton velocity can be written as

$$\beta = -2a\kappa. \tag{109}$$

As a result, the real part eq. (105) becomes

$$aH_s'' - (\omega + a\kappa^2)H_s + k_s H_s^2 + k_s H_s H_{s^*} + \ell_s H_s^3 + (m_s + n_s)H_s H_{s^*}^2 = 0. \tag{110}$$

Now, by the balancing procedure, we obtain

$$H_s = H_{s^*}. \tag{111}$$

Hence, eq. (110) is transformed to the following form:

$$aH_s'' - (\omega + a\kappa^2)H_s + 2k_s H_s^2 + (\ell_s + m_s + n_s)H_s^3 = 0. \tag{112}$$

Let us rewrite eq. (112) as

$$aH_s''(\hbar) - (\omega + a\kappa^2)H_s(\hbar) + \Gamma_{2s}H_s^2(\hbar) + \Gamma_{3s}H_s^3(\hbar) = 0, \tag{113}$$

where

$$\Gamma_{2s} = 2k_s, \quad \Gamma_{3s} = \ell_s + m_s + n_s. \tag{114}$$

Therefore, we are concerned to consider eq. (113) by using the proposed techniques of the current paper to extract the new exact solutions of the coupled systems (99) and (100) as follows:

4.3 *Solutions of eq. (113) via the extended simplest equation method*

Similarly, according to the regime of the extended simplest equation method and by the homogeneous balance technique in eq. (113), we thus have the balance number $M = 1$. Therefore, eq. (113) has the formal solution

$$H_s(\hbar) = A_0 + A_1\left(\frac{\tau'(\hbar)}{\tau(\hbar)}\right) + B_1\left(\frac{1}{\tau(\hbar)}\right), \tag{115}$$

where A_0, A_1 and B_1 are constants that should be calculated, provided $A_1B_1 \neq 0$ and the function $\tau(\hbar)$ satisfies the linear ODE (6). Accordingly, we acquire the following distinct types of solutions:

Type 1. If $\lambda \prec 0$. Here, we replace (115) into eq. (113) and using eqs (6) and (8), setting all the terms with the same order of $(1/\tau(\hbar))^j\left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ $(i = 0, 1, 2, 3, 4;\ j = 0, 1, 2, 3)$ to zero, we attain nonlinear algebraic equations. With the assistance of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$A_0 = -\frac{\Gamma_{2s}}{3\Gamma_{3s}}, \quad A_1 = \theta\sqrt{-\frac{2a}{\Gamma_{3s}}}, \quad B_1 = 0,$$
$$\kappa = \theta\sqrt{\frac{-3\omega + 2\Gamma_{2s}A_0}{3a}},$$
$$\lambda = \frac{-15a\Gamma_{2s}A_0 + 2\Gamma_{2s}^2A_1^2}{18a^2}, \quad \Lambda = 0, \tag{116}$$

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-3\omega + 2\Gamma_{2s}A_0) \succ 0$ and $\theta = \pm 1$.

From (7), (101), (102), (115) and (116), we obtain the following wave solutions of the hyperbolic type of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \left[-\dfrac{\Gamma_{21}}{3\Gamma_{31}}\left\{1 + \theta_1\left(\dfrac{C_1\sinh\left(\hbar\sqrt{-\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right) + C_2\cosh\left(\hbar\sqrt{-\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right) + C_2\sinh\left(\hbar\sqrt{-\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right)}\right)\right\}\right] \\ \qquad \times \mathrm{e}^{i\left[-\kappa x + \left(-\frac{2\Gamma_{21}^2}{9\Gamma_{31}} - a\kappa^2\right)t + \tau\right]}, \\ \phi(x,t) = \left[-\dfrac{\Gamma_{22}}{3\Gamma_{32}}\left\{1 + \theta_1\left(\dfrac{C_1\sinh\left(\hbar\sqrt{-\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right) + C_2\cosh\left(\hbar\sqrt{-\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right) + C_2\sinh\left(\hbar\sqrt{-\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right)}\right)\right\}\right] \\ \qquad \times \mathrm{e}^{i\left[-\kappa x + \left(-\frac{2\Gamma_{22}^2}{9\Gamma_{32}} - a\kappa^2\right)t + \tau\right]}, \end{cases} \tag{117}$$

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s}\Gamma_{2s}^2 \prec 0$ and $\theta_1 = \mp 1$.

Output 2.

$$A_0 = -\frac{\Gamma_{2s}}{3\Gamma_{3s}}, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0,$$
$$\kappa = \theta\sqrt{\frac{-3\omega + 2\Gamma_{2s}A_0}{3a}}, \quad \lambda = \frac{-2\Gamma_{2s}A_0}{3a},$$
$$\Lambda = \theta\lambda\sqrt{C_1^2 - C_2^2}, \tag{118}$$

provided $\Gamma_{2s}A_0 \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-3\omega+2\Gamma_{2s}A_0) \succ 0$ and $\theta = \pm 1$.

From (7), (101), (102), (115) and (118), we obtain the following wave solutions of the hyperbolic type of eqs (99) and (100):

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s}\Gamma_{2s}^2 \prec 0$, $\omega = -\frac{2\Gamma_{2s}^2}{9\Gamma_{3s}} - a\kappa^2$, $\theta_1 = \mp 1$ and $\theta = \pm 1$.

Output 3.

$$A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0, \quad \Gamma_{2s} = 0,$$
$$\lambda = \frac{2(\omega + a\kappa^2)}{a},$$
$$\Lambda = \theta\lambda\sqrt{C_1^2 - C_2^2}, \tag{120}$$

provided $\omega + a\kappa^2 \neq 0$, $a\Gamma_{3s} \prec 0$ and $\theta = \pm 1$.

$$\begin{cases} \psi(x,t) = \left[-\frac{\Gamma_{21}}{3\Gamma_{31}}\left\{1+\theta_1\left(\frac{C_1\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right)+C_2\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right)+C_2\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right)+\theta\sqrt{C_1^2-C_2^2}}\right)\right\}\right] \\ \quad\times \mathrm{e}^{i\left[-\kappa x+\left(-\frac{2\Gamma_{21}^2}{9\Gamma_{31}}-a\kappa^2\right)t+\tau\right]}, \\ \phi(x,t) = \left[-\frac{\Gamma_{22}}{3\Gamma_{32}}\left\{1+\theta_1\left(\frac{C_1\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right)+C_2\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right)+C_2\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right)+\theta\sqrt{C_1^2-C_2^2}}\right)\right\}\right] \\ \quad\times \mathrm{e}^{i\left[-\kappa x+\left(-\frac{2\Gamma_{22}^2}{9\Gamma_{32}}-a\kappa^2\right)t+\tau\right]}, \end{cases} \tag{119}$$

From (7), (101), (102), (115) and (120), we obtain the following wave solutions of the hyperbolic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\theta\sqrt{\frac{\omega+a\kappa^2}{\Gamma_{31}}}\left(\frac{C_1\sinh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+C_2\cosh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+C_2\sinh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+\theta\sqrt{C_1^2-C_2^2}}\right)\right] \\ \quad\times \mathrm{e}^{i[-\kappa x+\omega t+\tau]}, \\ \phi(x,t) = \left[\theta\sqrt{\frac{\omega+a\kappa^2}{\Gamma_{31}}}\left(\frac{C_1\sinh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+C_2\cosh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)}{C_1\cosh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+C_2\sinh\left(\hbar\sqrt{-\frac{2(\omega+a\kappa^2)}{a}}\right)+\theta\sqrt{C_1^2-C_2^2}}\right)\right] \\ \quad\times \mathrm{e}^{i[-\kappa x+\omega t+\tau]}, \end{cases} \tag{121}$$

provided $\Gamma_{3s}(\omega + a\kappa^2) \succ 0$, $a(\omega + a\kappa^2) \prec 0$, $\Gamma_{2s} = 0$ and $\theta = \pm 1$.

Output 4.

$$A_0 = A_0, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0,$$

$$\kappa = \theta\sqrt{\frac{-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2}{a}},$$

$$C_2 = \theta C_1, \quad \lambda = -\frac{A_0^2}{A_1^2}, \quad \Lambda = 0, \tag{122}$$

provided $A_0 \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2) \succ 0$ and $\theta = \pm 1$.

From (7), (101), (102), (115) and (122), we obtain the following wave solutions of the hyperbolic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[A_0\left\{1 + \theta\left(\dfrac{\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{31}}{a}}A_0\right) - \cosh\left(\hbar\sqrt{-\frac{2\Gamma_{31}}{a}}A_0\right)}{\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{31}}{a}}A_0\right) - \sinh\left(\hbar\sqrt{-\frac{2\Gamma_{31}}{a}}A_0\right)}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + (2\Gamma_{21}A_0 + 4\Gamma_{31}A_0^2 - a\kappa^2)t + \tau\right]}, \\ \phi(x,t) = \left[A_0\left\{1 + \theta\left(\dfrac{\sinh\left(\hbar\sqrt{-\frac{2\Gamma_{32}}{a}}A_0\right) - \cosh\left(\hbar\sqrt{-\frac{2\Gamma_{32}}{a}}A_0\right)}{\cosh\left(\hbar\sqrt{-\frac{2\Gamma_{32}}{a}}A_0\right) - \sinh\left(\hbar\sqrt{-\frac{2\Gamma_{31}}{a}}A_0\right)}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + (2\Gamma_{22}A_0 + 4\Gamma_{32}A_0^2 - a\kappa^2)t + \tau\right]}, \end{cases} \tag{123}$$

provided $A_0 \neq 0$, $a\Gamma_{3s} \prec 0$ and $\theta = \pm 1$.

Output 5.

$$A_0 = A_0, \quad A_1 = A_1, \quad B_1 = 0,$$

$$\kappa = \theta\sqrt{\frac{-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2}{a}},$$

$$C_2 = \theta C_1, \quad \lambda = -\frac{A_0^2}{A_1^2}, \quad \Lambda = 0, \tag{124}$$

provided $A_0 \neq 0$, $A_1 \neq 0$, $a(-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2) \succ 0$ and $\theta = \pm 1$.

From (7), (101), (102), (115) and (124), we obtain the following wave solutions of the hyperbolic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[A_0\left\{1 + \theta\left(\dfrac{\sinh\left(\hbar\frac{A_0}{A_1}\right) - \cosh\left(\hbar\frac{A_0}{A_1}\right)}{\cosh\left(\hbar\frac{A_0}{A_1}\right) - \sinh\left(\hbar\frac{A_0}{A_1}\right)}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + (2\Gamma_{21}A_0 + 4\Gamma_{31}A_0^2 - a\kappa^2)t + \tau\right]}, \\ \phi(x,t) = \left[A_0\left\{1 + \theta\left(\dfrac{\sinh\left(\hbar\frac{A_0}{A_1}\right) - \cosh\left(\hbar\frac{A_0}{A_1}\right)}{\cosh\left(\hbar\frac{A_0}{A_1}\right) - \sinh\left(\hbar\frac{A_0}{A_1}\right)}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + (2\Gamma_{22}A_0 + 4\Gamma_{32}A_0^2 - a\kappa^2)t + \tau\right]}, \end{cases} \tag{125}$$

provided $A_0 \neq 0$, $A_1 \neq 0$ and $\theta = \pm 1$.

Type 2. If $\lambda \succ 0$. Here, we substitute (115) into eq. (113) and using eqs (6) and (8), setting all terms with the same order of $(1/\tau(\hbar))^j\left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ ($i = 0, 1, 2, 3, 4$; $j = 0, 1, 2, 3$) to zero, we attain the nonlinear algebraic equations. With the help of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$A_0 = -\frac{\Gamma_{2s}}{3\Gamma_{3s}}, \quad A_1 = \theta\sqrt{-\frac{2a}{\Gamma_{3s}}}, \quad B_1 = 0, \quad \kappa = \theta\sqrt{\frac{-3\omega + 2\Gamma_{2s}A_0}{3a}},$$
$$\lambda = \frac{-15a\Gamma_{2s}A_0 + 2\Gamma_{2s}^2 A_1^2}{18a^2}, \quad \Lambda = 0, \tag{126}$$

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-3\omega + 2\Gamma_{2s}A_0) \succ 0$ and $\theta = \pm 1$.

From (9), (101), (102), (115) and (126), we attain the following wave solutions of the periodic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[-\frac{\Gamma_{21}}{3\Gamma_{31}}\left\{1 + \theta_1 i\left(\frac{-C_1\sin\left(\hbar\sqrt{\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right) + C_2\cos\left(\hbar\sqrt{\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right)}{C_1\cos\left(\hbar\sqrt{\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right) + C_2\sin\left(\hbar\sqrt{\frac{\Gamma_{21}^2}{18a\Gamma_{31}}}\right)}\right)\right\}\right] e^{i\left[-\kappa x + \left(-\frac{2\Gamma_{21}^2}{9\Gamma_{31}} - a\kappa^2\right)t + \tau\right]}, \\ \phi(x,t) = \left[-\frac{\Gamma_{22}}{3\Gamma_{32}}\left\{1 + \theta_1 i\left(\frac{-C_1\sin\left(\hbar\sqrt{\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right) + C_2\cos\left(\hbar\sqrt{\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right)}{C_1\cos\left(\hbar\sqrt{\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right) + C_2\sin\left(\hbar\sqrt{\frac{\Gamma_{22}^2}{18a\Gamma_{32}}}\right)}\right)\right\}\right] e^{i\left[-\kappa x + \left(-\frac{2\Gamma_{22}^2}{9\Gamma_{32}} - a\kappa^2\right)t + \tau\right]}, \end{cases} \tag{127}$$

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s}\Gamma_{2s}^2 \succ 0$ and $\theta_1 = \mp 1$.

Output 2.

$$A_0 = -\frac{\Gamma_{2s}}{3\Gamma_{3s}}, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0,$$
$$\kappa = \theta\sqrt{\frac{-3\omega + 2\Gamma_{2s}A_0}{3a}}, \quad \lambda = \frac{-2\Gamma_{2s}A_0}{3a}, \quad \Lambda = \theta\lambda\sqrt{C_1^2 + C_2^2}, \tag{128}$$

provided $\Gamma_{2s}A_0 \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-3\omega + 2\Gamma_{2s}A_0) \succ 0$ and $\theta = \pm 1$.

From (9), (101), (102), (115) and (128), we attain the following wave solutions of the periodic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[-\frac{\Gamma_{21}}{3\Gamma_{31}}\left\{1 + \theta_1 i\left(\frac{-C_1\sin\left(\hbar\sqrt{\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right) + C_2\cos\left(\hbar\sqrt{\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right)}{C_1\cos\left(\hbar\sqrt{\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right) + C_2\sin\left(\hbar\sqrt{\frac{2\Gamma_{21}^2}{9a\Gamma_{31}}}\right) + \theta\sqrt{C_1^2 + C_2^2}}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + \left(-\frac{2\Gamma_{21}^2}{9\Gamma_{31}} - a\kappa^2\right)t + \tau\right]}, \\ \phi(x,t) = \left[-\frac{\Gamma_{22}}{3\Gamma_{32}}\left\{1 + \theta_1 i\left(\frac{-C_1\sin\left(\hbar\sqrt{\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right) + C_2\cos\left(\hbar\sqrt{\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right)}{C_1\cos\left(\hbar\sqrt{\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right) + C_2\sin\left(\hbar\sqrt{\frac{2\Gamma_{22}^2}{9a\Gamma_{32}}}\right) + \theta\sqrt{C_1^2 + C_2^2}}\right)\right\}\right] \\ \quad \times e^{i\left[-\kappa x + \left(-\frac{2\Gamma_{22}^2}{9\Gamma_{32}} - a\kappa^2\right)t + \tau\right]}, \end{cases} \tag{129}$$

provided $\Gamma_{2s} \neq 0$, $a\Gamma_{3s}\Gamma_{2s}^2 \succ 0$, $\omega = -\frac{2\Gamma_{2s}^2}{9\Gamma_{3s}} - a\kappa^2$, $\theta_1 = \mp 1$ and $\theta = \pm 1$.

Output 3.

$$A_0 = 0, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0, \quad \Gamma_{2s} = 0, \quad \lambda = \frac{2(\omega + a\kappa^2)}{a}, \quad \Lambda = \theta\lambda\sqrt{C_1^2 + C_2^2}, \tag{130}$$

provided $\omega + a\kappa^2 \neq 0$, $a\Gamma_{3s} \prec 0$ and $\theta = \pm 1$.

From (9), (101), (102), (115) and (130), we attain the following wave solutions of the periodic type of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \left[\theta\sqrt{-\frac{\omega + a\kappa^2}{\Gamma_{31}}}\left(\frac{-C_1 \sin\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + C_2 \cos\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right)}{C_1 \cos\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + C_2 \sin\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + \theta\sqrt{C_1^2 + C_2^2}}\right)\right] \\ \qquad \times \mathrm{e}^{i[-\kappa x + \omega t + \tau]}, \\ \phi(x,t) = \left[\theta\sqrt{-\frac{\omega + a\kappa^2}{\Gamma_{31}}}\left(\frac{-C_1 \sin\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + C_2 \cos\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right)}{C_1 \cos\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + C_2 \sin\left(\hbar\sqrt{\frac{2(\omega+a\kappa^2)}{a}}\right) + \theta\sqrt{C_1^2 + C_2^2}}\right)\right] \\ \qquad \times \mathrm{e}^{i[-\kappa x + \omega t + \tau]}, \end{cases} \tag{131}$$

provided $\Gamma_{3s}(\omega + a\kappa^2) \prec 0$, $a(\omega + a\kappa^2) \succ 0$, $\Gamma_{2s} = 0$ and $\theta = \pm 1$.

Output 4.

$$A_0 = A_0, \quad A_1 = \theta\sqrt{-\frac{a}{2\Gamma_{3s}}}, \quad B_1 = 0, \quad \kappa = \theta\sqrt{\frac{-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2}{a}},$$
$$C_2 = \theta i C_1, \quad \lambda = -\frac{A_0^2}{A_1^2}, \quad \Lambda = 0, \tag{132}$$

provided $A_0 \neq 0$, $a\Gamma_{3s} \prec 0$, $a(-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2) \succ 0$ and $\theta = \pm 1$.

From (9), (101), (102), (115) and (132), we attain the following wave solutions of the periodic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[A_0\left\{1 + \theta i\left(\frac{-\sin\left(\hbar\sqrt{\frac{2\Gamma_{31}}{a}}A_0\right) + \theta i \cos\left(\hbar\sqrt{\frac{2\Gamma_{31}}{a}}A_0\right)}{\cos\left(\hbar\sqrt{\frac{2\Gamma_{31}}{a}}A_0\right) + \theta i \sin\left(\hbar\sqrt{\frac{2\Gamma_{31}}{a}}A_0\right)}\right)\right\}\right] \\ \qquad \times \mathrm{e}^{i[-\kappa x + (2\Gamma_{21}A_0 + 4\Gamma_{31}A_0^2 - a\kappa^2)t + \tau]}, \\ \phi(x,t) = \left[A_0\left\{1 + \theta i\left(\frac{-\sin\left(\hbar\sqrt{\frac{2\Gamma_{32}}{a}}A_0\right) + \theta i \cos\left(\hbar\sqrt{\frac{2\Gamma_{32}}{a}}A_0\right)}{\cos\left(\hbar\sqrt{\frac{2\Gamma_{32}}{a}}A_0\right) + \theta i \sin\left(\hbar\sqrt{\frac{2\Gamma_{32}}{a}}A_0\right)}\right)\right\}\right] \\ \qquad \times \mathrm{e}^{i[-\kappa x + (2\Gamma_{22}A_0 + 4\Gamma_{32}A_0^2 - a\kappa^2)t + \tau]}, \end{cases} \tag{133}$$

provided $A_0 \neq 0$, $a\Gamma_{3s} \succ 0$ and $\theta = \pm 1$.

Output 5.

$$A_0 = A_0, \quad A_1 = A_1, \quad B_1 = 0, \quad \kappa = \theta\sqrt{\frac{-\omega + 2\Gamma_{2s}A_0 + 4\Gamma_{3s}A_0^2}{a}},$$

$$C_2=\theta i C_1,\quad \lambda=-\frac{A_0^2}{A_1^2},\quad \Lambda=0, \tag{134}$$

provided $A_0\neq 0$, $A_1\neq 0$, $a(-\omega+2\Gamma_{2s}A_0+4\Gamma_{3s}A_0^2)\succ 0$ and $\theta=\pm1$.

From (9), (101), (102), (115) and (134), we obtain the following wave solutions of the periodic type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)=\left[A_0\left\{1+\theta i\left(\frac{-\sin\left(\hbar\frac{iA_0}{A_1}\right)+\theta i\cos\left(\hbar\frac{iA_0}{A_1}\right)}{\cos\left(\hbar\frac{iA_0}{A_1}\right)+\theta i\sin\left(\hbar\frac{iA_0}{A_1}\right)}\right)\right\}\right]\mathrm{e}^{i\left[-\kappa x+(2\Gamma_{21}A_0+4\Gamma_{31}A_0^2-a\kappa^2)t+\tau\right]},\\ \phi(x,t)=\left[A_0\left\{1+\theta i\left(\frac{-\sin\left(\hbar\frac{iA_0}{A_1}\right)+\theta i\cos\left(\hbar\frac{iA_0}{A_1}\right)}{\cos\left(\hbar\frac{iA_0}{A_1}\right)+\theta i\sin\left(\hbar\frac{iA_0}{A_1}\right)}\right)\right\}\right]\mathrm{e}^{i\left[-\kappa x+(2\Gamma_{22}A_0+4\Gamma_{32}A_0^2-a\kappa^2)t+\tau\right]},\end{cases} \tag{135}$$

provided $A_0\neq 0$, $A_1\neq 0$ and $\theta=\pm1$.

Type 3. If $\lambda=0$.
Here, we substitute (115) into eq. (113) and using eqs (6) and (12), setting all terms with the same order of $(1/\tau(\hbar))^j\left(\tau'(\hbar)/\tau(\hbar)\right)$ and $(1/\tau(\hbar))^i$ ($i=0,1,2,3,4$; $j=0,1,2,3$) to zero, we obtain nonlinear algebraic equations. With the help of Mathematica, the solutions of this resulting system are provided by the following outputs:

Output 1.

$$A_0=-\frac{\Gamma_{2s}}{2\Gamma_{3s}},\quad A_1=0,\quad B_1=\frac{6a\Lambda}{\Gamma_{2s}},$$
$$a=-\frac{\Gamma_{2s}^2(C_1^2-2\Lambda C_2)}{18\Lambda^2\Gamma_{3s}},$$
$$\kappa=\theta\sqrt{\frac{-2\omega+\Gamma_{2s}A_0}{2a}},\quad \Lambda=\Lambda, \tag{136}$$

provided $\Gamma_{2s}\neq 0$, $\Lambda\neq 0$, $a(-2\omega+\Gamma_{2s}A_0)\prec 0$ and $\theta=\pm1$.

From (11), (101), (102), (115) and (136), we obtain the following solutions of eqs (99) and (100):

$$\begin{cases}\psi(x,t)\\ \quad=\left[-\frac{\Gamma_{21}}{2\Gamma_{31}}\left(1+\frac{4(C_1^2-2\Lambda C_2)}{3\Lambda(\Lambda\hbar^2+2C_1\hbar+2C_2)}\right)\right]\\ \qquad\times\,\mathrm{e}^{i\left[-\kappa x+\left(\frac{-18\Lambda^2\Gamma_{21}^2+4\kappa^2\Gamma_{21}^2(C_1^2-2\Lambda C_2)}{72\Gamma_{31}\Lambda^2}\right)t+\tau\right]},\\ \phi(x,t)\\ \quad=\left[-\frac{\Gamma_{22}}{2\Gamma_{32}}\left(1+\frac{4(C_1^2-2\Lambda C_2)}{3\Lambda(\Lambda\hbar^2+2C_1\hbar+2C_2)}\right)\right]\\ \qquad\times\,\mathrm{e}^{i\left[-\kappa x+\left(\frac{-18\Lambda^2\Gamma_{22}^2+4\kappa^2\Gamma_{22}^2(C_1^2-2\Lambda C_2)}{72\Gamma_{32}\Lambda^2}\right)t+\tau\right]},\end{cases} \tag{137}$$

provided $\Gamma_{2s}\neq 0$ and $\Lambda\neq 0$.

Output 2.

$$A_0=0,\quad A_1=\theta\sqrt{-\frac{a}{2\Gamma_{3s}}},$$
$$B_1=\theta A_1\sqrt{C_1^2-2\Lambda C_2},\quad \Gamma_{2s}=0,$$
$$\kappa=\theta\sqrt{-\frac{\omega}{a}},\quad \Lambda=\Lambda, \tag{138}$$

provided $a\Gamma_{3s}\prec 0$ and $\theta=\pm1$.

From (11), (101), (102), (115) and (138), we attain the rational solution of eqs (99) and (100) as

$$\begin{cases}\psi(x,t)=\left[\theta\sqrt{-\frac{a}{2\Gamma_{31}}}\right.\\ \qquad\left.\times\left(\frac{2\left(\Lambda\hbar+C_1+\sqrt{C_1^2-2\Lambda C_2}\right)}{\Lambda\hbar^2+2C_1\hbar+2C_2}\right)\right]\\ \qquad\times\,\mathrm{e}^{i\left[-\kappa x+(-a\kappa^2)t+\tau\right]},\\ \phi(x,t)=\left[\theta\sqrt{-\frac{a}{2\Gamma_{32}}}\right.\\ \qquad\left.\times\left(\frac{2\left(\Lambda\hbar+C_1+\sqrt{C_1^2-2\Lambda C_2}\right)}{\Lambda\hbar^2+2C_1\hbar+2C_2}\right)\right]\\ \qquad\times\,\mathrm{e}^{i\left[-\kappa x+(-a\kappa^2)t+\tau\right]},\end{cases} \tag{139}$$

provided $a\Gamma_{3s}\prec 0$, $\omega=-a\kappa^2$, $\Gamma_{2s}=0$ and $\theta=\pm1$.

Output 3.

$$A_0=0,\quad A_1=0,\quad B_1=-\frac{3a\Lambda}{\Gamma_{2s}},\quad \kappa=\theta\sqrt{-\frac{\omega}{a}},$$
$$a=\frac{-2\Gamma_{2s}^2(C_1^2-2\Lambda C_2)}{9\Lambda^2\Gamma_{3s}},\quad \Lambda=\Lambda, \tag{140}$$

provided $a\omega\prec 0$, $\Gamma_{2s}\neq 0$, $\Lambda\neq 0$ and $\theta=\pm1$.

From (11), (101), (102), (115) and (140), we obtain the following solutions of the rational type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\dfrac{4\Gamma_{21}(C_1^2 - 2\Lambda C_2)}{3\Lambda\Gamma_{31}(\Lambda\hbar^2 + 2C_1\hbar + 2C_2)}\right] \\ \qquad \times\, \mathrm{e}^{i[-\kappa x + (-a\kappa^2)t + \tau]}, \\ \phi(x,t) = \left[\dfrac{4\Gamma_{22}(C_1^2 - 2\Lambda C_2)}{3\Lambda\Gamma_{32}(\Lambda\hbar^2 + 2C_1\hbar + 2C_2)}\right] \\ \qquad \times\, \mathrm{e}^{i[-\kappa x + (-a\kappa^2)t + \tau]}, \end{cases} \tag{141}$$

provided $\Gamma_{2s} \neq 0$, $\Lambda \neq 0$.

The procedures for taking C_1 and C_2 as particular values drawn in §3.3 can also be achieved to construct a further variety of other analytical solutions of eqs (99) and (100).

4.4 *Solutions of eq. (113) via the generalised sub-ODE method*

Now, balancing $H_s''(\hbar)$ with H_s^3 in eq. (113), by using (16), we get $n + 2q = 3n \Rightarrow n = q$. Hence, eq. (113) admits the formal solution

$$H_s(\hbar) = \Lambda\Psi(\hbar)^q. \tag{142}$$

Substituting (142) along with (14) into eq. (113) and equating all the coefficients of $\Psi(\hbar)^{jq}$ $(j = 0, 1, 2, 3)$ with zero, we get the following set of algebraic equations:

$$\Psi(\hbar)^0 : a\Lambda Bq^2 = 0, \tag{143a}$$

$$\Psi(\hbar)^q : a\Lambda Cq^2 - \Lambda(\omega + a\kappa^2) = 0, \tag{143b}$$

$$\Psi(\hbar)^{2q} : \frac{3}{2}a\Lambda Dq^2 + \Lambda^2\Gamma_{2s} = 0, \tag{143c}$$

$$\Psi(\hbar)^{3q} : 2a\Lambda Eq^2 + \Lambda^3\Gamma_{3s} = 0. \tag{143d}$$

On solving eqs (143a)–(143d), we get

$$\begin{aligned} &A = 0,\ B = 0, \\ &\Lambda = \theta q\sqrt{-\frac{2aE}{\Gamma_{3s}}}, \\ &C = \frac{(\omega + a\kappa^2)}{aq^2}, \\ &D = \frac{2\theta_1\Gamma_{2s}}{3q}\sqrt{-\frac{2E}{a\Gamma_{3s}}}, \end{aligned} \tag{144}$$

provided $aE\Gamma_{3s} \prec 0$, $\Gamma_{2s} \neq 0$, $\omega \neq -a\kappa^2$, $\theta_1 = \mp 1$ and $\theta = \pm 1$.

With regard to the solutions of eq. (14), we obtain the following types of the exact solutions of systems (99) and (100) as follows:

Type 1. Since $A = B = D = 0$, then the following cases arises:

Case I. Equation (14) has the solution

$$\Psi(\hbar) = \left[\theta\sqrt{-\frac{C}{E}}\,\mathrm{sech}(q\sqrt{C}\hbar)\right]^{\frac{1}{q}}, \quad C > 0,\ E < 0,\ \theta = \pm 1. \tag{145}$$

Then, we obtain the following solutions of the bright soliton type of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \left[\theta\sqrt{\dfrac{2(\omega + a\kappa^2)}{\Gamma_{31}}} \times \mathrm{sech}\left(\sqrt{\dfrac{\omega + a\kappa^2}{a}}(x + 2a\kappa t)\right)\right]\mathrm{e}^{i(-\kappa x + \omega t + \tau)}, \\ \phi(x,t) = \left[\theta\sqrt{\dfrac{2(\omega + a\kappa^2)}{\Gamma_{32}}} \times \mathrm{sech}\left(\sqrt{\dfrac{\omega + a\kappa^2}{a}}(x + 2a\kappa t)\right)\right]\mathrm{e}^{i(-\kappa x + \omega t + \tau)}, \end{cases} \tag{146}$$

provided $(\omega + a\kappa^2)\Gamma_{3s} \succ 0$, $a(\omega + a\kappa^2) \succ 0$, $\Gamma_{2s} = 0$ and $\theta = \pm 1$.

Case II. Equation (14) provides the following solution:

$$\Psi(\hbar) = \left[\theta\sqrt{-\frac{C}{E}}\,\sec(q\sqrt{-C}\hbar)\right]^{\frac{1}{q}}, \quad C < 0,\quad E > 0,\quad \theta = \pm 1. \tag{147}$$

Then, we obtain the following solutions of the periodic type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\theta\sqrt{\dfrac{2(\omega + a\kappa^2)}{\Gamma_{31}}} \times \sec\left(\sqrt{-\dfrac{(\omega + a\kappa^2)}{a}}(x + 2a\kappa t)\right)\right]\mathrm{e}^{i(-\kappa x + \omega t + \tau)}, \\ \phi(x,t) = \left[\theta\sqrt{\dfrac{2(\omega + a\kappa^2)}{\Gamma_{32}}} \times \sec\left(\sqrt{-\dfrac{(\omega + a\kappa^2)}{a}}(x + 2a\kappa t)\right)\right]\mathrm{e}^{i(-\kappa x + \omega t + \tau)}, \end{cases} \tag{148}$$

provided $(\omega + a\kappa^2)\Gamma_{3s} \succ 0$, $a(\omega + a\kappa^2) \prec 0$, $\Gamma_{2s} = 0$ and $\theta = \pm 1$.

Case III. Equation (14) provides the solution

$$\Psi(\hbar)=\left[\frac{\theta}{q\sqrt{E}\hbar}\right]^{\frac{1}{q}},\quad C=0,\quad E>0,\quad \theta=\pm 1. \tag{149}$$

Then, we obtain the following solutions of the rational type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)=\left[\theta\sqrt{-\frac{2a}{\Gamma_{31}}}\times\frac{1}{(x+2a\kappa t)}\right]\times \mathrm{e}^{i(-\kappa x-a\kappa^2 t+\tau)},\\ \phi(x,t)=\left[\theta\sqrt{-\frac{2a}{\Gamma_{32}}}\times\frac{1}{(x+2a\kappa t)}\right]\times \mathrm{e}^{i(-\kappa x-a\kappa^2 t+\tau)},\end{cases} \tag{150}$$

provided $a\Gamma_{3s}\prec 0$, $\Gamma_{2s}=0$ and $\theta=\pm 1$.

Type 2.

Case I: By using the conditions of $\Psi(\hbar)$ of eq. (73), we obtain the following solutions of Weierstrass elliptic function type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)\\ =\left[\theta\sqrt{\frac{-2[3a\wp(x+2a\kappa t,g_2,g_3)-(\omega+a\kappa^2)]}{3\Gamma_{31}}}\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\ \phi(x,t)\\ =\left[\theta\sqrt{\frac{-2[3a\wp(x+2a\kappa t,g_2,g_3)-(\omega+a\kappa^2)]}{3\Gamma_{32}}}\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{151}$$

provided

$$\Gamma_{3s}\prec 0,\quad g_2=\frac{4(\omega+a\kappa^2)^2}{3a^2},\quad g_3=-\frac{8(\omega+a\kappa^2)^3}{27a^3}$$

and

$\theta=\pm 1$.

Case II. Using the conditions of $\Psi(\hbar)$ of eq. (82), lead to the following solutions of Weierstrass elliptic function type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)\\ =\left[3\theta\sqrt{-\frac{2a^3}{\Gamma_{31}}}\frac{\wp(x+2a\kappa t,g_2,g_3)}{6a\wp(x+2a\kappa t,g_2,g_3)+\omega+a\kappa^2}\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\ \phi(x,t)\\ =\left[3\theta\sqrt{-\frac{2a^3}{\Gamma_{32}}}\frac{\wp(x+2a\kappa t,g_2,g_3)}{6a\wp(x+2a\kappa t,g_2,g_3)+\omega+a\kappa^2}\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{152}$$

provided

$$a\Gamma_{3s}\prec 0,\quad g_2=\frac{(\omega+a\kappa^2)^2}{12a^2},\quad g_3=-\frac{(\omega+a\kappa^2)^3}{216a^3}$$

and $\theta=\pm 1$.

Remark. Similarly, as drawn in §3.5, we can in particular rewrite the Weierstrass elliptic function solutions of (151) in terms of Jacobi elliptic functions and this is left to our readers.

Type 3. Since $B=E=0$, then, we obtain the following cases:

Case I: Equation (14) provides the following solution:

$$\Psi(\hbar)=\left[-\frac{C}{D}\operatorname{sech}^2\left(\frac{q\sqrt{C}}{2}\hbar\right)\right]^{\frac{1}{q}},\ C>0,\ D<0. \tag{153}$$

Since $A=0$, then, we obtain the following solutions of bright soliton type of eqs (99) and (100) as

$$\begin{cases}\psi(x,t)=\left[\frac{3(\omega+a\kappa^2)}{2\Gamma_{21}}\right.\\ \quad\left.\times\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{\omega+a\kappa^2}{a}}(x+2a\kappa t)\right)\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\ \phi(x,t)=\left[\frac{3(\omega+a\kappa^2)}{2\Gamma_{22}}\right.\\ \quad\left.\times\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{\omega+a\kappa^2}{a}}(x+2a\kappa t)\right)\right]\\ \quad\times \mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{154}$$

provided $a(\omega+a\kappa^2)\succ 0$, $\Gamma_{2s}\neq 0$ and $\Gamma_{3s}=0$.

Case II. Equation (14) provides the following solution:

$$\Psi(\hbar)=\left[-\frac{C}{D}\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)\right]^{\frac{1}{q}},\ C<0,\ D>0. \tag{155}$$

Since $A=0$, we obtain the following solutions of the periodic type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)=\left[\dfrac{3(\omega+a\kappa^2)}{2\Gamma_{21}}\right.\\ \quad\left.\times\sec^2\left(\dfrac{1}{2}\sqrt{-\dfrac{(\omega+a\kappa^2)}{a}}(x+2a\kappa t)\right)\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\ \phi(x,t)=\left[\dfrac{3(\omega+a\kappa^2)}{2\Gamma_{22}}\right.\\ \quad\left.\times\sec^2\left(\dfrac{1}{2}\sqrt{-\dfrac{(\omega+a\kappa^2)}{a}}(x+2a\kappa t)\right)\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{156}$$

provided $a(\omega+a\kappa^2)\prec 0$, $\Gamma_{2s}\neq 0$ and $\Gamma_{3s}=0$.

Type 4. Since $A=B=0$, then we attain the following cases:

Case I: Equation (14) has the solution

$$\Psi(\hbar)=\left[\frac{1}{2}\sqrt{\frac{C}{E}}\left(1+\theta\tanh\left(\frac{q\sqrt{C}}{2}\hbar\right)\right)\right]^{\frac{1}{q}},$$

$$C>0,\quad E>0,\quad D=-2\sqrt{CE},\quad \theta=\pm 1. \tag{157}$$

This yields the following solutions of the hyperbolic function type of eqs (99) and (100) as

$$\begin{cases}\psi(x,t)=\left[-\dfrac{\Gamma_{21}}{3\Gamma_{31}}\right.\\ \quad\left.\times\left\{1+\theta\tanh\left(\sqrt{\dfrac{-\Gamma_{21}^2}{18a\Gamma_{21}}}(x+2a\kappa t)\right)\right\}\right]\\ \quad\times\,\mathrm{e}^{i\left(-\kappa x+\left(-\frac{2\Gamma_{21}^2}{9\Gamma_{31}}-a\kappa^2\right)t+\tau\right)},\\ \phi(x,t)=\left[-\dfrac{\Gamma_{22}}{3\Gamma_{32}}\right.\\ \quad\left.\times\left\{1+\theta\tanh\left(\sqrt{\dfrac{-\Gamma_{22}^2}{18a\Gamma_{22}}}(x+2a\kappa t)\right)\right\}\right]\\ \quad\times\,\mathrm{e}^{i\left(-\kappa x+\left(-\frac{2\Gamma_{22}^2}{9\Gamma_{32}}-a\kappa^2\right)t+\tau\right)},\end{cases} \tag{158}$$

provided $a\Gamma_{3s}\prec 0$, $\Gamma_{2s}\neq 0$ and $\theta=\pm 1$.

Case II: Equation (14) provides the solution

$$\Psi(\hbar)=\left[\frac{\mathrm{sech}(q\sqrt{C}\hbar)}{1-\frac{D}{2C}\,\mathrm{sech}(q\sqrt{C}\hbar)}\right]^{\frac{1}{q}},$$

$$C>0,\quad D<2C,\quad E=\frac{D^2}{4C}-C. \tag{159}$$

This yields the soliton solutions of eqs (99) and (100) as

$$\begin{cases}\psi(x,t)=\left[\dfrac{-3\theta_1\sqrt{2}(\omega+a\kappa^2)\,\mathrm{sech}\left(\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar\right)}{\sqrt{\Xi}+\sqrt{2}\theta\Gamma_{21}\,\mathrm{sech}\left(\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar\right)}\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\ \phi(x,t)=\left[\dfrac{-3\theta_1\sqrt{2}(\omega+a\kappa^2)\,\mathrm{sech}\left(\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar\right)}{\sqrt{\Xi}+\sqrt{2}\theta\Gamma_{22}\,\mathrm{sech}\left(\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar\right)}\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{160}$$

provided $(\omega+a\kappa^2)a\succ 0$, $\Xi\succ 0$, $\theta_1=\mp 1$ and $\theta=\pm 1$, where $\Xi=2\Gamma_{2s}^2+9(\omega+a\kappa^2)\Gamma_{3s}$.

Case III. Equation (14) has the solution

$$\Psi(\hbar)=\left[\frac{4}{(q\hbar)^2-4E}\right]^{\frac{1}{q}},$$

$$C=0,\quad D=1,\quad E<0. \tag{161}$$

Then,

$$\begin{cases}\psi(x,t)=\left[-\dfrac{12a\Gamma_{21}}{2\Gamma_{21}^2(x+2a\kappa t)^2+9a\Gamma_{31}}\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+(-a\kappa^2)t+\tau)},\\ \phi(x,t)=\left[-\dfrac{12a\Gamma_{22}}{2\Gamma_{22}^2(x+2a\kappa t)^2+9a\Gamma_{32}}\right]\\ \quad\times\,\mathrm{e}^{i(-\kappa x+(-a\kappa^2)t+\tau)},\end{cases} \tag{162}$$

provided $\Gamma_{2s}\neq 0$.

Case IV. Equation (14) has the solution

$$\Psi(\hbar)=\left[\frac{2C\,\mathrm{sech}^2\left(\frac{q\sqrt{C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}+D)\,\mathrm{sech}^2\left(\frac{q\sqrt{C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$

$$C>0,\quad \Delta=D^2-4CE>0. \tag{163}$$

Thus, we obtain the following solutions of the soliton type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t)]\right)}{2\sqrt{\Xi}-\left(\sqrt{\Xi}+\theta\sqrt{2}\Gamma_{21}\right)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t)]\right)}{2\sqrt{\Xi}-\left(\sqrt{\Xi}+\theta\sqrt{2}\Gamma_{22}\right)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{164}$$

provided $a(\omega+a\kappa^2) \succ 0$, $\Xi \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega+a\kappa^2)\Gamma_{3s}$.

Case V. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{2C\,\mathrm{csch}^2\left(\frac{\theta q\sqrt{C}}{2}\hbar\right)}{2\sqrt{\Delta}+(\sqrt{\Delta}-D)\,\mathrm{csch}^2\left(\frac{q\sqrt{C}}{2}\hbar\right)}\right]^{\frac{1}{q}},$$

$$C>0,\quad \Delta = D^2-4CE>0,\quad \theta=\pm 1. \tag{165}$$

Thus, we obtain the following solutions of the soliton type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\mathrm{csch}^2\left(\frac{\theta}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\sqrt{\Xi}+\left(\sqrt{\Xi}-\theta\sqrt{2}\Gamma_{21}\right)\mathrm{csch}^2\left(\frac{\theta}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\mathrm{csch}^2\left(\frac{\theta}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\sqrt{\Xi}+\left(\sqrt{\Xi}-\theta\sqrt{2}\Gamma_{22}\right)\mathrm{csch}^2\left(\frac{\theta}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{166}$$

provided $a(\omega+a\kappa^2) \succ 0$, $\Xi \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega+a\kappa^2)\Gamma_{3s}$.

Case VI. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{2C\,\mathrm{sech}(q\sqrt{C}\hbar)}{\theta\sqrt{\Delta}-D\,\mathrm{sech}(q\sqrt{C}\hbar)}\right]^{\frac{1}{q}},$$

$$C>0,\quad \Delta = D^2-4CE>0,\quad \theta=\pm 1. \tag{167}$$

Therefore, we attain the solutions of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \\ \quad\left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\,\mathrm{sech}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{\sqrt{\Xi}-\sqrt{2}\theta\Gamma_{21}\,\mathrm{sech}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right] \\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \\ \quad\left[\dfrac{3\sqrt{2}\theta_1(\omega+a\kappa^2)\,\mathrm{sech}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{\sqrt{\Xi}-\sqrt{2}\theta\Gamma_{22}\,\mathrm{sech}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right] \\ \quad\times\,\mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{168}$$

provided $a(\omega+a\kappa^2) \succ 0$, $\Xi \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega+a\kappa^2)\Gamma_{3s}$.

Case VII. Equation (14) has the solution

$$\Psi(\hbar) = \left[-\frac{CD\,\mathrm{sech}^2(\frac{q\sqrt{C}}{2}\hbar)}{D^2-CE[1+\theta\tanh(\frac{q\sqrt{C}}{2}\hbar)]^2}\right]^{\frac{1}{q}},$$

$$C>0,\quad \theta=\pm 1. \tag{169}$$

Therefore, we obtain the following solutions of the soliton type of eqs (99) and (100):

$$\begin{cases} \psi(x,t) = \left[\dfrac{12(\omega + a\kappa^2)\Gamma_{21}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{8\Gamma_{21}^2 + 9(\omega+a\kappa^2)\Gamma_{31}\left(1+\theta\tanh\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)\right)^2} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{12(\omega + a\kappa^2)\Gamma_{22}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{8\Gamma_{22}^2 + 9(\omega+a\kappa^2)\Gamma_{32}\left(1+\theta\tanh\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)\right)^2} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{170}$$

provided $a(\omega + a\kappa^2) \succ 0$, $\Gamma_{2s} \neq 0$ and $\theta = \pm 1$.

Case VIII. Equation (14) provides the solution

$$\Psi(\hbar) = \left[\frac{CD\,\mathrm{csch}^2(\frac{q\sqrt{C}}{2}\hbar)}{D^2 - CE[1+\theta\coth(\frac{q\sqrt{C}}{2}\hbar)]^2} \right]^{\frac{1}{q}}, \quad C > 0, \quad \theta = \pm 1. \tag{171}$$

Therefore, we obtain the following solutions of the soliton type of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \left[\dfrac{-12(\omega + a\kappa^2)\Gamma_{21}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{8\Gamma_{21}^2 + 9(\omega+a\kappa^2)\Gamma_{31}\left(1+\theta\coth\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)^2} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{-12(\omega + a\kappa^2)\Gamma_{22}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{8\Gamma_{22}^2 + 9(\omega+a\kappa^2)\Gamma_{32}\left(1+\theta\coth\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)^2} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{172}$$

provided $a(\omega + a\kappa^2) \succ 0$, $\Gamma_{2s} \neq 0$ and $\theta = \pm 1$.

Case IX. Equation (14) provides the solution

$$\Psi(\hbar) = \left[\frac{2C\,\mathrm{csch}(q\sqrt{C}\hbar)}{\theta\sqrt{-\Delta} - D\,\mathrm{csch}(q\sqrt{C}\hbar)} \right]^{\frac{1}{q}}, \quad C > 0, \quad \Delta = D^2 - 4CE > 0, \quad \theta = \pm 1. \tag{173}$$

Thus, we attain the following solutions of eqs (99) and (100) as

$$\begin{cases} \psi(x,t) = \left[\dfrac{-3\sqrt{2}i(\omega + a\kappa^2)\,\mathrm{csch}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{\sqrt{\Xi} - i\sqrt{2}\Gamma_{21}\,\mathrm{csch}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{-3\sqrt{2}i(\omega + a\kappa^2)\,\mathrm{csch}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{\sqrt{\Xi} - i\sqrt{2}\Gamma_{22}\,\mathrm{csch}\left(\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{174}$$

provided $a(\omega+a\kappa^2)\succ 0$, $\Xi\succ 0$, where $\Xi=2\Gamma_{2s}^2+9(\omega+a\kappa^2)\Gamma_{3s}$ and $i=\sqrt{-1}$.

Case X. Equation (14) has the solution

$$\Psi(\hbar)=\left[-\frac{C\,\mathrm{sech}^2(\frac{q\sqrt{C}}{2}\hbar)}{D+2\theta\sqrt{CE}\tanh(\frac{q\sqrt{C}}{2}\hbar)}\right]^{\frac{1}{q}},\quad C>0,\quad E>0,\quad \theta=\pm 1. \tag{175}$$

Therefore, we obtain the following solutions of the soliton type of eqs (99) and (100) as

$$\begin{cases}\psi(x,t)=\left[\dfrac{3(\omega+a\kappa^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\Gamma_{21}+3\sqrt{-2(\omega+a\kappa^2)\Gamma_{31}}\tanh\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\[2ex] \phi(x,t)=\left[\dfrac{3(\omega+a\kappa^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\Gamma_{22}+3\sqrt{-2(\omega+a\kappa^2)\Gamma_{32}}\tanh\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{176}$$

provided $a(\omega+a\kappa^2)\succ 0$ and $(\omega+a\kappa^2)\Gamma_{3s}\prec 0$.

Case XI. Equation (14) has the solution

$$\Psi(\hbar)=\left[\frac{C\,\mathrm{csch}^2(\frac{q\sqrt{C}}{2}\hbar)}{D+2\theta\sqrt{CE}\coth(\frac{q\sqrt{C}}{2}\hbar)}\right]^{\frac{1}{q}},\quad C>0,\quad E>0,\quad \theta=\pm 1. \tag{177}$$

Therefore, we attain the following solutions of the soliton type of eqs (99) and (100):

$$\begin{cases}\psi(x,t)=\left[\dfrac{-3(\omega+a\kappa^2)\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\Gamma_{21}+3\sqrt{-2(\omega+a\kappa^2)\Gamma_{31}}\coth\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\\[2ex] \phi(x,t)=\left[\dfrac{-3(\omega+a\kappa^2)\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}{2\Gamma_{22}+3\sqrt{-2(\omega+a\kappa^2)\Gamma_{32}}\coth\left(\frac{1}{2}\sqrt{\frac{(\omega+a\kappa^2)}{a}}\,[x+2a\kappa t]\right)}\right]\mathrm{e}^{i(-\kappa x+\omega t+\tau)},\end{cases} \tag{178}$$

provided $a(\omega+a\kappa^2)\succ 0$ and $(\omega+a\kappa^2)\Gamma_{3s}\prec 0$.

Case XII. Equation (14) has the solutions for $C<0$, $\Delta=D^2-4CE>0$:

$$\Psi(\hbar)=\left[-\frac{2C\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}-D)\sec^2\left(\frac{q\sqrt{-C}}{2}\hbar\right)}\right]^{\frac{1}{q}}, \tag{179a}$$

$$\Psi(\hbar)=\left[\frac{2C\csc^2\left(\frac{\theta q\sqrt{-C}}{2}\hbar\right)}{2\sqrt{\Delta}-(\sqrt{\Delta}+D)\csc^2\left(\frac{\theta q\sqrt{-C}}{2}\hbar\right)}\right]^{\frac{1}{q}}, \tag{179b}$$

$$\Psi(\hbar)=\left[\frac{2C\sec(q\sqrt{-C}\hbar)}{\theta\sqrt{\Delta}-D\sec(q\sqrt{-C}\hbar)}\right]^{\frac{1}{q}}, \tag{179c}$$

$$\Psi(\hbar)=\left[\frac{2C\csc(q\sqrt{-C}\hbar)}{\theta\sqrt{\Delta}-D\csc(q\sqrt{-C}\hbar)}\right]^{\frac{1}{q}}. \tag{179d}$$

Therefore, we attain the solutions of eqs (99) and (100) in the following forms:

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\sec^2\left(\frac{1}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{2\sqrt{\Xi} - \left(\sqrt{\Xi} - \theta_1\sqrt{2}\Gamma_{21}\right)\sec^2\left(\frac{1}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\sec^2\left(\frac{1}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{2\sqrt{\Xi} - \left(\sqrt{\Xi} - \theta_1\sqrt{2}\Gamma_{22}\right)\sec^2\left(\frac{1}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{180}$$

provided $a(\omega + a\kappa^2) \prec 0$, $\Xi \succ 0$ and $\theta_1 = \mp 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega + a\kappa^2)\Gamma_{3s}$,

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\csc^2\left(\frac{\theta}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{2\sqrt{\Xi} - \left(\sqrt{\Xi} + \theta\sqrt{2}\Gamma_{21}\right)\csc^2\left(\frac{\theta}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\csc^2\left(\frac{\theta}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{2\sqrt{\Xi} - \left(\sqrt{\Xi} + \theta\sqrt{2}\Gamma_{22}\right)\csc^2\left(\frac{\theta}{2}\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{181}$$

provided $a(\omega + a\kappa^2) \prec 0$, $\Xi \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega + a\kappa^2)\Gamma_{3s}$,

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\sec\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{\sqrt{\Xi} - \sqrt{2}\theta\Gamma_{21}\sec\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\sec\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{\sqrt{\Xi} - \sqrt{2}\theta\Gamma_{22}\sec\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{182}$$

provided $a(\omega + a\kappa^2) \prec 0$, $\Xi \succ 0$, $\theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega + a\kappa^2)\Gamma_{3s}$,

$$\begin{cases} \psi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\csc\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{\sqrt{\Xi} - \sqrt{2}\theta\Gamma_{21}\csc\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) = \left[\dfrac{3\sqrt{2}\theta_1(\omega + a\kappa^2)\csc\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)}{\sqrt{\Xi} - \sqrt{2}\theta\Gamma_{22}\csc\left(\sqrt{-\frac{(\omega+a\kappa^2)}{a}}\,[x + 2a\kappa t]\right)} \right] \mathrm{e}^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{183}$$

provided $a(\omega+a\kappa^2) \prec 0, \Xi \succ 0, \theta_1 = \mp 1$ and $\theta = \pm 1$, where $\Xi = 2\Gamma_{2s}^2 + 9(\omega + a\kappa^2)\Gamma_{3s}$, respectively.

Case XIII. Equation (14) has the solution

$$\Psi(\hbar) = \left[\frac{4Cq^2 e^{\theta q\sqrt{C}\hbar}}{-1 + 4CEq^4 e^{2\theta q\sqrt{C}\hbar}}\right]^{\frac{1}{q}},$$
$$C > 0, \quad D = 0, \quad \theta = \pm 1. \tag{184}$$

Thus, the solutions of eqs (99) and (100) are in the following form:

$$\begin{cases} \psi(x,t) \\ = \left[\dfrac{4\theta q(\omega + a\kappa^2)\sqrt{-\frac{2aE}{\Gamma_{31}}}}{-a e^{-\theta\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar} + 4Eq^2(\omega + a\kappa^2)e^{\theta\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar}}\right], \\ \times e^{i(-\kappa x+\omega t+\tau)}, \\ \phi(x,t) \\ = \left[\dfrac{4\theta q(\omega + a\kappa^2)\sqrt{-\frac{2aE}{\Gamma_{32}}}}{-a e^{-\theta\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar} + 4Eq^2(\omega + a\kappa^2)e^{\theta\sqrt{\frac{\omega+a\kappa^2}{a}}\hbar}}\right], \\ \times e^{i(-\kappa x+\omega t+\tau)}, \end{cases} \tag{185}$$

provided $aE\Gamma_{3s} \prec 0$, $\Gamma_{2s} = 0$, $a(\omega + a\kappa^2) \succ 0$ and $\theta = \pm 1$.

5. Graphical results and discussion

In this section, to better understand the dynamical behaviour of the obtained results, we demonstrated the graphical configurations of some of them by providing appropriate values to the parameters. The generated kink wave, dark soliton, singular bright soliton, singular and doubly periodic wave solutions have been graphically illustrated in figures 1–11, using appropriate arbitrary parameters. The physical structure of new solutions demonstrates the usefulness and strength of these techniques.

In figure 1, the amplitude and intensity profiles of non-singular bright solitary wave solution (42) when $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$ are illustrated. In comparison to the dark solitons, bright solitons in optical fibres are relatively easy to generate and have a constant phase throughout the entire pulse.

In figure 2, the amplitude and intensity structures of singular solitary wave solution (45) with the selected parameters $\alpha_1 = 1.0, \omega = 1.0, \beta_1 = 2.0, \kappa = 1.0, \gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$ are depicted. It has been observed that the intensity profile takes the form of a singular bright soliton with selected values of parameters.

Figure 3 displays the amplitude and intensity structures of soliton solution (46) when $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$. It is seen that the amplitude profile of wave solution (46) is kink type (ascending from one state to another) and the intensity profile takes the form of a dark soliton to describe the localised intensity drops on a continuous wave background with selected parameters.

In figures 4 and 5, the evolution of the amplitude and intensity structures of soliton solution (48) for $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $S_1 = 1.10$, $S_2 = 1.0$, $\hbar_0 = 0$ and $\theta = \pm 1.0$ are depicted. It is noted from the figures that the amplitude structure of wave solution (46) is a breather-type solution for $\theta = 1.0$ and is of antikink type for $\theta = -1.0$.

Figure 6 illustrates the evolution of the amplitude and intensity structures of periodic wave solution (64) when $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = 1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

Evolution of the amplitude and intensity structures of singular periodic wave solution (89) when $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$ are depicted in figure 7.

Evolution of the amplitude structure of rogue wave solution (123) when $a = -1.0$, $\beta = 1.0$, $A_0 = 2.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\theta = 1.0$ are depicted in figure 8. It can be noted that from eq. (109), that the frequency κ of rogue wave solution (123) is half of its velocity when $a = -1.0$ and $\beta = 1.0$.

Figure 9 represents the evolution of the amplitude profile of rational wave solution (137) when $k_1 = -1.0$, $\beta = 1.0$, $C_1 = 2.0$, $C_2 = 1.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\Lambda = 1$. The movement of wave propagation is depicted at different positions.

Figure 10 represents the evolution of the amplitude and intensity of kink wave solution (158) when $k_1 = 1.0$, $\kappa = 1.0$, $a = -1.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\theta = 1$. It is noted from the figure that the amplitude structure of wave solution (158) is kink type while the intensity profile takes the form of antikink wave soliton. Also, solution (158) possesses some kind of symmetry for $\theta = \pm 1$ (figures not shown here). From symmetry, one can observe that $|\psi_{++}(x,t)| = |\psi_{--}(x,t)|$ represents the kink soliton and $|\psi_{+-}(x,t)| = |\psi_{-+}(x,t)|$ represents the antikink soliton.

Figure 11 represents the evolution of the amplitude of bright solution (176) when $k_1 = 1.0$, $\omega = 0.1$, $\kappa = 1.0$, $a = 1.0$ and $\ell_1 = m_1 = n_1 = -0.01$. It can be noted

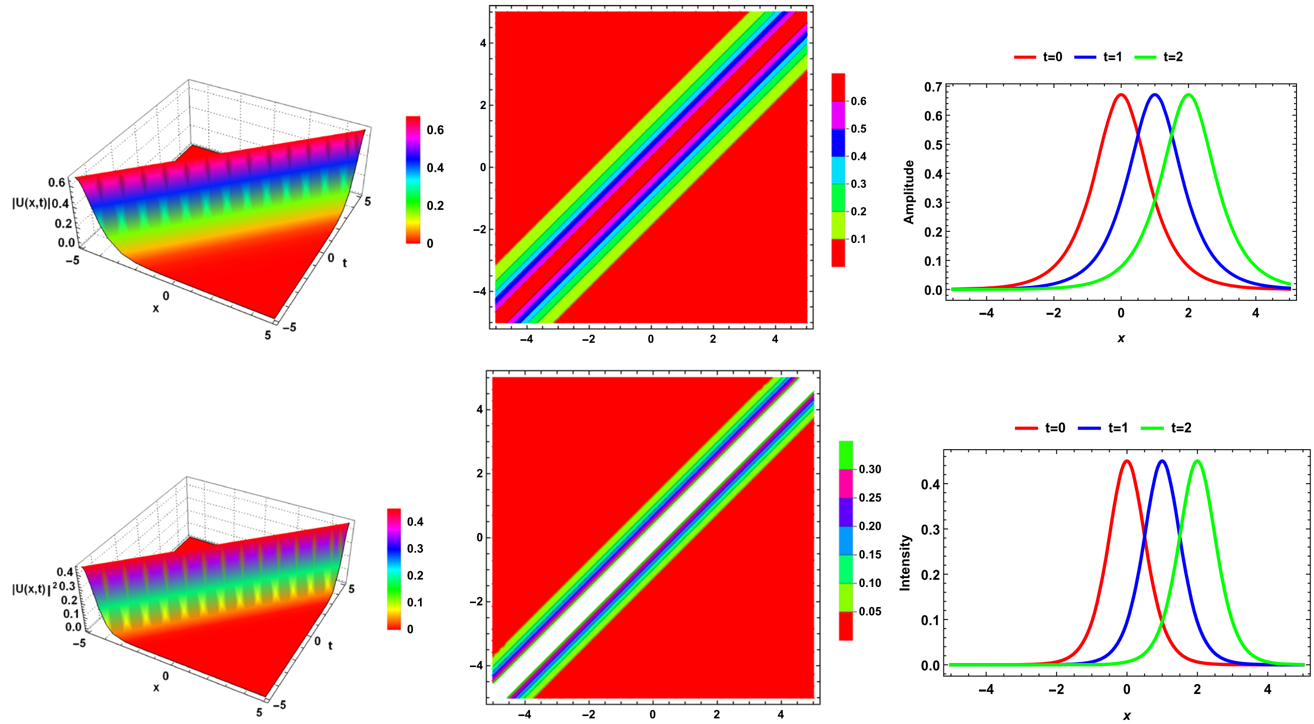

Figure 1. Evolution of the amplitude and intensity of solitary wave solution (42) with parameter values $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

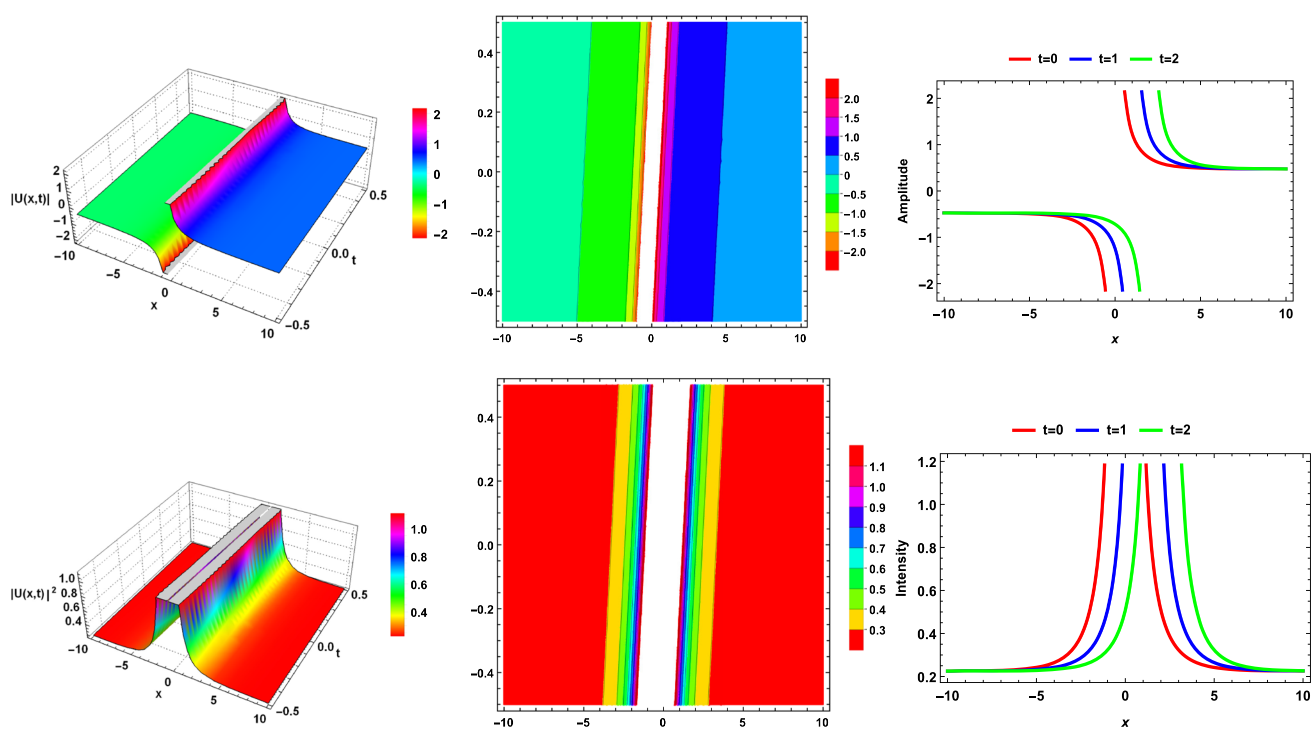

Figure 2. Evolution of the amplitude and intensity of the singular solitary wave solution (45) with parameter values $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

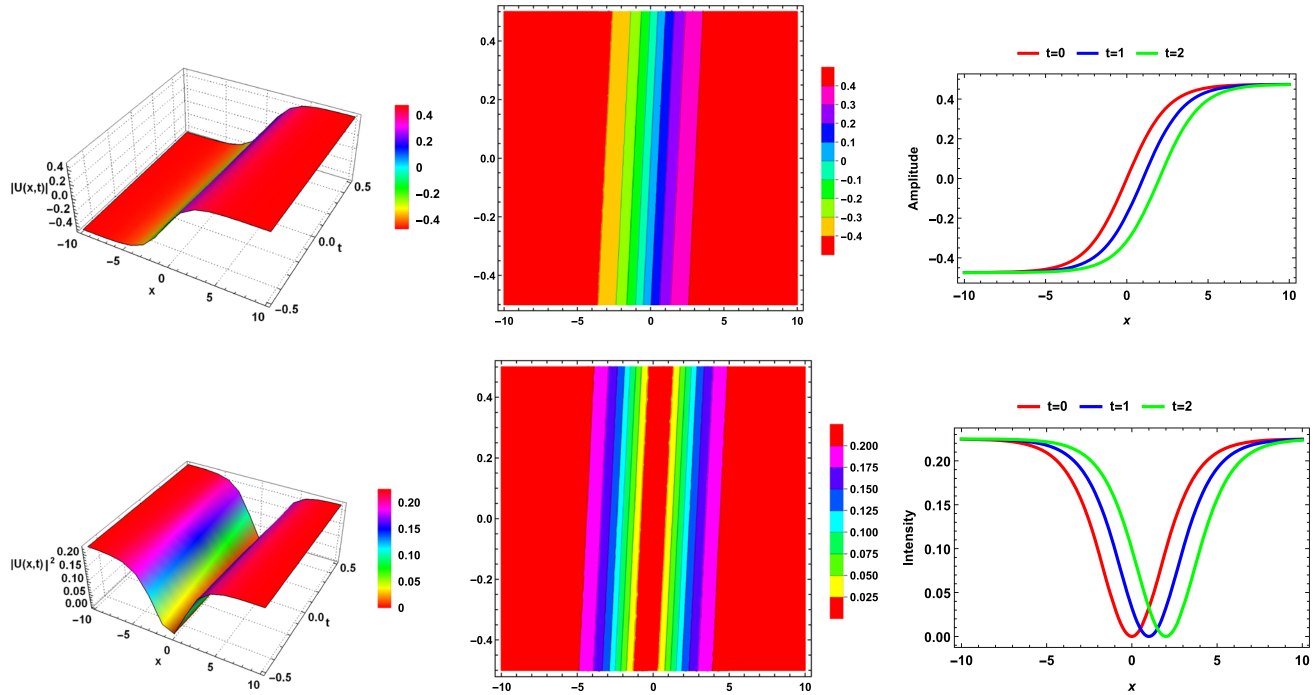

Figure 3. Evolution of the amplitude and intensity of solitary wave solution (46) with parameter values $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

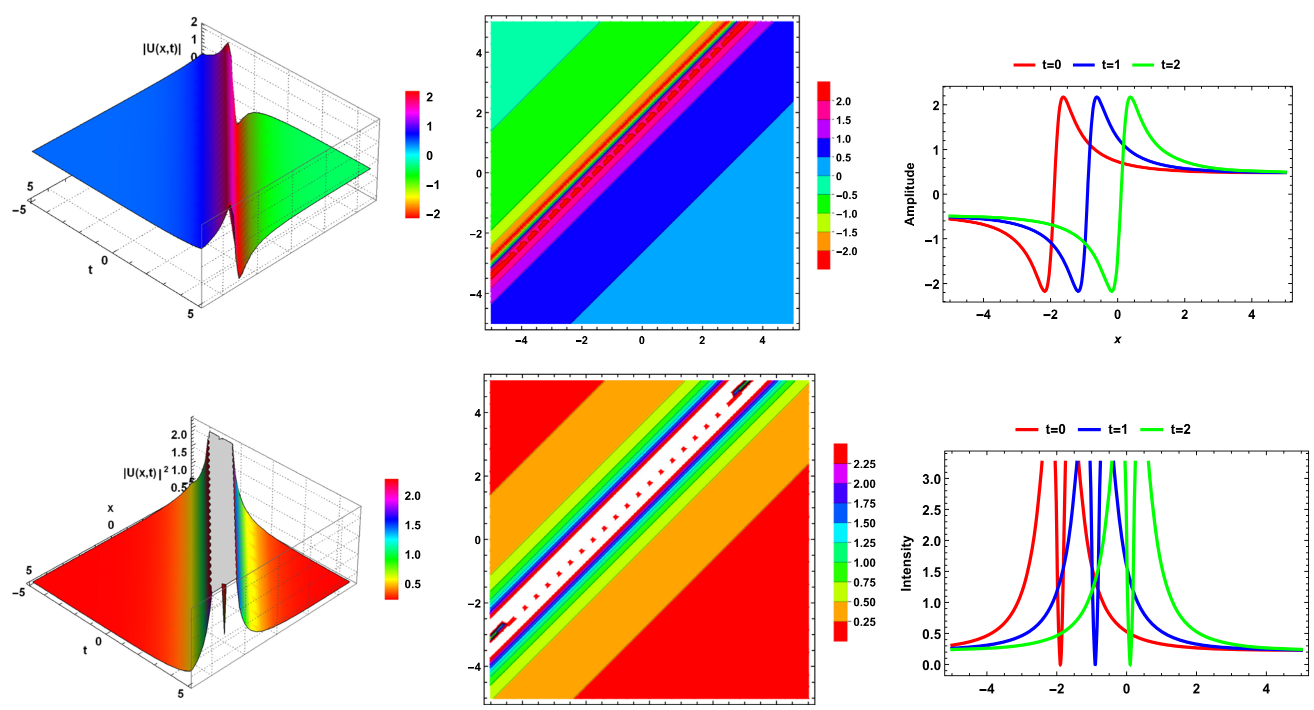

Figure 4. Evolution of the amplitude and intensity of solitary wave solution (48) with parameter values $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $S_1 = 1.10$, $S_2 = 1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

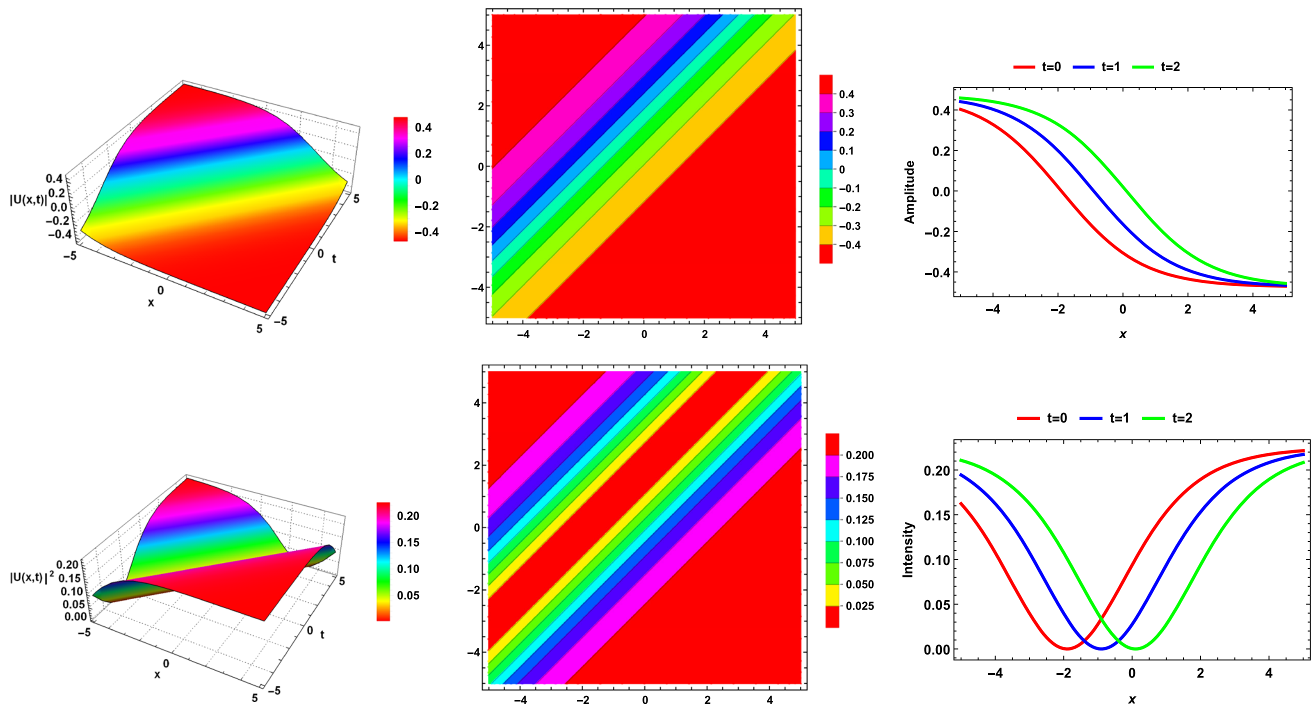

Figure 5. Evolution of the amplitude and intensity of solitary wave solution (48) with parameter values $\alpha_1 = 1.0$, $\omega = 1.0$, $\beta_1 = 2.0$, $\kappa = 1.0$, $\gamma_1 = 0.1$, $C_0 = 0.1$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $S_1 = 1.10$, $S_2 = 1.0$, $\hbar_0 = 0$ and $\theta = -1.0$.

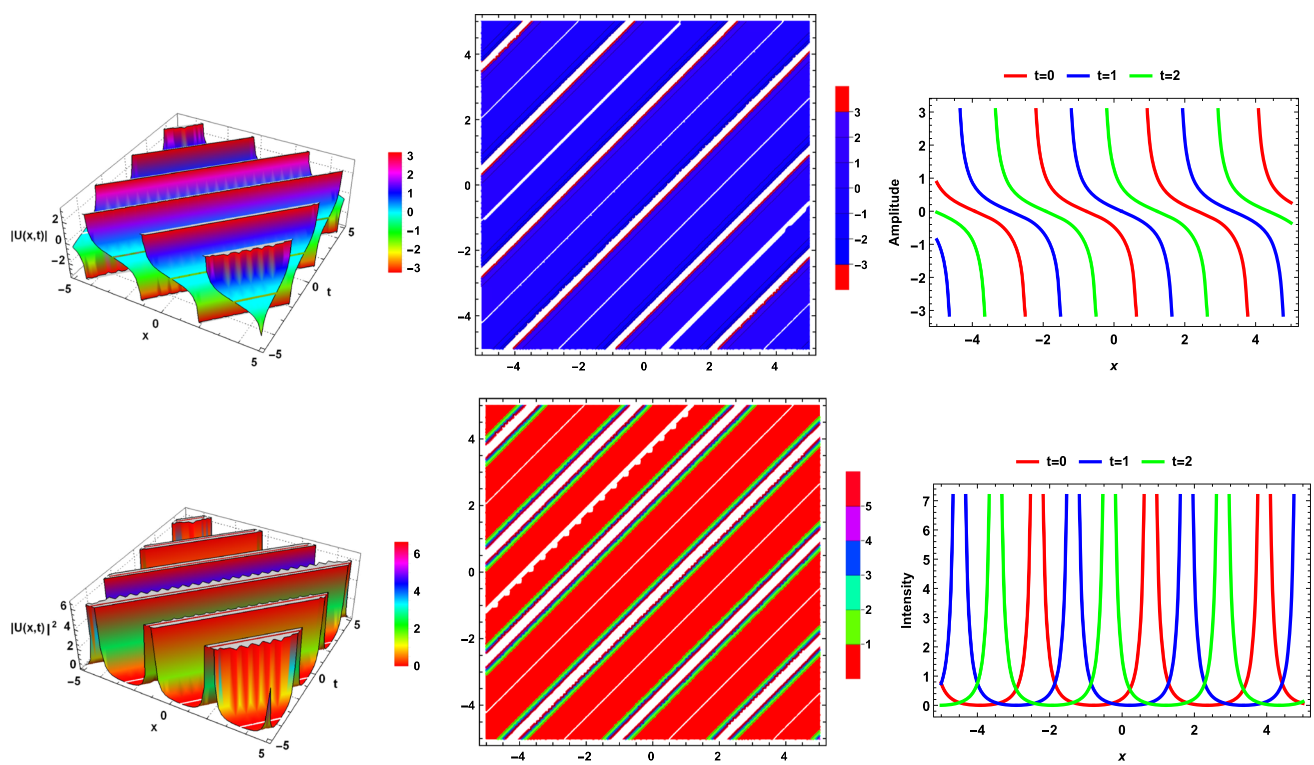

Figure 6. Evolution of the amplitude and intensity of periodic wave solution (64) with parameter values $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = 1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

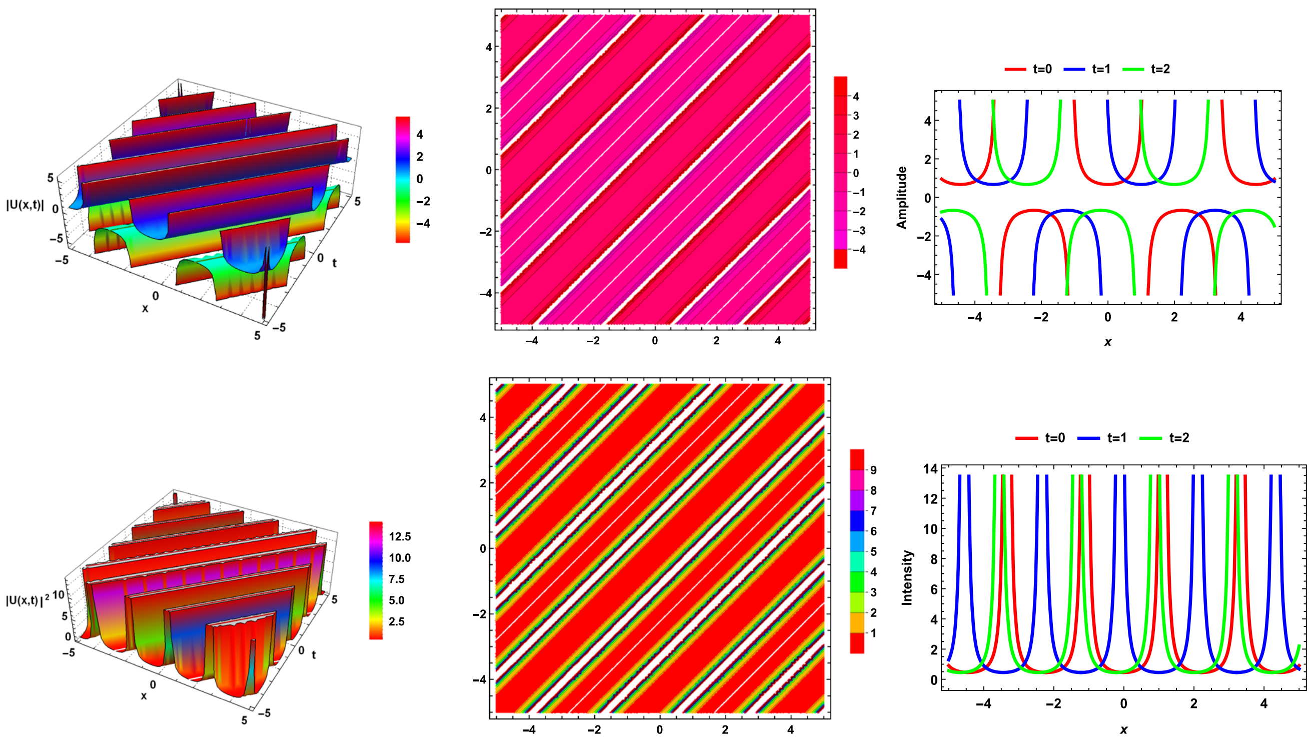

Figure 7. Evolution of the amplitude and intensity of periodic wave solution (89) with parameter values $\alpha_1 = 2.0$, $\omega = 1.0$, $\beta_1 = 1.0$, $\kappa = 1.0$, $\gamma_1 = 1.0$, $C_0 = 1.0$, $\Lambda_1 = 1.0$, $\delta_1 = \eta_1 = \sigma_1 = r_1 = -1.0$, $\hbar_0 = 0$ and $\theta = 1.0$.

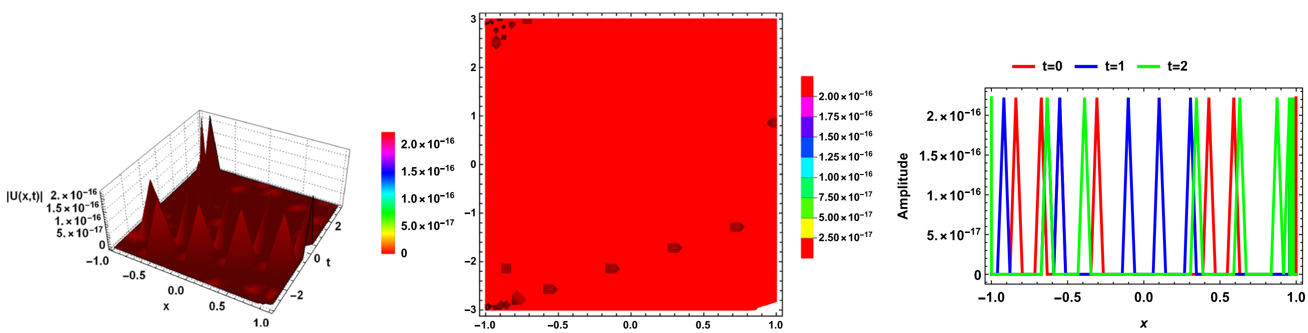

Figure 8. Evolution of the amplitude of rogue wave solution (123) with parameter values $a = -1.0$, $\beta = 1.0$, $A_0 = 2.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\theta = 1.0$.

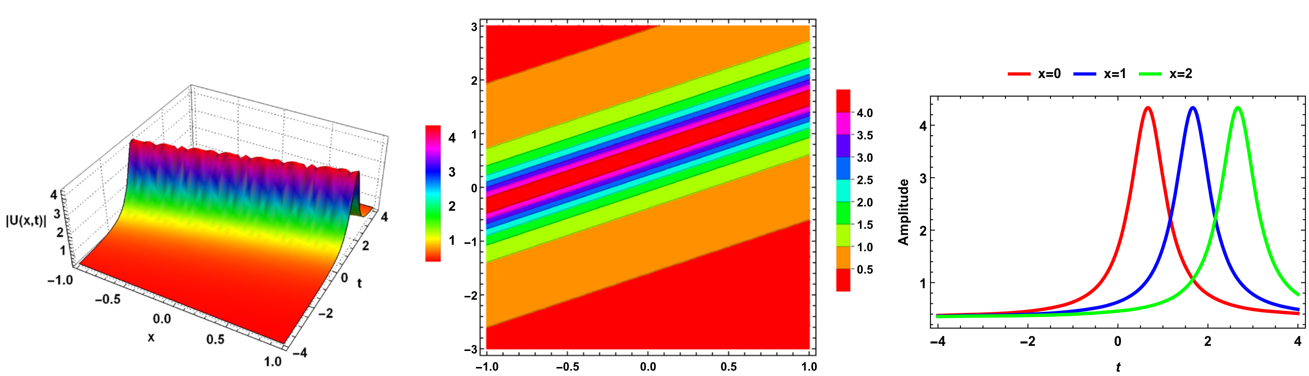

Figure 9. Evolution of the amplitude of rational wave solution (137) with parameter values $k_1 = -1.0$, $\beta = 1.0$, $C_1 = 2.0$, $C_2 = 1.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\Lambda = 1$.

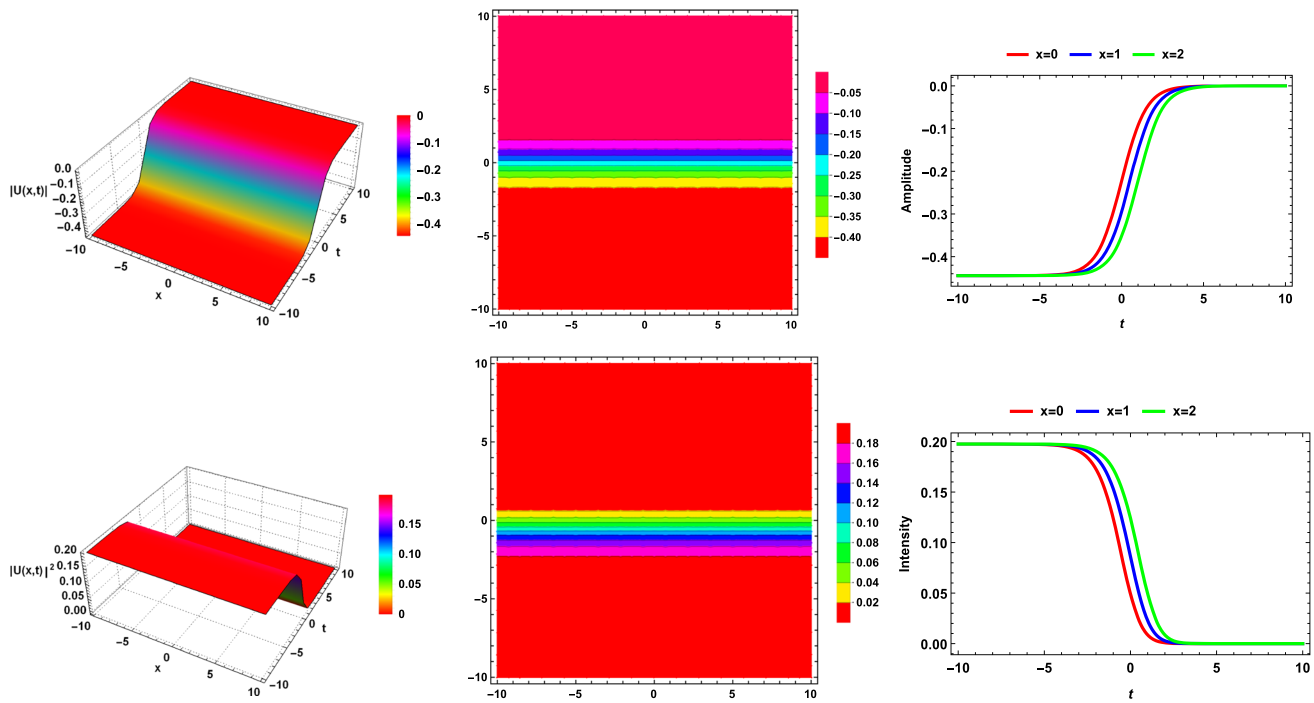

Figure 10. Evolution of the amplitude and intensity of kink wave solution (158) with parameter values $k_1 = 1.0$, $\kappa = 1.0$, $a = -1.0$, $\ell_1 = m_1 = n_1 = 1.0$ and $\theta = 1$.

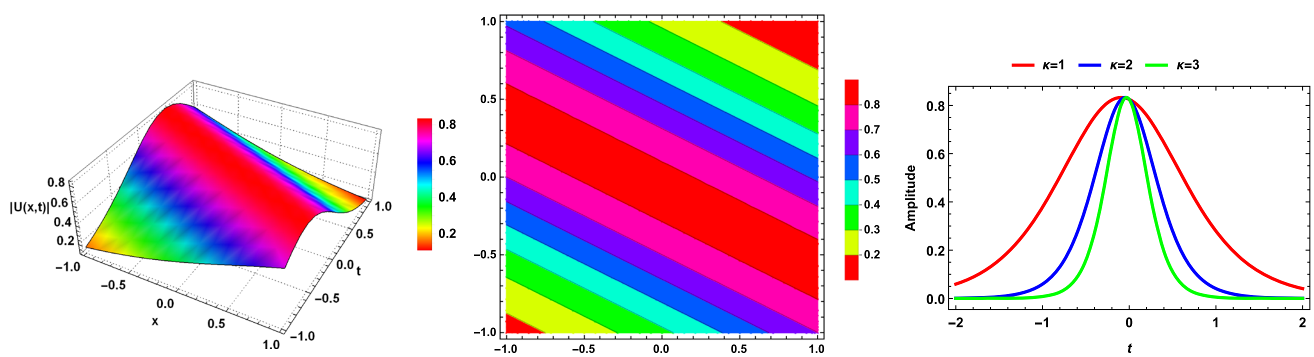

Figure 11. Evolution of the amplitude of bright solution (176) with parameter values $k_1 = 1.0$, $\omega = 0.1$, $\kappa = 1.0$, $a = 1.0$ and $\ell_1 = m_1 = n_1 = -0.01$.

from eq. (109) that when κ increases, wave speed β increases for a constant value of a. The amplitude profile of bright soliton (176) remains constant with increase in wave speed while there is a decrease in the width of the soliton as depicted in figure 11.

The graphical depiction shows the behaviour of the obtained solutions as a single soliton, periodic waves, kink waves, dark–bright solitons and breather waves. Solitons form as a result of cancelling nonlinear and dispersive processes in the medium. Soliton is a self-sustaining wave that advances at a constant speed while keeping its form. Soliton is formed in a variety of situations, including light propagation in fibres, energy transmission in hydrogen-bonded spines, physical plasma, stratified fluid flows, shallow water waves, and so on. The existence of arbitrary functions in the established results indicates that these solutions may be more effective and appropriate in describing the physical events at hand. The generated solutions are physically relevant and may be used to explain stability, nonlinear behaviour and dispersion effects in physics and mathematics fields. These acquired solutions can aid numerical solvers in certifying the accuracy of established solutions and allowing them to examine the behaviour graphically and physically. Furthermore, these explicit exact solutions may provide rich localised behaviours and provide us with a greater understanding of the underlying mechanisms of nonlinear complex systems. When considering the localisation-type appearances in solitons theory, periodic solutions have a highly confined formation with a temporally periodic structure. Rogue waves are dangerous events that occur in systems with a large number of waves. They are common in nature and can be seen in many situations such as nonlinear optics, water waves, liquid helium, microwave cavities and other areas.

6. Modulation instability (MI) analysis

Due to numerous applications such as in high-speed optical communications, time-resolved spectroscopy, ultrafast optical switching, super-continuum generation and so on [64–66], research into MI in optical fibre has stirred the curiosity of many researchers. MI is caused by the interaction of group velocity dispersion and nonlinear effects.

6.1 *MI of Biswas–Arshed model in birefringent fibres without FWM*

To investigate the MI, we begin with a steady-state solution of eqs (18) and (19) as

$$U(x,t)=a_1\mathrm{e}^{i\omega t},\quad V(x,t)=a_2\mathrm{e}^{i\omega t}. \tag{186}$$

Now, we determine whether or not this steady-state solution is stable in the presence of small perturbations. To accomplish this, the steady-state solution is perturbed in such a manner that

$$\begin{aligned}U(x,t)&=[a_1+\varphi_1(x,t)]\mathrm{e}^{i\omega t},\\ V(x,t)&=[a_2+\varphi_2(x,t)]\mathrm{e}^{i\omega t},\end{aligned} \tag{187}$$

where $\varphi_j(x,t)$, $j=1,2$, represent weak perturbations. Replacing eq. (187) into eqs (18) and (19) and subsequently linearising the resulting equations in φ_1 and φ_2, we get

$$\begin{aligned}&i\varphi_{1t}-\omega\varphi_1+(\alpha_1-\Lambda_1\omega)\varphi_{1xx}+\beta_1\varphi_{1xt}+i\gamma_1\varphi_{1xxx}\\ &\quad+i\Lambda_1\varphi_{1xxt}-2i\tau_1a_1^2\varphi_{2x}\\ &\quad-i[\delta_1a_1^2(\varphi_1^*+2\varphi_1)_x+\sigma_1a_1^2(\varphi_2^*+2\varphi_2)_x]\\ &\quad-i[(\eta_1+r_1+2\epsilon_1)a_1^2-\beta_1\omega]\varphi_{1x}=0,\end{aligned} \tag{188}$$

$$\begin{aligned}&i\varphi_{2t}-\omega\varphi_2+(\alpha_2-\Lambda_2\omega)\varphi_{2xx}+\beta_2\varphi_{2xt}+i\gamma_2\varphi_{2xxx}\\ &\quad+i\Lambda_2\varphi_{2xxt}-2i\tau_2a_2^2\varphi_{1x}\\ &\quad-i[\delta_2a_2^2(\varphi_2^*+2\varphi_2)_x+\sigma_2a_2^2(\varphi_1^*+2\varphi_1)_x]\\ &\quad-i[(\eta_2+r_2+2\epsilon_2)a_2^2-\beta_2\omega]\varphi_{2x}=0,\end{aligned} \tag{189}$$

where $*$ denotes the complex function's conjugate. We get perturbation of $\varphi_1(x,t)$ and one can have the MI analysis of the perturbation $\varphi_2(x,t)$ by using the same approach.

For solving the resulting linear equations, we assume the general solutions of the following form:

$$\begin{cases}\varphi_1(x,t)=u_1\mathrm{e}^{i(Kx-\Omega t)}+u_2\mathrm{e}^{-i(Kx-\Omega t)},\\ \varphi_2(x,t)=u_1\mathrm{e}^{i(Kx-\Omega t)}+u_2\mathrm{e}^{-i(Kx-\Omega t)},\end{cases} \tag{190}$$

where K denotes the normalised wave number and Ω specifies the frequency of perturbation. The two complex constants u_1 and u_2 satisfy the relation $|u_1|+|u_2|>0$. Substituting the assumed solutions (190) into eq. (188) and splitting the coefficients of $\mathrm{e}^{i(Kx-\Omega t)}$ and $\mathrm{e}^{-i(Kx-\Omega t)}$ provides the following dispersion relation:

$$\begin{aligned}&K^6\gamma_1^2-2K^5\Omega\Lambda_1\gamma_1+(A_1+\Lambda_1^2\Omega^2)K^4-2\Omega A_2K^3\\ &\quad+(A_3-(\beta_1^2+2\Lambda_1)\Omega^2)K^2\\ &\quad+2\Omega A_4K+\Omega^2-\omega^2=0,\end{aligned} \tag{191}$$

where

$$\begin{aligned}A_1&=2\gamma_1(2\tau_1+\eta_1+2\delta_1+2\epsilon_1+2\sigma_1+r_1)a_1^2\\ &\quad-\omega^2\Lambda_1^2+(2\alpha_1\Lambda_1-2\beta_1\gamma_1)\omega-\alpha_1^2,\\ A_2&=\Lambda_1(2\tau_1+\eta_1+2\delta_1+2\epsilon_1+2\sigma_1+r_1)a_1^2\\ &\quad-\gamma_1-\beta_1\alpha_1,\\ A_3&=(\eta_1+r_1+2\tau_1+\delta_1+\sigma_1+2\epsilon_1)(\eta_1+r_1\\ &\quad+2\tau_1+3\delta_1+3\sigma_1+2\epsilon_1)a_1^4\\ &\quad-2\omega\beta_1(2\tau_1+\eta_1+2\delta_1+2\epsilon_1+2\sigma_1+r_1)a_1^2\\ &\quad+(\beta_1^2+2\Lambda_1)\omega^2-2\omega\alpha_1,\\ A_4&=a_1^2\,(2\tau_1+\eta_1+2\delta_1+2\epsilon_1+2\sigma_1+r_1,).\end{aligned}$$

Solving the dispersion relation (191) for Ω, we get

$$\Omega(K)=\frac{K^5\Lambda_1\gamma_1-A_4K+A_2K^3-\sqrt{B_1K^8+B_2K^6+B_3K^4+B_4K^2+\omega^2}}{1+K^4\Lambda_1^2-\left(\beta_1^2+2\Lambda_1\right)K^2}, \tag{192}$$

where $B_1 = -\Lambda_1^2 A_1 + 2\gamma_1(\gamma_1 + A_2)\Lambda_1 + \beta_1^2\gamma_1^2$, $B_2 = -\Lambda_1^2 A_3 + (-2A_4\gamma_1 + 2A_1)\Lambda_1 + \beta_1^2 A_1 + A_2^2 - \gamma_1^2$, $B_3 = \Lambda_1^2\omega^2 - A_1 + \beta_1^2 A_3 - 2A_4A_2 + 2\Lambda_1 A_3$, $B_4 = \Lambda_1^2\omega^2 - A_1 + \beta_1^2 A_3 - 2A_4A_2 + 2\Lambda_1 A_3$.

From dispersion relation equation (192), it is revealed that $\Omega(K)$ is real for all K if $B_1K^8 + B_2K^6 + B_3K^4 + B_4K^2 + \omega^2 > 0$ and $1 + K^4\Lambda_1^2 - (\beta_1^2 + 2\Lambda_1)K^2 \neq 0$. Then, against small perturbations, the steady state is stable. In contrast, the steady-state solution is always unstable if $B_1K^8 + B_2K^6 + B_3K^4 + B_4K^2 + \omega^2 < 0$ and $1 + K^4\Lambda_1^2 - (\beta_1^2 + 2\Lambda_1)K^2 \neq 0$, which shows that the perturbation develops exponentially as $\Omega(K)$ contains the imaginary portion. Under this circumstance, for the occurrence of MI, it could be easily demonstrated that $B_1K^8 + B_2K^6 + B_3K^4 + B_4K^2 + \omega^2 < 0$ and $1 + K^4\Lambda_1^2 - (\beta_1^2 + 2\Lambda_1)K^2 \neq 0$. The rate of growth of the MI gives spectrum $g(K)$ as

$$g(K) = 2Im(K) = \frac{-2\sqrt{B_1K^8 + B_2K^6 + B_3K^4 + B_4K^2 + \omega^2}}{1 + K^4\Lambda_1^2 - (\beta_1^2 + 2\Lambda_1)K^2}. \quad (193)$$

The MI gain spectrum (200) is shown in figure 12 when $a_1 = 1.0, \tau_1 = 1.0, \eta_1 = 1.0, \delta_1 = 1.0, \epsilon_1 = 1.0, \sigma_1 = 1.0, r_1 = 1.0, \Lambda_1 = 1.0, \alpha_1 = 1.0, \beta_1 = 0.1, \gamma_1 = 1.0$ and $w = 1.0$.

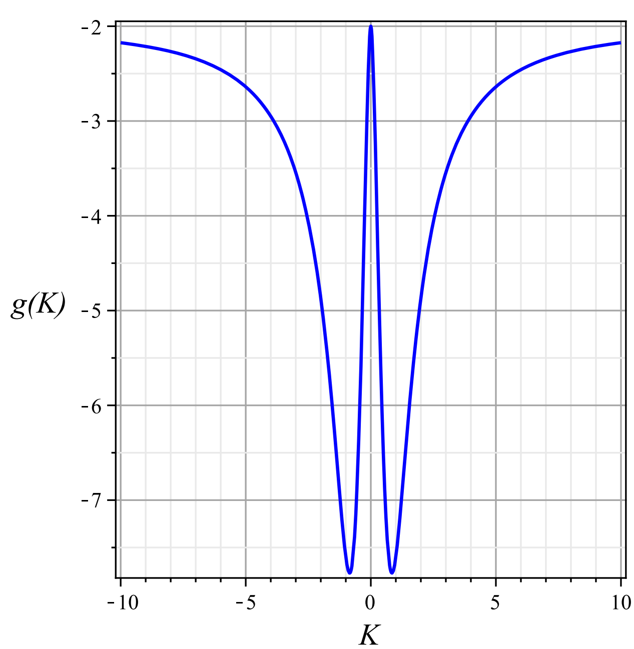

Figure 12. The MI gain spectrum (193) for parameter values $a_1 = 1.0, \tau_1 = 1.0, \eta_1 = 1.0, \delta_1 = 1.0, \epsilon_1 = 1.0, \sigma_1 = 1.0, r_1 = 1.0, \Lambda_1 = 1.0, \alpha_1 = 1.0, \beta_1 = 0.1, \gamma_1 = 1.0$ and $w = 1.0$.

6.2 *MI of NLSE with quadratic–cubic law of refractive index along with FWM*

Suppose that eqs (99) and (100) have the perturbed steady-state solutions in the following form:

$$\psi(x,t) = [b_1 + \chi_1(x,t)]e^{i\omega t},$$
$$\phi(x,t) = [b_2 + \chi_2(x,t)]e^{i\omega t}, \quad (194)$$

respectively, where $\chi_j(x,t)$, $j = 1, 2$, are weak perturbations. Substituting eq. (194) into eqs (99) and (100) and linearising the resulting equations in χ_1 and χ_2, we obtained the perturbation field as

$$i\chi_{1t} + [(m_1 + n_1)b_2^2 - \omega]\chi_1 + a_1\chi_{1xx} + [l_1b_1^2(\chi_1^* + 2\chi_1) + m_1b_1b_2(\chi_2^* + \chi_2)] + 2n_1b_1b_2\chi_2 = 0, \quad (195)$$

$$i\chi_{2t} + [(m_2 + n_2)b_1^2 - \omega]\chi_2 + a_2\chi_{2xx} + [l_2b_2^2(\chi_2^* + 2\chi_2) + m_2b_1b_2(\chi_1^* + \chi_1)] + 2n_2b_1b_2\chi_1 = 0. \quad (196)$$

We get perturbation of $\chi_1(x,t)$ and one can have the MI analysis of the perturbation $\chi_2(x,t)$ by using the same approach.

For solving these equations, consider the general solutions of the form

$$\begin{cases} \chi_1(x,t) = v_1 e^{i(Kx - \Omega t)} + v_2 e^{-i(Kx - \Omega t)}, \\ \chi_2(x,t) = v_1 e^{i(Kx - \Omega t)} + v_2 e^{-i(Kx - \Omega t)}, \end{cases} \quad (197)$$

where K denotes the normalised wave number and Ω represents the frequency of perturbation. The two complex constants v_1 and v_2 satisfy the relation $|v_1| + |v_2| > 0$. Replacing (197) into eq. (195) and decomposing the coefficients of $e^{i(Kx-\Omega t)}$ and $e^{-i(Kx-\Omega t)}$ provides the following dispersion relation:

$$\begin{aligned} &\Omega^2 - 2K^2a_1\omega - \omega^2 - (n_1 + m_1)^2 b_2^4 \\ &- 2b_1(m_1 + 2n_1)(n_1 + m_1)b_2^3 \\ &+ \left[\left(-m_1^2 - (4n_1 + 4l_1)m_1 - 4n_1(n_1 + l_1)\right)b_1^2 \right. \\ &\left. + 2(n_1 + m_1)\left(w + K^2a_1\right)\right]b_2^2 \\ &+ 2b_1\left[-\frac{3}{2}\left(\frac{8}{3}n_1 + m_1\right)l_1b_1^2 + \frac{1}{2}{m_1}^2 \right. \\ &\left. + \left(w + K^2a_1\right)m_1 + 2n_1\left(w + K^2a_1\right)\right]b_2 \\ &- 3b_1^4l_1^2 + 4\left(\frac{1}{4}m_1 + w + K^2a_1\right)l_1{b_1}^2 - K^4{a_1}^2 \\ &= 0. \end{aligned} \quad (198)$$

Solving dispersion relation (198) for Ω, we get

$$\Omega(K) = \pm\sqrt{K^4a_1^2 + C_1K^2 + C_2}, \quad (199)$$

where

$$C_1 = 2\left[(-n_1 - m_1) b_2^2 - b_1 (m_1 + 2n_1) b_2 - 2l_1 b_1^2 + \omega\right] a_1,$$

$$C_2 = (n_1 + m_1)^2 b_2^4 + 2b_1 (m_1 + 2n_1)(n_1 + m_1) b_2^3 + 3b_1^4 l_1^2 - 4l_1\left(\omega + \frac{1}{4}m_1\right) b_1^2 + \omega^2 + \left[\left({m_1}^2 + (4n_1 + 4l_1) m_1 + 4n_1 (n_1 + l_1)\right) b_1^2 - 2\omega (n_1 + m_1)\right] b_2^2 - 2\left[-\frac{3}{2}\left(\frac{8}{3}n_1 + m_1\right) l_1 b_1^2 + \omega m_1 + \frac{1}{2}m_1^2 + 2\omega n_1\right] b_1 b_2.$$

The dispersion relation equation (199) leads to $\Omega(K)$ being real for all K if $K^4 a_1^2 + C_1 K^2 + C_2 > 0$. Then, the steady state becomes stable against weak perturbations. In contrast, this becomes unstable if $K^4 a_1^2 + C_1 K^2 + C_2 < 0$, showing that $\Omega(K)$ is the imaginary part and this causes the perturbation to expand exponentially with intensity, resulting in MI. It is easy to see that MI occurs for $K^4 a_1^2 + C_1 K^2 + C_2 < 0$. The growth rate of the MI gain spectrum $g(K)$ is

$$g(K) = 2Im(K) = \pm 2\sqrt{K^4 a_1^2 + C_1 K^2 + C_2}. \quad (200)$$

The gain spectrum for MI frequency (200) is shown in figure 13 when $a_1 = 1.0, n_1 = 1.0, m_1 = 1.0, l_1 = -1.0, b_1 = 1.0, b_2 = -1.0$ and $w = 1.0$ with plus sign (blue curve) and minus sign (red curve) in the expression. It is evident from the above obtained dispersion relations that the MI gain spectrum depends on various coefficients of the considered model equations. Therefore, the effects of these coefficients on the MI gain spectrum require a separate deeper analysis and such detailed studies may be taken up in future.

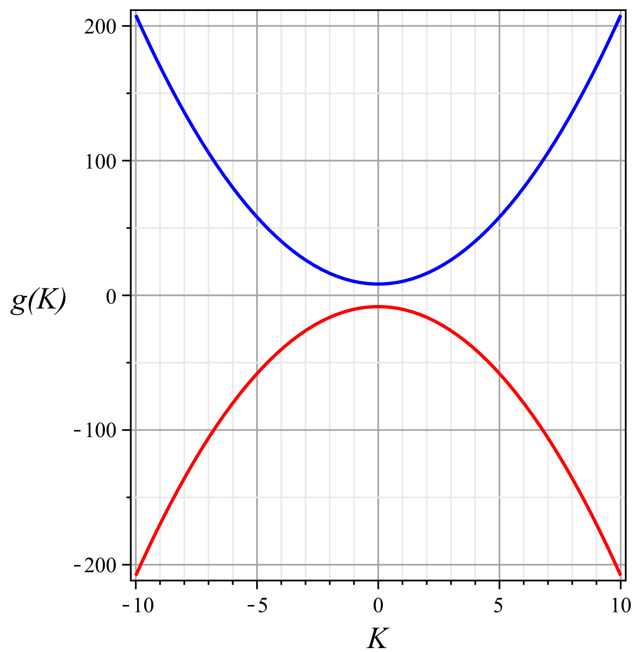

Figure 13. The MI gain spectrum (200) for parameter values $a_1 = 1.0, n_1 = 1.0, m_1 = 1.0, l_1 = -1.0, b_1 = 1.0, b_2 = -1.0$ and $w = 1.0$ with plus sign (blue curve) and minus sign (red curve) in the expression.

7. Conclusions

The extended simplest equation and the generalised sub-ODE methods have been successfully applied to extraction of optical soliton solutions and travelling wave solutions to the Biswas–Arshed model in birefringent fibres without FWM terms and the NLSE in birefringent fibres with quadratic cubic law of refractive index along with FWM. The obtained solutions include bright solitons, Weierstrass elliptic function solutions, hyperbolic function solutions, periodic function solutions and Jacobian elliptic function solutions. Those solutions should clarify some physical behaviours in birefringent fibres. Compared with the studies in [33–41], we see that our solutions are much broader and contain all the existing ones in the above literature and could be beneficial to explain the distinct physical behaviours. This comparison reflects the novelty of this paper and elucidate the difference between our results via the adopted approaches and the already published solutions. This, particularly, shows once more that the sub-ODE expansion idea is a very promising strategy for determining exact solutions for a wide range of nonlinear model equations, in birefringent fibres as studied in this paper. We will show in future studies that these approaches are also powerful in solving nonlinear models in polarisation-preserving fibres, Bragg gratings, highly dispersive solitons, photonic crystal fibres and DWDM technology with Kerr and non-Kerr media. Moreover, the linear stability analysis method has been used for exploring the MI of the obtained steady-state solutions to the two considered nonlinear models. Finally, Mathematica software is used to verify all the extracted outcomes by substituting them back into the proposed models.

References

[1] M Saha, A K Sarma and A Biswas, *Phys. Lett. A* **373**, 4438 (2009)

[2] A Biswas, *Prog. Electromagn. Res.* **96**, 1 (2009)
[3] Y Qiu, B A Malomed, D Mihalache, X Zhu, J Peng and Y He, *Phys. Lett. A* **383**, 2579 (2019)
[4] R Hirota, *J. Math. Phys.* **14**, 805 (1973)
[5] R Hirota and J Satsuma, *Phys. Lett. A* **85**, 407 (1981)
[6] Z Zhang, *Turk. J. Phys.* **32**, 235 (2008)
[7] Z Y Zhang *et al*, *Appl. Math. Comput.* **216**, 3064 (2010)
[8] Z Y Zhang *et al*, *Commun. Nonlinear Sci. Numer. Simul.* **16**, 3097 (2011)
[9] Z Y Zhang *et al*, *Z. Naturforschung.* **67**, 167 (2012)
[10] Z Y Zhang *et al*, *Phys. Lett. A* **375**, 1275 (2011)
[11] Z Y Zhang, X Y Gan and D M Yu, *Z. Naturforschung.* **66**, 721 (2011)
[12] A M Wazwaz, *Appl. Math. Comput.* **188**, 1467 (2007)
[13] S K Liu, Z T Fu, S D Liu and Q Zhao, *Phys. Lett. A* **289**, 69 (2001)
[14] S E Ganaini and M O Al-Amr, *Comput. Math. Appl.* **78**, 2094 (2019)
[15] J H He and X H Wu, *Chaos Solitons Fractals* **30**, 700 (2006)
[16] Y Zhou, M L Wang and Y M Wang, *Phys. Lett. A* **308**, 31 (2003)
[17] Y Z Peng, *Chin. J. Phys.* **41**, 103 (2003)
[18] E M E Zayed, R M A Shohib, M E M Alngar, A Biswas, M Asma, M Ekici, S P Moshokoa, A K Alzahrani and M R Belic, *Phys. Lett. A* **384**, 126697 (2020)
[19] Sirendaoreji, *Nonlinear Dyn.* **89**, 333 (2017)
[20] M O Al-Amr and S El-Ganaini, *Comput. Math. Appl.* **74**, 1274 (2017)
[21] H Kumar and S El-Ganaini, *Eur. Phys. J. Plus* **135**, 749 (2020)
[22] Z-L Li, *Z. Naturforsch.* **65a**, 649 (2010)
[23] Z Li-Hua and H Jin-Yu, *Commun. Theor. Phys.* **52**, 773 (2009)
[24] E M E Zayed and M E M Alngar, *Optik* **207**, 1643 (2020)
[25] G Xu, *Abstr. Appl. Anal.* **2014**, 541370 (2014)
[26] E M E Zayed and K A E Alurrfi, *Appl. Math. Comput.* **289**, 111 (2016)
[27] W X Ma and J H Lee, *Chaos Solitons Fractals* **42**, 1356 (2009)
[28] W X Ma, T W Huang and Y Zhang, *Phys. Scr.* **82**, 065003 (2010)
[29] W X Ma, *Opt. Quant. Electron.* **52**, 511 (2020)
[30] W X Ma, *Acta Math. Sci.* **42**, 127 (2022)
[31] W X Ma, *Physica D* **430**, 133078 (2022)
[32] W X Ma, *Partial Diff. Equ. Appl. Math.* **4**, 100190 (2021)
[33] S El Ganaini and H Kumar, *Chaos Solitons Fractals* **140**, 110218 (2020)
[34] S Kumar, S Malik, H Rezazadeh and L Akinyemi, *Nonlinear Dyn.* **107**, 2703 (2022)
[35] L Akinyemi, M Inc, M M A Khater and H Rezazadeh, *Opt. Quant. Elect.* **54**, 191 (2022)
[36] H Kumar and P Saravanan, *Sci. Iran. B* **24**, 2429 (2017)
[37] M O Al-Amr, H Rezazadeh, K K Ali and A Korkmazki, *Commun. Theor. Phys.* **72(6)**, 065503 (2020)
[38] H U Rehman, N Ullah and M A Imran, *Optik* **226**, 165378 (2021)
[39] A Malik, H Kumar, R P Chahal and F Chand, *Pramana – J. Phys.* **92**, 8 (2019)
[40] E M E Zayed and M E M Alngar, *Optik* **203**, 163922 (2020)
[41] A Biswas, M O Al-Amr, H Rezazadeh, M Mirzazadeh, M Eslami, Q Zhou, S P Moshokoa and M Belic, *Optik* **165**, 233 (2018)
[42] A Biswas and S Arshed, *Optik* **174**, 452 (2018)
[43] Y Yildirim, *Optik* **227**, 165960 (2021)
[44] M Tahir and A U Awan, *Optik* **207**, 164421 (2020)
[45] A Darwish, H M Ahmed, A H Arnous and M F Shehab, *Optik* **227**, 165385 (2021)
[46] A U Awan, M Tahir and H Ur-Rehman, *Optik* **210**, 164489 (2020)
[47] M Munawar, A Jhangeer, A Pervaiz and F Ibraheem, *Optik* **228**, 165790 (2021)
[48] Y Chen and Z Yan, *Chaos Solitons Fractals* **29**, 948 (2006)
[49] Y Yildirim, *Optik* **182**, 810 (2019)
[50] Y Yildirim, *Optik* **182**, 1149 (2019)
[51] M M A El-Sheikh, H M Ahmed, A H Arnous and W B Rabie, *Optik* **202**, 163546 (2020)
[52] H U Rehman, M Tahir, M Bibi and Z Ishfaq, *Optik* **218**, 164894 (2020)
[53] M Ozisik, A Secer and M Bayram, *Optik* **267**, 169650 (2022)
[54] M I Aszad, N Ullah, A Taskeen and F Jarad, *AIMS Math.* **7(b)**, 18603 (2022)
[55] M Tahir, A U Awan and H U Rehman, *Optik* **185**, 777 (2019)
[56] M Tahir and A U Awan, *Pramana – J. Phys.* **94**, 29 (2020)
[57] J F Gomez-Aguilar, M S Osman, N Raza, A Zubair, S Arshed, M E Goneim, E E Mahmoud and A H Abdel-Aty, *AIP Adv.* **11**, 025121 (2021)
[58] A Biswas, M Ekici, A Sonmezogluc and M Alfiras, *Optik* **178**, 178 (2019)
[59] M Asma, A Biswas, M Ekici, O Gonzalez-Gaxiola, A K Alzahrani and M R Belic, *Opt. Quantum Electron.* **53**, 138 (2021)
[60] A Biswas, M Ekici, A Sonmezoglu and M Belic, *Optik* **178**, 59 (2019)
[61] N Raza and A Zubair, *Commun. Theor. Phys.* **71**, 723 (2019)
[62] A Biswas, Y Yildirim, E Yasar, Q Zhou, S Khan, S Adesanya, S P Moshokoa and M Belic, *Optik* **179**, 927 (2019)
[63] L Wang, Z Luan, Q Zhou, A Biswas, A K Alzahrani and W Liu, *Nonlinear Dyn.* **104**, 2613 (2021)
[64] A Choudhuri and K Porsezian, *Phys. Rev. A* **85**, 033820 (2012)
[65] A Mohamadau and P H Tatsing, C G L Tiofack, C B Tabi and T C Kofane, *J. Mod. Opt.* **61**, 1670 (2014)
[66] N V Priya and M Senthilvelan, *Wave Motion* **54**, 125 (2015)